The energy-momentum of plane-fronted gravitational waves in the teleparallel equivalent of GR

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(Received 8 March 2008; published 1 August 2008)

We show that in the framework of the teleparallel equivalent of general relativity the gravitational energy-momentum of plane-fronted gravitational waves contained in an arbitrary three-dimensional volume V may be easily obtained and is non-negative in the frame of static observers.

DOI: 10.1103/PhysRevD.78.047502 PACS numbers: 04.20.Cv, 04.30.−w, 04.20.Fy

I. INTRODUCTION

The observation of gravitational waves is presently one of the main challenges of gravitational physics. Significant efforts are being undertaken to construct the necessary apparatus for the detection of the waves [1,2]. On the theoretical side there is no general agreement regarding the description of the energy-momentum carried by gravitational waves. The reason is that there is no generally accepted definition for the gravitational energy-momentum. The use of noncovariant energy-momentum pseudotensors totally obscures the analysis of the issue. In the framework of pseudotensors one concludes that the energy carried by a gravitational wave, considered as a gravitational fluctuation of the spacetime, is not gauge invariant, i.e., is coordinate dependent [3].

The energy-momentum of gravitational waves has been investigated with the help of the Bel-Robinson tensor [4]. The idea is to consider a gravitoelectromagnetic stress-energy tensor [5], whose properties are similar to the Maxwell stress-energy tensor in electrodynamics. Covariant approaches to the energy-momentum of gravitational waves have been investigated [6,7]. However, it must be noted that the Bel-Robinson tensor requires an additional multiplicative factor with physical dimensions to relate it to an acceptable energy-momentum tensor.

In this article we will show that in the spacetime of plane-fronted gravitational waves, the energy-momentum definition established in the teleparallel equivalent of general relativity (TEGR) allows us to obtain the gravitational energy-momentum enclosed in an arbitrary volume V of the three-dimensional space in an easy way. Moreover, it is straightforward to verify that the resulting expression is non-negative for any V. The resulting expression for the energy is naturally invariant under coordinate transformations of the three-dimensional space, and under time reparametrizations. In this sense, it is gauge invariant.

Notation: spacetime indices $\mu, \nu, \ldots$ and SO(3,1) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i$, $a = (0), (i)$. The tetrad field is denoted $e^a{}_\mu$, and the torsion tensor reads $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. The flat, Minkowski spacetime metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-+++)$. The determinant of the tetrad field is represented by $e = \det(e^a{}_\mu)$.

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II. THE ENERGY-MOMENTUM DEFINITION IN THE LAGRANGIAN FRAMEWORK

We assume that the spacetime geometry is defined by the tetrad field $e^a{}_\mu$ only. In this case we note that the only possible nontrivial definition for the torsion tensor is given by $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. This torsion tensor is related to the antisymmetric part of Cartan's connection $\Gamma^\lambda_{\mu\nu} = e^{a\lambda}\partial_\mu e_{a\nu}$, which is the connection of the Weitzenböck spacetime. However, the tetrad field also yields the metric tensor, which establishes the Riemannian geometry. Therefore, in the framework of a geometrical theory based only on the tetrad field one may use the concepts of both Riemannian and Weitzenböck geometries. $T_{a\mu\nu}$ is not covariant under local Lorentz transformations. The tetrad fields are interpreted as reference frames adapted to preferred fields of observers in spacetime. This interpretation is possible by identifying the $e_{(0)}{}^\mu$ components of the frame with the four-velocities u^μ of the observers, $e_{(0)}{}^\mu = u^\mu$ [8]. Therefore, two different sets of tetrad fields that yield the same metric tensor, and which are related by a local Lorentz transformation, represent different frames in the same spacetime.

The equivalence of the TEGR with Einstein's general relativity may be understood by means of an identity between the scalar curvature $R(e)$ constructed out of the tetrad field and a combination of quadratic terms of the torsion tensor,

$$eR(e) \equiv -e(\tfrac{1}{4}T^{abc}T_{abc} + \tfrac{1}{2}T^{abc}T_{bac} - T^aT_a) + 2\partial_\mu(eT^\mu). \tag{1}$$

The formulation of Einstein's general relativity in the context of the teleparallel geometry is presented in review articles [9–11] and in recent books [12,13].

The Lagrangian density of the TEGR is given by the combination of the quadratic terms on the right-hand side of Eq. (1),

$$L = -ke(\tfrac{1}{4}T^{abc}T_{abc} + \tfrac{1}{2}T^{abc}T_{bac} - T^aT_a) - L_M \equiv -ke\Sigma^{abc}T_{abc} - L_M, \quad (2)$$

where $k = c^3/16\pi G$, $T_a = T^b{}_{ba}$, $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$ and

$$\Sigma^{abc} = \tfrac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \tfrac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c). \quad (3)$$

L_M stands for the Lagrangian density for the matter fields. The field equations derived from (2) are equivalent to Einstein's equations. They read

$$e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu}) - e\left(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}\right) = \frac{1}{4k}eT_{a\mu}, \quad (4)$$

where $\delta L_M/\delta e^{a\mu} = eT_{a\mu}$. It is possible to show that the left-hand side of the equation above may be rewritten as $\frac{1}{2}e[R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)]$.

Equation (4) may be simplified as

$$\partial_\nu(e\Sigma^{a\lambda\nu}) = \frac{1}{4k}ee^a{}_\mu(t^{\lambda\mu} + T^{\lambda\mu}), \quad (5)$$

where $T^{\lambda\mu} = e_a{}^\lambda T^{a\mu}$ and $t^{\lambda\mu}$ is defined by

$$t^{\lambda\mu} = k(4\Sigma^{bc\lambda}T_{bc}{}^\mu - g^{\lambda\mu}\Sigma^{bcd}T_{bcd}). \quad (6)$$

In view of the antisymmetry property $\Sigma^{a\mu\nu} = -\Sigma^{a\nu\mu}$, it follows that

$$\partial_\lambda[ee^a{}_\mu(t^{\lambda\mu} + T^{\lambda\mu})] = 0. \quad (7)$$

The equation above yields the continuity (or balance) equation,

$$\frac{d}{dt}\int_V d^3x\, ee^a{}_\mu(t^{0\mu} + T^{0\mu}) = -\oint_S dS_j[ee^a{}_\mu(t^{j\mu} + T^{j\mu})]. \quad (8)$$

We identify $t^{\lambda\mu}$ as the gravitational energy-momentum tensor [14], and

$$P^a = \int_V d^3x\, ee^a{}_\mu(t^{0\mu} + T^{0\mu}), \quad (9)$$

as the total energy-momentum contained within a volume V of the three-dimensional space. In view of (5), Eq. (9) may be written as

$$P^a = -\int_V d^3x\,\partial_j\Pi^{aj}, \quad (10)$$

where $\Pi^{aj} = -4ke\Sigma^{a0j}$. Π^{aj} is the momentum canonically conjugated to e_{aj} [15]. The expression above is the definition for the gravitational energy-momentum presented in Ref. [16], obtained in the framework of the Hamiltonian vacuum field equations. We note that (7) is a true energy-momentum conservation equation.

By transforming the SO(3,1) index a in Eq. (4) into a spacetime index λ, say, we find that the left-hand side of (4) becomes the symmetric tensor density $\frac{1}{2}eR_{\lambda\mu}$. For spin 0 and spin 1 fields the energy-momentum tensor for the matter fields $T_{\lambda\mu}$ is symmetric. For Dirac spinor fields, $T_{\lambda\mu}$ is also symmetric provided the coupling of the spinor field with the gravitational field is constructed out of the torsion-free Levi-Civita connection ${}^o\omega_{\mu ab}$ [17]. In the evaluation of $T_{\lambda\mu}$ one first obtains a tensor that is not symmetric. However, it has been shown that the antisymmetric part of $T_{\lambda\mu}$ vanishes in view of the Dirac equation. This issue has been discussed in detail in Ref. [17] and references therein. As for the tensor $t^{\lambda\mu}$, in general it is not symmetric. So far it is not clear the meaning of the antisymmetric part of $t^{\lambda\mu}$.

The emergence of a nontrivial total divergence is a feature of theories with torsion. The integration of this total divergence yields a surface integral. If we consider the $a = (0)$ component of Eq. (10) and adopt asymptotic boundary conditions for the tetrad field we find [16] that the resulting expression is precisely the surface integral at infinity that defines the ADM energy [18]. This fact is a strong indication (but no proof) that Eq. (10) does indeed represent the gravitational energy-momentum.

The evaluation of definition (10) is carried out in the configuration space. The definition is invariant under (i) general coordinate transformations of the three-dimensional space, (ii) time reparametrizations, and (iii) global SO(3,1) transformations. The noninvariance of Eq. (10) under the local SO(3,1) group reflects the frame dependence of the definition. We have argued [8] that this dependence is a natural feature of P^a, since in the TEGR each set of tetrad fields is interpreted as a reference frame in spacetime.

It is worthwhile to recall a simple physical situation in which the frame dependence of the gravitational energy-momentum takes place. For this purpose we consider a black hole of mass m and an observer that is very distant from the black hole. The black hole will appear to this observer as a particle of mass m, with energy $cP^{(0)} = mc^2$ (m is the rest mass of the black hole, i.e., the mass of the black hole in the frame where the black hole is at rest). If, however, the black hole is moving at velocity v with respect to the observer, then its total gravitational energy will be $cP^{(0)} = \gamma mc^2$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. This example is a consequence of the special theory of relativity, and demonstrates the frame dependence of the gravitational energy-momentum. We note that the frame dependence is not restricted to observers at spacelike infinity. It holds for observers located everywhere in space.

III. THE ENERGY-MOMENTUM OF PLANE-FRONTED GRAVITATIONAL WAVES

Definition (10) for the gravitational energy-momentum may be easily applied to plane-fronted gravitational waves. We can show that these waves carry positive energy.

A plane-fronted gravitational wave is an exact solution of Einstein's equations. A wave that travels along the z direction may be described by the line element [19]

$$ds^2 = dx^2 + dy^2 + 2dudv + H(x, y, u)du^2, \quad (11)$$

where the function $H(x, y, u)$ satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)H(x,y,u)=0. \tag{12}$$

Transforming the (u, v) to (t, z) coordinates, where

$$u=\frac{1}{\sqrt{2}}(z-t),$$
$$v=\frac{1}{\sqrt{2}}(z+t),$$

we find

$$ds^2=\left(\frac{H}{2}-1\right)dt^2+dx^2+dy^2+\left(\frac{H}{2}+1\right)dz^2-Hdtdz. \tag{13}$$

The function H is only required to satisfy (12). However, it would be interesting to specify H such that it describes a wave-packet [19]. The inverse metric tensor reads

$$g^{\mu\nu}=\begin{pmatrix}-\frac{1}{2}H-1 & 0 & 0 & -\frac{1}{2}H\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\frac{1}{2}H & 0 & 0 & -\frac{1}{2}H+1\end{pmatrix}. \tag{14}$$

We will choose the tetrad field that is adapted to static observers in spacetime. Therefore the tetrad field must satisfy $e_{(0)}{}^{i}=0$. One suitable construction, adapted to the symmetry of the gravitational field, is

$$e_{a\mu}=\begin{pmatrix}-A & 0 & 0 & -B\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & C\end{pmatrix}, \tag{15}$$

where

$$A=\left(-\frac{H}{2}+1\right)^{1/2},\qquad AB=\frac{H}{2},\qquad AC=1. \tag{16}$$

In (15) a and μ label rows and columns, respectively. The geometrical interpretation of (15) is best understood if we consider the inverse components $e_a{}^{\mu}$. We verify that

$$e_{(0)}{}^{\mu}=(1/A,0,0,0), \tag{17}$$

and

$$e_{(1)}{}^{\mu}=(0,1,0,0),\qquad e_{(2)}{}^{\mu}=(0,0,1,0),$$
$$e_{(3)}{}^{\mu}=(-H/(2A),0,0,A). \tag{18}$$

Note that if $H\ll 1$ we have $A\cong 1-H/4$ and therefore $e_{(3)}{}^{i}=(0,0,A)\cong(0,0,1-H/4)$.

The frame is determined by fixing six conditions on $e_{a\mu}$. Equation (17) fixes the kinematic state of the frame, since the three velocity conditions $e_{(0)}{}^{i}=0$ ensure that the frame is static. Three other conditions fix the spatial sector of the frame. According to (18), $e_{(1)}{}^{\mu}$, $e_{(2)}{}^{\mu}$ and $e_{(3)}{}^{\mu}$ are unit vectors along the x, y, and z axis, respectively. Therefore (18) fixes the orientation of the frame in spacetime: the $e_{(3)}{}^{\mu}$ component is oriented along the direction of propagation of the wave. Alternatively, the frame may be determined by fixing the six components of the acceleration tensor $\phi_{ab}=-\phi_{ba}$ [8]. These are the inertial accelerations (translational and rotational) that are necessary to maintain the frame in a given inertial state in spacetime. However, in the present case the characterization of the frame by means of (17) and (18) seems to be more appropriate. Finally, it must be noted that by requiring $H=0$ we obtain $e_a{}^{\mu}=\delta_a^{\mu}$, and consequently $T_{a\mu\nu}=0$.

The nonvanishing components of $T_{\mu\nu\lambda}$ are

$$\begin{gathered}T_{001}=\tfrac{1}{2}\partial_1 A^2\qquad T_{002}=\tfrac{1}{2}\partial_2 A^2\\ T_{003}=\tfrac{1}{2}\partial_3 A^2-A\partial_0 B\qquad T_{013}=-A\partial_1 B\\ T_{023}=-A\partial_2 B\qquad T_{301}=B\partial_1 A\qquad T_{302}=B\partial_2 A\\ T_{303}=B\partial_3 A+\tfrac{1}{2}\partial_0(-B^2+C^2)\\ T_{313}=\tfrac{1}{2}\partial_1(-B^2+C^2)\qquad T_{323}=\tfrac{1}{2}\partial_2(-B^2+C^2).\end{gathered} \tag{19}$$

In the expressions above we have $(-B^2+C^2)=g_{33}$.

In order to calculate the gravitational energy-momentum we find it more convenient to transform expression (10) into a surface integral,

$$P^a=4k\oint_S dS_i(e\Sigma^{a0i}). \tag{20}$$

The determinant e is simply $e=AC=1$. After long but straightforward calculations we obtain

$$\begin{aligned}dS_i(e\Sigma^{(0)0i})&=-\frac{1}{8(-g_{00})^{1/2}}[dydz\partial_1 H+dzdx\partial_2 H],\\ dS_i(e\Sigma^{(1)0i})&=\frac{1}{8(-g_{00})}[dydz\partial_0 H+dxdy\partial_1 H],\\ dS_i(e\Sigma^{(2)0i})&=\frac{1}{8(-g_{00})}[dzdx\partial_0 H+dxdy\partial_2 H],\\ dS_i(e\Sigma^{(3)0i})&=-\frac{1}{8(-g_{00})^{1/2}}[dydz\partial_1 H+dzdx\partial_2 H].\end{aligned} \tag{21}$$

In the evaluation of $\Sigma^{(1)0i}$ and $\Sigma^{(2)0i}$ we have used the relation $\partial_3 g_{00}=-\partial_0 g_{00}$, since the metric quantities in (11) are functions of u.

We may now return to volume integrals. We have

$$\begin{aligned}\oint_S dS_i(e\Sigma^{(0)0i})&=\int_V d^3x\partial_i(e\Sigma^{(0)0i})\\ &=-\frac{1}{8}\int_V d^3x\partial_i\left[\frac{1}{(-g_{00})^{1/2}}\partial_i H\right].\end{aligned} \tag{22}$$

Taking into account that

$$\partial_i(\partial_i H)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)H=0,$$

we find

$$\oint_S dS_i(e\Sigma^{(0)0i})=\frac{1}{32}\int_V\frac{(\partial_i H)^2}{(-g_{00})^{3/2}}.$$

Therefore, we obtain

$$P^{(0)} = P^{(3)} = \frac{k}{8}\int_V d^3x \frac{(\partial_i H)^2}{(-g_{00})^{3/2}} \geq 0, \tag{23}$$

assuming that $(-g_{00}) > 0$. Next we calculate $P^{(1)}$. In view of (21) we have

$$\begin{aligned}\oint_S dS_i(e\Sigma^{(1)0i}) &= \int_V d^3x\partial_i(e\Sigma^{(1)0i}) = \int_V d^3x\bigg[\partial_1\bigg(\frac{1}{8(-g_{00})}\partial_0 H\bigg) + \partial_3\bigg(\frac{1}{8(-g_{00})}\partial_1 H\bigg)\bigg] \\ &= \frac{1}{8}\int_V\bigg[\frac{1}{(-g_{00})}\partial_1\partial_0 H - \frac{1}{(-g_{00})^2}\partial_0 H\partial_1\bigg(-\frac{H}{2}\bigg) + \frac{1}{(-g_{00})}\partial_3\partial_1 H - \frac{1}{(-g_{00})^2}\partial_1 H\partial_3\bigg(-\frac{H}{2}\bigg)\bigg].\end{aligned} \tag{24}$$

Considering that $\partial_0 H = -\partial_3 H$ we see that the expression above vanishes. The same result holds in the calculation of $P^{(2)}$. Therefore, we conclude that

$$P^{(1)} = P^{(2)} = 0. \tag{25}$$

Equations (23) and (25) imply

$$P^a P^b \eta_{ab} = 0. \tag{26}$$

IV. FINAL REMARKS

Equation (26) is a feature of a plane electromagnetic wave. If the similarity between plane gravitational and electromagnetic waves holds, we may assert that the plane-fronted gravitational wave transports the excitations of a massless field. Moreover, in view of Eq. (23) we see that the energy enclosed by an arbitrary volume V is non-negative. Therefore, the energy carried by a plane-fronted gravitational wave is positively defined and gauge invariant, i.e., coordinate independent (in addition we note that Eq. (26) is invariant under global SO(3,1) transformations of the frame). Therefore, we conclude that the noncovariant character of the energy carried by gravitational waves [3] is restricted to the investigation in the context of pseudotensors.

The analysis above shows that in the framework of the TEGR definition (10) allows a satisfactory treatment of the energy-momentum of plane-fronted waves. In fact the application of Eq. (10) to any configuration of gravitational field is conceptually simple. One has to specify the frame adapted to a field of observers in spacetime endowed with velocities $u^\mu = e_{(0)}{}^\mu$. The orientation of the frame in the three-dimensional space may be fixed according to the symmetry of the gravitational field. This choice of the tetrad field is not arbitrary. It is determined by the choice of the reference frame adapted to a field of observers in spacetime, exactly like in the special theory of relativity or in the Newtonian limit of general relativity.

ACKNOWLEDGMENTS

This work was supported in part by CNPQ.

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