Future evolution and finite-time singularities in $F(R)$ gravity unifying inflation and cosmic acceleration

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We study the future evolution of quintessence/phantom-dominated epoch in modified $F(R)$ gravity which unifies the early-time inflation with late-time acceleration and which is consistent with observational tests. Using the reconstruction technique it is demonstrated that there are models where any known (big rip, II, III, or IV type) singularity may classically occur. From another side, in Einstein frame (scalar-tensor description) only IV type singularity occurs. Near the singularity the classical description breaks up, and it is demonstrated that quantum effects act against the singularity and may prevent its appearance. The realistic $F(R)$ gravity which is future singularity free is proposed. We point out that additional modification of any $F(R)$ gravity by the terms relevant at the early universe is possible, in such a way that future singularity does not occur even classically.

I. INTRODUCTION

The current interest in modified $F(R)$ gravity is caused by its success as the gravitational alternative for dark energy (for a review, see [1]). The number of studies of dark energy aspects of $F(R)$ gravity [2–4] clarified its structure and possible cosmological applications. It has been demonstrated [2] that there appears the possibility of unifying the description of the inflation and late-time cosmic acceleration within the same theory which has a standard Newtonian regime in the Solar System. Hence, the whole universe expansion history may be obtained as the cosmological solution from some modified $F(R)$ gravity. Recently, the class of viable (nonlinear, analytical) $F(R)$ gravities was suggested in Ref. [5]. Such theories and their extensions [6,7] do pass the local tests and successfully describe the (almost) ΛCDM epoch. Generalizations of these theories proposed in Refs. [8–10] keep all their nice viability properties but may simultaneously describe the inflation, so that whole universe expansion history—inflation, radiation/matter dominance, dark energy epoch—follows from the same viable $F(R)$ gravity consistent with Solar System tests.

The observational data indicate that the current dark universe has the effective equation of state parameter w being very close to -1. When $w = -1$ the Universe passes through the ΛCDM epoch. If w is slightly less than -1 then we live in a phantom-dominated universe, and if w is slightly more than -1 the quintessence dark epoch occurs. In all these cases, part of (or all) the energy conditions are

violated. It is known that the dark energy universe with the effective phantom phase ends up in the future, finite-time singularity [11]. From another point, the effective quintessence universe may end up in a more general (soft) singularity. This is also true for the modified $F(R)$ gravity model [1] which leads to a dark universe with the corresponding (phantom/quintessence) effective w. One can always make the effective phantom phase transient, just by changing the classical structure of $F(R)$ gravity by the terms which are relevant at the very early universe (see the corresponding investigation in [12]). Recently, the appearance of future singularity in the specific example of $F(R)$ gravity was reobtained in Refs. [13,14].

In the present work we investigate the future evolution of modified gravity which unifies the inflation with late-time acceleration. Using the classification of finite-time future singularities proposed in Ref. [15], we describe how all types of finite-time singularities may occur in modified gravity. Comparing Jordan and Einstein frames we show that the transition of the singularity type occurs: whatever type future singularity appears in the Jordan frame [the original $F(R)$ gravity], in the Einstein frame (scalar-tensor description) it shows up as a type IV singularity. The classical mechanics analogy for future singularity is given. As classical description breaks up near to singularity, the quantum effects should be taken into account while approaching a future singularity. It is demonstrated how quantum effects prevent the future singularity in modified gravity. The realistic model of nonlinear $F(R)$ gravity which unifies the inflation with late-time cosmic acceleration without future singularity and which is consistent with local tests is proposed. The early-time modification of the gravitational theory may be always proposed

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in such a way that future singularity does not occur while local tests are not violated.

II. GENERAL STRUCTURE OF $F(R)$ GRAVITY AND TRACE EQUATION

Let us start from the general (Jordan frame) action of $F(R)$ gravity (for a review, see [1]) with matter:

$$S_{F(R)} = \int d^4x \sqrt{-g} \left\{ \frac{F(R)}{2\kappa^2} + \mathcal{L}_m \right\}. \quad (1)$$

The standard field equations are given by

$$\frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu} \Box F'(R) + \nabla_\mu \nabla_\nu F'(R)$$
$$= -\frac{\kappa^2}{2} T_{(m)\mu\nu}. \quad (2)$$

Here $F(R)$ is a proper function of the scalar curvature R and $\mathcal{L}_m$ is the matter Lagrangian. In (2), $T_{(m)\mu\nu}$ is the matter energy-momentum tensor. By separating $F(R)$ into the Einstein-Hilbert part and modified part as

$$F(R) = R + f(R), \quad (3)$$

the trace part of the equation of motion (2) has the following Klein-Gordon equationlike form:

$$3\Box f'(R) = \frac{dV_{\text{eff}}}{df'(R)} \equiv R + 2f(R) - Rf'(R) - \kappa^2 T. \quad (4)$$

The above trace equation can be interpreted as an equation of motion for the nontrivial "scalaron" $f'(R)$ with the effective potential V_{eff}. This means that the curvature itself propagates. The above trace equation is very convenient for studying the possible instabilities and Solar System tests [7]. Note also that the potential which appears in the trace equation is rather formal; it has no direct physical meaning, like energy density or pressure.

By introducing the auxiliary field A, one rewrites the action (1) of the $F(R)$ gravity in the following form:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \{ F'(A)(R - A) + F(A) + \mathcal{L}_m \}. \quad (5)$$

By the variation over A, one obtains $A = R$. Substituting $A = R$ into the action (5), one can reproduce the action in (1). Furthermore, we rescale the metric in the following way (conformal transformation):

$$g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu}, \qquad \sigma = -\ln F'(A). \quad (6)$$

Hence, the Einstein frame action is obtained:

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) + \mathcal{L}_m^A \right\},$$

$$V(\sigma) = e^\sigma G(e^{-\sigma}) - e^{2\sigma} f(G(e^{-\sigma})) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}. \quad (7)$$

Here $G(e^{-\sigma})$ is given by solving the equation $\sigma = -\ln(1 + f'(A)) = \ln F'(A)$ as $A = g(e^{-\sigma})$.

In the Hu-Sawicki model [5], when the curvature is large, $f(R)$ behaves as

$$f(R) \sim -2\Lambda + \frac{\alpha}{R^n}. \quad (8)$$

Here Λ, α, and n are positive constants. Then the potential V is given by

$$V = 2\Lambda + \mathcal{O}(A^{-n}). \quad (9)$$

Therefore, the curvature infinity ($R = A \rightarrow \infty$) surely corresponds to the finite value of the potential.

We may consider the case that $f(R)$ behaves as

$$f(R) \sim f_n R^n, \quad (10)$$

with positive n and f_n being a constant. The potential is found to be

$$V = \frac{n - 2}{f_n A^{n-2}}. \quad (11)$$

Then if $2 > n > 0$, V becomes infinite when curvature $R = A$ goes to infinity. Therefore, in this case the singularity could not be realized easily. In the model proposed in [9], $f(R)$ behaves as (10). Therefore, this case is qualitatively similar to the above behavior.

In the model [8],

$$f(R) = -\frac{(R - R_0)^{2n+1} + R_0^{2n+1}}{f_0 + f_1\{(R - R_0)^{2n+1} + R_0^{2n+1}\}}$$
$$= -\frac{1}{f_1} + \frac{f_0/f_1}{f_0 + f_1\{(R - R_0)^{2n+1} + R_0^{2n+1}\}}, \quad (12)$$

when R is large, $f(R)$ behaves as

$$f(R) \sim -\frac{1}{f_1} + \frac{f_0}{f_1^2 R^{2n+1}}, \quad (13)$$

which is almost the same with (8), and we obtain the expression of V, which is also similar to (9),

$$V = \frac{1}{f_1} + \mathcal{O}(A^{-n}). \quad (14)$$

In the case of Eq. (13), $1/f_1$ corresponds to the effective cosmological constant in the inflation epoch and therefore it is very large. Then, since V becomes positively large, the singularity could not be easily generated. Hence, from the general structure of $F(R)$ gravity in the Einstein frame one may deduce its behavior at large curvature.

We may include the effect of the matter with a constant equation of state (EoS) parameter w. Since the matter density behaves as $\rho = \rho_0 a^{-3(1+w)}$ ($\rho_0 > 0$) and ρ has mass dimension 4, by the scale transformation (6), ρ is transformed as

$$\tilde{\rho} = e^{-2\sigma} \rho = \rho_0 e^{-2\sigma} \tilde{a}^{-3(1+w)} e^{3(1+w)\sigma/2}$$
$$= \rho_0 \tilde{a}^{-3(1+w)} e^{-(1-3w)\sigma/2}. \quad (15)$$

Here $\tilde{a}$ is the scale factor in the Einstein frame, which is given by $\tilde{a} = e^{\sigma/2}a$. Note that the coordinates effectively transform as $x^\mu \to x^{\mu\prime} = e^{\sigma/2}$ under the scale transformation, general mass dimension n quantity Q transforms as $Q \to Q' = e^{-n\sigma/2}Q$. Since the energy density depends on the scalar field σ, in the Einstein frame, the potential could be shifted as

$$V \to \tilde{V} \equiv V + C\rho_0 \tilde{a}^{-3(1+w)}e^{-(1-3w)\sigma/2}$$
$$= V + \rho_0 \tilde{a}^{-3(1+w)}(F'(A))^{1-3w}. \qquad (16)$$

For the model (8), we find $F'(A) \to 1$ then the matter correction does not give a strong effect and the singularity could be easily realized. On the other hand, in the model (10), since $F'(R) \sim nf_n R^{n-1}$, the potential $\tilde{V}$ blows up as $\tilde{V} \sim R^{(1-3w)(n-1)}$ if $w < 1/3$ and $n > 1$. Then the singularity could be prevented if we include the matter.

III. FINITE-TIME SINGULARITIES IN $F(R)$ GRAVITY

In this section, we investigate $F(R)$ gravity models which generate several known types of finite-time singularities.

A. Big rip type singularity

As the first example, we consider the case of the big rip singularity, where H behaves as

$$H = \frac{h_0}{t_0 - t}. \qquad (17)$$

Here h_0 and t_0 are positive constants and H diverges at $t = t_0$. In order to find the $F(R)$ gravity which generates the big rip type singularity, we use the method of the reconstruction, that is, we construct the $F(R)$ model realizing *any* given cosmology using the techniques of Ref. [16]. The general $F(R)$ gravity action with general matter is given as

$$S = \int d^4x \sqrt{-g}\{F(R) + \mathcal{L}_{\text{matter}}\}. \qquad (18)$$

The action (18) can be rewritten by using proper functions $P(\phi)$ and $Q(\phi)$ of a scalar field ϕ:

$$S = \int d^4x \sqrt{-g}\{P(\phi)R + Q(\phi) + \mathcal{L}_{\text{matter}}\}. \qquad (19)$$

Since the scalar field ϕ has no kinetic term, one may regard ϕ as an auxiliary scalar field. By the variation over ϕ, we obtain

$$0 = P'(\phi)R + Q'(\phi), \qquad (20)$$

which could be solved with respect to ϕ as $\phi = \phi(R)$. By substituting $\phi = \phi(R)$ into the action (19), we obtain the action of $F(R)$ gravity where

$$F(R) = P(\phi(R))R + Q(\phi(R)). \qquad (21)$$

By the variation of the action (19) with respect to $g_{\mu\nu}$, the equation of motion follows:

$$0 = -\tfrac{1}{2}g_{\mu\nu}\{P(\phi)R + Q(\phi)\} - R_{\mu\nu}P(\phi) + \nabla_\mu\nabla_\nu P(\phi)$$
$$- g_{\mu\nu}\nabla^2 P(\phi) + \tfrac{1}{2}T_{\mu\nu}. \qquad (22)$$

In the Friedmann-Robertson-Walker (FRW) universe, Eq. (22) has the following form:

$$0 = -6H^2 P(\phi) - Q(\phi) - 6H\frac{dP(\phi(t))}{dt} + \rho$$
$$0 = (4\dot{H} + 6H^2)P(\phi) + Q(\phi) + 2\frac{d^2 P(\phi(t))}{dt} \qquad (23)$$
$$+ 4H\frac{dP(\phi(t))}{dt} + p.$$

By combining the two equations in (23) and deleting $Q(\phi)$, we obtain

$$0 = 2\frac{d^2 P(\phi(t))}{dt^2} - 2H\frac{dP(\phi(t))}{dt} + 4\dot{H}P(\phi) + p + \rho. \qquad (24)$$

Since one can redefine ϕ properly as $\phi = \phi(\varphi)$, we may choose ϕ to be a time coordinate: $\phi = t$. Then assuming ρ, p could be given by the corresponding sum of matter with a constant EoS parameters w_i and writing the scale factor $a(t)$ as $a = a_0 e^{g(t)}$ (a_0: constant), we obtain the second rank differential equation:

$$0 = 2\frac{d^2 P(\phi)}{d\phi^2} - 2g'(\phi)\frac{dP(\phi))}{d\phi} + 4g''(\phi)P(\phi)$$
$$+ \sum_i (1 + w_i)\rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)g(\phi)}. \qquad (25)$$

If one can solve Eq. (25), with respect to $P(\phi)$, one can also find the form of $Q(\phi)$ by using (23) as

$$Q(\phi) = -6(g'(\phi))^2 P(\phi) - 6g'(\phi)\frac{dP(\phi)}{d\phi}$$
$$+ \sum_i \rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)g(\phi)}. \qquad (26)$$

Thus, it follows that any given cosmology can be realized by some specific $F(R)$ gravity.

In the case of (17), if we neglect the contribution from the matter, the general solution of (25) is given by

$$P(\phi) = P_+(t_0 - \phi)^{\alpha_+} + P_-(t_0 - \phi)^{\alpha_-},$$
$$\alpha_\pm \equiv \frac{-h_0 + 1 \pm \sqrt{h_0^2 - 10h_0 + 1}}{2}, \qquad (27)$$

when $h_0 > 5 + 2\sqrt{6}$ or $h_0 < 5 - 2\sqrt{6}$ and

$$P(\phi) = (t_0 - \phi)^{-(h_0+1)/2}$$

$$\times \left(\hat{A} \cos\left((t_0 - \phi) \ln \frac{-h_0^2 + 10h_0 - 1}{2} \right) \right.$$

$$\left. + \hat{B} \sin\left((t_0 - \phi) \ln \frac{-h_0^2 + 10h_0 - 1}{2} \right) \right), \quad (28)$$

when $5 + 2\sqrt{6} > h_0 > 5 - 2\sqrt{6}$. Using (20), (21), and (26), we find the form of $F(R)$ when R is large as

$$F(R) \propto R^{1 - \alpha_-/2}, \quad (29)$$

for the $h_0 > 5 + 2\sqrt{6}$ or $h_0 < 5 - 2\sqrt{6}$ case and

$$F(R) \propto R^{(h_0+1)/4} \times (\text{oscillating parts}), \quad (30)$$

for the $5 + 2\sqrt{6} > h_0 > 5 - 2\sqrt{6}$ case. Then V (7) behaves as

$$V \sim R^{1 + \alpha_-/2}, \quad (31)$$

for the $h_0 > 5 + 2\sqrt{6}$ or $h_0 < 5 - 2\sqrt{6}$ case and

$$V \sim R^{(3-h_0)/4} \times (\text{oscillating parts}), \quad (32)$$

for the $5 + 2\sqrt{6} > h_0 > 5 - 2\sqrt{6}$ case. Hence, even if the curvature tends to infinity, the potential V does not always tend to infinity. Note also that the potential is often unbounded below.

B. More general singularities

Let us investigate a more general singularity

$$H \sim h_0 (t_0 - t)^{-\beta}. \quad (33)$$

Here h_0 and β are constants, h_0 is assumed to be positive and $t < t_0$ as it should be for the expanding universe. Even for noninteger $\beta < 0$, some derivative of H and therefore the curvature becomes singular. Since the case $\beta = 1$ corresponds to the big rip, whichwas investigated in the last subsection, we assume $\beta \neq 1$. Furthermore since $\beta = 0$ corresponds to de Sitter space, which has no singularity, it is assumed $\beta \neq 0$. When $\beta > 1$, the scalar curvature R behaves as

$$R \sim 12H^2 \sim 12h_0^2 (t_0 - t)^{-2\beta}. \quad (34)$$

On the other hand, when $\beta < 1$, the scalar curvature R behaves as

$$R \sim 6\dot{H} \sim 6h_0\beta(t_0 - t)^{-\beta-1}. \quad (35)$$

If we write

$$P(\phi) = e^{-h_0(t_0-\phi)^{-\beta+1}/2(1-\beta)} S(\phi), \quad (36)$$

Equation (25) without matter contribution has the following Schrödinger equationlike form:

$$0 = \frac{d^2 S(\phi)}{d\phi^2} + \left(\frac{5\beta h_0}{2} (t_0 - \phi)^{-\beta-1} - \frac{h_0^2}{4} (t_0 - \phi)^{-2\beta} \right) S. \quad (37)$$

When $\phi = t \to t_0$, in the case $\beta > 1$, one finds

$$\left| \frac{5\beta h_0}{2} (t_0 - \phi)^{-\beta-1} \right| \ll \left| \frac{h_0^2}{4} (t_0 - \phi)^{-2\beta} \right|. \quad (38)$$

On the other hand, in the case $\beta < 1$, we find

$$\left| \frac{5\beta h_0}{2} (t_0 - \phi)^{-\beta-1} \right| \gg \left| \frac{h_0^2}{4} (t_0 - \phi)^{-2\beta} \right|. \quad (39)$$

In any case, Eq. (37) reduces to the following form:

$$0 = \frac{d^2 S(\phi)}{d\phi^2} - V_0 (t_0 - \phi)^{-\alpha} S, \quad (40)$$

when $\phi = t \to t_0$. Here

$$V_0 = -\frac{5\beta h_0}{2}, \quad \alpha = \beta + 1 \quad \text{when } \beta < 1,$$

$$V_0 = \frac{h_0^2}{4}, \quad \alpha = 2\beta \quad \text{when } \beta > 1. \quad (41)$$

With further redefinition

$$y \equiv (t_0 - \phi)^{1-\alpha/2}, \quad S = y^{(\alpha/4)(1-\alpha/2)^{-1}} \varphi, \quad (42)$$

Eq. (40) has the following form:

$$0 = \frac{d^2 \varphi}{dy^2} - \left\{ \left(\frac{\alpha^2}{16} - \frac{\alpha}{4} \right) \frac{1}{y^2} + \frac{4V_0}{(2-\alpha)^2} \right\} \varphi. \quad (43)$$

Note that $y \to 0$ when $\phi \to t_0$ if $1 - \alpha/2 > 0$ but $y \to \infty$ when $\phi \to t_0$ if $1 - \alpha/2 < 0$. Then if $1 - \alpha/2 > 0$, Eq. (43) reduces to the following form when $\phi \to t_0$:

$$0 = \frac{d^2 \varphi}{dy^2} - \left(\frac{\alpha^2}{16} - \frac{\alpha}{4} \right) \frac{1}{y^2} \varphi, \quad (44)$$

whose general solution is given by

$$\varphi = Ay^{\alpha/4-1} + By^{-\alpha/4}. \quad (45)$$

Here A and B are constants of the integration. On the other hand, if $1 - \alpha/2 < 0$, Eq. (43) reduces to the following form when $\phi \to t_0$:

$$0 = \frac{d^2 \varphi}{dy^2} + \frac{4V_0}{(\alpha-2)^2} \varphi. \quad (46)$$

When $V_0 > 0$, the general solution of (46) is given by

$$y = \tilde{A} \cos(\omega y) + \tilde{B} \sin(\omega y), \quad \omega \equiv \frac{2\sqrt{V_0}}{\alpha - 2}. \quad (47)$$

Here $\tilde{A}$ and $\tilde{B}$ are constants of the integration. On the other hand, if $V_0 < 0$, the general solution has the following form:

$$y = \hat{A} e^{\hat{\omega} y} + \hat{B} e^{-\hat{\omega} y}, \quad \hat{\omega} \equiv \frac{2\sqrt{-V_0}}{\alpha - 2}. \quad (48)$$

From the above analysis, one may get the asymptotic solution for P when $\phi \to t_0$.

(i) $\beta > 1$ *case.*—From (41), $\alpha = 2\beta > 2$ and therefore $1 - \alpha/2 = 1 - \beta < 0$, which corresponds to (46). Since we also find $V_0 > 0$, the solution is given by (47). Then by combining (33), (36), (42), and (47), we find the following asymptotic expression of $P(\phi)$:

$$P(\phi) \sim e^{(h_0/2(\beta-1))(t_0-\phi)^{-\beta+1}}(t_0 - \phi)^{\beta/2}$$
$$\times (\tilde{A}\cos(\omega(t_0 - \phi)^{-\beta+1})$$
$$+ \tilde{B}\sin(\omega(t_0 - \phi)^{-\beta+1})), \quad (49)$$
$$\omega \equiv \frac{h_0}{2(\beta - 1)}.$$

When $\phi \to t_0$, $P(\phi)$ tends to vanish very rapidly. By using (20), (21), and (26), $F(R)$ looks like (at large R)

$$F(R) \propto e^{(h_0/2(\beta-1))(R/12h_0)^{(\beta-1)/2\beta}}R^{-1/4}$$
$$\times \text{(oscillating part)}, \quad (50)$$

which gives

$$V \sim e^{-(h_0/2(\beta-1))(R/12h_0)^{(\beta-1)/2\beta}}R^{9/4}$$
$$\times \text{(oscillating part)}. \quad (51)$$

Near the curvature singularity $R \to \infty$, the potential becomes small exponentially.

(ii) $1 > \beta > 0$ *case.*—From (41), we find $\alpha = \beta + 1$ and therefore $1 - \alpha/2 = 1/2 - \beta/2 > 0$, which corresponds to (44). Since

$$\frac{\alpha}{4} - 1 - \left(-\frac{\alpha}{4}\right) = \beta > 0, \quad (52)$$

the second term in (45) dominates when $\phi \to t_0$ if $B \neq 0$. Then by combining (33), (36), (42), and (45), we find the following asymptotic expression of $P(\phi)$:

$$P(\phi) \sim Be^{-(h_0/2(1-\beta))(t_0-\phi)^{1-\beta}}(t_0 - \phi)^{(\beta+1)/8}. \quad (53)$$

Therefore,

$$F(R) \sim e^{-(h_0/2(1-\beta))(-6\beta h_0 R)^{(\beta-1)/(\beta+1)}}R^{7/8}. \quad (54)$$

Equation (35) shows that when $\phi = t \to t_0$, $R \to \infty$ in the case $\beta > -1$ but $R \to 0$ in the case $\beta < -1$. Therefore we find the asymptotic behavior of the potential when $R \to \infty$ as

$$V \sim e^{(h_0/2(1-\beta))(-6\beta h_0 R)^{(\beta-1)/(\beta+1)}}R^{9/8} \sim R^{9/8}, \quad (55)$$

which diverges in the limit of $R \to \infty$.

(iii) $\beta < 0$ *case.*—As in the $1 > \beta > 0$ case, one gets $\alpha = \beta + 1$ and therefore $1 - \alpha/2 > 0$ but since

$$\frac{\alpha}{4} - 1 - \left(-\frac{\alpha}{4}\right) = \beta < 0, \quad (56)$$

the first term in (45) dominates when $\phi \to t_0$, if $A \neq 0$. Hence, the asymptotic expression of $P(\phi)$ follows:

$$P(\phi) \sim Ae^{-(h_0/2(1-\beta))(t_0-\phi)^{1-\beta}}(t_0 - \phi)^{-(\beta^2-6\beta+1)/8}. \quad (57)$$

Then $F(R)$ is given by

$$F(R) \sim (-6h_0\beta R)^{(\beta^2+2\beta+9)/8(\beta+1)}$$
$$\times e^{-(h_0/2(1-\beta))(-6h_0\beta R)^{(\beta-1)/(\beta+1)}}. \quad (58)$$

Note that $-6h_0\beta R > 0$ when h_0, $R > 0$. If $0 < \beta < -1$, $R \to \infty$ when $\phi = t \to t_0$ and therefore we find

$$V \sim R^{2-(\beta+1)^2/8-1/(\beta+1)}e^{(h_0/2(1-\beta))(-6h_0\beta R)^{(\beta-1)/(\beta+1)}}, \quad (59)$$

which diverges when $R \to \infty$. On the other hand, if $\beta < -1$, $R \to 0$ when $\phi = t \to t_0$ and the potential has the following form:

$$V \sim R^{2-(\beta+1)^2/8-1/(\beta+1)}, \quad (60)$$

which could vanish or diverge depending on the value of β.

When $\beta > 1$ in (33), R behaves as in (34), and when $\beta < 1$, the scalar curvature R behaves as in (35). Conversely, when R behaves as

$$R \sim 6\dot{H} \sim R_0(\beta + 1)(t_0 - t)^{-\gamma}, \quad (61)$$

if $\gamma > 2$, which corresponds to $\beta = \gamma/2 > 1$, H behaves as

$$H \sim \sqrt{\frac{R_0}{12}}(t_0 - t)^{-\gamma/2}, \quad (62)$$

if $2 > \gamma > 1$, which corresponds to $1 > \beta = \gamma - 1 > 0$, H is given by

$$H \sim \frac{R_0}{6(\gamma - 1)}(t_0 - t)^{-\gamma+1}, \quad (63)$$

and if $\gamma < 1$, which corresponds to $\beta = \gamma - 1 < 0$, one obtains

$$H \sim H_0 + \frac{R_0}{6(\gamma - 1)}(t_0 - t)^{-\gamma+1}. \quad (64)$$

Here H_0 is an arbitrary constant, which is chosen to vanish in (33). Then, since $H = \dot{a}(t)/a(t)$, if $\gamma > 2$, we find

$$a(t) \propto \exp\left(\left(\frac{2}{\gamma} - 1\right)\sqrt{\frac{R_0}{12}}(t_0 - t)^{-\gamma/2+1}\right), \quad (65)$$

when $2 > \gamma > 1$, $a(t)$ behaves as

$$a(t) \propto \exp\left(\frac{R_0}{6\gamma(\gamma - 1)}(t_0 - t)^{-\gamma}\right), \quad (66)$$

and if $\gamma < 1$,

$$a(t) \propto \exp\left(H_0 t + \frac{R_0}{6\gamma(\gamma-1)}(t_0 - t)^{-\gamma}\right). \quad (67)$$

In any case, there appears a sudden future singularity [17] at $t = t_0$.

Since the second term in (64) is smaller than the first term, we may solve (25) asymptotically as follows:

$$P \sim P_0\left(1 + \frac{2h_0}{1-\beta}(t_0 - \phi)^{1-\beta}\right), \quad (68)$$

with a constant P_0, which gives

$$F(R) \sim F_0 R + F_1 R^{2\beta/(\beta+1)}. \quad (69)$$

Here F_0 and F_1 are constant. When $0 > \beta > -1$, we find $2\beta/(\beta+1) < 0$, which corresponds to (8) by identifying $n = -2\beta/(\beta+1)$ or $\beta = -n/(n+2)$, which could be confirmed later in (98). On the other hand when $\beta < -1$, we find $2\beta/(\beta+1) > 2$. As we saw in (29), the $F(R)$ generates the big rip singularity when R is large. Then even if R is small, the $F(R)$ generates a singularity where higher derivatives H diverge.

Let us also investigate how an effective EoS parameter w_{eff} for (64) behaves when $t \sim t_0$. In the Einstein gravity, the FRW equations are given by

$$\frac{3}{\kappa^2}H^2 = \rho, \qquad -\frac{1}{\kappa^2}(2\dot{H} + 3H^2) = p. \quad (70)$$

Since the EoS parameter is defined by $w = p/\rho$, even for the $F(R)$ gravity, we may introduce the EoS parameter w_{eff} by

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}. \quad (71)$$

Then if $\beta = \gamma/2 > 1$, it follows

$$w_{\text{eff}} \sim -1 - \frac{2\beta}{3h_0}(t_0 - t)^{-1+\beta} \rightarrow -1, \quad (72)$$

when $t \rightarrow t_0$. If $1 > \beta = \gamma - 1 > 0$, we find

$$w_{\text{eff}} \sim -\frac{2\beta}{3h_0}(t_0 - t)^{-1+\beta} \rightarrow -\infty. \quad (73)$$

Finally if $\beta = \gamma - 1 < 0$, one gets the expression (73) when $H_0 = 0$. When $H_0 \neq 0$, on the other hand, we obtain

$$w_{\text{eff}} \sim -1 - \frac{2\beta h_0}{3H_0^2}(t_0 - t)^{-1-\beta}, \quad (74)$$

when H_0 vanishes. Then if $-1 < \beta < 0$, $w_{\text{eff}} \rightarrow +\infty$ when $t \rightarrow t_0$. On the other hand, if $\beta < -1$, $w_{\text{eff}} \rightarrow -1$.

Equation (70) also shows that, even for $F(R)$ gravity, we may define the effective energy density ρ_{eff} and the effective pressure p_{eff} by

$$\rho_{\text{eff}} \equiv \frac{3}{\kappa^2}H^2, \qquad p_{\text{eff}} \equiv -\frac{1}{\kappa^2}(2\dot{H} + 3H^2). \quad (75)$$

We now assume H behaves as (33). Then if $\beta > 1$, when $t \rightarrow t_0$, $a \sim \exp(h_0(t_0 - t)^{1-\beta}/(\beta - 1)) \rightarrow \infty$, and $\rho_{\text{eff}}, |p_{\text{eff}}| \rightarrow \infty$. If $\beta = 1$, we find $a \sim (t_0 - t)^{h_0} \rightarrow \infty$ and $\rho_{\text{eff}}, |p_{\text{eff}}| \rightarrow \infty$. If $0 < \beta < 1$, a goes to a constant but $\rho, |p| \rightarrow \infty$. If $-1 < \beta < 0$, we find a and ρ vanishes but $|p_{\text{eff}}| \rightarrow \infty$. When $\beta < 0$, instead of (33), as in (63), one may assume

$$H \sim H_0 + h_0(t_0 - t)^{-\beta}. \quad (76)$$

Hence, ρ_{eff} has a finite value $3H_0^2/\kappa^2$ in the limit $t \rightarrow t_0$ when $-1 < \beta < 0$. If $\beta < -1$ but β is not any integer, a is finite and ρ_{eff} and p_{eff} vanishes if $H_0 = 0$ or ρ_{eff} and p_{eff} are finite if $H_0 \neq 0$ but higher derivatives of H diverge.

In [15], there was suggested the classification of the finite-time singularities in the following way:

(i) Type I ("big rip"): For $t \rightarrow t_s$, $a \rightarrow \infty$, $\rho \rightarrow \infty$, and $|p| \rightarrow \infty$. This also includes the case of ρ, p being finite at t_s.

(ii) Type II ("sudden"): For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho \rightarrow \rho_s$, and $|p| \rightarrow \infty$

(iii) Type III: For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho \rightarrow \infty$, and $|p| \rightarrow \infty$

(iv) Type IV: For $t \rightarrow t_s$, $a \rightarrow a_s$, $\rho \rightarrow 0$, $|p| \rightarrow 0$, and higher derivatives of H diverge. This also includes the case when $p(\rho)$ or both of them tend to some finite values while higher derivatives of H diverge.

Here t_s, a_s, and ρ_s are constants with $a_s \neq 0$. We now identify t_s with t_0. The type I corresponds to the $\beta > 1$ or $\beta = 1$ case, type II to the $-1 < \beta < 0$ case, type III to the $0 < \beta < 1$ case, and type IV to the $\beta < -1$ case but β is not any integer case. Thus, we have constructed $F(R)$ gravity examples which show any type of above finite-time singularity. This is natural because it is known that modified gravity may lead to the effective phantom/quintessence phase [1] while the phantom/quintessence dominated universe may end up with a finite-time singularity.

The reconstruction method also tells us that there appears a type I singularity for $F(R) = R + \alpha R^n$ with $n > 2$ and a type III singularity for $F(R) = R - \beta R^{-n}$ with $n > 0$.

C. Classical mechanics analogy

Let us now consider the analogy with the classical mechanics in the description of finite-time singularities in modified gravity.

We start with the trace equation:

$$2F(R) - RF'(R) - 3\Box F'(R) = -\frac{\kappa^2}{2}T. \quad (77)$$

By writing $F(R)$ as $F(R) = R + f(R)$, Eq. (77) could be rewritten as

$$R + 2f(R) - Rf'(R) - 3\Box f'(R) = -\frac{\kappa^2}{2}T. \quad (78)$$

When the curvature is large, we assume $f(R)$ behaves as (8). Then Eq. (78) reduces to

$$R + 3\alpha\Box(R^{-n-1}) \sim 0. \tag{79}$$

If

$$\chi \equiv R^{-n-1}, \tag{80}$$

and the FRW metric with flat spatial part is chosen, Eq. (79) has the following form:

$$\ddot{\chi} + 3H\dot{\chi} = \frac{1}{3\alpha}\chi^{-1/(n+1)}. \tag{81}$$

Note $\chi = 0$ corresponds to the curvature singularity $R \to \infty$. Neglecting the second term in the left-hand side of (81), the equation looks very similar to the equation of the motion in the classical system with repulsive force $(1/3\alpha)\chi^{-1/(n+1)}$.

First we consider the classical equation of motion:

$$\ddot{x} = \frac{1}{3\alpha}x^{-1/(n+1)}. \tag{82}$$

For Eq. (82), one gets an exact solution:

$$x = C(t_0 - t)^{2(n+1)/(n+2)}. \tag{83}$$

Here C and t_0 are constants. Note $2 > 2(n + 1)/(n + 2) > 1$. Then x vanishes in a finite time $t = t_0$, which corresponds to the curvature singularity in (82).

We now investigate the asymptotic solution when the curvature is large, that is, χ is small. As there is a curvature singularity, one may assume (33). Since

$$R \sim \frac{12h_0^2}{(t_0 - t)^{2\beta}} - \frac{6h_0\beta}{(t_0 - t)^{\beta+1}}, \tag{84}$$

R diverges when $\beta > -1$ but $\beta = 0$.

We now consider three cases: (1) $\beta = 1$, (2) $\beta > 1$, (3) $0 < \beta < 1$, and (4) $0 > \beta > -1$.

(i) In case (1) $\beta = 1$, since

$$R \sim \frac{12h_0^2 + 6h_0}{(t_0 - t)^2}, \tag{85}$$

and therefore, from (80), we find

$$\chi \sim (t_0 - t)^{2(n+1)}, \tag{86}$$

and the left-hand side of (81) behaves as

$$\ddot{\chi} + 3H\dot{\chi} \sim (t_0 - t)^{2n}, \tag{87}$$

but the right-hand side behaves as

$$\frac{1}{3\alpha}\chi^{-1/(n+1)} \sim (t_0 - t)^{-2}, \tag{88}$$

which is inconsistent since the powers of both sides do not coincides with each other. Therefore, $\beta \neq 1$.

(ii) In case (2) $\beta > 1$, we find

$$R = 12H^2 + 6\dot{H} \sim 12H^2 \sim (t_0 - t)^{-2\beta}, \tag{89}$$

and therefore

$$\chi \sim (t_0 - t)^{2\beta(n+1)}. \tag{90}$$

In the left-hand side of (81), the second term dominates and the left-hand side behaves as

$$\ddot{\chi} + 3H\dot{\chi} \sim 3H\dot{\chi} \sim (t_0 - t)^{\beta(2n+1)-1}. \tag{91}$$

On the other hand, the right-hand side behaves as

$$\frac{1}{3\alpha}\chi^{-1/(n+1)} \sim (t_0 - t)^{-2\beta}. \tag{92}$$

Then by comparing the powers of both sides, one gets

$$\beta(2n + 1) - 1 = -2\beta, \tag{93}$$

which gives $\beta = 1/(2n + 3)$ but this conflicts with the assumption $\beta > 1$.

(iii) In case (3) $0 < \beta < 1$ or case (4) $0 > \beta > -1$, we find

$$R = 12H^2 + 6\dot{H} \sim \dot{H} \sim (t_0 - t)^{-\beta-1}, \tag{94}$$

and therefore

$$\chi \sim (t_0 - t)^{(\beta+1)(n+1)}. \tag{95}$$

Then in the left-hand side of (81), the first term dominates and the left-hand side behaves as

$$\ddot{\chi} + 3H\dot{\chi} \sim \ddot{\chi} \sim (t_0 - t)^{\beta(n+1)+n-1}. \tag{96}$$

On the other hand, the right-hand side behaves as

$$\frac{1}{3\alpha}\chi^{-1/(n+1)} \sim (t_0 - t)^{-\beta-1}. \tag{97}$$

Then by comparing the powers of the left-hand side and the right-hand side, the consistency tells

$$\begin{aligned} \beta(n + 1) + n - 1 = -\beta - 1 \quad \text{or} \\ \beta = -n/(n + 2). \end{aligned} \tag{98}$$

This conflicts with case (3) $0 < \beta < 1$ but is consistent with case (4) $0 > \beta > -1$, which could correspond to the case in [14] or to (55). In fact, by substituting (98) into (95), we get

$$\chi \sim (t_0 - t)^{2(n+1)/(n+2)}, \tag{99}$$

which corresponds to (83). Since $0 > \beta > -1$, this singularity corresponds to type III in [15]. Then we found the curvature singularity really appears in the Hu-Sawicki model in finite time as pointed out in [14].

We now briefly comment on what could happen near the singularity in the Einstein frame (7). Since $\sigma \sim 0$ and therefore $e^\sigma \sim 1$ in the scale transformation (6), we may identify the time coordinate in the Einstein frame and the time coordinate in the original $F(R)$ frame. Since (6) shows also

$$\sigma \sim \frac{n\alpha}{A^{n+1}} = \frac{n\alpha}{R^{n+1}}, \tag{100}$$

by using (94) and (98), one obtains

$$\sigma \sim \frac{n\alpha}{(6h_0)^{n+1}} (t_0 - t)^{2(n+1)/(n+2)}, \tag{101}$$

and therefore

$$\dot{\sigma} \sim \frac{2n(n+1)\alpha}{(6h_0)^{n+1}(n+2)} (t_0 - t)^{n/(n+2)}. \tag{102}$$

On the other hand, the potential $V(\sigma)$ in (7) has the following form:

$$V(\sigma) \sim 2\Lambda - \frac{(n+1)\alpha}{A^n} \sim 2\Lambda - \frac{(n+1)\alpha}{(6h_0)^n} (t_0 - t)^{2n/(n+2)}. \tag{103}$$

Then in the Einstein frame, the energy density ρ_σ and pressure p_σ are given by

$$\rho_\sigma = \tfrac{3}{2}\dot{\sigma}^2 + V(\sigma), \qquad p_\sigma = \tfrac{3}{2}\dot{\sigma}^2 - V(\sigma), \tag{104}$$

and therefore the curvature in the Einstein frame $R_E = -(\kappa^2/2)$, $T = (\kappa^2/2)(\rho - 3p)$ are finite even if $t \to t_0$ since $\dot{\sigma}^2 \sim V - 2\Lambda \sim (t_0 - t)^{2n/(n+2)}$ vanishes. As the exponent $2n/(n+2)$ is fractional in general, however, the higher derivative of H could diverge. Then in the Einstein frame, the finite-time singularity appears as a type IV singularity. Thus, we discovered the possibility of a change of a finite-time singularity nature with the change of frame in modified gravity. This is not strange. Indeed, it was shown some time ago [18] that the big rip singularity which appears in scalar-tensor theory qualitatively changes its form in the equivalent Jordan frame. Exactly in the time when the scalar-tensor theory becomes singular, the mathematically equivalent $F(R)$ gravity (which is not completely physically equivalent to scalar-tensor theory [19]) becomes complex [18].

An important remark is in order. The general (Jordan frame) action of $F(R)$ gravity with matter in (1) can be rewritten in the Einstein frame action (7). One should note that the sign in front of the kinetic term in (7) is always canonical, and therefore there never appears a phantom in the transformed Einstein frame. Note that this transformation is exact one.

In the Einstein frame, the FRW equations have the following form:

$$\frac{3}{\kappa^2} H^2 = \rho_\sigma + \rho, \qquad -\frac{1}{\kappa^2}(3H^2 + 2\dot{H}) = p_\sigma + p. \tag{105}$$

Here ρ and p express the contribution from the matter and ρ_σ and p_σ are defined by (104). If we neglect the contribution from the matter, the EoS parameter is given by $w = p_\sigma/\rho_\sigma$ and therefore $w > -1$ if we assume $V(\sigma) > 0$,

which corresponds to the fact that σ cannot be phantom in the Einstein frame.

When the Hubble rate H is given by

$$H \sim h_0(t_0 - t)^{-\beta}, \tag{106}$$

the effective EoS parameter looks like

$$w_{\text{eff}} \equiv -1 - \frac{2\dot{H}}{3H^2} \sim -1 - \frac{2\beta}{3h_0}(t_0 - t)^{\beta-1}. \tag{107}$$

Since we are considering the period $t < t_0$ and the Universe is expanding, we assume $h_0 > 0$. Then w_{eff} can be greater than -1 when $\beta < 0$, which corresponds to type II ($0 > \beta > -1$) or type IV ($\beta < -1$). Since the EoS parameter is greater than -1 in the Einstein frame, type I (big rip type $\beta \geq 1$) and III ($1 > \beta > 0$) singularities are not allowed (the phantom absence) but there can occur a type II or IV singularity.

Even if there is no singularity in the Einstein frame, there could appear a finite-time singularity in the Jordan frame. We may consider the following action in the Einstein frame (7), where the potential is given by

$$V(\sigma) = V_0 e^{\sigma\sqrt{3/2h_0}}. \tag{108}$$

Here V_0 and h_0 are positive constants. Then by assuming the FRW universe in the Einstein frame

$$ds_E^2 = -dt_E^2 + a_E(t_E)^2 \sum_{i=1,2,3} (dx^i)^2, \tag{109}$$

one obtains a solution:

$$H = \frac{h_0}{t_E} (a_E(t_E) \propto t_E^{h_0}), \qquad \sigma = \sqrt{\frac{2h_0}{3}} \ln \frac{t_E}{t_{E0}}. \tag{110}$$

Then there is no singularity although a becomes infinite in the limit of $t_E \to +\infty$. Since the metric tensor $g_{E\mu\nu}$ in the Einstein frame is related with the tensor $g_{J\mu\nu}$ in the Jordan frame by $g_{E\mu\nu} = e^{-\sigma} g_{J\mu\nu}$, the metric in the Jordan frame is given by

$$ds_J^2 = e^{-\sigma}\left(-dt_E^2 + a_E(t_E)^2 \sum_{i=1,2,3} (dx^i)^2\right). \tag{111}$$

Since the FRW universe in the Jordan frame is given by

$$ds_J^2 = -dt_J^2 + a_J(t_J)^2 \sum_{i=1,2,3} (dx^i)^2, \tag{112}$$

by comparing (111) with (112), the time coordinate t_J in the Jordan frame is given by

$$t_J - t_{J0} = \pm \int dt_E e^{-\sigma(t_E)/2} = \pm \frac{t_{E0}}{1 - \frac{h_0}{6}} \left(\frac{t_E}{t_{E0}}\right)^{1-(h_0/6)}. \tag{113}$$

One can now choose the sign $\pm$ so that the directions of the time in the Einstein frame and the Jordan frame could not be changed. If $1 - \frac{h_0}{6} < 0$, i.e., $h_0 > 6$, the limit $t \to \infty$

corresponds to $t_J \rightarrow t_{J0}$. Then we find

$$a_J(t_J) = e^{-\sigma/2}a_E(t_E) \propto t_E^{h_0-(h_0/6)}$$
$$\propto (t_{J0} - t_J)^{h_0-(h_0/6)/1-(h_0/6)}. \quad (114)$$

Hence, if $h_0 > 6$, $a_J(t_J)$ diverges at $t_J \rightarrow t_{J0}$, that is there appears a finite singularity even if there is no singularity in the Einstein frame.

D. Quantum effects near finite-time singularity

Near the future singularity at $t = t_0$, the curvature becomes large in general. As a result, near the singularity the quantum fields/quantum gravity effects become very important again. All classical considerations are not valid; all speculations about future cosmic doomsday cannot restrict the classical theory structure because quantum effects can stop (or shift) the future singularity [20]. Moreover, the quantum corrections usually contain the curvature powers, which become important near the singularity. Hence, any claim about the appearance of the effective phantom/quintessence phase in modified gravity which subsequently enters the future singularity is not justified without the quantum effects' account near to singularity. One may include the massless quantum effects by taking into account the conformal anomaly contribution as a backreaction near the singularity [20]. The conformal anomaly T_A has the following well-known form:

$$T_A = b\left(F + \frac{2}{3}\Box R\right) + b'G + b''\Box R, \quad (115)$$

where F is the square of the 4d Weyl tensor and G is Gauss-Bonnet invariant, which are given by

$$F = \frac{1}{3}R^2 - 2R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},$$
$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \quad (116)$$

In general, with N scalar, $N_{1/2}$ spinor, N_1 vector fields, $N_2(= 0$ or $1)$ gravitons, and N_{HD} higher derivative conformal scalars, b and b' are given by

$$b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{HD}}{120(4\pi)^2}, \quad (117)$$
$$b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{HD}}{360(4\pi)^2}.$$

As is seen $b > 0$ and $b' < 0$ for the usual matter except the higher derivative conformal scalars. Notice that b'' can be shifted by the finite renormalization of the local counterterm R^2, so b'' can be an arbitrary coefficient.

By including the trace anomaly, Eq. (78) is modified as

$$R + 2f(R) - Rf'(R) - 3\Box f'(R) = -\frac{\kappa^2}{2}(T + T_A). \quad (118)$$

For the FRW universe, we find

$$F = 0, \qquad G = 24(\dot{H}H^2 + H^4). \quad (119)$$

We now assume H behaves as (33) and neglect the contribution from matter by putting $T = 0$. Then in case of the Hu-Sawicki model, we assume $f(R)$ behaves as (8) and (118) becomes

$$R + 3\Box\left(\frac{\alpha n}{R^{n+1}}\right) = -\frac{\kappa^2}{2}T_A. \quad (120)$$

First we consider the case that $2b/3 + b'' = 0$ and therefore $T_A = G$. If $-1 < \beta < 0$, R behaves as $(t_0 - t)^{-\beta-1}$ and G behaves as $G \sim 24\dot{H}H^2 \sim (t_0 - t)^{-3\beta-1}$. Since $-3\beta - 1 > -\beta - 1$ when $\beta < 0$, T_A can be neglected compared with R. Here, $\Box(\alpha n/R^{n+1}) \sim (t_0 - t)^{(\beta+1)(n+1)-2}$. Then by comparing the right-hand side with the left-hand side in (120), one reobtains (98). Therefore, the curvature singularity appears in finite time and the quantum correction does not prevent the singularity when $2b/3 + b'' = 0$. We may, however, consider the case $2b/3 + b'' \neq 0$. In this case, T_A behaves as

$$T_A \sim \left(\frac{2}{3}b + b''\right)\Box R \sim (t_0 - t)^{-\beta-3}. \quad (121)$$

Since $R \sim (t_0 - t)^{-\beta-1}$ and $\Box(\alpha n/R^{n+1}) \sim (t_0 - t)^{(\beta+1)(n+1)-2}$, the terms in the left-hand side of (120) are always less singular than T_A since

$$-\beta - 1, \qquad (\beta + 1)(n + 1) - 2 > -\beta - 3. \quad (122)$$

This indicates that Eq. (120) does not allow the singular solution and the curvature singularity does not appear. Therefore, in the case $2b/3 + b'' \neq 0$, the quantum effects prevent the singularity appearance.

In the above analysis, the $\Box R$ term acts against the singularity. The $\Box R$ term is generated by a local term R^2, which shows that if one modifies $F(R)$ or $f(R)$ by adding the R^2 term as

$$F(R) \rightarrow F(R) + \gamma R^2, \quad (123)$$

with a constant γ, the curvature singularity could not be generated. Note also that near to singularity, quantum gravity effects become dominant. Then, that which is quantum gravity which is not constructed yet should define the Universe behavior near to singularity.

Actually, this observation was made some time ago in Ref. [12] where it was shown that in the presence of the R^2 term (or higher powers of curvature) as well as negative power curvature terms the effective phantom phase is transient and future singularity never occurs. This suggests the scenario to avoid the future singularity by the modification of the early-time relevant part of $F(R)$ theory, for instance, by powers of curvature. Of course, such modification should not destroy the inflationary phase and its exit as well as perturbations structure. It is also important that by construction such modification does not violate the local tests viability of modified gravity.

IV. REALISTIC MODEL WITHOUT FUTURE SINGULARITY

It has been shown that if $f(R)$ behaves as (8) for large curvature there could occur the curvature singularity in a finite time. In the Hu-Sawicki model, the behavior of $f(R)$ in (8) generates the late-time acceleration.

Note, however, the late-time acceleration occurs if $f(R_0)$ becomes almost constant in the present universe:

$$f(R_0) = -2\tilde{R}_0, \qquad f'(R_0) \sim 0. \qquad (124)$$

Here R_0 is the current curvature of the Universe and we assume $R_0 > \tilde{R}_0$. Almost constant $f(R)$ plays the role of the effective small cosmological constant: $\Lambda_l \sim -f(R_0) = 2\tilde{R}_0$, which could generate the accelerating expansion in the present universe.

In the Hu-Sawicki model [5] the following condition is also satisfied:

$$\lim_{R \to 0} f(R) = 0. \qquad (125)$$

The condition shows that there occurs a flat spacetime solution, where $R = 0$.

Then we need not to require the behavior (8) when the curvature is really large $R \gg R_0$. For example, if $f(R)$ behaves as

$$f(R) = -(1 - \eta)R - 2\eta\Lambda_I + \mathcal{O}(R^{-1}), \qquad (126)$$

we find

$$F(R) = \eta(R - 2\Lambda_I) + \mathcal{O}(R^{-1}). \qquad (127)$$

In (126), η is a positive constant. Equation (127) tells us that the effective gravitational coupling κ_{eff} is given by $\kappa_{\text{eff}}^2 = \eta\kappa^2$ and the effective cosmological constant, which could generate the inflation, is given by Λ_I. For (127), the potential V in (7) is given by

$$V \sim \frac{2\Lambda_I}{\eta}. \qquad (128)$$

Then if Λ_I is large enough, the curvature singularity could not be easily realized. Hence, if $F(R)$ goes to the Einstein action (the gravitational constant could be changed) with the large cosmological constant, the value of the potential for large R becomes the order of the cosmological constant. Then if the cosmological constant is large enough, the value of the potential becomes large and the value cannot be easily realized.

An example satisfying the conditions (124)–(126) could be given by (compare with general viable construction [10])

$$f(R) = f_I(R) + f_L(R),$$

$$f_I(R) = -(1 - \eta)R - 2\eta\Lambda_I + 2\eta\Lambda_I \\ \times e^{(1-\eta)R/2\eta\Lambda_I - (1+\xi)(1-\eta)^2 R^2/8\eta^2\Lambda_I^2}, \qquad (129)$$

$$f_L(R) = -2\alpha R^2 e^{-c\alpha^2 R^2}.$$

Here c, α, and ξ are positive constants satisfying the conditions $\Lambda_I \gg 1/\alpha$ and $c \sim 1$. When the curvature has the order of $1/\alpha$, $f_I(R)$ behaves as $f_I(R) = R \times \mathcal{O}(R/\Lambda_I)$ and therefore $f_I(R)$ could be neglected compared with $f_L(R) \sim R$. Then since

$$f'(R) \sim f_L'(R) = -4\alpha R(1 - c\alpha^2 R^2)e^{-c\alpha^2 R^2}, \qquad (130)$$

compared with (124), we find

$$R_0 \sim \frac{1}{\sqrt{c}\alpha}, \qquad \tilde{R}_0 \sim \frac{1}{c\alpha}e^{-1}. \qquad (131)$$

Since about 70% of the energy density in the Universe is dark energy, we find $\tilde{R}_0 \sim 0.7R_0$. Then one has

$$c \sim (0.7)^2 e^{-2} \sim 0.07. \qquad (132)$$

We should note that the model (129) unifies the late acceleration and the inflation in the early universe like models in [8,9] and the future singularity does not appear.

We now check that the model (129) could satisfy the constraints on the correction to the Newton law. It is convenient to work in the Einstein frame (7). The potential $V(\sigma)$ in (7) gives a mass m_σ for the scalar field σ:

$$m_\sigma^2 \equiv \frac{1}{2}\frac{d^2V(\sigma)}{d\sigma^2} = \frac{1}{2}\left\{\frac{A}{F'(A)} - \frac{4F(A)}{(F'(A))^2} + \frac{1}{F''(A)}\right\}. \qquad (133)$$

In order that the correction to the Newton law could be small, the mass m_σ should be large enough. The Newton law has been checked on the Earth and in the Solar System. In air on the Earth, the scalar curvature could be given by $A = R \sim 10^{-50}$ eV^2. On the other hand, in the Solar System, we find $A = R \sim 10^{-61}$ eV^2. Since $R_0 \sim 10^{-66}$ eV^2, one may assume $\Lambda_I \gg R \gg 1/\alpha$. Then

$$f_I(R) \sim -\frac{\xi(1 - \eta)^2 R^2}{4\eta\Lambda_I}, \qquad (134)$$

and therefore

$$|f_I(R)|, \qquad |f_L(R)| \ll R, \qquad |f_I'(R)|, \qquad |f_L'(R)| \ll 1. \qquad (135)$$

Note that $f_I''(R)$ is negative and if $\Lambda_I \sim (10^{15-18}\text{GeV})^2 = 10^{48-54}$ eV^2, we have

$$f_I''(R) \sim (10^{48-54} \text{ eV}^2)^{-1}. \qquad (136)$$

We also find

$$f_L''(R) \sim -\frac{8}{\sqrt{c}R_0}\left(\frac{R}{R_0}\right)^4 e^{-R^2/R_0^2}, \qquad (137)$$

which is negative. Then in the air of the earth, we find

$$f_L''(R) \sim (10^{10^{32}} \text{ eV})^{-1}, \qquad (138)$$

which is extremely small and in the Solar System,

$$f_L''(R) \sim (10^{10^{10}} \text{ eV})^{-1}, \qquad (139)$$

which is also extremely small. Then the mass of scalar field

σ could be given by

$$m_\sigma^2 \sim -\frac{1}{2F''(R)} \sim -\frac{1}{2f_I''(R)}$$
$$\sim 10^{48-54} \text{ eV}^2 = (10^{15-18} \text{ GeV})^2, \qquad (140)$$

which is very large and the correction to the Newton law is beyond observational capacities. Hence, the viable modified gravity which unifies the inflation with late-time acceleration and which belongs to a general class of Ref. [10] is free of future singularity.

As it was shown in the arguments around (10) and (11) or (123), instead of (126), if we add the term R^n ($0 < n \le$

2), the curvature singularity could be avoided in the same way as in [12].

Equation (11) shows that there could be finite future singularity. We also find that the big rip singularity could be realized in $F(R) \sim R^n$ ($n > 2$) theory as shown in (29). This could be confirmed by the trace equation (4). We now assume (10) with $n > 2$ and therefore $F(R) \sim f_n R^n$ with $n > 2$ when the curvature is large. Then the trace equation (4) is reduced in the following form:

$$-3n\left(\frac{d^2}{dt^2} + 3H\frac{d}{dt}\right)R^{n-1} \sim (2-n)R^n. \qquad (141)$$

The explicit solution of (141) is given by a big rip solution:

$$H = \frac{h_\pm^{(n)}}{t_0 - t}, \qquad h_\pm^{(n)} = \frac{3n^2 - 4n + 2 \pm \sqrt{(3n^2 - 4n + 2)^2 + 8(n-2)(n-1)n(2n-1)}}{4(n-2)}. \qquad (142)$$

Since $3n^2 - 4n + 2 = 3(n-2)^2 + 8(n-2) + 4$, $h_+^{(n)}$ is surely real and positive. Then there could appear a finite future big rip singularity in $F(R) \sim R^n$ ($n > 2$) theory. However, considering several such terms with different powers of curvature may often lead to a stable de Sitter solution which prevents the evolution to singularity.

V. DISCUSSION

In summary, we studied the future evolution in the models of $F(R)$ gravity consistent with local tests and unifying the universe expansion history. Working in both (Einstein and Jordan) frames we demonstrated how big rip or type II, III, and IV future singularities appear. Using the reconstruction method [16] it is shown which models may lead to any specific type of future singularity. It is remarkable that whatever type singularity may occur in the Jordan frame, in the corresponding Einstein frame the only IV type singularity appears. It is also interesting that as in the scalar-tensor theory, the appearing singularity does not lead to the violation of local tests when modified gravity is consistent with them. Nevertheless, some indications to possible future singularity may be expected to be found in the current observational bounds [9,21].

Even if the $F(R)$ gravity under investigation develops the future singularity, there is always a way to modify its structure by terms relevant at the early universe. Such terms, like in the model in [12], prevent the development

of the singularity. The explicit example of realistic $F(R)$ gravity which does not lead to future singularity even classically is proposed. From another point, the classical description breaks down near to singularity. The account of quantum effects becomes necessary in this situation. Taking into account the quantum effects of conformal fields, we show that such quantum effects naturally act against the singularity appearance. Of course, in order to understand how realistic our quantum description is of future singularity, one should take into account quantum gravity.

As a final remark, let us note that other modified gravities (Gauss-Bonnet gravity, string-inspired gravity, theory with inhomogeneous equation of state) [1] also naturally describe the quintessence/phantom-dominated universe which may evolve to future singularity. It would be of interest to study the structure of future singularities in such models too.

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