New string vacua from twistor spaces

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We find a new family of AdS$_4$ vacua in IIA string theory. The internal space is topologically either the complex projective space $\mathbb{CP}^3$ or the "flag manifold" SU(3)/(U(1) × U(1)), but the metric is in general neither Einstein nor Kähler. All known moduli are stabilized by fluxes, without using quantum effects or orientifold planes. The analysis is completely ten-dimensional and does not rely on assumptions about Kaluza-Klein reduction.

I. INTRODUCTION

The search for realistic string vacua usually proceeds in steps, by mixing and matching different features of the theory. For example, achieving a positive cosmological constant Λ is not easy [1]. For that reason, one usually starts with a vacuum with a vanishing or negative Λ, and then attempts to modify it using ingredients that evade the no-go argument in [1].

One popular construction of this kind [2] actually proceeds in three steps. It starts from a class of Minkowski vacua [3–5]; it then gives mass to the only massless scalar in those vacua, by using quantum corrections, and in the process introducing a *negative* Λ; it finally makes Λ positive, and breaks supersymmetry, by using brane-antibrane pairs. Even if this construction seems to produce very large numbers of vacua with pleasing features, it should encourage us to look further and to ask whether there are different families of vacua, or maybe different constructions, that can later be made realistic. For example, the Minkowski vacua from which [2] starts are obtained by compactifying IIB on Calabi-Yau manifolds. One can ask if there are other manifolds that give supersymmetric flux vacua; the answer turns out to be remarkably simple [6,7], and to involve interesting geometrical concepts. A case study, however, shows [8] that finding concrete examples to the conditions in [6] is slower than one would like.

Other than this geometrical simplicity, there is no reason, however, not to start with Λ negative to begin with. There are indeed several examples of supersymmetric AdS$_4$ with moduli stabilized. For example, [9] find a simple IIA family with no moduli and in which possible corrections are parametrically under control, using orientifold planes and again Calabi-Yau manifolds. As later shown in [10], from a purely ten-dimensional perspective these vacua are found by using a low-energy approximation in which the orientifold sources are effectively smeared. Although I think this approximation is correct, there is no reason one should use orientifold planes at all to find AdS$_4$ vacua (whereas they are a prominent way of evading the arguments in [1] for Minkowski vacua). It

should then be possible to find many vacua even without them.

The oldest construction of AdS$_4$ vacua in IIA is from M theory via the so-called Freund-Rubin choice of fluxes (for a review see [11]). The internal space M_7 is, in this case, an Einstein manifold. Some of those vacua can also be reduced to (or directly found in) IIA. For example, AdS$_4$ × S^7 in M theory can also be understood dually as AdS$_4$ × $\mathbb{CP}^3$ in IIA with F_2 flux [12–14] (although some subtlety about the amount of perturbative supersymmetry arises because the reduction does not preserve all the supercharges [15]). Another embarrassingly simple IIA construction was found in [16,17]. The idea is to consider metrics which are not Calabi-Yau, but whose deviations from the Calabi-Yau condition is [in a sense to be reviewed later, in terms of an internal "SU(3) structure"] parametrized by a single real number W_1. These metrics are called *nearly Kähler*. They are also Einstein, with the scalar curvature being proportional to $|W_1|^2$. By a suitable choice of the internal fluxes [i.e., by taking them to be singlets under the internal SU(3) structure], all the supersymmetry equations then reduce to easily solvable algebraic equations involving scalars.[1] It turns out that a nearly Kähler metric exists on $\mathbb{CP}^3$; it is different from the usual Fubini-Study metric.

In this paper I will generalize both of these two constructions of vacua on AdS$_4$ × $\mathbb{CP}^3$, in a way that in a sense interpolates between the constructions in [12–14,16,17]. These metrics are in general not Einstein and, in particular, not nearly Kähler, nor Kähler. (Not surprisingly to flux compactifications aficionados, the almost complex structure is not integrable, because of the cosmological constant.) Nor are the fluxes simply singlets of the internal SU(3) structure.

The way I found these vacua is by considering $\mathbb{CP}^3$ as a twistor fibration (that has fiber S^2) on S^4, with a slightly unusual choice of nonintegrable almost complex structure that turns out to have vanishing c_1. The metrics are obtained by varying the relative factor between the metric on

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[1]Nearly Kähler manifolds have also appeared in heterotic string theory [18,19].

the fiber and the one on the base; one can think of it as a "squashing parameter." The construction can be repeated with few changes for the twistor space $SU(3)/(U(1) \times U(1))$ of $\mathbb{CP}^2$, but we will focus mostly on $\mathbb{CP}^3$.

There are infinitely many of these vacua; because of flux quantization, all the known moduli are stabilized. In a sense, instead of starting with many geometrical moduli and finding then a way of stabilizing them (as one does with Calabi-Yau manifolds), we start with a space that has very few moduli to begin with. Just as in [9], dilaton and internal curvature can be made parametrically small.

Given that our computations are always purely ten dimensional, we have nothing to say in this paper about the low-energy effective action describing excitations around these vacua.[2] It should not be difficult, however, if need be, to compute these effective theories, perhaps using an alternative description of these metrics in terms of group cosets [23].

These examples also illustrate a limitation inherent to the usual approach of finding first an effective theory by Kaluza-Klein (KK) reducing on a space, and finding then vacua for this effective theory. While this looks physically very reasonable, KK reducing on a general manifold is in fact not easy, in general: so far, the only examples fully understood are Calabi-Yau's, parallelizable manifolds (like the so-called "twisted tori," used in Scherk-Schwarz constructions)[3] or cosets. Proposals exist on how to understand more general manifolds (see, for example, [27]), but they are plagued by many geometrical issues [28], which so far seem to be under control only in simple cases [20,29] (although one can show that four- and ten-dimensional supersymmetry are equivalent [30,31]). Given this state of affairs, one might want to look for vacua first, and only later for effective theories.

On a different note, it would be interesting to know the conformal field theory duals to these vacua; as remarked in [32], the Romans mass F_0 should give rise to a Chern-Simons theory, perhaps of the type discussed in [33].

After reviewing in Sec. II the conditions (2.2) and (2.6) for supersymmetric vacua, and the geometrical information we need in Sec. III, we will show the existence of the new vacua in Sec. IV.

II. REVIEW OF ANTI–DE SITTER VACUA IN TYPE IIA

We will start by reviewing the conditions imposed by supersymmetry on the internal geometry, and by special-

izing them to the case in which no sources (branes or orientifold planes) are present.

This computation has been carried out in [34]; in [8], Sec. 7 it has been rederived using the techniques of $SU(3) \times SU(3)$ structures. Since the last presentation seems smoother to me (no doubt because of personal bias), I will use the notation in [8] (save for one minor difference to be noted later). Rather than reviewing here the machinery of generalized complex structures, I will cut to the chase and describe the final result of that analysis in terms of (hopefully) lighter mathematics.

We do need, however, the concept of an SU(3) structure. This is just the type of structure that we are familiar with from Calabi-Yau manifolds, but without the differential equations. Namely, an SU(3) structure is a pair of forms (J, Ω) such that

  (i) J is a real two-form, Ω is a complex three-form and *decomposable* (locally the wedge product of three complex one-forms), and Ω then determines an almost complex structure I;
 (ii) $\frac{4}{3}J^3 = i\Omega \wedge \bar{\Omega} \neq 0$ everywhere;
(iii) $J \wedge \Omega = 0$;
 (iv) the tensor $g = JI$ (which is symmetric because of the conditions above) is positive definite.

Notice that with respect to I, it is easy to see from the above conditions that J is (1, 1) and Ω is (3, 0).

A Calabi-Yau manifold can be defined then as a manifold on which

$$dJ = 0 = d\Omega \qquad \text{(Calabi-Yau)}. \qquad (2.1)$$

In this paper we are not interested in Calabi-Yau manifolds, however, and we will shortly see why.

The supersymmetry conditions in IIA for an AdS$_4$ vacuum with SU(3) structure read[4]

$$dJ = 2\tilde{m}\,\mathrm{Re}\Omega, \qquad d\Omega = i(W_2^- \wedge J - \tfrac{4}{3}\tilde{m}J^2),$$

$$H = 2m\,\mathrm{Re}\Omega; \qquad g_sF_0 = 5m,$$

$$g_sF_2 = -W_2^- + \tfrac{1}{3}\tilde{m}J, \qquad g_sF_4 = \tfrac{3}{2}mJ^2, \qquad (2.2)$$

$$g_sF_6 = -\tfrac{1}{2}\tilde{m}J^3.$$

Here, m and $\tilde{m}$ are two real numbers, W_2^- is a *primitive* (1, 1)-form (the strange notation comes from [35,36], primitive means that $W_2^- \wedge J^2 = 0$), and g_s is the constant string coupling. Notice that the parameter m is related, but not exactly equal, to the Romans mass F_0. The cosmological constant in four dimensions is given by

$$\Lambda = -3(m^2 + \tilde{m}^2). \qquad (2.3)$$

In (2.2) the F_i's are the *internal* fluxes. There are also "external" fluxes that span the AdS$_4$ directions as well as

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some of the internal directions. These are determined by the internal fluxes by ten-dimensional Hodge duality. For example, there is also a flux extended along the AdS$_4$ directions only

$$F_4^{\text{ext}} = *_{10}F_6 = \text{vol}_4 *_6 F_6 = \frac{3\tilde{m}}{g_s}\text{vol}_4. \quad (2.4)$$

For the same reason, there are also fluxes of the form vol$_4$ wedge, an internal two-form, four-form, and six-form. We will never mention again the external fluxes F_i^{ext}; we will always use the internal $F_i^{\text{int}} \equiv F_i$.

If one wants to attach a name to the geometrical part of (2.2), one could say that they describe a "half-flat" manifold (as also noticed in [10]), namely, one such that $d\,\text{Re}\,\Omega = 0 = dJ^2$, although it is a very particular one, and hence the name is probably not very useful. Notice also that, even for more general solutions with SU(3) × SU(3) structure, one still gets that all vacua are "generalized half-flat" manifolds, as explained in [8]. (We will see at the end of this paper how that a more general analysis should be relevant for the vacua in [9].)

In any case, the supersymmetry conditions have to be supplemented by the Bianchi identities for the fluxes. If we impose that there are no sources, these read

$$dF_k = H \wedge F_{k-2}. \quad (2.5)$$

In fact, we have already used the $k = 0$ case, $dF_0 = 0$: this is how it was derived in [8] that the warping A and the dilaton ϕ must be constant. One would also need, *a priori*, to impose the equations of motion for the fluxes, $d *_6 F_k = -H \wedge *_6 F_{k+2}$ and $d * H = -g^2\sum_k F_k \wedge *F_{k+2}$. However, both have been shown to be implied by the supersymmetry equations, in [8,37,38], respectively [for all supersymmetric vacua, and not only for the class of AdS SU(3) structure vacua reviewed here]. So we can forget about them and impose (2.5) alone. Since $\text{Re}\,\Omega \wedge J = 0$, the only nontrivial case is $k = 2$. We get

$$dW_2^- = \tfrac{2}{3}(\tilde{m}^2 - 15m^2)\,\text{Re}\,\Omega. \quad (2.6)$$

Summing up, we have reviewed in this section the conditions for an AdS$_4$ vacuum with internal SU(3) structure: they are given by Eqs. (2.2) and (2.6). It would be rather easy to find solutions to (2.2) alone; the real problems come when trying to solve (2.6) as well. We will now review a family of SU(3) structures for which it is possible to compute dW_2, and then show in Sec. IV that some of them support string vacua.

III. GEOMETRY OF TWISTOR SPACES

The twistor bundle on a manifold M_k of dimension k is the bundle of all almost complex structures compatible with a metric on M_k. The fiber is hence given by SO(k)/U(k/2). For $k = 4$, this is SO(4)/U(2) = $\mathbb{CP}^1$ = S^2. Hence the twistor bundle on a four-manifold is an S^2 fibration, and its total space Tw(M_4) has dimension 6.

We will now review some aspects of this fibration: its topology, complex structures, and metrics.

A. Topology

For the topology, we will first focus on the case in which $M_4 = S^4$. It can be shown then that the total space of the twistor fibration is actually $\mathbb{CP}^3$:

$$S^2 \hookrightarrow \mathbb{CP}^3 = \text{Tw}(S^4) \atop \big\downarrow \atop S^4 \quad . \quad (3.1)$$

One way to see this is to think of S^4 as of the quaternionic projective line $\mathbb{HP}^1$. Then the projection map can be given as

$$\mathbb{CP}^3 \ni (z_1, z_2, z_3, z_4) \overset{p}{\mapsto} (z_1 + jz_2, z_3 + jz_4) \in \mathbb{HP}^1. \quad (3.2)$$

$\mathbb{CP}^3$ has Betti numbers $b_0 = b_2 = b_4 = b_6 = 1$ and $b_1 = b_3 = b_5 = 0$. In terms of the fibration , the two-cycle is just the fiber. One might get confused, however, in trying to identify the four-cycle. The twistor fibration cannot in this case have a global section, because that would be a globally defined almost complex structure on S^4, and it is known that none exists. So the base cannot be literally used as a cycle.

The answer can be found by looking at the map p in (3.2). Think of a hyperplane $\mathbb{CP}^2 \subset \mathbb{CP}^3$ as the union $\mathbb{C}^2 \cup \mathbb{CP}^1$, where $\mathbb{CP}^1$ is the line at infinity of the projective plane $\mathbb{CP}^2$. Then, the projection map p is one to one on $\mathbb{C}^2$, but projects $\mathbb{CP}^1$ to a point. The result is a one-point compactification of $\mathbb{C}^2$, which is topologically S^4.

So far we have looked at Tw(S^4) = $\mathbb{CP}^3$. Although we will devote less attention to it, there is another manifold to which the computations of Sec. IV apply, namely, Tw($\mathbb{CP}^2$). In that case, the fibration is

$$S^2 \hookrightarrow \frac{\text{SU(3)}}{\text{U(1)} \times \text{U(1)}} = \text{Tw}(\mathbb{CP}^2) \atop \big\downarrow \atop \mathbb{CP}^2 \quad . \quad (3.3)$$

Another notation used for the total space so obtained is $\mathbb{F}(1, 2; 3)$, often called "flag manifold." It is the space of complex planes and lines in $\mathbb{C}^3$ such that the line belongs to the plane. (The line is the "pole" and the plane is the "flag.") In equations

$$\mathbb{F}(1, 2; 3) = \left\{(z^i, \tilde{z}^i) \in \mathbb{CP}^2 \times \mathbb{CP}^2 \quad \text{such that} \sum_{i=1}^{3} z^i\tilde{z}^i = 0\right\}. \quad (3.4)$$

One can fiber this space over either of the two $\mathbb{CP}^2$ factors, by the map that forgets either the z^i or the $\tilde{z}^i$. The fiber is a $\mathbb{CP}^1$. Finally, one can use, for example, the Gysin exact

sequence to compute that the Betti numbers are $b_0 = b_6$, $b_1 = b_3 = b_5 = 0$, and $b_2 = b_4 = 2$. Intuitively, the two two-cycles are the $\mathbb{CP}^1$ in each of the $\mathbb{CP}^2$ in (3.4).

B. Almost complex structures

Having clarified somewhat the topology of this fibration, we now look at what almost complex structures can be defined on the total space $\text{Tw}(M_4)$, going back to a general M_4. Let the twistor fiber have coordinates σ^i, $i = 1, 2$, and 3, so that $\sum (\sigma^i)^2 = 1$. Since it is by definition the space of almost complex structure compatible with a given metric, we can "tautologically" write $I_4(\sigma^i)$, which means that there is an almost complex structure I_4 on the base M_4 for any choice of the coordinates σ^i on the fiber. This is by definition a tensor on the total space of the fibration $\text{Tw}(M_4)$. We cannot call it an almost complex structure on $\text{Tw}(M_4)$, however, because it has rank four. To promote it to rank six, we have to choose an action on vectors along the fiber; since the fiber is S^2, we can take the usual Riemann complex structure I_2 on it [explicitly, $I_2(\partial_{\sigma^i}) = \epsilon^{ijk} \sigma^j \partial_{\sigma^k}$]. So we can now combine the two in an almost complex structure on $\text{Tw}(M_4)$. On a local basis of vectors,

$$\tilde{I} = \begin{pmatrix} I_2 & 0 \\ 0 & I_4(\sigma) \end{pmatrix}. \tag{3.5}$$

Actually, we could have also combined them with a different sign:

$$I = \begin{pmatrix} -I_2 & 0 \\ 0 & I_4(\sigma) \end{pmatrix}. \tag{3.6}$$

The difference between these two almost complex structures $\tilde{I}$ and I on the total space was stressed in [39,40]. The first, $\tilde{I}$, is the most popular one because it is integrable (namely, it is a complex structure, and not just an "almost" complex structure) whenever [41] the anti–self-dual part of the Weyl tensor W_-[5] of M_4 is zero—that is, when M_4 is self-dual. In contrast, I is never integrable. But it has a nice feature of its own: its first Chern class is actually zero.

To highlight the difference, let us look at the particular case once again. $\tilde{I}$ is the usual complex structure for $\mathbb{CP}^3$; it has $c_1 = 4$, and so, in particular, there is no globally defined (3, 0)-form for it (let alone one in cohomology). This complex structure does not look very promising for us, because there is no Ω, but also because of another fact. If one *does* have a (3, 0)-form for an *almost* complex structure, the latter is integrable if and only if

$$(d\Omega)_{2,2} = 0. \tag{3.7}$$

Looking back at (2.2), we see that the almost complex structure we are looking for is only integrable if $W_2^- = 0$ and $\tilde{m} = 0$. Looking at (2.6), we also conclude that $m = 0$, and hence all fluxes are zero, the manifold is a Calabi-Yau [see (2.1)], and the cosmological constant is zero [see

[2.3)]. So an integrable complex structure would take us back to the usual Calabi-Yau compactifications.

So there are good reasons to focus on I instead, which has $c_1 = 0$ [hence a globally defined (3, 0)-form Ω exists] and which is not integrable.

C. SU(3) structure

To make progress, we need to complement the complex structure and its associated Ω with a two-form J that forms an SU(3) structure with it. This is always possible [because Ω alone defines a $\text{Sl}(3, \mathbb{C})$ structure, and $\text{Sl}(3, \mathbb{C})$ is homotopically equivalent to $U(3)$]. Explicitly, let us introduce a holomorphic vielbein e^a, $a = 1$, 2, and 3, namely, a basis of one-forms such that

$$I^t e^a = i e^a. \tag{3.8}$$

(The transposition ^{t} is because I is acting on one-forms.) More specifically, let us take e^3 along the fiber, and $e^{1,2}$ to be a pullback of forms on the base. Hence we also have

$$\tilde{I}^t e^{1,2} = i e^{1,2}, \qquad \tilde{I}^t e^3 = -i e^3. \tag{3.9}$$

In the case in which M_4 is self-dual (as defined above) and Einstein, [42] showed that

$$d \begin{pmatrix} e^1 \\ e^2 \\ e^3 \end{pmatrix} = \begin{pmatrix} -\alpha & \\ & \text{Tr}(\alpha) \end{pmatrix} \wedge \begin{pmatrix} e^1 \\ e^2 \\ e^3 \end{pmatrix} + \frac{1}{R} \begin{pmatrix} \bar{e}^2 \wedge \bar{e}^3 \\ \bar{e}^3 \wedge \bar{e}^1 \\ \sigma \bar{e}^1 \wedge \bar{e}^2 \end{pmatrix}. \tag{3.10}$$

Here, α is an anti-Hermitian 2×2 matrix of one-forms ($\alpha_{ij} + \bar{\alpha}_{ji} = 0$) that acts on $e^{1,2}$, R is an overall length scale, and σ parametrizes the curvature of M_4 relative to the one of the fiber S^2, as we will see more explicitly later.

The reason (3.10) is useful is that it allows us to check explicitly the properties of J and Ω that we need. Let us define the SU(3) structure and metric

$$J = \frac{i}{2} e^i \wedge \bar{e}^i, \qquad \Omega = i e^1 \wedge e^2 \wedge e^3, \qquad g_6 = e^i \bar{e}^i. \tag{3.11}$$

[The metric is actually determined by the SU(3) structure (J, Ω), since SU(3) $\subset$ SO(6).]

It is easy, then, to use (3.10) to compute

$$dJ = -\frac{1}{R}(\sigma + 2)\,\text{Re}\Omega,$$

$$d\Omega = i\left(W_2^- \wedge J + \frac{2}{3R}(\sigma + 2)J^2\right),$$

$$W_2^- = \frac{2}{3R} i(\sigma - 1)(e^1 \wedge \bar{e}^1 + e^2 \wedge \bar{e}^2 - 2e^3 \wedge \bar{e}^3). \tag{3.12}$$

Notice that W_2 is (1, 1) and primitive with respect to J. One can also compute

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[5]This is not to be confused with the form W_2^- in (2.2).

$$dW_2^- = \frac{8}{3R^2}(\sigma - 1)^2 \operatorname{Re}\Omega. \qquad (3.13)$$

These equations will become useful in the next section, to solve (2.2) and (2.6).[6]

As a cross-check of (3.10), we can also define *locally* a three-form $\tilde{\Omega} = ie^1 \wedge e^2 \wedge \bar{e}^3$ for the complex structure (3.5) [compare (3.9)], and a two-form $\tilde{J} = (i/2)(e^1 \wedge \bar{e}^1 + e^2 \wedge \bar{e}^2 + \bar{e}^3 \wedge e^3)$. One gets

$$d\tilde{\Omega} = 2\tilde{\Omega} \wedge \operatorname{Tr}\alpha \qquad (3.14)$$

which implies that $(d\tilde{\Omega})_{2,2} = 0$, in agreement with our earlier statement [see (3.7)] that $\tilde{I}$ is integrable when M_4 is self-dual. When $\sigma = 2$, one can also see that $d\tilde{J} = 0$, which reproduces the fact that $\mathbb{CP}^3$ admits a Kähler metric. (We will see later again how the value $\sigma = 2$ is special.)

In fact, if on top of the assumptions already made on M_4 to derive (3.10) (namely, that M_4 be self-dual and Einstein) we also impose that it have positive scalar curvature, we are left with only two nonsingular examples: S^4 and $\mathbb{CP}^2$ (see, for example, [43]). Even if we do not need to restrict to $\sigma = 2$ [we are not using the complex structure $\tilde{I}$, after all, nor do we want $\operatorname{Tw}(M_4)$ to be Kähler], we will see in Sec. IV that we still need $\sigma > 0$, so that we will only be left with $\mathbb{CP}^3$ and $\mathbb{F}(1, 2; 3)$.

D. Metric

Both almost complex structures I and $\tilde{I}$ are compatible with the same metric defined in (3.11). We end this section by reviewing some features of this metric. We can take the relevant computations from [44], Sec. 1.[7] Let us define

$$g_4 \equiv e^1 \bar{e}^1 + e^2 \bar{e}^2, \qquad g_2 \equiv e^3 \bar{e}^3. \qquad (3.15)$$

Then we have[8]

$$\operatorname{Ric}_4 = \frac{\sigma(6 - \sigma)}{R^2} g_4, \qquad \operatorname{Ric}_2 = \frac{\sigma^2 + 4}{R^2} g_2. \qquad (3.16)$$

We see that the metric g is Einstein if and only if

$$\sigma = 1 \quad \text{or} \quad \sigma = 2 \qquad \text{(Einstein)}. \qquad (3.17)$$

We will see that both these cases have already been used to construct vacua (in [12–14,16,17], respectively).

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[6]While this paper was in preparation, the paper [22] worked out many geometrical details about existing solutions on $\mathbb{CP}^3$. With a little more work along those lines, one can actually use their computations to the case with general σ, and find (3.12) and (3.13) in an alternative way. Their parameters are then mapped to ours as $\sigma = 2\lambda^2$, $R = 2\lambda$.

[7]For $R = 1$, $\lambda_{\text{GPP}} = \sqrt{\sigma/2}$.

[8]One could also derive these formulas directly from the machinery of SU(3) structures, as, for example, in [45]; the Ricci scalar is particularly easy to cross-check in this way.

IV. FINDING VACUA

We now have all the ingredients we need to solve the supersymmetry equations (2.2) and Bianchi identities (2.6). If we do, we will have found a IIA supergravity solution. We will first do so and then worry about possible string theory corrections.

A. Supergravity

We argued in Sec. III C that a good candidate for a flux vacuum is the twistor space $\operatorname{Tw}(M_4)$, when M_4 is self-dual and Einstein. Specifically, we proposed the almost complex structure I given in (3.6); and we derived in (3.12) and (3.13) some relevant geometrical quantities. The SU(3) structure and the metric depend on a squashing parameter σ, and on the overall scale R.

First of all, by comparing dJ in (3.12) with (2.2), we get

$$\tilde{m} = -\frac{1}{2R}(\sigma + 2). \qquad (4.1)$$

Next, comparing dW_2^- in (3.13) with (2.6), we get, after some manipulation,

$$m = \frac{1}{2R}\sqrt{\left(\sigma - \frac{2}{5}\right)(2 - \sigma)} \equiv \frac{1}{2R}m_0(\sigma), \qquad (4.2)$$

in particular,

$$\tfrac{2}{5} \le \sigma \le 2 \qquad (4.3)$$

as shown in Fig. 1.

Since σ then has to be positive, we have (as commented at the end of Sec. III C) that the only two manifolds on which we can apply the methods of this paper are $\mathbb{CP}^3$ and $\mathbb{F}(1, 2; 3)$. On each of these, however, we will find infinitely many vacua.

At this point, as far as IIA supergravity is concerned, we are done. We have satisfied the equations for dJ and $d\Omega$ in (2.2), and the one for dW_2^- in (2.6), by taking the parame-
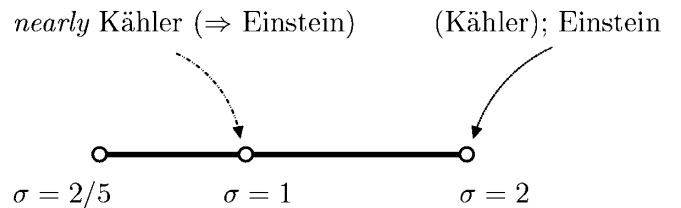
FIG. 1. This sketch shows the allowed interval for σ in (4.3), along with the three special cases already used for string vacua before this paper. This is *not* a moduli space, because of flux quantization, as discussed in Sec. IV B. In the two extrema [12–14], the Romans mass vanishes [see (2.2) and (4.2)]; the solution can hence be lifted to M theory. The resulting seven-dimensional metric on S^7 is Einstein in both cases. The metric at $\sigma = 2$ admits a Kähler structure, but supersymmetry uses another almost complex structure. The case $\sigma = 1$ was used in [16,17].

ters $\tilde{m}$ and m to be given by (4.1) and (4.2). The fluxes are then given in (2.2).

We also know from the general theory (as commented in Sec. II) that the equations of motion will be automatically satisfied. It is also not difficult to check them directly, by using the expressions for the fluxes in (2.2) and (3.16).

Since we want, however, to find *string theory* vacua and not just supergravity solutions, we have to now turn to flux quantization effects and to possible stringy corrections.

B. Flux quantization

The fluxes in (2.2) cannot be quantized. H is actually exact:

$$H = d\left(\frac{m}{\tilde{m}}J\right), \tag{4.4}$$

so its periods are zero; as for the F_k, they are not closed, because of (2.5), and hence their periods are not well defined. Thanks to (4.4), however, we can define "page charges"

$$\tilde{F}_k \equiv e^{-B\wedge}F_k, \qquad B = \frac{m}{\tilde{m}}J + B_0, \tag{4.5}$$

where B_0 is closed. We will set B_0 to zero in what follows, since this choice will be enough for finding vacua. We can then compute explicitly:

$$\tilde{F}_2 = \frac{4}{g_s R}\frac{(\sigma - 1)}{(\sigma + 2)}[-\sigma j_4 + 2j_2],$$

$$\tilde{F}_4 = -2\frac{m_0}{g_s R(\sigma + 2)^2}[(\sigma^2 - 2\sigma - 2)j_4^2 - 6\sigma j_2 \wedge j_4],$$

$$\tilde{F}_6 = \frac{8}{15 g_s R}\frac{(1 + 2\sigma)(-\sigma^2 + 12\sigma + 4)}{(\sigma + 2)^3}J^3, \tag{4.6}$$

where $m_0(\sigma)$ has been defined in (4.2) and

$$j_4 = \frac{i}{2}(e^1 \wedge \bar{e}^1 + e^2 \wedge \bar{e}^2), \qquad j_2 = \frac{i}{2}e^3 \wedge \bar{e}^3. \tag{4.7}$$

We can now impose flux quantization. To some extent, the proper understanding of what this means is still a work in progress (see, for example, [46]). For example, which fluxes are quantized depends on our choice of electric basis of field strengths. The choice should cancel in the partition function, but it does matter when trying to decide whether a single given configuration is a solution or not. In the present situation, the wisest course of action would seem to just impose that the internal fluxes F_k be quantized according to the formula $\text{ch}(x)\sqrt{\hat{A}}$, where x is an element of the K-theory group [47]. This formula gives rise to several subtleties, such as F_6 being actually half-integral or integral depending on the value of F_4. Working this out carefully seems to be beyond the scope of the present paper, since, as we will see, it does not affect the existence of solutions. We will impose, schematically,

$$\int_{C_a} \tilde{F}_k = n_k(2\pi l_s)^{k-1} \tag{4.8}$$

on all the internal F_k. To fix ideas, we can keep in mind the "naive" reduction of the half-quantization of [48] from M theory. Last, notice that allowing a nonzero B_0 [as in (4.5)] rather than setting it to zero as we did, will allow us even more freedom in the quantization, since it will alter the formula in [47] to $\text{ch}(x)e^{B_0}\sqrt{\hat{A}}$.

On $\mathbb{CP}^3$ there are four equations to be imposed. In $\tilde{F}_2$, the relevant term is the second that integrates on the fiber. In $\tilde{F}_4$, it is the first term that we are interested in: as we remarked in Sec. III A, even if the base is not a cycle, a $\mathbb{CP}^2 \subset \mathbb{CP}^3$ projects to the base by collapsing the line at infinity.

After imposing (4.8), from the equations for n_4 and n_0 one can derive

$$g_s = n_0^{-3/4}n_4^{-1/4}\frac{m_0(\sigma)}{\sqrt{\sigma(\sigma + 2)}}\left(\frac{125\pi^2}{6}(1 - \sigma)(2\sigma + 1)\right)^{1/4},$$

$$\frac{R}{2\pi l_s} \equiv r$$

$$= n_0^{-1/4}n_4^{1/4}\sqrt{\sigma(\sigma + 2)}\left(\frac{8\pi^2}{15}(1 - \sigma)(2\sigma + 1)\right)^{-1/4}. \tag{4.9}$$

We chose to derive g_s and r from the equations for n_0 and n_4 because the functions of σ that they contain are both positive and bounded within the allowed interval (4.3); this will be useful shortly. In particular, we have $n_0 > 0$ and $n_4 > 0$. (We will assume from now on that σ is not one of the special values 0, 1 or 2/5.)

We can then determine σ by

$$\frac{n_2}{\sqrt{n_0 n_4}} = \sqrt{\frac{24}{5}}\frac{\sigma}{m_0(\sigma)}\sqrt{\frac{\sigma - 1}{2\sigma + 1}} \tag{4.10}$$

and there is one σ in the allowed interval (4.3) for any integer n_2, negative or positive. So we have now fixed the three moduli g_s, σ, and r in terms of the (half) integers $n_{0,2,4}$. It would seem, however, that we are going to run into trouble when we impose the quantization of F_6. Fortunately, the relevant equation is

$$n_6 = \left(\frac{n_2 n_4}{n_0}\right)\frac{\sigma^2 - 12\sigma - 4}{8(\sigma - 1)^2}. \tag{4.11}$$

The fact that the function on the right-hand side is a rational function with rational coefficients is what saves us. Here is why. Let us first of all restrict our attention to σ rational. Before (4.11), one can choose any $n_{0,2,4}$ and determine g_s, r, and σ. Let us now give up a bit of that freedom and choose $n_{0,4}$ so that they cancel the square root that will appear in the denominator of the function on the right-hand side of (4.10). So far we have three particular integers n_0^0, n_2^0, and n_4^0 of (4.9) and (4.10) with σ rational

and some g_s and r. Let us now look at the right-hand side of (4.11). It will read at this point $(n_2^0 n_4^0 / n_0^0)(N_1 / N_2)$, for some integers N_1, N_2 (since σ is rational). It is now sufficient to take $n_{0,2,4} = (n_0^0 N_2) n_{0,2,4}^0$ [so that the solution for σ to (4.10) does not change; g_s and r will change, but so be it], and $n_6 = n_2 n_4 N_1$.

In the discussion so far, we have set B_0 in (4.5) to zero. Had we allowed it to be nonzero, we would have had one more parameter to vary (since on $\mathbb{CP}^3$ there is one harmonic two-form), which would have resulted in a system of four equations for four unknowns. To find solutions to this more general system one clearly does not have to work as hard as we had to for $B_0 = 0$.[9]

Once one has found a particular solution $\{n_k = \bar{n}_k\}$, one can find infinitely many others by rescaling. While we are at it, we can choose the rescaling so as to make r parametrically large and g_s parametrically small:

$$\{n_0 = N^2 \bar{n}_0, n_2 = N^3 \bar{n}_2, n_4 = N^4 \bar{n}_4, n_6 = N^5 \bar{n}_6\} \quad (4.12)$$

under which σ remains invariant, $g_s \sim N^{-5/2}$, and $r \sim N^{1/2}$. (We should take N odd so that it preserves any half-integrality.) Other choices of rescalings are of course possible.

With this rescaling we have made sure that both l_s and g_s corrections are under control, but one might worry about the fact that we are introducing ever larger quanta of flux. One might think that this would make large any corrections to the action in which the flux appears with high powers, for example. As remarked in [9], such corrections should be functions of $s_k \equiv (g_s F_k)^2$, where the square is actually a contraction of the indices, which involves k inverse metrics. Suppose, for example, that we are looking at the behavior of s_k under the rescaling (4.12), so that we can forget about the dependence on σ. The flux density (as opposed to the integral) F_k goes like $1/(g_s R)$, so s_k goes like $(1/r)^2 r^{-2k} = r^{-2(k+1)}$, taking into account the k inverse metrics. This means that s_k is small when r is large, and, in particular, that it gets smaller under (4.12).

The discussion for $\mathbb{F}(1, 2; 3)$ is very similar. Some numerical factors are different (essentially because of the different metric on the base). More importantly, there is an additional two-cycle (coming from a $\mathbb{CP}^1$ in the base) and an additional four-cycle (coming from the restriction of the fibration to that $\mathbb{CP}^1$). This might sound worrying, because we are then imposing two more equations. But, in fact, if we call $\tilde{n}_2$ and $\tilde{n}_4$ the two new integers, we can see from (4.6), with some work, that $\tilde{n}_2/n_2$ and $\tilde{n}_4/n_4$ are rational functions with rational coefficients. One can then perform the rescaling (4.12) until $\tilde{n}_{2,4}$ can be taken to be integer.

C. Comments and possible extensions

Now that we have convinced ourselves that the supergravity vacua found in Sec. IV A survive the gauntlet of flux quantization and possible stringy corrections, we can ask whether they are in fact interesting physically.

The first feature that springs to mind is the fact that at this point there are no *known* moduli left. There were only two geometrical moduli in our metric, R and g, and they have been stabilized along with the dilaton in Sec. IV B. Often, additional moduli can come from potentials, but in this case the Ramond-Ramond (RR) potentials are odd forms and have no cycles to be integrated on; B_0 can be integrated on the two-cycle, but it is not a modulus, since, for example, it shifts $F_2 \rightarrow F_2 - B_0 F_0$, which does not respect (4.8). It would have been suspicious anyway if there had been moduli coming from potentials, since these moduli are typically supersymmetry partners of geometrical moduli, and one cannot stabilize a field and not its partner without breaking supersymmetry.

Unfortunately, without having performed the whole KK reduction, we cannot be sure yet that there are no other moduli that we have not thought about. It is not even enough to know the spectrum of the Laplacian, because the internal fluxes mix with it (for an example, see [11], Table 5).

It would be interesting at this point to know more about the mass matrix around these vacua. For the subset found at $\sigma = 1$ by [16,17], the effective theory for a subset of fields is now known [20], and the masses are positive (not just about the stability bound).

This would be interesting in view of a possible uplifting of these vacua.[10] If one wants to uplift an AdS vacuum which has some masses over the stability bound but negative, the uplifting term in the potential is unlikely to make them positive unless it has itself a minimum at the vacuum. Hence having positive masses from the beginning appears desirable.

The uplifting would hopefully also cure one unpleasant feature of the vacua in this paper that we have not remarked about so far. Namely, there is no separation of scales between the four-dimensional cosmological constant and the Kaluza-Klein scale. Indeed, from (2.3), and using (4.1) and (4.2),

$$\Lambda R^2 = -3(m^2 + \tilde{m}^2)R^2 = \frac{12}{5}(2\sigma + 1) \quad (4.13)$$

whereas one would have liked this number to be small [it is proportional to $(H/m_{\text{KK}})^2$, where H is the Hubble scale]. This is unlike the vacua in [9], where, in the notation of this paper, $R \sim n_4/n_0$, $g_s \sim n_4^{-3/4} n_0^{-5/4}$, and $\Lambda R^2 \sim g_s^2 R^2 \sim n_4^{-1} n_0^{-3}$. The crucial difference appears to be the presence, in their case, of orientifold sources; we will have some

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[9]I thank Amir Kashani-Poor for discussions on this point.

[10]Notice that the no-go argument in [49] about de Sitter vacua only applies to Calabi-Yau spaces.

speculative comments about this at the end of this section. In any case, as already mentioned, the position taken in this paper is that this kind of question should be asked only after the uplifting.

Another question we are not answering concerns the gauge group of the effective theory. For example, the metrics considered here for $\mathbb{CP}^3$ have isometries Sp(2), but the fluxes might break some of them, and mix the survivors with the vector in [20] (that comes from the RR potential A_3) in a semidirect product, similar to what happens in Scherk-Schwarz reductions.[11]

It appears possible to answer all these questions with a reasonable amount of work. In any case, the point of this paper is less in the features of the vacua than in the techniques utilized to obtain them that hopefully might become of more general use.

Concretely, here is a more speculative possibility. So far we have not introduced any RR source, because they usually make the equations much more difficult to solve. There is a brutal approximation that in many cases seems to reproduce vacua that one has otherwise good control on: it consists of replacing the source say for an O6 plane that would look locally like

$$-\mu\,\delta(x^1)\delta(x^2)\delta(x^3)dx^1 \wedge dx^2 \wedge dx^3, \qquad (4.14)$$

with x^i being the transverse coordinates, with a nonsingular form. Acharya *et al.* [10] propose taking

$$-\frac{\mu}{R^3}\,\mathrm{Re}\Omega \qquad (4.15)$$

with the obvious motivation that it would then be comparable to the existing terms in (2.6).[12] One possible way to think about it is to expand (4.14) in eigenforms of the Laplacian and keep the lowest mode.

In any case, if one believes in this approximation, one can try to combine it with the computations in this paper. After adding (4.15) to the right-hand side of (2.6), one finds that (4.1) gets modified to

$$m = \sqrt{\frac{1}{4R^2}\Big(\sigma - \frac{2}{5}\Big)(2 - \sigma) + \frac{g\,\mu}{10R^3}}. \qquad (4.16)$$

As expected, the introduction of the O6 plane makes the equations more forgiving: it becomes possible *a priori* to have negative σ, which would correspond to the twistor space of a hyperbolic M_4 (such as quotients of hyperbolic four-space). It then also becomes possible to make σ close to $-1/2$, which would introduce a hierarchy of scales between the four-dimensional cosmological constant and the KK scale, as discussed above. However, we will not further investigate this possibility here.

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[11]In trying to understand better the supersymmetry of the effective action, the superspace constructions of [50] might turn out to be useful.

[12]Notice that, with a nonsmeared source such as in (4.14), one would expect a nontrivial warping and dilaton, whereas we have seen in Sec. II that this is not possible for vacua with SU(3) structure. This presumably means that one has to consider vacua with SU(3) × SU(3) structure instead.

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