Some mass relations for mesons and baryons in Regge phenomenology

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In the quasilinear Regge trajectory ansatz, some useful linear mass inequalities, quadratic mass inequalities, and quadratic mass equalities are derived for mesons and baryons. Based on these relations, mass ranges of some mesons and baryons are given. The masses of $\bar{b}c$ and $s\bar{s}$ belonging to the pseudoscalar, vector, and tensor meson multiplets are also extracted. The J^P of the baryon $\Xi_{cc}^{+}(3520)$ is assigned to be $\frac{1}{2}^{+}$. The parameters of the $\frac{1}{2}^{+}$ and $\frac{3}{2}^{+}$ SU(4) baryon trajectories are extracted and the masses of the orbital excited baryons lying on the $\frac{1}{2}^{+}$ and $\frac{3}{2}^{+}$ trajectories are estimated. The J^P assignments of baryons $\Xi_c(2980)$, $\Xi_c(3055)$, $\Xi_c(3077)$, and $\Xi_c(3123)$ are discussed. The predictions are in reasonable agreement with the existing experimental data and those suggested in many other different approaches. The mass relations and the predictions may be useful for the discovery of the unobserved meson and baryon states and the J^P assignment of these states.

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I. INTRODUCTION

The study of hadronic physics has been a subject of intense interest. There are many hadronic states reported in recent years: B_2^* [1], B_{s2}^* [2], $\Xi_{cc}^{+}(3520)$ [3], $\Lambda_c^{+}(2880)$ [4–6], $\Lambda_c^{+}(2940)$ [5,6], $\Xi_c^{0,+}(2980, 3077)$ [7,8], $\Xi_c^{+}(3055, 3123)$ [9], $\Sigma_b^{(*)\pm}$ [10], and Ξ_b^{-} [11]. More and more states will be discovered in the near future. However, the properties of some states such as $\Xi_{cc}^{+}(3520)$ are still not very clear. $\Xi_{cc}^{+}(3520)$ was reported as the doubly charmed baryon state by SELEX in two different decay modes [3], but the J^P number has not been determined. Moreover, it has not been confirmed by other experiments (notably by *BABAR* [12], BELLE [13], and FOCUS [14]). According to the Particle Data Group's "Review of Particle Physics" in 2006 [15], many hadrons, especially heavy hadrons, are still absent from the summary tables. Obviously, there is still a lot of work to be done both theoretically and experimentally.

The eightfold way and the standard SU(3) Gell-Mann–Okubo (GMO) formula [16] have played an important role in the historical progress in particle physics. However, the direct generalization of the GMO formula to the charmed and bottom hadrons cannot agree well with experimental data due to higher-order breaking effects. Consequently, there are many works focused on the mass relations, including inequalities [17–20] and equalities [21–36].

Quantum chromodynamics (QCD) has been verified as an appropriate theory to describe strong interaction at short distances. However, the application of QCD to the processes of hadronic interactions at large distances is still limited by the unsolved confinement problem. Nowadays calculations of hadronic properties, which are related to the nonperturbative effects, are frequently carried out with the help of phenomenological models. Regge phenomenology (which was derived from the analysis of the properties of the scattering amplitude in the complex angular momentum plane [37]) is one of the simplest ones among these phenomenological models. Regge theory is concerned with almost all aspects of strong interactions, including the particle spectra, the forces between particles, and the high energy behavior of scattering amplitudes [38]. The quasilinear Regge trajectory ansatz, which is one of the most effective and popular approaches for studying hadron spectra, can (at least at present) give a reasonable description for the hadron spectroscopy [21–23,39,40], although some suggestions that the realistic Regge trajectories could be nonlinear exist [41].

As pointed out in Refs. [21,42], Regge intercepts and slopes are useful for many spectral and nonspectral purposes, for example, in the recombination [43] and fragmentation [44] models. Therefore, as pointed out in Ref. [45], the slopes and intercepts of the Regge trajectories are fundamental constants of hadron dynamics, perhaps in general more important than the masses of particular states. Thus, the determination of slopes and intercepts of hadrons is of great importance since this provides opportunities for a better understanding of the dynamics of strong interactions [42].

In the quasilinear Regge trajectory ansatz, the numerical values of the parameters of the Regge trajectories were extracted for mesons of different flavors [21,22,39,40,46]. Under the approximation that mesons or baryons in the

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light quark sector have the common Regge slopes, Burakovsky *et al.* derived two 6th power and one 14th power meson mass relations in Ref. [22] and derived some new quadratic Gell-Mann–Okubo–type baryon mass equalities in Ref. [23]. Using those new quadratic baryon mass relations they predicted the masses of $\frac{1}{2}^+$ and $\frac{3}{2}^+$ charmed baryon states absent from the baryon summary table then. (Here and below, $\frac{1}{2}^+$ and $\frac{3}{2}^+$ multiplets refer to the ground multiplets in which the total orbital angular momenta $L = 0$.) However, the numerical values for the parameters of the charmed baryon Regge trajectories were not given in Ref. [23].

In the present work, under the assumption that the quasilinear Regge trajectory ansatz is suitable to describe meson spectra and baryon spectra with the requirements of the additivity of intercepts and inverse slopes, the relations between slope ratios and masses of hadrons with different flavors and the mass relations among hadrons will be studied. We will show that the linear mass GMO formula is virtually an inequality and the quadratic mass GMO formula is also an inequality with the sign opposite to the linear case. We will get a high-power mass equation which is very useful to predict the masses of $\bar{b}c$ states and the masses of pure $s\bar{s}$ states. We will also get some useful quadratic mass equations for baryons. The J^P assignment of $\Xi_{cc}^+(3520)$, $\Xi_c(2980)$, $\Xi_c(3055)$, $\Xi_c(3077)$, and $\Xi_c(3123)$ baryons will be discussed. The numerical values for the parameters of the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories will be extracted and the masses of the baryon states lying on the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories will be estimated.

The remainder of this paper is organized as follows. In Sec. II we briefly introduce the quasilinear Regge trajectory ansatz. Then, we extract the mass inequalities and mass equalities for mesons and baryons. In Sec. III we present some applications of the relations derived in Sec. II and discuss the J^P assignment of $\Xi_{cc}^+(3520)$, $\Xi_c(2980)$, $\Xi_c(3055)$, $\Xi_c(3077)$, and $\Xi_c(3123)$ baryons. The parameters of the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories are extracted and the masses of the baryon states lying on the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories are estimated. Finally, we give a discussion and conclusion in Sec. IV.

II. FRAMEWORK

It is known from Regge theory that all mesons and baryons are associated with Regge poles which move in the complex angular momentum plane as a function of energy. The trajectory of a particular pole (Regge trajectory) is characterized by a set of internal quantum numbers (baryon number $\mathcal{B}$, intrinsic parity P, strangeness $\mathcal{S}$, charmness C, bottomness B, etc.) and by the evenness or oddness of the total spin J for mesons ($J - \frac{1}{2}$ for baryons) [47]. The plots of Regge trajectories of hadrons in the (J, M^2) plane are usually called Chew-Frautschi plots (where J and M are, respectively, the total spins and the masses of the hadrons). In Fig. 1, we draw the Chew-Frautschi plots for some meson and baryon Regge trajectories.

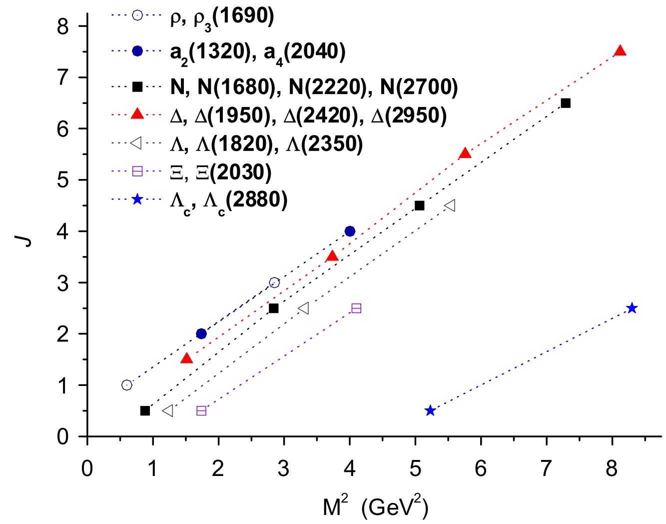

FIG. 1 (color online). Chew-Frautschi plots in the (J, M^2) plane for some mesons and baryons.

Assuming the existence of the quasilinear Regge trajectories for both light and heavy hadrons, one can have

$$J = \alpha(M) = a(0) + \alpha' M^2, \tag{1}$$

where $a(0)$ and α' are, respectively, the intercept and slope of the trajectory on which the particles lie. Hadrons lying on the same Regge trajectory which have the same internal quantum numbers are classified into the same family. The difference between the total spins of these hadrons is $2n$ ($n = 1, 2, 3, \ldots$), e.g., mesons with the quantum numbers $\mathcal{N}^{2S+1}L_J$, $\mathcal{N}^{2S+1}(L+2)_{J+2}$, $\mathcal{N}^{2S+1}(L+4)_{J+4}, \ldots$ (where $\mathcal{N}$, L, and S denote the radial excited quantum number, the orbital quantum number, and the intrinsic spin, respectively) lying on the same Regge trajectory. These features can be seen from the well-known Chew-Frautschi plots (Fig. 1).

For a meson multiplet with spin-parity J^P (more exactly speaking, with quantum numbers $\mathcal{N}^{2S+1}L_J$), the parameters for different quark constituents can be related by the following relations:

the additivity of intercepts [21,22,42,46,48–52],

$$a_{i\bar{i}}(0) + a_{j\bar{j}}(0) = 2a_{i\bar{j}}(0), \tag{2}$$

the additivity of inverse slopes [21,22,42,46],

$$\frac{1}{\alpha'_{i\bar{i}}} + \frac{1}{\alpha'_{j\bar{j}}} = \frac{2}{\alpha'_{i\bar{j}}}, \tag{3}$$

where i and j represent quark flavors. Equations (2) and (3) were derived in a model based on the topological expansion and the $q\bar{q}$-string picture of hadrons [46]. This model provides a microscopic approach to describe Regge phenomenology in terms of quark degrees of freedom [53]. In fact, Eq. (2) was first derived for light quarks in the dual-resonance model [48], and was found to be satisfied in two-dimensional QCD [49], the dual-analytic model [50], and the quark bremsstrahlung model [51]. Also, it saturates the inequality for Regge intercepts [52] which follows from the Schwarz inequality and the unitarity relation. The above two relations are usually generalized to the baryon case [23,42,51], in which one has

$$a_{iiq}(0) + a_{jjq}(0) = 2a_{ijq}(0), \tag{4}$$

$$\frac{1}{\alpha'_{iiq}} + \frac{1}{\alpha'_{jjq}} = \frac{2}{\alpha'_{ijq}}, \tag{5}$$

where q represents a quark.

There are also relations about the factorization of slopes for mesons [54,55] and baryons [55]:

$$\alpha'_{i\bar{i}} \cdot \alpha'_{j\bar{j}} = \alpha'^2_{i\bar{j}}, \tag{6}$$

$$\alpha'_{iiq} \cdot \alpha'_{jjq} = \alpha'^2_{ijq}, \tag{7}$$

which follow from the factorization of residues of the t-channel poles. The paper by Burakovsky and Goldman [42] showed that only the additivity of inverse Regge slopes is consistent with the formal chiral and heavy quark limits for both mesons and baryons, and that the factorization of Regge slopes, although consistent with the formal chiral limit, fails in the heavy quark limit. Besides, in Sec. III B, we will show that the high-power equation (63) derived from the relations (1), (2), and (6) is not as good as the high-power equation (16) derived from the relations (1)–(3) compared with the well-established meson multiplets. Therefore, we will use the relations (3) and (5) (the additivity of inverse slopes) rather than the relations (6) and (7) (the factorization of slopes) in this study. There are also studies about the relations between the ground state and its radial excited states [39,56,57] and there are suggestions that the radial excited states lie on daughter trajectories of the ground state [38]. However, we do not discuss these relations in the present work.

A. Relations between slope ratios and hadron masses

For mesons, using Eqs. (1) and (2), one obtains

$$\alpha'_{i\bar{i}}M^2_{i\bar{i}} + \alpha'_{j\bar{j}}M^2_{j\bar{j}} = 2\alpha'_{i\bar{j}}M^2_{i\bar{j}}, \tag{8}$$

where the meson states $i\bar{i}$, $j\bar{j}$, and $i\bar{j}$ belong to the same $\mathcal{N}^{2S+1}L_J$ multiplet. This relation can be reduced to the quadratic Gell-Mann–Okubo–type formula by assuming that all the slopes are independent of flavors ($\alpha'_{i\bar{i}} = \alpha'_{i\bar{j}} = \alpha'_{j\bar{j}}$). Combining the relations (3) and (8), one can get two pairs of solutions. The first pair of solutions are

$$\begin{cases} \frac{\alpha'_{j\bar{j}}}{\alpha'_{i\bar{i}}} = \frac{1}{2M^2_{j\bar{j}}} \times [(4M^2_{i\bar{j}} - M^2_{i\bar{i}} - M^2_{j\bar{j}}) + \sqrt{(4M^2_{i\bar{j}} - M^2_{i\bar{i}} - M^2_{j\bar{j}})^2 - 4M^2_{i\bar{i}}M^2_{j\bar{j}}}], \\ \frac{\alpha'_{i\bar{j}}}{\alpha'_{i\bar{i}}} = \frac{1}{4M^2_{i\bar{j}}} \times [(4M^2_{i\bar{j}} + M^2_{i\bar{i}} - M^2_{j\bar{j}}) + \sqrt{(4M^2_{i\bar{j}} - M^2_{i\bar{i}} - M^2_{j\bar{j}})^2 - 4M^2_{i\bar{i}}M^2_{j\bar{j}}}], \end{cases} \tag{9}$$

while the second pair of solutions are

$$\begin{cases} \frac{\alpha'_{j\bar{j}}}{\alpha'_{i\bar{i}}} = \frac{1}{2M^2_{j\bar{j}}} \times [(4M^2_{i\bar{j}} - M^2_{i\bar{i}} - M^2_{j\bar{j}}) - \sqrt{(4M^2_{i\bar{j}} - M^2_{i\bar{i}} - M^2_{j\bar{j}})^2 - 4M^2_{i\bar{i}}M^2_{j\bar{j}}}], \\ \frac{\alpha'_{i\bar{j}}}{\alpha'_{i\bar{i}}} = \frac{1}{4M^2_{i\bar{j}}} \times [(4M^2_{i\bar{j}} + M^2_{i\bar{i}} - M^2_{j\bar{j}}) - \sqrt{(4M^2_{i\bar{j}} - M^2_{i\bar{i}} - M^2_{j\bar{j}})^2 - 4M^2_{i\bar{i}}M^2_{j\bar{j}}}]. \end{cases} \tag{10}$$

From Eq. (1), one has

$$\alpha' = \frac{(J+2) - J}{M^2_{J+2} - M^2_J}. \tag{11}$$

It is obvious that the Regge slope α' should be a single positive real number. Thus, $\alpha'_{j\bar{j}}/\alpha'_{i\bar{i}}$ should take only one value for a multiplet with certain i and j. Since the relations (3) and (8) are symmetric under the exchange of the quark flavors i and j, we only consider the case in which quark masses satisfy $m_i < m_j$ for mesons here and after.

From Eqs. (9) and (10), we have the values of $\alpha'_{c\bar{c}}/\alpha'_{n\bar{n}}$ and $\alpha'_{b\bar{b}}/\alpha'_{n\bar{n}}$ (n denotes u or d quark) for the well-established multiplets. In the calculation, we do not consider the small mass splittings caused by isospin breaking effects due to electromagnetic interaction. Here and below, all the masses of hadrons used in calculation are taken from PDG2006 [15] except for the newly observed hadrons. The results are shown in Table I.

TABLE I. The values of $\alpha'_{c\bar{c}}/\alpha'_{n\bar{n}}$ and $\alpha'_{b\bar{b}}/\alpha'_{n\bar{n}}$($n$ denotes u or d quark) obtained from Eqs. (9) and (10).

	$\mathcal{N}^{2S+1}L_J$	(9)	(10)
$\alpha'_{c\bar{c}}/\alpha'_{n\bar{n}}$	$1\,^1S_0$	0.5636	0.0038
	$1\,^1P_1$	0.5433	0.2238
	$1\,^3S_1$	0.4921	0.1274
	$1\,^3P_2$	0.5041	0.2726
$\alpha'_{b\bar{b}}/\alpha'_{n\bar{n}}$	$1\,^1S_0$	0.2880	0.0008
	$1\,^3S_1$	0.2361	0.0290
	$1\,^3P_2$	0.2562	0.0690

The values of $\alpha'_{n\bar{n}}$ for light nonstrange meson trajectories of different multiplets are in the range 0.7–0.9 GeV^{-2} [21,22,39,46]. The values of $\alpha'_{c\bar{c}}$ and $\alpha'_{b\bar{b}}$ for charmonium and bottomonium trajectories of different multiplets are in the ranges 0.3–0.5 GeV^{-2} and 0.18–0.25 GeV^{-2}, respectively [21,22,46,57]. Then, we have $\alpha'_{c\bar{c}}/\alpha'_{n\bar{n}} \sim 0.5$ and $\alpha'_{b\bar{b}}/\alpha'_{n\bar{n}} \sim 0.27$. From Table I, one can see that the values of $\alpha'_{c\bar{c}}/\alpha'_{n\bar{n}}$ $(\alpha'_{b\bar{b}}/\alpha'_{n\bar{n}})$ given by Eq. (9) are approximately the same for different multiplets as they should be. However, the values of $\alpha'_{c\bar{c}}/\alpha'_{n\bar{n}}$ $(\alpha'_{b\bar{b}}/\alpha'_{n\bar{n}})$ given by Eq. (10) are quite different for different multiplets. Furthermore, the values of $\alpha'_{c\bar{c}}/\alpha'_{n\bar{n}}$ and $\alpha'_{b\bar{b}}/\alpha'_{n\bar{n}}$ given by Eq. (10) are too small to be accepted. Therefore, we take the first pair of solutions [Eq. (9)] and discard the second pair of solutions [Eq. (10)].

For baryons, using Eqs. (1) and (4), one obtains

$$\alpha'_{iiq}M^2_{iiq} + \alpha'_{jjq}M^2_{jjq} = 2\alpha'_{ijq}M^2_{ijq}, \tag{12}$$

where q denotes an arbitrary light or heavy quark. Combining the relations (5) and (12), one can get two pairs of solutions,

$$\begin{cases} \frac{\alpha'_{jjq}}{\alpha'_{iiq}} = \frac{1}{2M^2_{jjq}} \times [(4M^2_{ijq} - M^2_{iiq} - M^2_{jjq}) + \sqrt{(4M^2_{ijq} - M^2_{iiq} - M^2_{jjq})^2 - 4M^2_{iiq}M^2_{jjq}}], \\ \frac{\alpha'_{ijq}}{\alpha'_{iiq}} = \frac{1}{4M^2_{ijq}} \times [(4M^2_{ijq} + M^2_{iiq} - M^2_{jjq}) + \sqrt{(4M^2_{ijq} - M^2_{iiq} - M^2_{jjq})^2 - 4M^2_{iiq}M^2_{jjq}}], \end{cases} \tag{13}$$

and

$$\begin{cases} \frac{\alpha'_{jjq}}{\alpha'_{iiq}} = \frac{1}{2M^2_{jjq}} \times [(4M^2_{ijq} - M^2_{iiq} - M^2_{jjq}) - \sqrt{(4M^2_{ijq} - M^2_{iiq} - M^2_{jjq})^2 - 4M^2_{iiq}M^2_{jjq}}], \\ \frac{\alpha'_{ijq}}{\alpha'_{iiq}} = \frac{1}{4M^2_{ijq}} \times [(4M^2_{ijq} + M^2_{iiq} - M^2_{jjq}) - \sqrt{(4M^2_{ijq} - M^2_{iiq} - M^2_{jjq})^2 - 4M^2_{iiq}M^2_{jjq}}]. \end{cases} \tag{14}$$

From the Chew-Frautschi plots (Fig. 1), it is obvious that the Regge slope α' should be a single positive real number. Thus, $\alpha'_{jjq}/\alpha'_{iiq}$ should take only one value for a multiplet with certain i, j, and q. Since the relations (5) and (12) are symmetric under the exchange of the quark flavors i and j, we only consider the case in which quark masses satisfy $m_i < m_j$ for baryons here and after.

For the $\frac{1}{2}^+$ multiplet, when $i = n$, $j = s$, and $q = n$, we have $M_{nnn} = M_{N(939)}$, $M_{nss} = M_{\Xi}$, and $M^2_{nns} = \frac{1}{4}(3M^2_{\Lambda} + M^2_{\Sigma})$ [23]. Then, we have $\alpha'_{\Xi}/\alpha'_{N} = 0.89$ from Eq. (13) and $\alpha'_{\Xi}/\alpha'_{N} = 0.57$ from Eq. (14). For the $\frac{3}{2}^+$ multiplet, when $i = n$, $j = s$, and $q = n$, we have $M_{nnn} = M_{\Delta}$, $M_{nns} = M_{\Sigma^*}$, and $M_{nss} = M_{\Xi^*}$. Then, we have $\alpha'_{\Xi^*}/\alpha'_{\Delta} = 0.89$ from Eq. (13) and $\alpha'_{\Xi^*}/\alpha'_{\Delta} = 0.72$ from Eq. (14). Since the Regge trajectories of light baryons are approximately parallel, the values of $\alpha'_{\Xi}/\alpha'_{N}$ and $\alpha'_{\Xi^*}/\alpha'_{\Delta}$ should be close to 1. Therefore, Eqs. (14) should be discarded in the case of quark masses $m_i < m_j$. Furthermore, Eqs. (13) and (14) can be considered as the generalization of Eqs. (9) and (10), respectively, from the meson case to the baryon case. Therefore, we take Eq. (13) and discard Eq. (14).

B. High-power mass equalities

From Eqs. (9) and (13), high-power mass equalities can be derived for mesons and baryons, respectively. For mesons, using

$$\frac{\alpha'_{j\bar{j}}}{\alpha'_{i\bar{i}}} = \frac{\alpha'_{k\bar{k}}}{\alpha'_{i\bar{i}}} \times \frac{\alpha'_{j\bar{j}}}{\alpha'_{k\bar{k}}}, \tag{15}$$

and Eq. (9), when $m_i < m_j < m_k$, we have

$$\frac{(4M^2_{i\bar{j}} - M^2_{i\bar{i}} - M^2_{j\bar{j}}) + \sqrt{(4M^2_{i\bar{j}} - M^2_{i\bar{i}} - M^2_{j\bar{j}})^2 - 4M^2_{i\bar{i}}M^2_{j\bar{j}}}}{2M^2_{j\bar{j}}} = \frac{[(4M^2_{i\bar{k}} - M^2_{i\bar{i}} - M^2_{k\bar{k}}) + \sqrt{(4M^2_{i\bar{k}} - M^2_{i\bar{i}} - M^2_{k\bar{k}})^2 - 4M^2_{i\bar{i}}M^2_{k\bar{k}}}]/2M^2_{k\bar{k}}}{[(4M^2_{j\bar{k}} - M^2_{j\bar{j}} - M^2_{k\bar{k}}) + \sqrt{(4M^2_{j\bar{k}} - M^2_{j\bar{j}} - M^2_{k\bar{k}})^2 - 4M^2_{j\bar{j}}M^2_{k\bar{k}}}]/2M^2_{k\bar{k}}}. \tag{16}$$

For baryons, using

$$\frac{\alpha'_{jjq}}{\alpha'_{iiq}} = \frac{\alpha'_{kkq}}{\alpha'_{iiq}} \times \frac{\alpha'_{jjq}}{\alpha'_{kkq}}, \tag{17}$$

and Eq. (13), when $m_i < m_j < m_k$, we have

$$\frac{(4M_{ijq}^2 - M_{iiq}^2 - M_{jjq}^2) + \sqrt{(4M_{ijq}^2 - M_{iiq}^2 - M_{jjq}^2)^2 - 4M_{iiq}^2 M_{jjq}^2}}{2M_{jjq}^2}$$
$$= \frac{[(4M_{ikq}^2 - M_{iiq}^2 - M_{kkq}^2) + \sqrt{(4M_{ikq}^2 - M_{iiq}^2 - M_{kkq}^2)^2 - 4M_{iiq}^2 M_{kkq}^2}]/2M_{kkq}^2}{[(4M_{jkq}^2 - M_{jjq}^2 - M_{kkq}^2) + \sqrt{(4M_{jkq}^2 - M_{jjq}^2 - M_{kkq}^2)^2 - 4M_{jjq}^2 M_{kkq}^2}]/2M_{kkq}^2}, \tag{18}$$

where q denotes an arbitrary light or heavy quark.

Relations (16) and (18) are the high-power mass equalities among one J^P multiplet. They can be used to predict the masses of unobserved states. In Sec. III, we will apply Eq. (16) to predict the masses of $\bar{b}c$ meson states and the masses of the pure $s\bar{s}$ meson states.

C. Linear mass inequalities and quadratic mass inequalities

From Eqs. (9) and (13), two kinds of interesting inequalities can be derived for mesons and baryons, respectively. For mesons, as mentioned in the above discussion, $\alpha'_{j\bar{j}}$ and $\alpha'_{i\bar{i}}$ ought to be positive real numbers. Thus $\alpha'_{j\bar{j}}/\alpha'_{i\bar{i}}$ should also be a real number. Then from Eq. (9), we have

$$|4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2| \geq 2M_{i\bar{i}}M_{j\bar{j}}. \tag{19}$$

When $i = j$, $4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2 \leq 0$ cannot be held; when $i \neq j$, $4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2 \leq 0$ can be easily ruled out by the data of the well established meson multiplets. Therefore, $4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2 \geq 0$. Thus, Eq. (19) can be written as the following:

$$4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2 \geq 2M_{i\bar{i}}M_{j\bar{j}}. \tag{20}$$

This relation can be simplified to

$$2M_{i\bar{j}} \geq M_{i\bar{i}} + M_{j\bar{j}}. \tag{21}$$

If $i = j$, $M_{i\bar{i}} = M_{i\bar{j}} = M_{j\bar{j}}$, then we have $2M_{i\bar{j}} = M_{i\bar{i}} + M_{j\bar{j}}$. On the other hand, if $2M_{i\bar{j}} = M_{i\bar{i}} + M_{j\bar{j}}$, using Eq. (9), we have

$$\frac{\alpha'_{j\bar{j}}}{\alpha'_{i\bar{i}}} = \frac{M_{i\bar{i}}}{M_{j\bar{j}}}. \tag{22}$$

From the derivation of Eq. (22), we can see that this equation is valid for hadrons belonging to the same multiplet. Since hadrons lying on the same Regge trajectory (which have the total angular momenta $J, J+2, J+4, \ldots$) have the same slope, we have

$$\frac{\alpha'_{j\bar{j}}}{\alpha'_{i\bar{i}}} = \frac{M_{i\bar{i},J}}{M_{j\bar{j},J}} = \frac{M_{i\bar{i},J+2}}{M_{j\bar{j},J+2}}. \tag{23}$$

From Eq. (11), we have

$$\alpha'_{i\bar{i}} = \frac{2}{M_{i\bar{i},J+2}^2 - M_{i\bar{i},J}^2}, \qquad \alpha'_{j\bar{j}} = \frac{2}{M_{j\bar{j},J+2}^2 - M_{j\bar{j},J}^2}. \tag{24}$$

Combining Eqs. (23) and (24), we have

$$\frac{\alpha'_{j\bar{j}}}{\alpha'_{i\bar{i}}} = \frac{M_{i\bar{i},J+2} + M_{i\bar{i},J}}{M_{j\bar{j},J+2} + M_{j\bar{j},J}} \times \frac{M_{i\bar{i},J+2} - M_{i\bar{i},J}}{M_{j\bar{j},J+2} - M_{j\bar{j},J}} = \left(\frac{\alpha'_{j\bar{j}}}{\alpha'_{i\bar{i}}}\right)^2. \tag{25}$$

As mentioned before, the Regge slope α' is a positive real number. Therefore, $\alpha'_{j\bar{j}}/\alpha_{i\bar{i}'} = 1$ when $2M_{i\bar{j}} = M_{i\bar{i}} + M_{j\bar{j}}$. Consequently we have $M_{i\bar{i},J} = M_{j\bar{j},J}$ and $M_{i\bar{i},J+2} = M_{j\bar{j},J+2}$ from Eq. (23). This leads to $i = j$ since the $i\bar{i}$ and $j\bar{j}$ states have the same J^P.

From the above analysis, we can conclude that if and only if $i = j$, $2M_{i\bar{j}} = M_{i\bar{i}} + M_{j\bar{j}}$. Therefore, when $i \neq j$, we have

$$2M_{i\bar{j}} > M_{i\bar{i}} + M_{j\bar{j}}. \tag{26}$$

Many authors argued recently that the slopes of Regge trajectories decrease with quark mass increase [21,22,40,41,45,46,55,58,59]. Therefore, $\alpha'_{j\bar{j}}/\alpha'_{i\bar{i}} < 1$ when the j quark is heavier than the i quark. Then, from Eq. (9) one can have

$$\frac{1}{2M_{j\bar{j}}^2} \times [(4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2) + \sqrt{(4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2)^2 - 4M_{i\bar{i}}^2 M_{j\bar{j}}^2}] < 1. \tag{27}$$

From this relation, we obtain

$$\begin{cases} 2M_{j\bar{j}}^2 - (4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2) > 0, \\ (4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2)^2 - 4M_{i\bar{i}}^2 M_{j\bar{j}}^2 < [2M_{j\bar{j}}^2 - (4M_{i\bar{j}}^2 - M_{i\bar{i}}^2 - M_{j\bar{j}}^2)]^2. \end{cases} \tag{28}$$

These two inequalities can be simplified to

$$2M_{i\bar{j}}^2 < M_{i\bar{i}}^2 + M_{j\bar{j}}^2. \quad (29)$$

The relation (29) can also be derived in the same way if we use the second equation in Eq. (9) considering $\alpha'_{i\bar{j}}/\alpha'_{i\bar{i}} < 1$.

The baryon mass inequalities can be extracted in the same way as that in the meson case. Then, we have

$$2M_{ijq} > M_{iiq} + M_{jjq}, \quad (30)$$

$$2M_{ijq}^2 < M_{iiq}^2 + M_{jjq}^2. \quad (31)$$

It is very interesting that the inequalities (26) and (29)–(31) are the concave and convex relations. These mass inequalities can be used to give constrains (lower limit and upper limit) for masses of hadrons which have not been discovered. For example, we have from the inequalities (26) and (29) that

$$\frac{M_{i\bar{i}} + M_{j\bar{j}}}{2} < M_{i\bar{j}} < \sqrt{\frac{M_{i\bar{i}}^2 + M_{j\bar{j}}^2}{2}}, \quad (32)$$

in which one inequality gives an upper limit while the other gives a lower limit for $M_{i\bar{j}}$. For baryons, we have from the inequalities (30) and (31) that

$$\frac{M_{iiq} + M_{jjq}}{2} < M_{ijq} < \sqrt{\frac{M_{iiq}^2 + M_{jjq}^2}{2}}. \quad (33)$$

We will use Eqs. (32) and (33) to give mass ranges for mesons and baryons in Sec. III.

D. Quadratic mass equalities

To evaluate the deviations of relations (29) and (31) from the equalities that would be obtained by changing the signs of inequalities to equal signs, we introduce a parameter δ, which is denoted by δ_{ij}^m for mesons,

$$\delta_{ij}^m = M_{i\bar{i}}^2 + M_{j\bar{j}}^2 - 2M_{i\bar{j}}^2, \quad (34)$$

and by δ_{ij}^b for baryons,

$$\delta_{ij}^b = M_{iiq}^2 + M_{jjq}^2 - 2M_{ijq}^2, \quad (35)$$

where i, j, and q are arbitrary light or heavy quarks. From relations (29) and (31), we know $\delta^{m(b)} > 0$. It will be shown later that δ_{ij}^b is independent of q.

For mesons, from Eqs. (2) and (3), we have

$$a_{i\bar{i}}(0) - a_{i\bar{j}}(0) = a_{i\bar{j}}(0) - a_{j\bar{j}}(0), \quad (36)$$

$$\frac{1}{\alpha'_{i\bar{i}}} - \frac{1}{\alpha'_{i\bar{j}}} = \frac{1}{\alpha'_{i\bar{j}}} - \frac{1}{\alpha'_{j\bar{j}}}. \quad (37)$$

Let

$$\lambda_i \equiv a_{n\bar{n}}(0) - a_{n\bar{i}}(0), \qquad \gamma_i \equiv \frac{1}{\alpha'_{n\bar{i}}} - \frac{1}{\alpha'_{n\bar{n}}}, \quad (38)$$

where n denotes light nonstrange quark u or d. Using Eqs. (36)–(38) we have

$$\lambda_i = a_{n\bar{n}}(0) - a_{n\bar{i}}(0) = a_{n\bar{i}}(0) - a_{i\bar{i}}(0), \quad (39)$$

$$\gamma_i = \frac{1}{\alpha'_{n\bar{i}}} - \frac{1}{\alpha'_{n\bar{n}}} = \frac{1}{\alpha'_{i\bar{i}}} - \frac{1}{\alpha'_{n\bar{i}}}. \quad (40)$$

Hence,

$$a_{i\bar{i}}(0) = a_{n\bar{n}}(0) - 2\lambda_i, \quad (41)$$

$$\frac{1}{\alpha'_{i\bar{i}}} = \frac{1}{\alpha'_{n\bar{n}}} + 2\gamma_i. \quad (42)$$

With the help of Eqs. (41) and (42), we have from Eqs. (2) and (3)

$$a_{i\bar{j}}(0) = \tfrac{1}{2}[a_{i\bar{i}}(0) + a_{j\bar{j}}(0)] = a_{n\bar{n}}(0) - \lambda_i - \lambda_j, \quad (43)$$

$$\frac{1}{\alpha'_{i\bar{j}}} = \frac{1}{2}\left(\frac{1}{\alpha'_{i\bar{i}}} + \frac{1}{\alpha'_{j\bar{j}}}\right) = \frac{1}{\alpha'_{n\bar{n}}} + \gamma_i + \gamma_j. \quad (44)$$

Similarly for baryons, from Eqs. (4) and (5), we have

$$a_{ijq}(0) = a_{nnn}(0) - \lambda_i - \lambda_j - \lambda_q, \quad (45)$$

$$\frac{1}{\alpha'_{ijq}} = \frac{1}{\alpha'_{nnn}} + \gamma_i + \gamma_j + \gamma_q, \quad (46)$$

where $\lambda_x \equiv a_{nnn}(0) - a_{nnx}(0)$, $\gamma_x \equiv \frac{1}{\alpha'_{nnx}} - \frac{1}{\alpha'_{nnn}}$ (x denotes i, j, or q). It should be pointed out that the values of λ_x and γ_x can be different for different multiplets.

For $n\bar{n}$ and $i\bar{j}$ states in a meson multiplet, from Eq. (1), we have

$$J = a_{n\bar{n}}(0) + \alpha'_{n\bar{n}} M_{n\bar{n}}^2, \quad (47)$$

$$J = a_{i\bar{j}}(0) + \alpha'_{i\bar{j}} M_{i\bar{j}}^2. \quad (48)$$

With the help of Eqs. (43), (44), and (47), we have from Eq. (48)

$$M_{i\bar{j}}^2 = (\alpha'_{n\bar{n}} M_{n\bar{n}}^2 + \lambda_i + \lambda_j)\left(\frac{1}{\alpha'_{n\bar{n}}} + \gamma_i + \gamma_j\right). \quad (49)$$

Therefore, from Eqs. (34) and (49), we have

$$\delta_{ij}^m = (\alpha'_{n\bar{n}} M_{n\bar{n}}^2 + 2\lambda_i)\left(\frac{1}{\alpha'_{n\bar{n}}} + 2\gamma_i\right) + (\alpha'_{n\bar{n}} M_{n\bar{n}}^2 + 2\lambda_j)\left(\frac{1}{\alpha'_{n\bar{n}}} + 2\gamma_j\right) - 2(\alpha'_{n\bar{n}} M_{n\bar{n}}^2 + \lambda_i + \lambda_j)\left(\frac{1}{\alpha'_{n\bar{n}}} + \gamma_i + \gamma_j\right)$$
$$= 2(\lambda_i - \lambda_j)(\gamma_i - \gamma_j). \quad (50)$$

For baryons, in the same way, we have

$$\begin{aligned}\delta^b_{ij} &= M^2_{iiq} + M^2_{jjq} - 2M^2_{ijq}\\ &= (\alpha'_{nnn}M^2_{nnn} + 2\lambda_i + \lambda_q)\left(\frac{1}{\alpha'_{nnn}} + 2\gamma_i + \gamma_q\right) + (\alpha'_{nnn}M^2_{nnn} + 2\lambda_j + \lambda_q)\left(\frac{1}{\alpha'_{nnn}} + 2\gamma_j + \gamma_q\right)\\ &\quad - 2(\alpha'_{nnn}M^2_{nnn} + \lambda_i + \lambda_j + \lambda_q)\left(\frac{1}{\alpha'_{nnn}} + \gamma_i + \gamma_j + \gamma_q\right) = 2(\lambda_i - \lambda_j)(\gamma_i - \gamma_j).\end{aligned} \tag{51}$$

It can be seen that δ^b_{ij} is independent of q from Eq. (51).

From Eq. (38), we know that $\lambda_n = \gamma_n = 0$. Since we choose $m_i < m_j$, $\alpha'_{ii} > \alpha'_{jj}$. Hence from the definition of γ_i [Eq. (38)], we have $\gamma_i < \gamma_j$. Therefore, $0 = \gamma_n < \gamma_s < \gamma_c < \gamma_b$. From Eqs. (9) and (26), we know that $\frac{\alpha'_{jj}}{\alpha'_{ii}} > \frac{M_{ii}}{M_{jj}}$. Hence $\alpha'_{jj}M_{jj} \cdot M_{jj} > \alpha'_{ii}M_{ii} \cdot M_{ii}$. With the help of Eqs. (1) and (41), we have $\lambda_i < \lambda_j$. Therefore, $0 = \lambda_n < \lambda_s < \lambda_c < \lambda_b$. Consequently, we have $0 < \delta^m_{ns} < \delta^m_{nc} < \delta^m_{nb}$, $0 < \delta^m_{cb} < \delta^m_{sb} < \delta^m_{nb}$, $0 < \delta^m_{sc} < \delta^m_{nc}$, and $0 < \delta^m_{sc} < \delta^m_{sb}$. If we assume that $\gamma_s < \frac{1}{2}\gamma_c < \frac{1}{4}\gamma_b$ and $\lambda_s < \frac{1}{2}\lambda_c < \frac{1}{4}\lambda_b$, with the above analysis, we can have $\delta^m_{ns} < \delta^m_{sc} < \delta^m_{nc} < \delta^m_{cb} < \delta^m_{sb} < \delta^m_{nb}$. We will show later that these relations indeed hold. For baryons, we can have $\delta^b_{ns} < \delta^b_{sc} < \delta^b_{nc} < \delta^b_{cb} < \delta^b_{sb} < \delta^b_{nb}$ in the same way.

Inserting the corresponding masses into relation (34), we have the values of δ^m_{ij} for some meson multiplets which are shown in Table II. From Table II, we can see that the relation $\delta^m_{ns} < \delta^m_{sc} < \delta^m_{nc} < \delta^m_{cb} < \delta^m_{sb} < \delta^m_{nb}$ is indeed satisfied for different meson multiplets. These inequalities imply that the higher-order breaking effects become more pronounced with the quark mass increase.

TABLE II. The values of δ^m_{ij} for some multiplets (in units of GeV^2).

	δ^m_{ns}	δ^m_{sc}	δ^m_{nc}	δ^m_{cb}	δ^m_{sb}	δ^m_{nb}
$1\,^1S_0$	0.016	1.623	1.931	16.898	29.179	30.769
$1\,^3S_1$	0.015	1.682	2.125	18.294	31.930	33.387
$1\,^3P_2$	0.018	1.785	2.281	18.042	32.434	34.018
$1\,^1P_1$			2.198			

1. Mass relations for the $\frac{3}{2}^+$ multiplet

For the $\frac{3}{2}^+$ multiplet, noticing that $\delta^{(3/2)^+}_{ij}$ in the above relation (51) is independent of q, we have some equalities which are given in the following:

(1) When $i = n$, $j = s$, $q = n, s, c, b$,

$$\begin{aligned}\delta^{(3/2)^+}_{ns} &= M^2_{\Delta} + M^2_{\Xi^*} - 2M^2_{\Sigma^*}\\ &= M^2_{\Sigma^*} + M^2_{\Omega} - 2M^2_{\Xi^*}\\ &= M^2_{\Sigma^*_c} + M^2_{\Omega^*_c} - 2M^2_{\Xi^*_c}\\ &= M^2_{\Sigma^*_b} + M^2_{\Omega^*_b} - 2M^2_{\Xi^*_b}.\end{aligned} \tag{52a}$$

(2) When $i = n$, $j = c$, $q = n, s, c, b$,

$$\begin{aligned}\delta^{(3/2)^+}_{nc} &= M^2_{\Delta} + M^2_{\Xi^*_{cc}} - 2M^2_{\Sigma^*_c}\\ &= M^2_{\Sigma^*} + M^2_{\Omega^*_{cc}} - 2M^2_{\Xi^*_c}\\ &= M^2_{\Sigma^*_c} + M^2_{\Omega_{ccc}} - 2M^2_{\Xi^*_{cc}}\\ &= M^2_{\Sigma^*_b} + M^2_{\Omega^*_{bcc}} - 2M^2_{\Xi^*_{bc}}.\end{aligned} \tag{52b}$$

(3) When $i = s$, $j = c$, $q = n, s, c, b$,

$$\begin{aligned}\delta^{(3/2)^+}_{sc} &= M^2_{\Xi^*} + M^2_{\Xi^*_{cc}} - 2M^2_{\Xi^*_c}\\ &= M^2_{\Omega} + M^2_{\Omega^*_{cc}} - 2M^2_{\Omega^*_c}\\ &= M^2_{\Omega^*_c} + M^2_{\Omega_{ccc}} - 2M^2_{\Omega^*_{cc}}\\ &= M^2_{\Omega^*_b} + M^2_{\Omega^*_{bcc}} - 2M^2_{\Omega^*_{bc}}.\end{aligned} \tag{52c}$$

(4) When $i = n$, $j = b$, $q = n, s, c, b$,

$$\begin{aligned}\delta^{(3/2)^+}_{nb} &= M^2_{\Delta} + M^2_{\Xi^*_{bb}} - 2M^2_{\Sigma^*_b}\\ &= M^2_{\Sigma^*} + M^2_{\Omega^*_{bb}} - 2M^2_{\Xi^*_b}\\ &= M^2_{\Sigma^*_c} + M^2_{\Omega^*_{bbc}} - 2M^2_{\Xi^*_{bc}}\\ &= M^2_{\Sigma^*_b} + M^2_{\Omega_{bbb}} - 2M^2_{\Xi^*_{bb}}.\end{aligned} \tag{52d}$$

(5) When $i = s$, $j = b$, $q = n, s, c, b$,

$$\begin{aligned}\delta^{(3/2)^+}_{sb} &= M^2_{\Xi^*} + M^2_{\Xi^*_{bb}} - 2M^2_{\Xi^*_b}\\ &= M^2_{\Omega} + M^2_{\Omega^*_{bb}} - 2M^2_{\Omega^*_b}\\ &= M^2_{\Omega^*_c} + M^2_{\Omega^*_{bbc}} - 2M^2_{\Omega^*_{bc}}\\ &= M^2_{\Omega^*_b} + M^2_{\Omega_{bbb}} - 2M^2_{\Omega^*_{bb}}.\end{aligned} \tag{52e}$$

(6) When $i = c$, $j = b$, $q = n, s, c, b$,

$$\begin{aligned}\delta^{(3/2)^+}_{cb} &= M^2_{\Xi^*_{cc}} + M^2_{\Xi^*_{bb}} - 2M^2_{\Xi^*_{bc}}\\ &= M^2_{\Omega^*_{cc}} + M^2_{\Omega^*_{bb}} - 2M^2_{\Omega^*_{bc}}\\ &= M^2_{\Omega_{ccc}} + M^2_{\Omega^*_{bbc}} - 2M^2_{\Omega^*_{bcc}}\\ &= M^2_{\Omega^*_{bcc}} + M^2_{\Omega_{bbb}} - 2M^2_{\Omega^*_{bbc}}.\end{aligned} \tag{52f}$$

From Eqs. (52a)–(52c), one can get the quadratic mass Eqs. (25)–(29) in Ref. [23] derived by Burakovsky *et al.* The linear forms of Eqs. (52a)–(52c) were obtained by Hendry and Lichtenberg in the quark model [26], by Verma and Khanna considering the second-order effects arising from the 84 representation of SU(4) [27] and in the framework of SU(8) symmetry [28], and by Singh *et al.* studying SU(4) second-order mass-breaking effects with a dynamical consideration [29] (bottom baryons were not included in Refs. [23,26–29]). The linear forms of Eqs. (52a)–(52f) were derived by Singh and Khanna in the nonrelativistic additive quark model [30] and by Singh using broken SU(6) internal symmetry including second-order mass contributions [31]. We will show some arguments in Sec. IV which support the quadratic form mass formulas for mesons and baryons rather than the linear form.

2. Mass relations for the $\frac{1}{2}^+$ multiplet

For the $\frac{1}{2}^+$ multiplet, it is very different from the $\frac{3}{2}^+$ multiplet because there are different ways for the spins of the constituent quarks to form the total spin $S = \frac{1}{2}$. Three constituent quarks in a $\frac{1}{2}^+$ baryon can be regarded as a quark and a scalar diquark or regarded as a quark and an axial-vector diquark. Regge slopes of Λ, Λ_c, Λ_b, Ξ_c, and Ξ_b are slightly bigger than those of Σ, Σ_c, Σ_b, Ξ'_c, and Ξ'_b, respectively, although sometimes they can be considered to be approximately equal [23,60]. Regge intercepts of Λ, Λ_c, Λ_b, Ξ_c, and Ξ_b are much bigger than those of Σ, Σ_c, Σ_b, Ξ'_c, and Ξ'_b, respectively. However, these cannot be reflected from Eqs. (45) and (46). Therefore, some of the $\frac{1}{2}^+$ baryons may not be related as the $\frac{3}{2}^+$ baryons.

The Qqq' and $QQ'q$ (where q and q' denote the light quarks and Q and Q' denote the heavy quarks c or b) baryon states are believed to be described by the quark-diquark picture: Two light quarks qq' are bound into a color antitriplet system with the size comparable to the QCD scale in the Qqq' baryon state [61,62]; two heavy quarks QQ' are bound into a small (compared with the QCD scale) color antitriplet system in the $QQ'q$ baryon state [61,63]. The heavy baryons which are composed of a heavy quark and a light axial-vector diquark (Σ_Q, Ξ_Q, and Ω_Q) belong to a **6** representation of flavor SU(3) [15]. Therefore, $\delta_{ns}^{(1/2)^+}$ can be expressed as

$$\delta_{ns}^{(1/2)^+} = M^2_{\Sigma_c} + M^2_{\Omega_c} - 2M^2_{\Xi'_c} = M^2_{\Sigma_b} + M^2_{\Omega_b} - 2M^2_{\Xi'_b}. \tag{53}$$

For the doubly heavy baryons which are composed of a light quark and a heavy axial-vector diquark, $\delta_{bc}^{(1/2)^+}$ can be expressed as

$$\delta_{bc}^{(1/2)^+} = M^2_{\Xi_{cc}} + M^2_{\Xi_{bb}} - 2M^2_{\Xi'_{bc}} = M^2_{\Omega_{cc}} + M^2_{\Omega_{bb}} - 2M^2_{\Omega'_{bc}}. \tag{54}$$

Since $\delta^b_{qq'}$ is determined by the dynamics of the light diquark system qq' inside a heavy baryon Qqq' and since this dynamics is independent of flavor and spin of the heavy quark due to the $SU(2)_f \otimes SU(2)_s$ symmetry in the heavy quark limit [64], we assume that $\delta_{ns}^{(1/2)^+}$ for the $\frac{1}{2}^+$ charmed (bottom) sextet equals $\delta_{ns}^{(3/2)^+}$ for the $\frac{3}{2}^+$ charmed (bottom) sextet, $\delta_{ns}^{(1/2)^+} = \delta_{ns}^{(3/2)^+}$. This relation holds exactly when the masses of charmed and bottom quarks are taken to be infinitely large. Deviations from this relation are due to $\frac{1}{m_c}$ and $\frac{1}{m_b}$ corrections. Then, one can have

$$M^2_{\Sigma_c} + M^2_{\Omega_c} - 2M^2_{\Xi'_c} = M^2_{\Sigma^*_c} + M^2_{\Omega^*_c} - 2M^2_{\Xi^*_c}, \tag{55}$$

$$M^2_{\Sigma_b} + M^2_{\Omega_b} - 2M^2_{\Xi'_b} = M^2_{\Sigma^*_b} + M^2_{\Omega^*_b} - 2M^2_{\Xi^*_b}. \tag{56}$$

There are two linear mass equations similar to the above quadratic mass equations,

$$M_{\Sigma_c} + M_{\Omega_c} - 2M_{\Xi'_c} = M_{\Sigma^*_c} + M_{\Omega^*_c} - 2M_{\Xi^*_c}, \tag{57}$$

$$M_{\Sigma_b} + M_{\Omega_b} - 2M_{\Xi'_b} = M_{\Sigma^*_b} + M_{\Omega^*_b} - 2M_{\Xi^*_b}, \tag{58}$$

which were extracted by Jenkins in the $1/m_Q$ and $1/N_c$ expansions [33]. Similarly, assuming that $\delta_{bc}^{(1/2)^+} = \delta_{bc}^{(3/2)^+}$, one can have

$$M^2_{\Xi_{cc}} + M^2_{\Xi_{bb}} - 2M^2_{\Xi'_{bc}} = M^2_{\Omega_{cc}} + M^2_{\Omega_{bb}} - 2M^2_{\Omega'_{bc}} = M^2_{\Xi^*_{cc}} + M^2_{\Xi^*_{bb}} - 2M^2_{\Xi^*_{bc}} = M^2_{\Omega^*_{cc}} + M^2_{\Omega^*_{bb}} - 2M^2_{\Omega^*_{bc}}. \tag{59}$$

From Eq. (52), we can have a relation for the $\frac{3}{2}^+$ baryons,

$$(M^2_{\Omega^*_{cc}} - M^2_{\Xi^*_{cc}}) + (M^2_{\Xi^*} - M^2_{\Sigma^*}) = (M^2_{\Omega^*_c} - M^2_{\Sigma^*_c}). \tag{60}$$

Its corresponding relation for the $\frac{1}{2}^+$ baryons is

$$(M^2_{\Omega_{cc}} - M^2_{\Xi_{cc}}) + (M^2_{\Xi} - M^2_{\Sigma}) = (M^2_{\Omega_c} - M^2_{\Sigma_c}). \tag{61}$$

The linear form of Eq. (61) can satisfy the instanton model [25] and has been given by Verma and Khanna considering the second-order effects arising from the 84 representation of SU(4) [27]. A different relation,

$$(M^2_{\Omega_{cc}} - M^2_{\Xi_{cc}}) + \left(\frac{3M^2_\Lambda + M^2_\Sigma}{4} - M^2_N\right) = 2\left(\frac{M^2_{\Xi_c} + M^2_{\Xi'_c}}{2} - \frac{3M^2_{\Lambda_c} + M^2_{\Sigma_c}}{4}\right), \tag{62}$$

has been proposed in Ref. [23]. However, the linear form of Eq. (62) cannot satisfy the instanton model [25]. Furthermore, the value of $(M^2_{\Omega_{cc}} - M^2_{\Xi_{cc}})$ given by Eq. (61) (~ 0.94 GeV^2) is close to the value of $(M^2_{\Omega^*_{cc}} - M^2_{\Xi^*_{cc}})$ given by Eq. (60) (~ 0.89 GeV^2), while the value of $(M^2_{\Omega_{cc}} - M^2_{\Xi_{cc}})$ given by Eq. (62) (~ 1.39 GeV^2) is much larger. We will use Eq. (61) rather than Eq. (62) to extract the mass of Ω_{cc} in Sec. III.

III. SOME APPLICATIONS

In this section, we will apply the relations we have obtained in Sec. II to discuss the mass ranges of mesons and baryons, the masses of the $\bar{b}c$ and $s\bar{s}$ meson states, the properties of $\Xi^+_{cc}(3520)$, the parameters of the Regge trajectories for the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ multiplet, and the properties of the charm-strange baryons (some of which have just been observed).

A. Mass ranges of mesons and baryons

Using Eqs. (32) and (33), we calculate the upper and lower mass limits for some meson states ($s\bar{s}$, $c\bar{n}$, $\bar{b}n$, $\bar{b}c$, $c\bar{s}$, and $\bar{b}s$) of different multiplets and some baryon states of $\frac{1}{2}^+$ and $\frac{3}{2}^+$ multiplets. The results for mesons are shown in Tables III and IV in comparison with the measured meson masses [15]. The results for baryons are shown in Table V in comparison with the measured baryon masses [15].

The masses of the pure $s\bar{s}$ states cannot be directly measured experimentally because of the usual mixing of the pure isoscalar $n\bar{n}$ and $s\bar{s}$ states. The way to extract masses of the pure $s\bar{s}$ states will be displayed in the next section. In calculating the mass limits about the $c\bar{s}$ and $\bar{b}s$ states in Table IV, we approximately use the values of $\sqrt{2M_K^2 - M_\pi^2}$ (given by the quadratic GMO formula $M_\pi^2 + M^2_{s\bar{s}(1^1S_0)} = 2M_K^2$), M_ϕ and $M_{f_2'(1525)}$ to replace $M_{s\bar{s}(1^1S_0)}$, $M_{s\bar{s}(1^3S_1)}$, and $M_{s\bar{s}(1^3P_2)}$, respectively. $f_2'(1525)$ was proved

TABLE III. The numerical results for upper and lower limits for the masses of mesons ($s\bar{s}$, $c\bar{n}$, and $\bar{b}n$) obtained from Eqs. (26) and (29) in comparison with the experimental data (in units of GeV).

$\mathcal{N}^{2S+1}L_J$	Inequalities	Lower and upper limits
$s\bar{s}$ sector	$\sqrt{2M^2_{n\bar{s}} - M^2_{n\bar{n}}} < M_{s\bar{s}} < 2M_{n\bar{s}} - M_{n\bar{n}}$	
$1\,^1S_0$	$\sqrt{2M^2_K - M^2_\pi} < M_{s\bar{s}} < 2M_K - M_\pi$	$0.687 < M_{s\bar{s}} < 0.854$
$1\,^3S_1$	$\sqrt{2M^2_{K^*} - M^2_\rho} < M_{s\bar{s}} < 2M_{K^*} - M_\rho$	$0.998 < M_{s\bar{s}} < 1.012$
$1\,^3P_2$	$\sqrt{2M^2_{K_2^*} - M^2_{a_2(1320)}} < M_{s\bar{s}} < 2M_{K_2^*} - M_{a_2(1320)}$	$1.538 < M_{s\bar{s}} < 1.547$
$1\,^1D_2$	$\sqrt{2M^2_{K_2(1770)} - M^2_{\pi_2(1670)}} < M_{s\bar{s}} < 2M_{K_2(1770)} - M_{\pi_2(1670)}$	$1.868 < M_{s\bar{s}} < 1.874$
$1\,^3D_3$	$\sqrt{2M^2_{K_3^*} - M^2_{\rho_3}} < M_{s\bar{s}} < 2M_{K_3^*} - M_{\rho_3}$	$1.859 < M_{s\bar{s}} < 1.863$
$c\bar{n}$ sector	$(M_{n\bar{n}} + M_{c\bar{c}})/2 < M_{c\bar{n}} < \sqrt{(M^2_{n\bar{n}} + M^2_{c\bar{c}})/2}$	
$1\,^1S_0$	$(M_\pi + M_{\eta_c(1S)})/2 < M_D < \sqrt{(M^2_\pi + M^2_{\eta_c(1S)})/2}$	$1.559 < 1.867(\text{exp}) < 2.110$
$1\,^3S_1$	$(M_\rho + M_{J/\psi(1S)})/2 < M_{D^*} < \sqrt{(M^2_\rho + M^2_{J/\psi(1S)})/2}$	$1.936 < 2.008(\text{exp}) < 2.257$
$1\,^3P_2$	$(M_{a_2(1320)} + M_{\chi_{c2}(1P)})/2 < M_{D_2^*} < \sqrt{(M^2_{a_2(1320)} + M^2_{\chi_{c2}(1P)})/2}$	$2.437 < 2.460(\text{exp}) < 2.682$
$1\,^1P_1$	$(M_{b_1(1235)} + M_{h_c(1P)})/2 < M_{D_1(2420)} < \sqrt{(M^2_{b_1(1235)} + M^2_{h_c(1P)})/2}$	$2.378 < 2.423(\text{exp}) < 2.640$
$1\,^3P_1$	$(M_{a_1(1260)} + M_{\chi_{c1}(1P)})/2 < M_{D_1(1^3P_1)} < \sqrt{(M^2_{a_1(1260)} + M^2_{\chi_{c1}(1P)})/2}$	$2.370 < M_{D_1(1^3P_1)} < 2.630$
$1\,^3D_1$	$(M_{\rho(1700)} + M_{\psi(3770)})/2 < M_{D^*(1^3D_1)} < \sqrt{(M^2_{\rho(1700)} + M^2_{\psi(3770)})/2}$	$2.746 < M_{D^*(1^3D_1)} < 2.931$
$2\,^1S_0$	$(M_{\pi(1300)} + M_{\eta_c(2S)})/2 < M_{D(2^1S_0)} < \sqrt{(M^2_{\pi(1300)} + M^2_{\eta_c(2S)})/2}$	$2.419 < M_{D(2^1S_0)} < 2.756$
$2\,^3S_1$	$(M_{\rho(1450)} + M_{\psi(2S)})/2 < M_{D^*(2^3S_1)} < \sqrt{(M^2_{\rho(1450)} + M^2_{\psi(2S)})/2}$	$2.573 < M_{D^*(2^3S_1)} < 2.803$
$\bar{b}n$ sector	$(M_{n\bar{n}} + M_{b\bar{b}})/2 < M_{\bar{b}n} < \sqrt{(M^2_{n\bar{n}} + M^2_{b\bar{b}})/2}$	
$1\,^1S_0$	$(M_\pi + M_{\eta_b(1S)})/2 < M_B < \sqrt{(M^2_\pi + M^2_{\eta_b(1S)})/2}$	$4.719 < 5.279(\text{exp}) < 6.577$
$1\,^3S_1$	$(M_\rho + M_{\Upsilon(1S)})/2 < M_{B^*} < \sqrt{(M^2_\rho + M^2_{\Upsilon(1S)})/2}$	$5.118 < 5.325(\text{exp}) < 6.712$
$1\,^3P_2$	$(M_{a_2(1320)} + M_{\chi_{b2}(1P)})/2 < M_{B_2^*} < \sqrt{(M^2_{a_2(1320)} + M^2_{\chi_{b2}(1P)})/2}$	$5.615 < 5.743(\text{exp}) < 7.071$
$1\,^3P_1$	$(M_{a_1(1260)} + M_{\chi_{b1}(1P)})/2 < M_{B_1(1^3P_1)} < \sqrt{(M^2_{a_1(1260)} + M^2_{\chi_{b1}(1P)})/2}$	$5.561 < M_{B_1(1^3P_1)} < 7.049$
$2\,^3S_1$	$(M_{\rho(1450)} + M_{\Upsilon(2S)})/2 < M_{B^*(2^3S_1)} < \sqrt{(M^2_{\rho(1450)} + M^2_{\Upsilon(2S)})/2}$	$5.741 < M_{B^*(2^3S_1)} < 7.162$
$2\,^3P_2$	$(M_{a_2(1700)} + M_{\chi_{b2}(2P)})/2 < M_{B_2^*(2^3P_2)} < \sqrt{(M^2_{a_2(1700)} + M^2_{\chi_{c2}(2P)})/2}$	$6.000 < M_{B_2^*(2^3P_2)} < 7.363$

TABLE IV. The numerical results for upper and lower limits for the masses of mesons ($\bar{b}c$, $c\bar{s}$, and $\bar{b}s$) obtained from Eqs. (26) and (29) in comparison with the experimental data (in units of GeV).

$\mathcal{N}^{2S+1}L_J$	Inequalities	Lower and upper limits
$\bar{b}c$ sector	$(M_{c\bar{c}}+M_{b\bar{b}})/2<M_{\bar{b}c}<\sqrt{(M_{c\bar{c}}^2+M_{b\bar{b}}^2)/2}$	
$1\,^1S_0$	$(M_{\eta_c(1S)}+M_{\eta_b(1S)})/2<M_{B_c}<\sqrt{(M_{\eta_c(1S)}^2+M_{\eta_b(1S)}^2)/2}$	$6.140<6.286(\text{exp})<6.906$
$1\,^3S_1$	$(M_{J/\psi(1S)}+M_{\Upsilon(1S)})/2<M_{B_c^*}<\sqrt{(M_{J/\psi(1S)}^2+M_{\Upsilon(1S)}^2)/2}$	$6.279<M_{B_c^*}<7.039$
$1\,^3P_2$	$(M_{\chi_{c2}(1P)}+M_{\chi_{b2}(1P)})/2<M_{B_{c2}^*}<\sqrt{(M_{\chi_{c2}(1P)}^2+M_{\chi_{b2}(1P)}^2)/2}$	$6.734<M_{B_{c2}^*}<7.446$
$1\,^3P_0$	$(M_{\chi_{c0}(1P)}+M_{\chi_{b0}(1P)})/2<M_{B_{c0}^*}<\sqrt{(M_{\chi_{c0}(1P)}^2+M_{\chi_{b0}(1P)}^2)/2}$	$6.637<M_{B_{c0}^*}<7.378$
$1\,^3P_1$	$(M_{\chi_{c1}(1P)}+M_{\chi_{b1}(1P)})/2<M_{B_{c1}(1^3P_1)}<\sqrt{(M_{\chi_{c1}(1P)}^2+M_{\chi_{b1}(1P)}^2)/2}$	$6.702<M_{B_{c1}(1^3P_1)}<7.423$
$1\,^3S_1$	$(M_{\psi(2S)}+M_{\Upsilon(2S)})/2<M_{B_c^*(2^3S_1)}<\sqrt{(M_{\psi(2S)}^2+M_{\Upsilon(2S)}^2)/2}$	$6.855<M_{B_c^*(2^3S_1)}<7.552$
$c\bar{s}$ sector	$(M_{c\bar{c}}+M_{s\bar{s}})/2<M_{c\bar{s}}<\sqrt{(M_{c\bar{c}}^2+M_{s\bar{s}}^2)/2}$	
$1\,^1S_0$	$(M_{\eta_c(1S)}+M_{s\bar{s}(1^1S_0)})/2<M_{D_s}<\sqrt{(M_{\eta_c(1S)}^2+M_{s\bar{s}(1^1S_0)}^2)/2}$	$1.834<1.968(\text{exp})<2.163$
$1\,^3S_1$	$(M_{J/\psi(1S)}+M_{s\bar{s}(1^3S_1)})/2<M_{D_s^*}<\sqrt{(M_{J/\psi(1S)}^2+M_{s\bar{s}(1^3S_1)}^2)/2}$	$2.058<2.112(\text{exp})<2.305$
$1\,^3P_2$	$(M_{\chi_{c2}(1P)}+M_{s\bar{s}(1^3P_2)})/2<M_{D_{s2}^*}<\sqrt{(M_{\chi_{c2}(1P)}^2+M_{s\bar{s}(1^3P_2)}^2)/2}$	$2.541<2.574(\text{exp})<2.736$
$\bar{b}s$ sector	$(M_{b\bar{b}}+M_{s\bar{s}})/2<M_{\bar{b}s}<\sqrt{(M_{b\bar{b}}^2+M_{s\bar{s}}^2)/2}$	
$1\,^1S_0$	$(M_{\eta_b(1S)}+M_{s\bar{s}(1^1S_0)})/2<M_{B_s}<\sqrt{(M_{\eta_b(1S)}^2+M_{s\bar{s}(1^1S_0)}^2)/2}$	$4.994<5.368(\text{exp})<6.594$
$1\,^3S_1$	$(M_{\Upsilon(1S)}+M_{s\bar{s}(1^3S_1)})/2<M_{B_s^*}<\sqrt{(M_{\Upsilon(1S)}^2+M_{s\bar{s}(1^3S_1)}^2)/2}$	$5.240<5.413(\text{exp})<6.728$
$1\,^3P_2$	$(M_{\chi_{b2}(1P)}+M_{s\bar{s}(1^3P_2)})/2<M_{B_{s2}^*}<\sqrt{(M_{\chi_{b2}(1P)}^2+M_{s\bar{s}(1^3P_2)}^2)/2}$	$5.719<5.840(\text{exp})<7.091$

to be a nearly pure tensor $s\bar{s}$ state ($\sim 98.2\%$) [65]. These approximations shift the mass limits of the $c\bar{s}$ and $\bar{b}s$ states only a few MeV.

It can be seen from Tables III, IV, and V that the inequalities (32) and (33) [which were given from the inequalities (26) and (29)–(31)] agree well with the existing experimental data [15]. The inequalities (32) and (33) also give predictions for the mass ranges of some hadrons which have not been observed. More detailed discussions

TABLE V. The numerical results for upper and lower limits for the masses of baryons obtained from Eqs. (30) and (31) in comparison with the experimental data (in units of GeV).

$J^P=\frac{1}{2}^+$ inequalities	Lower and upper limits
$(M_N+M_\Xi)/2<(3M_\Lambda+M_\Sigma)/4<\sqrt{(M_N^2+M_\Xi^2)/2}$	$1.128<1.135(\text{exp})<1.144$
$(M_{\Sigma_c}+M_{\Omega_c})/2<M_{\Xi_c}<\sqrt{(M_{\Sigma_c}^2+M_{\Omega_c}^2)/2}$	$2.576<2.577(\text{exp})<2.578$
$(M_N+M_{\Xi_{cc}})/2<(3M_{\Lambda_c}+M_{\Sigma_c})/4<\sqrt{(M_N^2+M_{\Xi_{cc}}^2)/2}$	$3.156<M_{\Xi_{cc}}<3.718$
$(M_N+M_{\Xi_{bb}})/2<(3M_{\Lambda_b}+M_{\Sigma_b})/4<\sqrt{(M_N^2+M_{\Xi_{bb}}^2)/2}$	$7.965<M_{\Xi_{bb}}<10.403$
$J^P=\frac{3}{2}^+$ inequalities	Lower and upper limits
$(M_\Delta+M_{\Xi^*})/2<M_{\Sigma^*}<\sqrt{(M_\Delta^2+M_{\Xi^*}^2)/2}$	$1.383<1.385(\text{exp})<1.391$
$(M_{\Sigma^*}+M_\Omega)/2<M_{\Xi^*}<\sqrt{(M_{\Sigma^*}^2+M_\Omega^2)/2}$	$1.529<1.533(\text{exp})<1.535$
$\sqrt{2M_{\Xi_c^*}^2-M_{\Sigma_c^*}^2}<M_{\Omega_c^*}<2M_{\Xi_c^*}-M_{\Sigma_c^*}$	$2.766<2.768(\text{exp})<2.778$
$\sqrt{2M_{\Sigma_c^*}^2-M_\Delta^2}<M_{\Xi_{cc}^*}<2M_{\Sigma_c^*}-M_\Delta$	$3.341<M_{\Xi_{cc}^*}<3.804$
$\sqrt{2M_{\Xi_c^*}^2-M_\Xi^2}<M_{\Xi_{cc}^*}<2M_{\Xi_c^*}-M_\Xi$	$3.414<M_{\Xi_{cc}^*}<3.759$
$\sqrt{2M_{\Xi_c^*}^2-M_\Sigma^2}<M_{\Omega_{cc}^*}<2M_{\Xi_c^*}-M_\Sigma$	$3.477<M_{\Omega_{cc}^*}<3.908$
$\sqrt{2M_{\Omega_c}^2-M_\Omega^2}<M_{\Omega_{cc}^*}<2M_{\Omega_c}-M_\Omega$	$3.544<M_{\Omega_{cc}^*}<3.869$
$\sqrt{2M_{\Sigma_b^*}^2-M_\Delta^2}<M_{\Xi_{bb}^*}<2M_{\Sigma_b^*}-M_\Delta$	$8.156<M_{\Xi_{bb}^*}<10.433$

about the inequalities derived in this work and those in Refs. [17–20] will be given in Sec. IV.

B. Masses of the $\bar{b}c$ and $s\bar{s}$ meson states

1. Masses of the $\bar{b}c$ meson states

The $\bar{b}c$ (or $b\bar{c}$) meson states are special systems with two heavy quarks of different flavors. The presence of both such quarks impacts the production, decay, and mass properties of the $\bar{b}c$ mesons. Until recently, only the pseudoscalar mesons $B_c^{\pm}$ have been observed experimentally [15,66]. The copious productions of B_c mesons and their radial and orbital excitations are expected at the experimental facilities such as the Large Hadron Collider (LHC) at CERN. The masses of $\bar{b}c$ mesons have been predicted in many different approaches [21,67–78]. In the following, we will use Eq. (16) to calculate the masses of B_c, B_c^*, and B_{c2}^* meson states and compare the results with those given in Refs. [21,67–78].

For the 1^1S_0 multiplet, when $i = n$, $j = c$, and $k = b$, inserting the masses of π, $\eta_c(1S)$, $\eta_b(1S)$, D, and B into Eq. (16), the mass of B_c can be extracted. For the 1^3S_1 multiplet, when $i = n$, $j = c$, and $k = b$, inserting the masses of ρ, $J/\psi(1S)$, $\Upsilon(1S)$, D^*, and B^* into Eq. (16), the mass of B_c^* can be extracted. For the 1^3P_2 multiplet, when $i = n$, $j = c$, and $k = b$, inserting the masses of $a_2(1320)$, $\chi_{c2}(1P)$, $\chi_{b2}(1P)$, $D_2^*(2460)$, and $B_2^*(5740)$ which was observed recently [1] into Eq. (16), the mass of B_{c2}^* can be extracted. Comparison of the masses of B_c, B_c^*, and B_{c2}^* extracted in the present work and those given by other references is shown in Table VI. The application of Eq. (18) (baryon case) will be performed in Sec. III D.

If Eq. (2) (the additivity of inverse slopes) were replaced by Eq. (6) (the factorization of slopes) in the derivation of Eq. (16), we would have the following equation instead of Eq. (16):

$$\frac{(2M_{i\bar{j}}^4 - M_{i\bar{i}}^2M_{j\bar{j}}^2) + 2M_{i\bar{j}}^2\sqrt{(2M_{i\bar{j}}^4 - M_{i\bar{i}}^2M_{j\bar{j}}^2)}}{M_{j\bar{j}}^4} = \frac{[(2M_{i\bar{k}}^4 - M_{i\bar{i}}^2M_{k\bar{k}}^2) + 2M_{i\bar{k}}^2\sqrt{(2M_{i\bar{k}}^4 - M_{i\bar{i}}^2M_{k\bar{k}}^2)}]/M_{k\bar{k}}^4}{[(2M_{j\bar{k}}^4 - M_{j\bar{j}}^2M_{k\bar{k}}^2) + 2M_{j\bar{k}}^2\sqrt{(2M_{j\bar{k}}^4 - M_{j\bar{j}}^2M_{k\bar{k}}^2)}]/M_{k\bar{k}}^4}. \tag{63}$$

Applying this equation to the 1^1S_0, 1^3S_1, and 1^3P_2 multiplets we would extract the masses of B_c, B_c^*, and B_{c2}^* which are also shown in Table VI.

In Ref. [22], under the approximation that mesons in the light quark sector have the common Regge slopes, a 14th power meson mass relation,

$$\begin{aligned}
&[(M_{s\bar{s}}^2 - M_{n\bar{n}}^2)(M_{c\bar{c}}^2M_{n\bar{b}}^2(M_{c\bar{s}}^2 - M_{c\bar{n}}^2) + M_{b\bar{b}}^2M_{c\bar{n}}^2(M_{s\bar{b}}^2 - M_{n\bar{b}}^2)) - M_{n\bar{n}}^2(M_{c\bar{c}}^2 + M_{b\bar{b}}^2)(M_{c\bar{s}}^2 - M_{c\bar{n}}^2)(M_{s\bar{b}}^2 - M_{n\bar{b}}^2)] \\
&\quad\times [(M_{s\bar{s}}^2 - M_{n\bar{n}}^2)(M_{n\bar{b}}^2(M_{c\bar{s}}^2 - M_{c\bar{n}}^2) + M_{c\bar{n}}^2(M_{s\bar{b}}^2 - M_{n\bar{b}}^2)) - 2M_{n\bar{n}}^2(M_{c\bar{s}}^2 - M_{c\bar{n}}^2)(M_{s\bar{b}}^2 - M_{n\bar{b}}^2)] \\
&= 4M_{b\bar{c}}^2(M_{c\bar{s}}^2 - M_{c\bar{n}}^2)(M_{s\bar{b}}^2 - M_{n\bar{b}}^2)(M_{c\bar{n}}^2M_{s\bar{s}}^2 - M_{c\bar{s}}^2M_{n\bar{n}}^2)(M_{n\bar{b}}^2M_{s\bar{s}}^2 - M_{s\bar{b}}^2M_{n\bar{n}}^2),
\end{aligned} \tag{64}$$

was derived to predict the mass of B_c^* with the value $M_{B_c^*} = 6.285$ GeV. The results of applying Eq. (64) with the existing experimental data [15] for the 1^1S_0, 1^3S_1, and 1^3P_2 multiplets to extract the masses of B_c, B_c^*, and B_{c2}^* are also shown in Table VI.

2. Masses of the pure $s\bar{s}$ states

The masses of the pure $s\bar{s}$ states cannot be directly measured experimentally because of the usual mixing of the pure isoscalar $n\bar{n}$ and $s\bar{s}$ states. However, the comparison of the mass of the pure $s\bar{s}$ state with that of the physical

TABLE VI. The masses of B_c, B_c^*, and B_{c2}^* (in units of GeV).

States ($\mathcal{N}^{2S+1}L_J$)	Present work	Eq. (63)	Eq. (64)	Exp	[21]	[67]	[68]	[69]	[70]
B_c ($1\,^1S_0$)	6.264	6.404	6.142	6.276[a]	6.263	6.270	6.253	6.264	6.247
B_c^* ($1\,^3S_1$)	6.356	6.502	6.292		6.354	6.332	6.317	6.337	6.308
B_{c2}^* ($1\,^3P_2$)	6.814	6.940	6.767		6.781	6.762	6.743	6.747	6.773
States ($\mathcal{N}^{2S+1}L_J$)	Present work	[71]	[72]	[73]	[74]	[75]	[76]	[77]	[78]
B_c ($1\,^1S_0$)	6.264	6.271	6.286	6.310	6.255	6.280	6.255	6.258	6.28
B_c^* ($1\,^3S_1$)	6.356	6.338	6.341	6.355	6.320	6.321	6.333	6.334	6.35
B_{c2}^* ($1\,^3P_2$)	6.814	6.768	6.772	6.773	6.770	6.783			

[a]The CDF Collaboration confirms their earlier report [79] with higher statistical samples with a significance greater than 8σ [66].

TABLE VII. The masses of the pure $s\bar{s}$ states in pseudoscalar, vector, and tensor meson multiplets given by Eqs. (16) and (65) (in units of GeV).

$\mathcal{N}^{2S+1}L_J$	Eq. (16), $i, j, k = n, s, c$	Eq. (16), $i, j, k = n, s, b$	Eq. (65), $Q = c$	Eq. (65), $Q = b$
$1\,^1S_0$	0.697	0.698	0.761 or 0.157	0.927 or 0.147
$1\,^3S_1$	1.009	1.006	0.891 or 1.079	0.841 or 1.145
$1\,^3P_2$	1.546	1.544	1.492 or 1.582	1.423 or 1.627

state can help us to understand the mixing of the two isoscalar states of a meson nonet.

The masses of the pure $s\bar{s}$ states can be calculated from Eq. (16). When $i = n$, $j = s$, $k = b$ or c, inserting the corresponding masses into Eq. (16), the masses of $s\bar{s}$ for the 1^1S_0, 1^3S_1, and 1^3P_2 multiplets are extracted and shown in Table VII.

In Ref. [22], under the approximation that mesons in the light quark sector have the common Regge slopes, two 6th power meson mass relations were derived to predict the masses of $c\bar{c}$ and $b\bar{b}$ meson states, respectively. Those two 6th power meson mass relations can be written as follows:

$$(M^2_{s\bar{s}}M^2_{n\bar{Q}} - M^2_{n\bar{n}}M^2_{s\bar{Q}})(M^2_{s\bar{s}} - M^2_{n\bar{n}}) + M^2_{QQ}(M^2_{s\bar{Q}} - M^2_{n\bar{Q}})(M^2_{s\bar{s}} - M^2_{n\bar{n}}) = 4(M^2_{s\bar{s}}M^2_{n\bar{Q}} - M^2_{n\bar{n}}M^2_{s\bar{Q}})(M^2_{s\bar{Q}} - M^2_{n\bar{Q}}), \qquad (65)$$

where Q denotes c or b. The results of applying Eq. (65) for the 1^1S_0, 1^3S_1, and 1^3P_2 multiplets to extract the masses of the $s\bar{s}$ states are also shown in Table VII.

From Table VI, one can see that the masses of B_c, B_c^*, and B_{c2}^* given by Eq. (63) are bigger than those given in Refs. [21,67–78]. The mass of the B_c meson given by Eq. (16) (present work) is better than those given by Eqs. (63) and (64) comparing with experimental data. The masses of B_c, B_c^*, and B_{c2}^* given by Eq. (16) (present work) are in reasonable agreement with those given in Refs. [21,67–78]. From Table VII, one can see that the masses of the pure $s\bar{s}$ state in the same multiplet given by Eq. (16) are approximately the same when we choose $k = c$ and $k = b$ and they all satisfy the mass ranges shown in Table III which are given by the linear mass inequality (26) and quadratic mass inequality (29). However, the masses of the pure $s\bar{s}$ states given by Eq. (65) do not satisfy these constrains.

As mentioned above, Eq. (65) was derived under the approximation that mesons in the light quark sector have the common Regge slopes and was applied for predicting the masses of charmonium and bottomonium [22]. Obviously, Eq. (65) may be limited by this approximation while predicting the masses of light hadrons. Equation (64) was extracted under the same arguments on which Eq. (65) is based [22]. When $i = n$, $j = s$, and $k = Q$, Eq. (16) can be reduced to Eq. (65) if we choose $\frac{\alpha_{s\bar{s}}}{\alpha_{n\bar{n}}} = 1$. Furthermore, with Eq. (16) one needs less meson states than those in the case of Eq. (64) to predict the masses of $\bar{b}c$ states. Therefore, Eq. (16) can properly describe the present meson spectroscopy [15].

C. Doubly charmed baryon $\Xi_{cc}^+(3520)$

The doubly charmed baryon $\Xi_{cc}^+(3520)$, which is composed of two charm quarks and one down quark, was first reported in the charged decay mode $\Xi_{cc}^+ \to \Lambda_c^+ K^- \pi^+$ (SELEX 2002) and confirmed in the decay mode $\Xi_{cc}^+ \to pD^+K^-$ (SELEX 2005). These reports were adopted by the Particle Data Group [15] with the average mass 3518.9 ± 0.9 MeV. However, the J^P number has not been determined experimentally. Moreover, it has not been confirmed by other experiments (notably by *BABAR* [12], BELLE [13] and FOCUS [14]), even though they have O(10) (FOCUS) and O(100) (*BABAR*, Belle) more reconstructed charm baryons than SELEX. This experimental puzzle raised many theoretical discussions [80–82]. It was suggested that $\Xi_{cc}^+(3520)$ should be the ground state ($L = 0$) with $J^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$ due to its mass [80–82].

Now we will see whether the state $\Xi_{cc}^+(3520)$ could be assigned as a $\frac{3}{2}^+$ doubly charmed baryon. Let us first assume that $\Xi_{cc}^+(3520)$ belongs to the $\frac{3}{2}^+$ multiplet. When $j = c$, $i = n$, and $q = n$, from Eq. (13), we have

$$\frac{\alpha'_{\Xi^*_{cc}}}{\alpha'_{\Delta}} = \frac{1}{2M^2_{\Xi_{cc}(3520)}} \times [(4M^2_{\Sigma^*_c} - M^2_{\Delta} - M^2_{\Xi_{cc}(3520)}) + \sqrt{(4M^2_{\Sigma^*_c} - M^2_{\Delta} - M^2_{\Xi_{cc}(3520)})^2 - 4M^2_{\Delta}M^2_{\Xi_{cc}(3520)}}], \qquad (66)$$

$$\frac{\alpha'_{\Sigma^*_c}}{\alpha'_{\Delta}} = \frac{1}{4M^2_{\Sigma^*_c}} \times [(4M^2_{\Sigma^*_c} + M^2_{\Delta} - M^2_{\Xi_{cc}(3520)}) + \sqrt{(4M^2_{\Sigma^*_c} - M^2_{\Delta} - M^2_{\Xi_{cc}(3520)})^2 - 4M^2_{\Delta}M^2_{\Xi_{cc}(3520)}}]. \qquad (67)$$

When $j = c$, $i = s$, and $q = n$, from Eq. (13), we have

$$\frac{\alpha'_{\Xi^*_{cc}}}{\alpha'_{\Xi^*}} = \frac{1}{2M^2_{\Xi_{cc}(3520)}} \times [(4M^2_{\Xi^*_c} - M^2_{\Xi^*} - M^2_{\Xi_{cc}(3520)}) + \sqrt{(4M^2_{\Xi^*_c} - M^2_{\Xi^*} - M^2_{\Xi_{cc}(3520)})^2 - 4M^2_{\Xi^*}M^2_{\Xi_{cc}(3520)}}]. \tag{68}$$

From Eq. (46), we have

$$\frac{1}{\alpha'_\Delta} + \frac{2}{\alpha'_\Omega} = \frac{3}{\alpha'_{\Xi^*}}. \tag{69}$$

Inserting the masses of Δ, Σ^*_c, and $\Xi^+_{cc}(3520)$ into Eq. (67), we have

$$\alpha'_{\Sigma^*_c} = 0.867\alpha'_\Delta.$$

Inserting the masses of Δ, Σ^*_c, Ξ^*, Ξ^*_c, and $\Xi^+_{cc}(3520)$ into Eqs. (66) and (68), with the aid of Eq. (69), we have

$$\alpha'_\Omega = 0.860\alpha'_\Delta.$$

Therefore, $\alpha'_\Omega \lesssim \alpha'_{\Sigma^*_c}$. This does not agree with the usual belief that the slopes of charmed baryons should be much smaller than the slopes of light noncharmed baryons. We have calculated the numerical results of $\alpha'_\Omega/\alpha'_{\Sigma^*_c}$ and find that it increases with the mass increase of Ξ^*_{cc}. Therefore, the mass of Ξ^*_{cc} should be much bigger than the mass of $\Xi^+_{cc}(3520)$. In other words, the mass of $\Xi^+_{cc}(3520)$ is too small to be assigned as the $\frac{3}{2}^+$ doubly charmed baryons.

According to the quark model, the lowest lying baryon states should be the ground states ($L = 0$) including the $J = \frac{1}{2}^+$ and $J = \frac{3}{2}^+$ doublets. In the above discussion, we have manifested that the mass of $\Xi^+_{cc}(3520)$ is too small to be assigned as the $\frac{3}{2}^+$ doubly charmed baryons in Regge phenomenology. Therefore, we can conclude that $\Xi^+_{cc}(3520)$ should be the ground state with its J^P as $\frac{1}{2}^+$. This assignment coincides with the fact that $\Xi^+_{cc}(3520)$ is observed to decay only weakly [3] [if the J^P of $\Xi^+_{cc}(3520)$ were $\frac{3}{2}^+$, it should decay electromagnetically [80]].

Inserting the masses of Σ, Ξ, Σ_c, Ω_c, and $\Xi^+_{cc}(3520)$ into Eq. (60), we can get the mass of Ω_{cc}, $M_{\Omega_{cc}} = 3650.4 \pm 6.3$ GeV, where the uncertainty comes from the errors of the input data. Comparison of the masses of Ξ_{cc} and Ω_{cc} extracted in the present work and those given in other references is shown in Table VIII.

D. Parameters of Regge trajectories for the $\frac{3}{2}^+$ SU(4) multiplet

In Ref. [21], the parameters of Regge trajectories for different meson multiplets and the masses of the meson states lying on those Regge trajectories were estimated. In this section, we will first extract the masses of the $\frac{3}{2}^+$ SU(4) baryons absent from the baryon summary table so far. And then, with all the $\frac{3}{2}^+$ SU(4) baryon masses and the value of α_Δ, we will calculate all the parameters (Regge slopes and intercepts) for the $\frac{3}{2}^+$ baryon trajectories. After that, we will estimate the masses of the orbital excited baryons lying on these Regge trajectories.

All the masses of $\frac{3}{2}^+$ light baryons and charmed baryons are known experimentally. We need to know one of the masses of the baryons Ξ^*_{cc}, Ω^*_{cc}, and Ω_{ccc} to calculate the masses of the other two states using the quadratic mass equalities (52). First, we apply Eq. (18) to extract the mass of Ξ^*_{cc} or Ω^*_{cc}. When $i = n$, $j = c$, and $q = s$, we could

TABLE VIII. The masses of doubly and triply charmed baryons (in units of MeV). The numbers in boldface are the experimental values taken as the input.

	Ξ_{cc}	Ω_{cc}	Ξ^*_{cc}	Ω^*_{cc}	Ω_{ccc}
Pre.	**3518.9±0.9**	3650.4 ± 6.3	3684.4 ± 4.4	3808.4 ± 4.3	4818.9 ± 6.8
[23]	3610 ± 3	3804 ± 8	3735 ± 17	3850 ± 25	4930 ± 45
[83]	3511	3664	3630	3764	4747
[84]	3524	3524	3548	3548	4632
[82]	3510	3719	3548	3746	4803
[85]	3642	3732	3723	3765	4473
[86]	3676	3815	3753	3876	4965
[87]	3635	3800	3695 ± 60	3840 ± 60	4925 ± 90
[88]	3549 ± 13 ± 19 ± 92	3663 ± 11 ± 17 ± 95	3641 ± 18 ± 8 ± 95	3734 ± 14 ± 8 ± 97	
[89]	3660 ± 70	3740 ± 80	3740 ± 70	3820 ± 80	
[90]	3620	3778	3727	3872	
[91]	3520	3619	3630	3721	
[92]	3478	3594	3610	3730	
[93]		3737		3797	4787
[94]	3550 ± 80	3650 ± 80			
[95]					4760 ± 60
[96]					4790

insert the masses of Δ, Σ^*, Ξ^*, Σ_c^*, and Ξ_c^* into the relation (18) to calculate $M_{\Xi_{cc}^*}$. When $i = s$, $j = c$, and $q = s$, we could insert the masses of Σ^*, Ξ^*, Ω, Ξ_c^*, and Ω_c^* into the relation (18) to calculate $M_{\Omega_{cc}^*}$. However, we find that the numerical results of $M_{\Xi_{cc}^*}$ and $M_{\Omega_{cc}^*}$ are very sensitive to the errors of the light baryon masses. Therefore, another way is needed to calculate the mass of Ξ_{cc}^* or Ω_{cc}^*. In Sec. III C, $\Xi_{cc}^+(3520)$ was assigned as the ground $\frac{1}{2}^+$ doubly charmed baryon. This may open a window to extract the masses of $\frac{3}{2}^+$ doubly charmed baryons.

The first-order GMO formula for the baryon octet,

$$2(M_N + M_\Xi) = (3M_\Lambda + M_\Sigma), \tag{70}$$

is usually generalized to charmed cases by replacing the s quark with the c quark,

$$2(M_N + M_{\Xi_{cc}}) = 3M_{\Lambda_c} + M_{\Sigma_c}. \tag{71}$$

The quadratic form of Eq. (71) is

$$2(M_N^2 + M_{\Xi_{cc}}^2) = 3M_{\Lambda_c}^2 + M_{\Sigma_c}^2. \tag{72}$$

However, the existence of high-order breaking effects in Eqs. (71) and (72) is obvious [23]. We use $\delta_{nc}^{(1/2)^+}$ to denote this effect in Eq. (72),

$$\delta_{nc}^{(1/2)^+} = M_N^2 + M_{\Xi_{cc}}^2 - 2\left(\frac{3M_{\Lambda_c}^2 + M_{\Sigma_c}^2}{4}\right). \tag{73}$$

Assuming that $\delta_{nc}^{(1/2)^+} = \delta_{nc}^{(3/2)^+}$, we have

$$\delta_{nc}^{(1/2)^+} = M_N^2 + M_{\Xi_{cc}}^2 - 2\left(\frac{3M_{\Lambda_c}^2 + M_{\Sigma_c}^2}{4}\right) = \delta_{nc}^{(3/2)^+} = M_\Delta^2 + M_{\Xi_{cc}^*}^2 - 2M_{\Sigma_c^*}^2. \tag{74}$$

Inserting the masses of N, Λ_c, Σ_c, $\Xi_{cc}^+(3520)$, Δ, and Σ_c^* into Eq. (74), we have $M_{\Xi_{cc}^*} = 3684.4 \pm 4.4$ MeV, where the uncertainty comes from the errors of the input data.

Then, inserting the masses of Δ, Ω, Σ_c^*, Ξ_c^*, and Ξ_{cc}^* into Eqs. (18) and (52a), we have

$$\frac{(4M_{\Sigma^*}^2 - M_\Delta^2 - M_{\Xi^*}^2) + \sqrt{(4M_{\Sigma^*}^2 - M_\Delta^2 - M_{\Xi^*}^2)^2 - 4M_{\Delta^*}^2 M_{\Xi^*}^2}}{2M_{\Xi^*}^2} = \frac{[(4M_{\Xi_c^*}^2 - M_\Delta^2 - M_{\Xi_{cc}^*}^2) + \sqrt{(4M_{\Xi_c^*}^2 - M_\Delta^2 - M_{\Xi_{cc}^*}^2)^2 - 4M_\Delta^2 M_{\Xi_{cc}^*}^2}]/2M_{\Xi_{cc}^*}^2}{[(4M_{\Xi_c^*}^2 - M_{\Xi^*}^2 - M_{\Xi_{cc}^*}^2) + \sqrt{(4M_{\Xi_c^*}^2 - M_{\Xi^*}^2 - M_{\Xi_{cc}^*}^2)^2 - 4M_{\Xi^*}^2 M_{\Xi_{cc}^*}^2}]/2M_{\Xi_{cc}^*}^2}, \tag{75}$$

$$M_\Delta^2 + M_{\Xi^*}^2 - 2M_{\Sigma^*}^2 = M_{\Sigma^*}^2 + M_\Omega^2 - 2M_{\Xi^*}^2. \tag{76}$$

Then, we have the masses of Σ^* and Ξ^*. Inserting the masses of Σ^*, Ξ^*, Ω, Σ_c^*, Ξ_c^*, and Ξ_{cc}^* into the quadratic mass equations in Eq. (52), we have the masses of Ω_c^*, Ω_{cc}^*, and Ω_{ccc}.

In this way, all the masses of $\frac{3}{2}^+$ SU(4) baryons are known. With these masses and the value $\alpha'_\Delta = 2/(M_{\Delta(1950)}^2 - M_\Delta^2) = 0.9022 \pm 0.0285\ \mathrm{GeV}^{-2}$ [where the uncertainty comes from the errors of the input masses of $\Delta(1950)$ and Δ], we have all the Regge slopes of $\frac{3}{2}^+$ trajectories from Eq. (13). Then, with these masses and the obtained Regge slopes, we have all the Regge intercepts of $\frac{3}{2}^+$ trajectories from Eq. (1).

From Eq. (1), one has

$$M_{J+2} = \sqrt{M_J^2 + \frac{2}{\alpha'}}. \tag{77}$$

Then, using this equation, the masses of the orbital excited baryons ($J^P = \frac{7}{2}^+, \frac{11}{2}^+$) lying on the $\frac{3}{2}^+$ trajectories can be calculated. The Regge intercepts and the Regge slopes of the $\frac{3}{2}^+$ trajectories are shown in Table IX. The masses of light baryons, charmed baryons, and doubly and triply charmed baryons lying on the $\frac{3}{2}^+$ trajectories are shown in Tables X, XI, and XII, respectively.

TABLE IX. The Regge slopes (in units of GeV^{-2}) and the Regge intercepts of the $\frac{3}{2}^+$ trajectories.

	Δ	Σ^*	Ξ^*	Ω	Σ_c^*	Ξ_c^*	Ω_c^*	Ξ_{cc}^*	Ω_{cc}^*	Ω_{ccc}
α'	0.902 $\pm$0.029	0.862 $\pm$0.036	0.825 $\pm$0.042	0.791 $\pm$0.047	0.644 $\pm$0.023	0.623 $\pm$0.026	0.604 $\pm$0.029	0.501 $\pm$0.019	0.488 $\pm$0.021	0.410 $\pm$0.016
$a(0)$	0.131 $\pm$0.046	−0.151 $\pm$0.074	−0.432 $\pm$0.102	−0.713 $\pm$0.133	−2.583 $\pm$0.147	−2.864 $\pm$0.174	−3.145 $\pm$0.203	−5.296 $\pm$0.249	−5.577 $\pm$0.276	−8.009 $\pm$0.351

TABLE X. The masses of the light baryons lying on the $\frac{3}{2}^+$ trajectories (in units of MeV). The numbers in boldface are the experimental values taken as the input.

	M_Δ			M_{Σ^*}		
	$J = 3/2$	$J = 7/2$	$J = 11/2$	$J = 3/2$	$J = 7/2$	$J = 11/2$
Pre.	**1232±1**	**1932.5±17.5**	2440 ± 28	1383.9 ± 2.3	2058 ± 22	2560 ± 36
Exp	1232 ± 1	1915 ∼ 1950	2300 ∼ 2500	1384.6 ± 2.6	2015 ∼ 2040	
[85]	1261	1951	2442	1411	2027	
[97]	1232	1921	2175			
[98]	1232	1950	2467	1394	2056	
[99]	1290	1954		1377	2029	
[100]	1232.9 ± 1.2	1923.3 ± 0.5				
[101]	1230	1940	2450	1370	2060	
[102]	1240	1915		1390	2015	
	M_{Ξ^*}			M_Ω		
	$J = 3/2$	$J = 7/2$	$J = 11/2$	$J = 3/2$	$J = 7/2$	$J = 11/2$
Pre.	1530.2 ± 1.9	2183 ± 27	2681 ± 45	**1672.45±0.29**	2308 ± 32	2802 ± 54
Exp	1533.4 ± 2.1			1672.45 ± 0.29		
[85]	1539	2169		1636	2292	
[97]						
[98]	1540	2157		1672		
[99]	1502	2142		1665	2293	
[101]	1505	2180		1635	2295	
[102]	1530			1675		

TABLE XI. The masses of the charmed baryons lying on the $\frac{3}{2}^+$ trajectories (in units of MeV).

	$M_{\Sigma_c^*}$			$M_{\Xi_c^*}$			$M_{\Omega_c^*}$		
	$J = 3/2$	$J = 7/2$	$J = 11/2$	$J = 3/2$	$J = 7/2$	$J = 11/2$	$J = 3/2$	$J = 7/2$	$J = 11/2$
Pre.	**2518.0±1.9**	3073 ± 18	3543 ± 30	**2646.4±1.6**	3196 ± 22	3664 ± 37	2774.1 ± 5.5	3318 ± 28	3784 ± 46
Exp	2518.0 ± 1.9			2646.6 ± 1.4			2768.3 ± 3		
[83]	2481			2642			2764		
[85]	2539			2651			2721		
[86]	2519	3015		2650	3100		2776	3206	
[89]	2520 ± 20			2650 ± 20			2770 ± 30		
[90]	2518	3015		2654	3136		2768	3237	
[101]	2495	3090							
[102]	2510	3010							

TABLE XII. The masses of the doubly and triply charmed baryons lying on the $\frac{3}{2}^+$ trajectories (in units of MeV).

	$M_{\Xi_{cc}^*}$			$M_{\Omega_{cc}^*}$			$M_{\Omega_{ccc}}$		
	$J = 3/2$	$J = 7/2$	$J = 11/2$	$J = 3/2$	$J = 7/2$	$J = 11/2$	$J = 3/2$	$J = 7/2$	$J = 11/2$
Pre.	3684.4 ± 4.4	4192 ± 19	4644 ± 32	3808.4 ± 4.3	4313 ± 23	4765 ± 39	4818.9 ± 6.8	5302 ± 21	5744 ± 34
Exp									
[86]	3753	4097		3876	4230		4965	5331	
[92]	3610	4089		3730					

The masses of Ξ_{cc}^*, Ω_{cc}^*, and Ω_{ccc} extracted in the present work and those given in other references are also shown in Table VIII. From Table VIII, we can see that the masses of $\frac{1}{2}^+$ and $\frac{3}{2}^+$ doubly and triply charmed baryons predicted by us agree well with those given in most other references. The predictions in Ref. [23] are bigger than ours because of the approximation adopted there that baryons in the light quark sector have common Regge slopes. The mass splitting obtained in the framework of nonrelativistic effective field theories of QCD, $M_{\Xi_{cc}^*} - M_{\Xi_{cc}} = 120 \pm 40$ MeV (see Ref. [103], and references therein), agrees with our present results shown Table VIII.

E. Parameters of Regge trajectories for the $\frac{1}{2}^+$ SU(4) multiplet

Up to now, all the masses of ground $\frac{1}{2}^+$ SU(4) baryons are known. We will determine the Regge slopes and intercepts of the $\frac{1}{2}^+$ SU(4) multiplet and give predictions for masses of the $\frac{5}{2}^+$ and $\frac{9}{2}^+$ baryon states lying on these Regge trajectories.

Recently, the spin parity of the $\Lambda_c^+(2880)$ baryon was determined by experiment. $\Lambda_c^+(2880)$ was observed by CLEO in the $\Lambda_c\pi^+\pi^-$ mode [4] and then confirmed by *BABAR* in the D^0p mode recently [6]. From the analysis of the angular distribution in its $\Sigma_c(2455)\pi$ decays and the small ratio, $\Gamma_{\Sigma_c(2520)\pi}/\Gamma_{\Sigma_c(2455)\pi} \simeq 0.23$, measured by BELLE it is concluded that the J^P of $\Lambda_c^+(2880)$ is $\frac{5}{2}^+$ [5]. This spin-parity assignment is in agreement with the theoretical investigation that $\Lambda_c^+(2880)$ is the orbital ($L = 2$) excitation of Λ_c^+ [90,104]. Therefore, $\Lambda_c^+(2880)$ and Λ_c^+ lie on the common Regge trajectory. We can have the Regge slope of Λ_c^+ from Eq. (11),

$$\alpha'_{\Lambda_c} = \frac{\frac{5}{2} - \frac{1}{2}}{M^2_{\Lambda_c^+(2880)} - M^2_{\Lambda_c^+}} = 0.650 \pm 0.005 \text{ GeV}^{-2}. \quad (78)$$

From Eq. (11), we also have

$$\alpha'_N = \frac{2}{M^2_{N(1680)} - M^2_N} = 1.022 \pm 0.009 \text{ GeV}^{-2},$$
$$\alpha'_\Lambda = \frac{2}{M^2_{\Lambda(1820)} - M^2_\Lambda} = 0.967 \pm 0.009 \text{ GeV}^{-2}. \quad (79)$$

We assume that $\alpha'_\Sigma = \alpha'_{\Sigma^*}$, $\alpha'_\Xi = \alpha'_{\Xi^*}$, $\alpha'_{\Sigma_c} = \alpha'_{\Sigma_c^*}$, $\alpha'_{\Xi'_c} = \alpha'_{\Xi_c^*}$, $\alpha'_{\Omega_c} = \alpha'_{\Omega_c^*}$, $\alpha'_{\Xi_{cc}} = \alpha'_{\Xi_{cc}^*}$, and $\alpha'_{\Omega_{cc}} = \alpha'_{\Omega_{cc}^*}$. Although the slopes of a heavy baryon containing a scalar diquark and that containing an axial-vector diquark are different, we assume that γ_s for the heavy baryons containing scalar diquarks is approximately the same as γ_s for heavy baryons containing axial-vector diquarks, i.e.,

$$\frac{1}{\alpha'_{\Xi_c}} - \frac{1}{\alpha_{\Lambda_c}} = \frac{1}{\alpha'_{\Xi'_c}} - \frac{1}{\alpha_{\Sigma_c}}.$$

Then, all the Regge slopes of $\frac{1}{2}^+$ SU(4) baryons are known and shown in Table XIII.

With the masses and the obtained Regge slopes for the $\frac{1}{2}^+$ baryons, we have all the Regge intercepts of $\frac{1}{2}^+$ trajectories from Eq. (1). Then, using Eq. (77), the masses of orbital excited baryons ($J^P = \frac{5}{2}^+, \frac{9}{2}^+$) lying on the $\frac{1}{2}^+$ trajectories can be calculated. The Regge intercepts of the $\frac{1}{2}^+$ trajectories are also shown in Table XIII. The masses of light baryons, charmed baryons, and doubly charmed baryons lying on the $\frac{1}{2}^+$ trajectories are shown in Tables XIV, XV, and XVI, respectively.

F. Charm-strange baryons

There are five charm-strange baryons presented in PDG 2006 [15]: Ξ_c, Ξ'_c, Ξ^*_c, $\Xi_c(2790)$, and $\Xi_c(2815)$. $\Xi_c(2790)$ and $\Xi_c(2815)$ were assigned as the first orbital (1P) excitations of Ξ_c with $J^P = \frac{1}{2}^-$ and $J^P = \frac{3}{2}^-$, respectively.

TABLE XIII. The Regge intercepts and Regge slopes of the $\frac{1}{2}^+$ trajectories.

	N	Λ	Σ	Ξ	Λ_c	Σ_c	Ξ_c	Ξ'_c	Ω_c	Ξ_{cc}	Ω_{cc}
$a(0)$	−0.401 ±0.010	−0.704 ±0.011	−0.727 ±0.059	−0.933 ±0.082	−2.900 ±0.003	−3.377 ±0.137	−3.337 ±0.043	−3.638 ±0.184	−3.892 ±0.217	−5.699 ±0.228	−6.002 ±0.291
α'	1.022 ±0.009	0.967 ±0.009	0.862 ±0.036	0.825 ±0.042	0.650 ±0.005	0.644 ±0.022	0.629 ±0.006	0.623 ±0.026	0.604 ±0.029	0.501 ±0.018	0.488 ±0.020

TABLE XIV. The masses of the light baryons lying on the $\frac{1}{2}^+$ trajectories (in units of MeV). The numbers in boldface are the experimental values taken as the input.

	M_N			M_Λ			M_Σ			M_Ξ		
	$J=1/2$	$J=5/2$	$J=9/2$	$J=1/2$	$J=5/2$	$J=9/2$	$J=1/2$	$J=5/2$	$J=9/2$	$J=1/2$	$J=5/2$	$J=9/2$
Pre.	**938.92 ±0.65**	**1685 ±5**	2190 ±8.0	**1115.683 ±0.006**	**1820 ±5**	2319 ±7.8	**1193.17 ±4.11**	1935 ±27	2463 ±41	**1318.07 ±4.31**	2040 ±33	2566 ±50
Exp	938.92 ±0.65	1680 ∼1690	2200 ∼2300	1115.683 ±0.006	1815 ∼1825	2340 ∼2370	1193.17 ±4.11	1900 ∼1935		1318.07 ±4.31	2025 ±5	
[85]	939	1723	2221	1108	1834	2340	1190	1956		1310	2013	
[97]	940	1722	2378									
[98]	939	1779	2334	1144	1895	2424	1144	1895	2424	1317	2004	2510
[99]	990	1744		1115	1844		1192	1906		1317	2014	
[100]		1683.2 ± 0.7	2270 ± 11									
[101]	960	1770	2345	1115	1890		1190	1955		1305	2045	
[102]	940	1715		1110	1815		1915	1940		1320		

TABLE XV. The masses of the charmed baryons lying on the $\frac{1}{2}^+$ trajectories (in units of MeV). The numbers in boldface are the experimental values taken as the input.

	M_{Λ_c} $J=1/2$	$J=5/2$	$J=9/2$	M_{Σ_c} $J=1/2$	$J=5/2$	$J=9/2$	M_{Ξ_c} $J=1/2$	$J=5/2$	$J=9/2$	$M_{\Xi'_c}$ $J=1/2$	$J=5/2$	$J=9/2$	M_{Ω_c} $J=1/2$	$J=5/2$	$J=9/2$
Pre.	**2286.46 ±0.14**	**2881.5 ±0.3**	3737 ±0.61	**2453.56 ±0.85**	3021 ±18	3497 ±31	**2469.5 ±2.0**	3046 ±7	3529 ±10	**2576.9 ±4.2**	3138 ±24	3614 ±40	**2697.5 ±2.6**	3254 ±26	3729 ±44
Exp	2286.46 ±0.14	2881.5 0.3		2453.56 ±0.85			2469.5 ±1.2			2576.9 ±4.2			2697.5 ±2.6		
[83]	2243			2380			2425			2530			2678		
[85]	2272			2459			2469			2595			2688		
[86]	2268	2887		2455	3003		2492	2995		2592	3100		2718	3196	
[89]	2285 ± 1			2453 ± 3			2468 ± 3			2580 ± 20			2710 ± 30		
[90]	2294	2883		2439	2960		2481	3042		2578	3087		2698	3187	
[101]	2265	2910		2440	3065										
[102]	2260	2810		2440	3010										

TABLE XVI. The masses of the doubly charmed baryons lying on the $\frac{1}{2}^+$ trajectories (in units of MeV). The numbers in boldface are the experimental values taken as the input.

	$M_{\Xi_{cc}}$ $J=1/2$	$J=5/2$	$J=9/2$	$M_{\Omega_{cc}}$ $J=1/2$	$J=5/2$	$J=9/2$
Pre.	**3518.9±0.9**	4047 ± 19	4514 ± 33	3650.4 ± 6.3	4174 ± 26	4639 ± 41
Exp	3518.9 ± 0.9					
[86]	3676	4047		3815	4202	
[92]	3478	4050		3594		

Recently, $\Xi_c(2980)$ and $\Xi_c(3077)$ were first reported by Belle [8] and then confirmed by BABAR [7]. *BABAR* also reported the observation of $\Xi_c^+(3055)$ and $\Xi_c^+(3123)$ [9]. The J^P of $\Xi_c(2980)$, $\Xi_c(3055)$, $\Xi_c(3077)$, and $\Xi_c(3123)$ have not been measured. The masses of these states imply that they could be the states with the total quark orbital angular momentum $L = 2$. Here we attempt to study which Regge trajectory these states may lie on.

From Table XV, it can be seen that the mass of $\Xi_c(3123)$ coincides with the mass of $\Xi'_c(\frac{5}{2}^+)$. Therefore, $\Xi_c(3123)$ probably lies on the Regge trajectory of Ξ'_c. In other words, $\Xi_c(3123)$ may be the orbital excited ($J^P = \frac{5}{2}^+$) state of Ξ'_c containing an axial-vector diquark. This assignment is in agreement with Ebert's assignment in the relativistic quark model [90]. We can also see that both the masses of $\Xi_c(3055)$ and $\Xi_c(3077)$ are near the mass of $\Xi_c(\frac{5}{2}^+)$. The mass of $\Xi_c(2980)$ is lower compared with that of $\Xi_c(\frac{5}{2}^+)$ or $\Xi'_c(\frac{5}{2}^+)$.

The above comments can be seen more clearly when combining with the slopes of these baryons. As mentioned above, the slopes of Regge trajectories decrease with quark mass increase. Therefore, the slope of Ξ_c (Ξ'_c, Ξ^*_c) is less than the slope of Λ_c,

$$\alpha'_{\Xi_c^{(\prime,*)}} < 0.650\ \mathrm{GeV}^{-2}. \tag{80}$$

Assuming that $\Xi_c(2980)$, $\Xi_c(3055)$, $\Xi_c(3077)$, or $\Xi_c(3123)$ lies on the same Regge trajectory with $\Xi_c^{(\prime,*)}$, respectively, so that the difference between the angular momenta of these baryons with those of $\Xi_c^{(\prime,*)}$ is $\Delta L = 2$, we obtain the values of the Regge slopes for $\Xi_c^{(\prime,*)}$ shown in Table XVII.

From the relation (80), Tables XIII, XV, and XVII, we can conclude that $\Xi_c(2980)$ cannot lie on the Regge trajectory of Ξ_c, Ξ'_c, and Ξ^*_c. [$\Xi_c(2980)$ can be interpreted in the relativistic quark as the first radial (2S) excitation of the Ξ_c with $J^P = \frac{1}{2}^+$ containing the light axial-vector diquark [90].] Both $\Xi_c(3055)$ and $\Xi_c(3077)$ can be assigned as the $J^P = \frac{5}{2}^+$ state. $\Xi_c(3123)$ probably lies on the Regge trajectory of Ξ'_c. In other words, $\Xi_c(3123)$ may be the orbital excited ($\Delta L = 2$) state of Ξ'_c with $J^P = \frac{5}{2}^+$ containing an axial-vector diquark. Further study is needed to determine the J^P of these states more accurately.

TABLE XVII. The values (in units of GeV^{-2}) of the Regge slope for $\Xi_c^{(\prime,*)}$ given from Eq. (1) under the assumption that $\Xi_c(2790)$, $\Xi_c(2815)$, $\Xi_c^+(3055)$, or $\Xi_c(3123)$ lies on the same Regge trajectory with $\Xi_c^{(\prime,*)}$, respectively.

	$\Xi_c(2980)$	$\Xi_c(3055)$	$\Xi_c(3077)$	$\Xi_c(3123)$
α'_{Ξ_c}	0.728	0.619	0.591	0.547
$\alpha'_{\Xi'_c}$	0.907	0.944	0.703	0.643
$\alpha'_{\Xi^*_c}$	1.086	0.860	0.806	0.727

IV. DISCUSSION AND CONCLUSION

In this work, under the main assumption that the quasi-linear Regge trajectory ansatz is suitable to describe meson spectra and baryon spectra, with the requirements of the additivity of intercepts and inverse slopes, some useful linear mass inequalities, quadratic mass inequalities, and quadratic mass equalities are derived for mesons and baryons.

Based on these relations, we have given upper limits and lower limits for some mesons and baryons. The masses of $\bar{b}c$ and $s\bar{s}$ belonging to the pseudoscalar (1^1S_0), vector (1^3S_1), and tensor (1^3P_2) meson multiplets are also extracted. We suggest that the J^P of $\Xi_{cc}^+(3520)$ should be $\frac{1}{2}^+$. The parameters of the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ SU(4) baryon trajectories are extracted and the masses of the orbital excited baryons lying on the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ trajectories are estimated. We propose that $\Xi_c(3123)$ may be a candidate for the orbital excited ($\Delta L = 2$) state of Ξ_c' with $J^P = \frac{5}{2}^+$ containing an axial-vector diquark. The predictions are in reasonable agreement with the existing experimental data and those suggested in many other different approaches.

In Sec. II C, we showed that the linear mass GMO formula is an inequality in fact and the quadratic mass GMO formula is also an inequality with the sign opposite to the linear case. Encouragingly, the linear meson mass inequalities (26) and the linear baryon mass inequalities (30) are similar to those derived from a general illation in QCD for the ground hadron states [18–20] [The authors of Ref. [20] also point out that the linear mass inequalities (26) and (30) hold for many potentials, although the linear baryon mass inequality (30) does not hold for some special potentials.] In Ref. [19], Nussinov and Lampert showed that the linear meson mass inequality (26) satisfies the experimental data of the well-established multiplets (vector 1^3S_1, tensor 1^3P_2, axial-vector 1^3P_1, and scalar 1^3P_0) with different flavor combinations of i and j, and the linear baryon mass inequality (30) satisfies the experimental data of the baryon octet and the baryon decuplet. They gave the lower limits for the masses of some unobserved mesons and baryons with the linear mass inequalities. In our work, in addition to the lower limits, we also give the upper limits for the masses of hadrons. We can see from Tables III, IV, and V that these limits agree with the existing data. The mass ranges in Tables III, IV, and V are narrow (smaller than 0.5 GeV) for hadrons which do not contain a b quark. These mass ranges will be useful for the discovery of the unobserved hadron states. When a b quark is involved, the mass ranges in Tables III, IV, and V become large (could be as large as 1 to 2 GeV) and consequently, the constraints become weaker. However, since many hadrons containing a b quark have not been observed in experiments, these mass ranges may also provide helpful guidance for the discovery of these hadrons.

As far as we know, there is only one work to study the quadratic meson mass inequalities. In Ref. [17], with the current-algebra technique, corrections to the GMO quadratic mass formula due to second-order SU(4) breaking was discussed by Simard and Suzuki. They gave a quadratic mass inequality for pseudoscalar mesons,

$$\frac{1}{2}[M_\pi^2 + (\tfrac{2}{3}M_\eta^2 + \tfrac{1}{3}M_{\eta'}^2)] + M_{\eta_c(1S)}^2 - 2M_D^2 > 0, \tag{81}$$

and two quadratic mass inequalities for vector mesons,

$$\frac{1}{2}(M_\rho^2 + M_\omega^2) + M_{J/\psi(1S)}^2 - 2M_{D^*}^2 < 0, \tag{82}$$

$$M_\phi^2 + M_{J/\psi(1S)}^2 - 2M_{D_s^*}^2 < 0. \tag{83}$$

The sign of the quadratic mass inequality (81) is the same as that of our quadratic mass inequality (29), but the signs of the quadratic mass inequalities (82) and (83) are opposite to that of our quadratic mass inequality (26). The calculations (shown in Tables III and IV) manifest that the quadratic mass inequalities (29) and (81) do satisfy the present experimental data [15] while the quadratic mass inequalities (82) and (83) do not.

We stress that quadratic baryon mass inequality (31) has not been given before. From Tables III, IV, and V, we can see that the inequalities (26) and (29)–(31) agree well with the existing experimental data [15]. These inequalities (26) and (29)–(31) indicate the existence of higher-order breaking effects.

For the Regge slopes of $\frac{3}{2}^+$ SU(4) baryons, from Table IX, we can see that $\alpha'_\Delta > \alpha'_{\Sigma^*} > \alpha'_{\Xi^*} > \alpha'_\Omega > \alpha'_{\Sigma_c^*} > \alpha'_{\Xi_c^*} > \alpha'_{\Omega_c^*} > \alpha'_{\Xi_{cc}^*} > \alpha'_{\Omega_{cc}^*} > \alpha'_{\Omega_{ccc}}$ and $a_\Delta(0) > a_{\Sigma^*}(0) > a_{\Xi^*}(0) > a_\Omega(0) > a_{\Sigma_c^*}(0) > a_{\Xi_c^*}(0) > a_{\Omega_c^*}(0) > a_{\Xi_{cc}^*}(0) > a_{\Omega_{cc}^*}(0) > a_{\Omega_{ccc}}(0)$. These inequalities coincide with the expectation that the slopes of Regge trajectories decrease with quark mass increase (flavor dependent).

From Table II, we can see that the values of δ_{ij}^m are very sensitive to quark flavors i and j. For the same i and j, δ_{ij}^m are approximately a constant (only a little different among different multiplets). This character may be used to predict meson masses approximately in some cases. The calculations (Table II) show that $\delta_{ns} < \delta_{sc} < \delta_{nc} < \delta_{cb} < \delta_{sb} < \delta_{nb}$. For the light mesons and baryons, δ_{ns} is close to zero. Letting $\delta \to 0$, one can get the usual Gell-Mann–Okubo quadratic relations, namely, the first order of Gell-Mann–Okubo relations. For the heavy mesons or baryons, δ_{Qq} are large. In this case, the quadratic mass inequalities are far from equalities. These features imply that the higher-order breaking effects arise with the quark mass increase.

To the second order, for baryons, as shown by Okubo long ago [34], both the well-known mass relation for the baryon octet [Eq. (70)] and the equal spacing rule for the baryon decuplet ($M_\Omega - M_{\Xi^*} = M_{\Xi^*} - M_{\Sigma^*} = M_{\Sigma^*} - M_\Delta$) do not hold. Only one relation remains,

$$M_\Omega - M_\Delta = 3(M_{\Xi^*} - M_{\Sigma^*}). \tag{84}$$

This second-order linear mass equation was given by Morpurgo in the relativistic field theory [35] and by

Lebed in the chiral perturbation theory [36] and was also given in Refs. [26–31] mentioned above.

A special equation among the masses of baryons involving only two flavors can be derived by taking $\delta^b_{ij}|_{q=i} = \delta^b_{ij}|_{q=j}$ in Eq. (51)

$$\delta^b_{ij}|_{q=i} = M^2_{iii} + M^2_{jji} - 2M^2_{iij} = \delta^b_{ij}|_{q=j} = M^2_{iij} + M^2_{jjj} - 2M^2_{ijj}, \tag{85}$$

namely,

$$M^2_{jjj} - M^2_{iii} = 3(M^2_{ijj} - M^2_{iij}). \tag{86}$$

In the light quark sector, when $i = n$, $j = s$, for the $\frac{3}{2}^+$ multiplet, we have

$$M^2_{\Omega} - M^2_{\Delta} = 3(M^2_{\Xi^*} - M^2_{\Sigma^*}). \tag{87}$$

The quadratic equation (87) was also given by Tait in the study of the unification SO(6, 1) as a spectrum generating algebra [32].

In the light sector, both the linear mass equation, Eq. (84), and the quadratic mass equation, Eq. (87), can be satisfied by the experimental data. The deviations from both of them are not more than 2%.

However, generally speaking, the linear mass relation and the quadratic mass relation may not be held at the same time. On the other hand, the quadratic mass equation (86) and the linear form of Eq. (86) should give very different mass values for heavy baryons. The masses of the charmed and bottom particles discovered in the near future will numerically test which of them is realized in nature.

Theoretically, we also have some reasons besides the Regge theory to believe that mass formulas for mesons and baryons should take the quadratic form rather than the linear form: (1) The square of the mass operator (M^2) is the Casimir invariant of the Poincare group independent of any certain frame [105]; (2) formulas given by asymptotic chiral symmetry are indeed in quadratic form [106]; (3) in the infinite-momentum frame, formulas between energy eigenvalues of hadrons spontaneously lead to quadratic mass formulas [107]; (4) analysis on the algebraic approach indeed leads to quadratic mass formulas [32,108]. It was pointed out that the quadratic mass formula can be approximately written as the relevant linear mass formula when the mass splittings between the hadrons of the formula are small compared with the hadron masses [105,107].

To sum up, we conclude that quasilinear Regge trajectory and the additivity of intercepts and inverse slopes are indeed suitable to describe meson spectra and baryon spectra at present. The mass relations and the predictions may be useful for the discovery of the unobserved meson and baryon states and the J^P assignment of the meson and baryon states which will be observed in the future.

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