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Perturbative exponential expansion and matter neutrino oscillations

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We derive an analytical description of neutrino oscillations in matter based on the Magnus exponential representation of the time evolution operator. Our approach is valid in a wide range of the neutrino energies and properly accounts for the modifications that the respective probability transitions suffer when neutrinos originated in different sources traverse the Earth. The present approximation considerably improves over other perturbative treatments existing in the current literature. Furthermore, the analytical expressions derived inside the Magnus framework are remarkably simple, which facilitates their practical use. When applied to the calculation of the day-night asymmetry in the solar neutrino flux our result reproduces the numerical calculation with an accuracy better than 1% for the first-order approximation. When the approximation is extended to the second order, the accuracy of the method is further improved by almost 1 order of magnitude, and it is still better than 5% even for neutrino energies as large as 100 MeV. In the GeV regime characteristic of atmospheric and accelerator neutrinos this accuracy is complemented by a good reproduction of the position of the maxima in the flavor transition probabilities.

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I. INTRODUCTION

Neutrino physics has experienced a spectacular progress in the last decade. Many experiments with neutrinos from different natural and artificial sources have provided convincing evidence on the existence of neutrino oscillations, a remarkable quantum interference phenomenon taking place at macroscopic distance. Experimental results can be satisfactorily accommodated within a scheme where at least two neutrinos are massive and there exist a leptonic mixing analogous to the one in the quark sector. From the present data set, two neutrino mass-squared differences and two mixing angles have been determined [1]: ($\delta m_{21}^2 \equiv m_2^2 - m_1^2 \approx 8.0 \times 10^{-5}$ eV^2, $\theta_{12} \approx 35°$) driving solar and reactor neutrino oscillations and ($|\delta m_{32}^2| \approx 2.5 \times 10^{-3}$ eV^2, $\theta_{23} \approx 45°$) which drives atmospheric and long baseline neutrino oscillations. The third angle θ_{13} and the CP-violating phase remain undetermined. The determination of these parameters, as well as the determination of the neutrino mass hierarchy, will be the main goals of the next generation of experiments. We are thus entering into a new stage characterized by high precision measurements. In turn, the interpretation of the forthcoming results will require more careful theoretical descriptions of neutrino oscillations that incorporate subleading processes.

A subject of particular interest within this context, refers to the matter effects on the flavor transformations for neutrinos propagating through the Earth. The problem has been investigated by direct numerical integration of the equation that governs flavor evolution in a medium. Yet, analytic calculations have been implemented to simplify the numerical computations greatly and also to gain a better understanding of the underlying physics. Many of these studies have been carried out under the assumption of one or several layers of constant density. Extensions for a varying density have been developed on the basis of the perturbation theory for oscillations, both in the low-energy [2–4] and high-energy regime [5–8]. For low energy neutrinos the perturbative solutions were found in the basis of the mass eigenstates, while in the high-energy limit the method was formulated in the flavor basis. In this work, we present a novel analytic description of the effect based on the Magnus exponential expansion of the time-displacement operator $\mathcal{U}(t, t_0)$, which makes possible a unified treatment of the problem and gives us precise simple formulas for both energy ranges.

The evolution of the flavor amplitudes of a neutrino system may be conveniently described in terms of the operator $\mathcal{U}$, which satisfies the Schrödinger-like equation [9]

$$i\hbar \frac{d\mathcal{U}}{dt}(t, t_0) = H(t)\mathcal{U}(t, t_0), \tag{1}$$

with the initial condition $\mathcal{U}(t_0, t_0) = I$. Later we shall give the explicit form for the matrix $H(t)$ in the Mikheev-Smirnov-Wolfenstein (MSW) theory. The Magnus expansion [10] supplies a method for finding a *true* exponential solution of Eq. (1) of the form $\mathcal{U} = \exp(\Omega)$ (i.e., without time ordering). The operator Ω satisfies its own differential equation which in turn is solved through a series expansion: $\Omega = \sum_{n=1}^{\infty} \Omega_n$, where Ω_n is of order $\hbar^{-n}$. The first two terms are explicitly given by

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$$\Omega_1 = -\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t'),$$
$$\Omega_2 = -\frac{1}{2\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''[H(t'), H(t'')]. \tag{2}$$

Because of the anti-Hermitian character of every operator Ω_n, truncating the series for Ω at any order gives a unitary approximation to $\mathcal{U}$. This is briefly what we shall need to know about the Magnus expansion for its present application; further details about the formalism and recursive procedures for building up the successive terms can be found in the specific literature [11–13].

Here, we use the first- and second-order Magnus approximation to seek solutions to the problem of 2ν oscillations in a medium with an arbitrary density profile, which is symmetric with respect to the middle point of the neutrino trajectory. The method is based on a formalism that was developed several years ago in order to incorporate nonadiabatic effects in the flavor transitions of neutrinos that propagate trough a matter-enhanced oscillation region [14]. The main idea is to follow the time development of the system in the adiabatic basis of the instantaneous energy eigenstates and to incorporate the corrections to adiabaticity through the Magnus expansion. In [15] the Magnus approximation was used to deal with the same problem but in the base of the (nonevolving) mass eigenstate. When applied to the calculation of the day-night asymmetry for solar neutrinos the method renders a simple formula for the regeneration factor, which has a better agreement with numerical calculations than those derived by using perturbation theory. The approach we are presenting now is not only more accurate than the one developed in [15] but is also valid in a much wider energy interval, allowing for a unified description of the Earth effect on the oscillations of both low- and high-energy neutrinos.

The paper is organized as follows. In the next section we describe the basic ingredients of the formalism and derive the formula for the flavor transition probability in a medium with varying density. In Sec. III we present two applications of physical interest. In the first one it is shown how the regeneration phenomenon of solar neutrinos traversing the Earth can be conveniently accounted for by our present approach. In the second application, we examine the influence of the terrestrial matter on the probabilities for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ transitions. Section IV contains the conclusions.

II. FORMALISM

Typically, the quantity of interest is the probability P_{ν_e} of observing an electron neutrino at a distance $L \simeq t_f - t_0$ from a source ($\hbar = c = 1$). If $|\nu(t_f)\rangle$ represents the neutrino state at time t_f, then $P_{\nu_e} = |\langle\nu_e|\nu(t_f)\rangle|^2 = |\langle\nu_e|\mathcal{U}(t_f, t_0)|\nu(t_0)\rangle|^2$, where $|\nu(t_0)\rangle$ denotes a certain initial state. We consider oscillations between two neutrino flavors, say ν_e and ν_a. In the relativistic limit and after discarding an overall phase, the Hamiltonian of the system in the flavor basis $\{|\nu_e\rangle, |\nu_a\rangle\}$ can be written as

$$H(t) = \frac{\Delta_0}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \frac{V(t)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

where θ is the mixing angle in vacuum and we have defined $\Delta_0 \equiv \delta m^2/2E$, with E the neutrino energy and δm^2 the squared mass difference. The effect of the medium is accounted for by means of V, the difference of the potential energies V_e and V_a. In normal matter, to lowest order in the Fermi constant G_F, we have $V(t) = V_e(t) - V_a(t) = \sqrt{2}G_F n_e(t)$, where n_e is the number density of electrons along the neutrino path.

The evolution operator in the flavor basis can be expressed as $\mathcal{U}(t_f, t_0) = U_m(t_f)\mathcal{U}^{\mathcal{A}}(t_f, t_0)U_m^\dagger(t_0)$, in terms of the corresponding operator $\mathcal{U}^{\mathcal{A}}(t, t_0)$ in the adiabatic basis of the (instantaneous) eigenstates $\{|\nu_{1m}(t)\rangle, |\nu_{2m}(t)\rangle\}$ of $H(t)$. Here,

$$U_m(t) = \begin{pmatrix} \cos\theta_m(t) & \sin\theta_m(t) \\ -\sin\theta_m(t) & \cos\theta_m(t) \end{pmatrix} \tag{4}$$

is the orthogonal transformation that, at each time, diagonalizes the matrix in Eq. (3). The mixing angle in matter $\theta_m(t)$ is given by

$$\sin 2\theta_m(t) = \frac{\Delta_0 \sin 2\theta}{\Delta_m(t)}, \tag{5}$$

where

$$\Delta_m(t) = \Delta_0\sqrt{(\varepsilon(t) - \cos 2\theta)^2 + \sin^2 2\theta} \tag{6}$$

stands for the difference between the energy eigenvalues and we have introduced the nondimensional quantity $\varepsilon(t) = V(t)/\Delta_0 = 2EV(t)/\delta m^2$.

If $V(t)$ is symmetric with respect to the middle point of the neutrino trajectory $\bar{t} = (t_f + t_0)/2$, then $\theta_m(t_f) = \theta_m(t_0) \equiv \theta_m^0$ and

$$\mathcal{U}(t_f, t_0) = U_m(t_0)\mathcal{U}^{\mathcal{A}}(t_f, t_0)U_m^\dagger(t_0). \tag{7}$$

This is the situation for the Earth, in which case θ_m^0 is the angle evaluated at the surface. In what follows, we restrict ourselves to such a case and find an analytical expression for $\mathcal{U}(t_f, t_0)$ in terms of $\mathcal{U}^{\mathcal{A}}(t_f, t_0)$ calculated by means of the first-order Magnus approximation. We follow the procedure presented in Ref. [14] adapted to the present situation. To make the work self contained we repeat here some of the steps presented there.

The evolution operator in the adiabatic basis is a 2×2 matrix that obeys Eq. (1), with the Hamiltonian

$$H^{\mathcal{A}}(t) = H_D(t) - iU_m^\dagger(t)\dot{U}_m(t), \tag{8}$$

where $H_D(t) = -\frac{1}{2}\Delta_m(t)\sigma_z$ is a diagonal matrix whose elements are the eigenvalues of Eq. (3) and $U_m^\dagger(t)\dot{U}_m(t) = i\dot{\theta}_m(t)\sigma_y$. Here, dot means differentiation with respect to time and σ_z and σ_y are Pauli matrices.

Neglecting the second term in Eq. (8) corresponds to solving the problem in the adiabatic approximation. In any case, the time dependence generated by $H_D(t)$ can be integrated exactly by a change of the representation, which is readily accomplished by means of the unitary transformation $\mathcal{U}^{\mathcal{A}}(t, t_0) = \mathcal{P}(t, t_0)\mathcal{U}^{\mathcal{A}}_{\mathcal{P}}(t, t_0)$, where

$$\mathcal{P}(t, t_0) = \exp\left(-i\int_{t_0}^{t} dt' H_D(t')\right) = \begin{pmatrix} e^{-(i/2)\phi_{t_0\to t}} & 0 \\ 0 & e^{i/2\phi_{t_0\to t}} \end{pmatrix}, \tag{9}$$

with

$$\phi_{x\to y} = \int_x^y dt'\Delta_m(t'). \tag{10}$$

In the new picture the evolution operator obeys

$$i\frac{d\mathcal{U}^{\mathcal{A}}_{\mathcal{P}}}{dt} = H^{\mathcal{A}}_{\mathcal{P}}(t)\mathcal{U}^{\mathcal{A}}_{\mathcal{P}}, \tag{11}$$

where

$$H^{\mathcal{A}}_{\mathcal{P}}(t) = i\dot{\theta}_m(t)\begin{pmatrix} 0 & -e^{-i\phi_{t_0\to t}} \\ e^{i\phi_{t_0\to t}} & 0 \end{pmatrix}. \tag{12}$$

Thus, we have removed not only the diagonal part, but the remainder of the Hamiltonian gets a simple structure which facilitates the algebraic manipulations that follows.

In general, it is not possible to solve (11) exactly and one has to rest on some approximation in order to determine $\mathcal{U}^{\mathcal{A}}_{\mathcal{P}}$. We employ here, with this purpose, the Magnus expansion and write $\mathcal{U}^{\mathcal{A}}_{\mathcal{P}} = e^{\Omega}$. Without loss of generality one can take $\det\mathcal{U}^{\mathcal{A}}_{\mathcal{P}} = 1$ and, therefore, to any order the Magnus operator has to be of the form $\Omega = -i\vec{\sigma}.\vec{\xi}$, where the components of vector $\vec{\sigma}$ are the Pauli matrices and ξ_x, ξ_y, and ξ_z are real coefficients whose specific forms depend on the order of the approximation used to determine Ω in terms of $H^{\mathcal{A}}_{\mathcal{P}}$. Consequently, we have

$$\mathcal{U}^{\mathcal{A}}_{\mathcal{P}} = \cos\xi I + \frac{\sin\xi}{\xi}\Omega, \tag{13}$$

where I is the identity matrix and $\xi = \sqrt{\xi_x^2, +\xi_y^2 + \xi_z^2}$. From Eqs. (9) and (13) it can be shown that $\mathcal{U}^{\mathcal{A}}$ is of the general form

$$\mathcal{U}^{\mathcal{A}} = \begin{pmatrix} \mathcal{U}^{\mathcal{A}}_{11} & \mathcal{U}^{\mathcal{A}}_{12} \\ -\mathcal{U}^{\mathcal{A}*}_{12} & \mathcal{U}^{\mathcal{A}*}_{11} \end{pmatrix}, \tag{14}$$

with the condition $|\mathcal{U}^{\mathcal{A}}_{11}|^2 + |\mathcal{U}^{\mathcal{A}}_{12}|^2 = 1$. The evolution operator in the flavor basis has the same matrix structure as it is easily checked by substituting (14) into Eq. (7).

Subsequently, we put $\Omega \cong \Omega_1 + \Omega_2$ and find $\Omega_{1,2}$ by means of the formulas given in Eq. (2) evaluated with the Hamiltonian of Eq. (12). Proceeding in this manner, and after some algebraic manipulations, we arrive at

$$\mathcal{U}^{\mathcal{A}}(t_f, t_0) \cong \begin{pmatrix} (\cos\xi - i\sin\xi\frac{\xi_{(2)}}{\xi})e^{i\phi_{\bar{t}\to t_f}} & i\sin\xi\frac{\xi_{(1)}}{\xi} \\ i\sin\xi\frac{\xi_{(1)}}{\xi} & (\cos\xi + i\sin\xi\frac{\xi_{(2)}}{\xi})e^{-i\phi_{\bar{t}\to t_f}} \end{pmatrix}, \tag{15}$$

where $\xi = \sqrt{\xi_{(1)}^2 + \xi_{(2)}^2}$. Nonadiabatic effects on the evolution of the flavor amplitudes are incorporated through the quantities $\xi_{(1)}$ and $\xi_{(2)}$, which come from the first- and the second-order Magnus approximations, respectively. They are given by

$$\xi_{(1)} = 2\int_{\bar{t}}^{t_f} dt'\frac{d\theta_m}{dt'}\sin\phi_{\bar{t}\to t'}, \tag{16}$$

$$\xi_{(2)} = \int_{t_0}^{t_f} dt'\int_{t_0}^{t'} dt'\frac{d\theta_m}{dt'}\frac{d\theta_m}{dt''}\sin\phi_{t'\to t''}. \tag{17}$$

The above expression for $\xi_{(1)}$ was obtained by taking into account that $V(t) = V(2\bar{t} - t)$ for a potential that is symmetric with respect to the middle point of the neutrino trajectory. In this case, $\dot{\theta}_m(t) = -\dot{\theta}_m(2\bar{t} - t)$ and $\int_{t_0}^{t_f} dt'\dot{\theta}_m(t')\sin\phi_{\bar{t}\to t'} = 2\int_{\bar{t}}^{t_f} dt'\dot{\theta}_m(t')\sin\phi_{\bar{t}\to t'}$, while $\int_{t_0}^{t_f} dt'\dot{\theta}_m(t')\cos\phi_{\bar{t}\to t'} = 0$. By integrating by parts, Eq. (16) can be rewritten as

$$\xi_{(1)} = 2\theta_m(t_f)\sin\phi_{\bar{t}\to t_f} - 2\int_{\bar{t}}^{t_f} dt'\theta_m(t')\Delta_m(t')\cos\phi_{\bar{t}\to t'}. \tag{18}$$

We see that $\mathcal{U}^{\mathcal{A}}$, as approximated by Eq. (15), has the form of the general matrix given in Eq. (14). This guarantee that the unitary condition $\mathcal{U}^{\mathcal{A}-1} = \mathcal{U}^{\mathcal{A}\dagger}$ is verified to second order. As mentioned in the introduction, this is an important quality of the Magnus expansion that remains true at every order. In addition, the off-diagonal elements of matrix (15) are purely imaginary, i.e., $\mathcal{U}^{\mathcal{A}*}_{12} = -\mathcal{U}^{\mathcal{A}}_{12}$, but in general this will not be verified when contributions of higher order are included. The same considerations apply to matrix $\mathcal{U}$.

Suppose that $|\nu(t_0)\rangle = \alpha|\nu_e\rangle + \beta|\nu_a\rangle$, with α and β non-negative (real) numbers satisfying $\alpha^2 + \beta^2 = 1$ then, taking into account the relations between the $\mathcal{U}_{\ell\ell'}$ ($\ell, \ell' = e, a$) just indicated, we find

$$P_{\nu_e} = \alpha^2 + (\beta^2 - \alpha^2)(\mathrm{Im}\mathcal{U}_{ea})^2 + 2\alpha\beta(\mathrm{Im}\mathcal{U}_{ee}) \times (\mathrm{Im}\mathcal{U}_{ea}), \tag{19}$$

with

$$\begin{aligned} \mathrm{Im}\,\mathcal{U}_{ee} &= \cos 2\theta_m^0\,\mathrm{Im}\mathcal{U}_{11}^{\mathcal{A}} + \sin 2\theta_m^0\,\mathrm{Im}\mathcal{U}_{12}^{\mathcal{A}}, \\ \mathrm{Im}\mathcal{U}_{ea} &= -\sin 2\theta_m^0\,\mathrm{Im}\mathcal{U}_{11}^{\mathcal{A}} + \cos 2\theta_m^0\,\mathrm{Im}\mathcal{U}_{12}^{\mathcal{A}}, \end{aligned} \tag{20}$$

where, according to Eq. (15),

$$\begin{aligned} \mathrm{Im}\,\mathcal{U}_{11}^{\mathcal{A}} &= \cos\xi \sin\phi_{\bar{t}\to t_f} - \sin\xi \frac{\xi_{(2)}}{\xi}\cos\phi_{\bar{t}\to t_f}, \\ \mathrm{Im}\mathcal{U}_{12}^{\mathcal{A}} &= \sin\xi\frac{\xi_{(1)}}{\xi}. \end{aligned} \tag{21}$$

As we see, to this order, only the imaginary parts of the matrix elements of the evolution operator are relevant to the calculation of P_{ν_e}. The result for the lowest-order Magnus approximation is obtained by putting $\xi_{(2)} = 0$ in the previous expressions for the imaginary parts of $\mathcal{U}_{11}^{\mathcal{A}}$ and $\mathcal{U}_{12}^{\mathcal{A}}$.

Formula (19), with the imaginary parts of $\mathcal{U}_{ee}$ and $\mathcal{U}_{ea}$ given by Eqs. (20) and (21), represents our main result. It provides an elegant and systematic description of neutrino oscillations in a medium with a symmetric, but otherwise arbitrary, density profile, which is valid for a wide range of energies. In order to illustrate its usefulness, in the next section we will apply it to two situations of physical interest where the 2ν oscillations are suitable to account for the leading process: (i) the regeneration effect of solar neutrinos when they go through the Earth, and (ii) the oscillations of high-energy neutrinos in the Earth.

III. APPLICATIONS

A. Day-night neutrino asymmetry

The relevant quantity in connection with the solar neutrinos is the probability for a neutrino born as a ν_e in the interior of the Sun, to remain as a ν_e at the Earth. The oscillation parameters controlling the leading effects are $\theta = \theta_{12}$ and $\delta m^2 = \delta m_{12}^2$ [16]. If the phase information is lost, as will typically happen for neutrinos traveling a long distance to the detection point, then according to the large mixing angle MSW solution the averaged survival probability for the electron neutrinos can be written as [17]

$$\bar{P}(\nu_e \to \nu_e) = \sin^2\theta + \cos 2\theta \cos^2\theta_\odot^0 - \cos 2\theta_\odot^0 f_{\mathrm{reg}}, \tag{22}$$

where $\theta_\odot^0$ denotes the matter mixing angle at the production point in the interior or the Sun. The regeneration factor $f_{\mathrm{reg}} = P_{2e} - \sin^2\theta$ represents the terrestrial matter effects expressed as the difference between the probability for ν_2 to become ν_e after traversing the Earth $P_{2e} \equiv P(\nu_2 \to \nu_e) = |\langle\nu_e|\mathcal{U}(t_f, t_0)|\nu_2\rangle|^2$ and the same probability in vacuum $|\langle\nu_e|\nu_2\rangle|^2 = \sin^2\theta$.

We will determine f_{reg} by calculating P_{2e} in terms of Eq. (19), with $|\nu(t_0)\rangle = |\nu_2\rangle = \sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle$. Accordingly, we get

$$P_{2e} = \sin^2\theta + \cos 2\theta(\mathrm{Im}\mathcal{U}_{e\mu})^2 + \sin 2\theta\,\mathrm{Im}\mathcal{U}_{ee}\,\mathrm{Im}\mathcal{U}_{e\mu}, \tag{23}$$

and from this result

$$\begin{aligned} f_{\mathrm{reg}} = {} & \cos 2\tilde{\theta}_m^0 \cos 2\theta_m^0 (\mathrm{Im}\mathcal{U}_{12}^{\mathcal{A}})^2 \\ & + \sin 2\tilde{\theta}_m^0 \sin 2\theta_m^0 (\mathrm{Im}\mathcal{U}_{11}^{\mathcal{A}})^2 \\ & - \sin(2\tilde{\theta}_m^0 + 2\theta_m^0)(\mathrm{Im}\mathcal{U}_{12}^{\mathcal{A}})(\mathrm{Im}\mathcal{U}_{12}^{\mathcal{A}}). \end{aligned} \tag{24}$$

Here, $\tilde{\theta}_m^0 = \theta_m^0 - \theta$ is the rotation angle that relates the basis of the mass eigenstates $\{|\nu_1,\rangle, |\nu_2\rangle\}$ with the adiabatic one, evaluated on the surface of the Earth. For a constant potential $\xi = 0$ and, taking into account that $\sin 2\tilde{\theta}_m = \varepsilon \sin 2\theta_m$, we recover the exact expression for the regeneration factor in a uniform medium

$$f_{\mathrm{reg}} = \varepsilon_0 \sin^2 2\theta_m^0 \sin^2\left[\frac{\Delta_m}{2}(t_f - t_0)\right]. \tag{25}$$

On the other hand, for the large mixing angle parameters of the solar neutrinos $\varepsilon \ll 1$ within the Earth. In this limit,

$$2\theta_m(t) = 2\theta + \sin 2\theta\varepsilon(t) + O(\varepsilon^2), \tag{26}$$

as can be shown by using Eq. (5) and

$$\frac{d\theta_m}{d\varepsilon} = \frac{\sin^2 2\theta_m}{2\sin 2\theta}. \tag{27}$$

Substituting Eq. (26) into Eq. (18) we find $\xi = -I + O(\varepsilon)$, where

$$I = \sin 2\theta \int_{\bar{t}}^{t_f} dt' V(t') \cos\phi_{\bar{t}\to t'}. \tag{28}$$

In this way, neglecting quantities of $O(\varepsilon)$ and higher everywhere, except in the adiabatic face $\phi_{\bar{t}\to t'}$, we arrive at

$$f_{\mathrm{reg}} = \frac{1}{2}\sin 2I \sin 2\theta \sin\phi_{\bar{t}\to t} + \sin^2 I \cos 2\theta, \tag{29}$$

which coincides with the expression for f_{reg} that was derived in Ref. [15] by applying the Magnus approximation to solve the equation for the evolution operator in the basis of the mass eigenstates. As pointed out there, by keeping the lowest terms of the expansion in I, Eq. (29) reduces to the result obtained by means of the perturbation theory.

In order to compare our results with those corresponding to the first- and second-order in the ε-perturbative expansion, we consider a simplified model for the electron density inside the Earth, the so-called mantle-core-mantle [18]. In this model the electron density is approximated by a step function and the radius of the core and the thickness of the mantle are assumed to be half of the Earth's radius $R_\oplus$:

$$n_e(r) = N_A \begin{cases} 5.95\ \mathrm{cm}^{-3}, & r \le R_\oplus/2 \\ 2.48\ \mathrm{cm}^{-3}, & R_\oplus/2 < r \le R_\oplus. \end{cases} \tag{30}$$

Following Ref. [4], we introduce the function

$$\delta(E) = \frac{1}{\bar{f}_{\mathrm{reg}}(E)}[f_{\mathrm{reg}}^{(\mathrm{appr})}(E) - f_{\mathrm{reg}}^{(\mathrm{exact})}(E)], \tag{31}$$

where $f_{\mathrm{reg}}^{(\mathrm{appr})}$ is given by a certain (approximated) analytical expression, $f_{\mathrm{reg}}^{(\mathrm{exact})}$ is obtained from the exact (numerical) solution, and $\bar{f}_{\mathrm{reg}}(E) = 1/2\varepsilon_0 \sin^2\theta$ is the average regeneration factor evaluated at the surface layer. Essentially, $\delta(E)$ represents the relative error of the approximated expression.

In Fig. 1 we show $\delta(E)$ as a function of the energy for neutrinos that cross the Earth through its center. For the "solar" oscillation parameters we take $\delta m_{21}^2 = 8 \times 10^{-5}\ \mathrm{eV}^2$ and $\tan^2\theta_{12} = 0.4$. As shown there, $\delta(E)$ for the different Magnus approximations is always smaller than those corresponding to the perturbative calculations. As already pointed out in Ref. [15], although the error associated with Eq. (29) increases with energy, it remains smaller than $\sim 2\%$ for $E \lesssim 14$ eV. The lowest-order adiabatic result derived by doing $\xi_{(2)} = 0$ in Eq. (24) works even better, reducing the relative error to less than 0.5% within the same energy interval. When the calculations in this basis are carried out up to the second order, the accuracy improves notably and the error is reduced by almost an order of magnitude as compared to the one for the first-order formula and remains less the 5% for energies up to 100 MeV. The last interval comprises the energy values that are typical for neutrinos originated in supernovae explosions. It should be noticed that our treatment works comparatively well in the whole range of energies, whenever the two neutrino approximation remains valid, which requires $E \ll \delta m_{31}^2/(2V \sin\theta_{13} \cos\theta_{13} \sin\theta_{12})$. In order to illustrate this point, in Fig. 2 we plot $\delta(E)$ for energies as large as 10 GeV, both for the first- and second-order calculations corresponding to the adiabatic Magnus expansion and the perturbative approach.

B. High-energy neutrinos

In this subsection we apply the present formalism to the oscillations of high-energy ($E \gtrsim 1$ GeV) neutrinos that go across a material medium with a symmetric density profile. If we assume that θ_{13} is not very small, then the quantity $\delta m_{21}^2/2E$ can be safely discarded in the equation governing the flavor evolution of a 3ν-system [5]. In this case, the mixing angle θ_{12} does not play any role and the problem reduces to an effective one of two states $|\nu_e\rangle$ and $|\nu_a\rangle = \sin\theta_{23}|\nu_\mu\rangle + \cos\theta_{23}|\nu_\tau\rangle$, where the matter oscillations are driven by the parameters $\delta m^2 = \delta m_{31}^2$ and $\theta = \theta_{31}$.

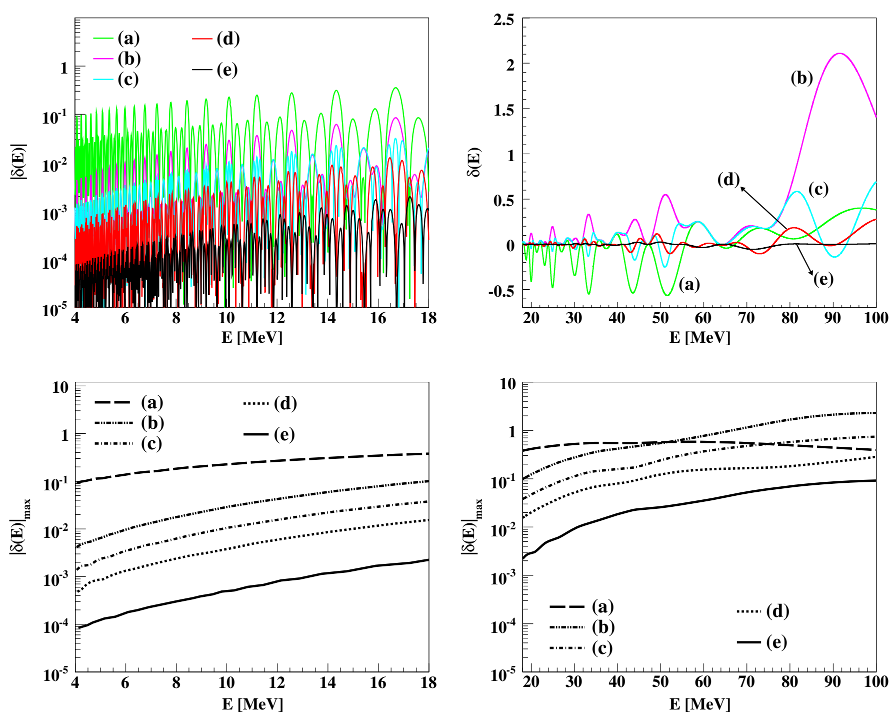

FIG. 1 (color online). The relative error δ [Eq. (31)] as a function of the energy for a neutrino crossing the center of the Earth. The lower panels correspond to the envelopes of $|\delta|$, i.e., to the maximum error to be expected at a given energy. The oscillation parameters are $\delta m_{21}^2 = 8 \times 10^{-5}\ \mathrm{eV}^2$ and $\tan^2\theta_{12} = 0.4$ and the density profile has been approximated by the core-mantle-core model [18]. (a) and (b) correspond to first and second order of the perturbative approach, respectively, (c) corresponds to Eq. (29), and (d) and (e) correspond to the first- and second-order Magnus calculation in the adiabatic basis, respectively.

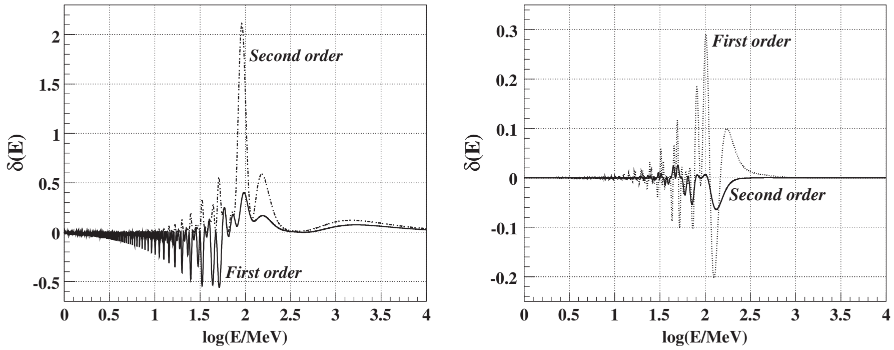

FIG. 2. The relative error δ [Eq. (31)] as a function of the energy (up to 10 GeV) for a neutrino crossing the center of the Earth. The oscillation parameters are $\delta m_{21}^2 = 8\times 10^{-5}\ \mathrm{eV}^2$ and $\tan^2\theta_{12} = 0.4$ and the density profile has been approximated by the core-mantle-core model. Curves plotted in the left panel correspond to first and second order of the perturbative approach and the ones in the right panel correspond to the first- and second-order Magnus calculation in the adiabatic basis. Note the different y-axis scales used in the graphics.

We focus hereafter in the transition probabilities $P(\nu_\mu \to \nu_e) = \sin^2\theta_{23} P(\nu_a \to \nu_e)$ and $P(\nu_\tau \to \nu_e) = \cos^2\theta_{23} P(\nu_a \to \nu_e)$. Now, $|\nu(t_0)\rangle = |\nu_a\rangle$ and according to Eq. (19), with $\alpha = 0$ and $\beta = 1$ we have

$$P(\nu_a \to \nu_e) = (\mathrm{Im}\,\mathcal{U}_{ae})^2 = (\cos 2\theta_m^0\,\mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}} - \sin 2\theta_m^0\,\mathrm{Im}\,\mathcal{U}_{11}^{\mathcal{A}})^2, \quad (32)$$

where $\mathrm{Im}\,\mathcal{U}_{11}^{\mathcal{A}}$ and $\mathrm{Im}\,\mathcal{U}_{12}^{\mathcal{A}}$ are determined from Eq. (21).

Suppose that $V \gg \Delta_0$; then, $\varepsilon \gg 1$ and we can implement a perturbative expansion in $1/\varepsilon$ for a varying potential. Accordingly,

$$2\theta_m \cong \pi - \frac{1}{\varepsilon}\sin 2\theta \quad (33)$$

and

$$\xi_{(1)} \cong \Delta_0 \sin 2\theta \int_{\bar{t}}^{t_f} dt' \cos\phi_{\bar{t}\to t'} + O\left(\frac{1}{\varepsilon}\right). \quad (34)$$

Using the last two equations and keeping at most terms of $O(1)$ in $1/\varepsilon$ (except in the phase $\phi_{\bar{t}\to t'}$), Eq. (32) becomes

$$P(\nu_a \to \nu_e) = \left[\sin\left(\Delta_0 \sin 2\theta \int_{\bar{t}}^{t_f} dt' \cos\phi_{\bar{t}\to t'}\right)\right]^2 \quad (35)$$

in the first-order Magnus approximation ($\xi_{(2)} = 0$). It is pertinent to note that the perturbative result presented in Ref. [5] $P(\nu_a \to \nu_e) = \Delta_0^2 \sin^2 2\theta [\int_{\bar{t}}^{t_f} dt' \cos\phi_{\bar{t}\to t'}]^2$ follows immediately from Eq. (35) when the sine function is replaced by its linear approximation.

The expression in Eq. (35) corresponds to the result derived by working directly in the flavor basis, following an approach similar to the one we used in [15]. This requires the factorization of the evolution operator as $\mathcal{U}(t, t_0) = \mathcal{P}^\dagger(t, t_0)\mathcal{U}_{\mathcal{P}}(t, t_0)$, where $\mathcal{P}$ is the same diagonal matrix given in Eq. (9), and the determination of $\mathcal{U}_{\mathcal{P}}(t, t_0)$ in terms of the lowest-order Magnus approximation $\mathcal{U}_{\mathcal{P}}(t, t_0) \cong \exp[-i\int_{t_0}^{t} dt' H_{\mathcal{P}}(t')]$, with the Hamiltonian

$$H_{\mathcal{P}}(t) = \mathcal{P}(t, t_0)[H(t) - H_D(t)]\mathcal{P}^\dagger(t, t_0) \cong \frac{\Delta_0}{2}\sin 2\theta \begin{pmatrix} 0 & e^{i\phi_{t_0\to t}} \\ e^{-i\phi_{t_0\to t}} & 0 \end{pmatrix}. \quad (36)$$

In the above equation, H_D is again a diagonal matrix whose elements are the eigenvalues of Eq. (3) and the second line has been obtained by using $\Delta_m(t) \cong \frac{1}{2}[V(t) - \Delta_0 \cos 2\theta]$. Proceeding in this way, the matrix representation for $\mathcal{U}(t_f, t_0)$ becomes

$$\mathcal{U}(t_f, t_0) = \begin{pmatrix} \cos\xi_{(1)} e^{i\phi_{\bar{t}\to t_f}} & i\sin\xi_{(1)} \\ i\sin\xi_{(1)} & \cos\xi_{(1)} e^{-i\phi_{\bar{t}\to t_f}} \end{pmatrix}, \quad (37)$$

with $\xi_{(1)}$ calculated according to Eq. (34). From the last expression we see that $\mathrm{Im}\,\mathcal{U}_{ae} = \sin\xi_{(1)}$ and formula (35) follows immediately when this result is substituted into $P(\nu_a \to \nu_e) = (\mathrm{Im}\,\mathcal{U}_{ae})^2$.

In Fig. 3 we plot $P(\nu_a \to \nu_e)$ as a function of E, for the same model of the Earth's density profile used in the previous section. We show the numerical calculation together with the analytical approximations corresponding to the Magnus expansion and to the perturbation theory at first order in $1/\varepsilon$. From the figures, it becomes clear that the formula derived by means of the Magnus expansion implemented in the adiabatic basis gives a better approximation than the perturbative method. Moreover, they never give probabilities higher than one, a pathology presented by the perturbative expressions as can be seen in the left panel of Fig. 3. This behavior remains true also for the formula given in Eq. (35), but in this case the approximation breaks down numerically for energies $E \approx$ (5–10) GeV, that corresponds to the resonance condition $V \approx \Delta_0$, for $\delta m_{31}^2 = 2.5\times 10^{-3}\ \mathrm{eV}^2$. The same limitation applies to the perturbative result quoted above.

If the mixing angle θ_{13} were vanishingly small or zero, then the problem is also described in terms of an effective

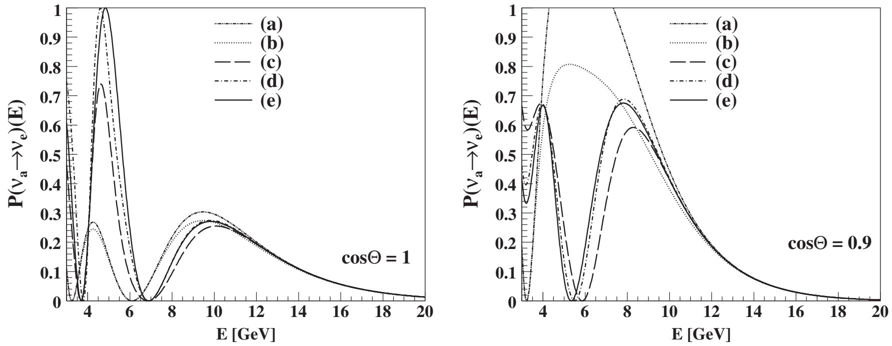

FIG. 3. $P(\nu_a \to \nu_e)$ as a function of the energy for a neutrino crossing the Earth passing by its center (left panel) and for a trajectory of Nadir angle $\Theta \cong 26°$ ($\cos\Theta = 0.9$) (right panel). The oscillation parameters are $\delta m^2_{31} = 2.5 \times 10^{-3}$ eV^2 and $\theta_{13} = 10°$. (a) corresponds to the perturbative approach (see Ref. [5]), (b) to Eq. (35), (c) and (d) correspond to the Magnus approximation implemented in the adiabatic basis for the first- and second-order, respectively, and (e) numerical calculation. Our approximation reproduces very well both the value and the position of the maxima of the numerical calculation.

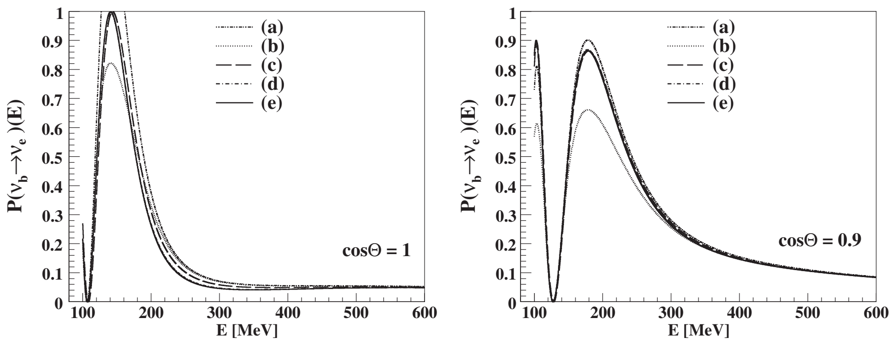

FIG. 4. $P(\nu_b \to \nu_e)$ as a function of the energy for a neutrino crossing the Earth passing by its center (left panel) and for a trajectory of Nadir angle $\Theta \cong 26°$ ($\cos\Theta = 0.9$) (right panel). The oscillation parameters are $\delta m^2_{21} = 8 \times 10^{-5}$ eV^2, $\tan^2\theta_{12} = 0.4$, and $\theta_{13} = 0$. (a) corresponds to the perturbative approach, (b) to Eq. (35), (c) and (d) correspond to the Magnus approximation implemented in the adiabatic basis for the first- and second-order respectively, and (e) numerical calculation.

two-state system $\{|\nu_e\rangle, |\nu_b\rangle\}$, with $|\nu_b\rangle = \cos\theta_{23}|\nu_\mu\rangle - \sin\theta_{23}|\nu_\tau\rangle$. In this case, the transition probabilities are $P(\nu_\mu \to \nu_e) = \cos^2\theta_{23} P(\nu_b \to \nu_e)$ and $P(\nu_\tau \to \nu_e) = \sin^2\theta_{23} P(\nu_b \to \nu_e)$, where $P(\nu_b \to \nu_e)$ can be computed by the same expression given in Eq. (32), but with the oscillation parameters $\delta m^2 = \delta m^2_{21}$ and $\theta = \theta_{12}$. From the curves plotted in Fig. 4, it is again evident that the analytical expression derived by means of the adiabatic Magnus expansion gives the best approximation to the exact (numerical) result.

IV. CONCLUSIONS

We have shown that the Magnus expansion for the evolution operator implemented in the basis of the instantaneous energy eigenvalues provides an elegant and, at the same time, efficient formalism to describe neutrino oscillations in a medium with an arbitrarily varying density profile. This approach incorporates in a simple way the Earth matter effects on the transition probabilities for neutrinos with a wide interval of energies, making possible a systematic description of such effects in the case of solar and atmospheric neutrinos. In both cases, the results are considerably more accurate than those derived by different perturbative calculations in the low and high-energy regimes. The same formalism can be applied without additional difficulties to the study of other situations of physical interest, like supernova neutrinos or long baseline experiments with accelerator neutrinos.

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