Towards the core of the quantum monopole

A. Campbellová* and R. von Unge+

Institute for Theoretical Physics and Astrophysics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

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We study monopole solutions of the quantum exact low-energy effective $N = 2$ super Yang-Mills theories of Seiberg and Witten. We find a first-order differential equation for the spatial dependence of the moduli and show that it can be interpreted as an attractor equation. Numerically integrating this equation, we try to address the question of what happens when one approaches the quantum core of the monopole where the low-energy effective theory breaks down or, alternatively, where there are modified monopole solutions that do not have a strongly coupled quantum core so that one may trust the solution not only asymptotically.

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I. INTRODUCTION

In the paper [1], quantum-corrected Bogomol'nyi-Prasad-Sommerfield (BPS) monopole solutions in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory were found. Using the exact low-energy effective Hamiltonian derived from the Seiberg-Witten solution of the effective action [2] it was shown that as one moves towards the center of the monopole, the moduli of the solution change in exactly such a way as to keep the phase of the local central charge constant. As was shown in [3,4], this result can be thought of as a rigid, nongravitational counterpart of the attractor mechanism [5,6] and as such it serves as a toy model for the full gravitational attractor mechanism.

However, while the original attractor mechanism is purely classical and perturbative quantum corrections have only recently been considered [7–10], in the gauge theory case the theory is fully quantum, both perturbative and nonperturbative. One might therefore hope to learn more about quantum corrections to the attractor mechanism by studying this toy model.

There is also another motivation for further studying the description of quantum-corrected monopoles given in [1]. As one approaches the center of the monopole the moduli reaches the strong coupling region where the effective action description ceases to be valid and one cannot trust the solution anymore. In this paper we investigate possible ways out of this dilemma. One possibility would be to mimic the enhancon idea [11] so that one cuts off the solution at some finite radius and replaces the center with another weakly coupled solution. Another possible solution would be to use a duality transformation to change the strongly coupled description in the core to a weakly coupled dual description.

Indeed in this paper we argue that a scenario somewhat analogous to the enhancon mechanism is possible to realize. By appropriately choosing an integration constant of the BPS equation (which is forced to be zero in the classical 't Hooft-Polyakov solution), we are able to find a natural cutoff point in the weak coupling region. At the cutoff radius both electric and magnetic fields are zero and there is no contribution to the mass of the monopole from the inner boundary. The center gets replaced by a bubble of Higgs vacuum. The similarity to the enhancon mechanism is only partial however; in our scenario we see no sign of symmetry enhancement. In fact, while the Abelian parts of the gauge field goes to zero at the cut off radius, the non-Abelian parts (W bosons) stay massive and nonzero. Also, one might have hoped that the quantum corrections would have made the geometry completely nonsingular (along the lines of [12]) but the energy density will be discontinuous indicating the presence of a shell-like singularity.

This paper is organized as follows. In Sec. II we review the basic results of [1]. In Sec. III we derive a differential equation for the spatial dependence of the moduli of the solution and show how this equation is related to the attractor mechanism. In Secs. IV and V we discuss the general behavior of the various fields as well as the energy density when one moves towards the center of the monopole. In Sec. VI we then give numerical results for various special cases and discuss the various scenarios that appear and the possibility to enhance them to solutions valid everywhere. Finally we conclude in Sec. VII. In the appendix various useful expansions around the strong coupling singularity (the attractor point) $u = 1$ are discussed.

II. REVIEW

The leading term of the low-energy effective action of $\mathcal{N} = 2$ super Yang-Mills theory is determined by a holomorphic function $\mathcal{F}(\mathcal{W})$ of the $\mathcal{N} = 2$ gauge superfield W

$$\mathcal{S}_{\mathcal{F}} = \frac{1}{2\pi} \operatorname{Im} \int d^4x d^4\theta \mathcal{F}(W). \tag{1}$$

Reducing the $\mathcal{N} = 2$ action to $\mathcal{N} = 1$ formulation and then to $\mathcal{N} = 0$ language we find for the bosonic part the

*campbell@physics.muni.cz
+unge@physics.muni.cz

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action

$$\mathcal{S}_{\mathcal{F}} = -\frac{1}{4\pi}\,\mathrm{Im}\int d^4x\mathcal{F}_{AB}\bigg[\frac{1}{2}(B_i^A + iE_i^A)(B_i^B + iE_i^B) + \nabla_\mu\phi^A\nabla^\mu\bar{\phi}^B + \frac{1}{2}[\phi,\bar{\phi}]^A[\phi,\bar{\phi}]^B\bigg],$$

where the magnetic and electric field are components of the field strength tensor $B_i^A = \epsilon_{ijk}F^{jkA}$ and $E_i^A = F_{0i}^A$, ϕ is the complex scalar, $A = 1, 2, 3$ is the $SU(2)$ group index, and $\mathcal{F}_A$ and $\mathcal{F}_{AB}$ are the first and second derivative of the prepotential $\mathcal{F}_A = \frac{\partial\mathcal{F}}{\partial\phi^A}$, $\mathcal{F}_{AB}(\phi) = \frac{\partial\mathcal{F}}{\partial\phi^A\partial\phi^B}$.

We consider only static configurations and choose the gauge $\nabla_0\phi^A = 0$.

In order to have finite energy configurations the vacuum expectation value for $r\to\infty$ must approach the Higgs vacuum, i.e. the potential must be zero and the scalar field must commute with its conjugate $[\phi,\bar{\phi}] = 0$. We can write the expectation value using a unit vector in the form

$$\langle\phi^A\rangle = a\mathbf{e}^A.$$

Then since $\mathcal{F}$ must be gauge invariant, it can depend only on $\phi^2 = \Sigma_{A=1}^3\phi^A\phi_A$ and the derivative $\mathcal{F}_A$ is simply

$$\mathcal{F}_A = \frac{\partial\mathcal{F}(\phi)}{\partial\phi}\frac{\phi^A}{\phi} = \mathcal{F}'\frac{\phi^A}{\phi}. \tag{2}$$

The expectation value of the dual field $\phi_{DA} = \mathcal{F}_A$ is

$$\langle\phi_{DA}\rangle = \mathcal{F}'\mathbf{e}_A \equiv a_D\mathbf{e}_A.$$

The parameter a is not gauge invariant; under the Weyl group it changes its sign $a \leftrightarrow -a$. Classically, a suitable gauge invariant parameter to distinguish the various Higgs vacua is $u = \langle\frac{1}{2}\phi^2\rangle$.

It was shown in [1] that as long as the imaginary part of $\mathcal{F}_{AB}$ is positive it is possible to write the effective Hamiltonian as a positive definite part plus a total derivative. Moreover, putting the positive definite part to zero one gets a BPS equation for the general monopole which is the same as for the classical theory

$$B_j^A + iE_j^A + e^{i\alpha}\sqrt{2}\nabla_j\phi^A = 0, \tag{3}$$

where $e^{i\alpha}$ is a constant phase. When the BPS equation is satisfied the whole contribution to the energy is from the total derivative term and solutions satisfying this equation are called BPS solutions.

We define the electric and magnetic quantum numbers by integrals at spatial infinity as

$$\begin{aligned} n_e a &= -\int d\vec{\Sigma}\cdot\vec{\Pi}_A\phi^A, \\ n_m a_D &= -\frac{1}{4\pi}\int d\vec{\Sigma}\cdot\vec{B}^A\phi_A^D, \end{aligned} \tag{4}$$

where $\vec{\Pi}_A$ is the momentum conjugate to the gauge field $\vec{A}^A$

$$\vec{\Pi}_A = -\frac{1}{4\pi}\,\mathrm{Re}\{\mathcal{F}_{AB}(\vec{B}^B + i\vec{E}^B)\}. \tag{5}$$

The contribution to the mass from the total derivative term can be written as

$$\mathrm{Im}\,(e^{i\alpha}Z), \tag{6}$$

for $Z = n_e a + n_m a_D$. This contribution to the energy will have its maximum value if we choose the phase $e^{i\alpha}$ to be related to the central charge Z by

$$e^{i\alpha} = i\frac{\bar{Z}}{|Z|}. \tag{7}$$

Then the BPS bound for the total energy is

$$E \geq |Z|. \tag{8}$$

As has been shown by Seiberg and Witten in [2], the vacuum expectation values $a(u)$ and $a_D(u)$ depend on the gauge invariant complex parameter u, which labels the different vacua

$$a(u) = \frac{4}{\pi q}\mathrm{E}(q), \qquad a_D(u) = -i\frac{4}{\pi q}[\mathrm{E}(q') - \mathrm{K}(q')], \tag{9}$$

where $q^2 = \frac{2}{1+u}$, $q'^2 = 1 - q^2$ and where $\mathrm{E}(q)$ and $\mathrm{K}(q)$ are complete elliptic integrals of the first and second kind. The complexified coupling constant in this case is

$$\tau = \frac{\partial^2}{\partial a^2}\mathcal{F} = \frac{\partial a_D}{\partial a} = i\frac{\mathrm{K}(q')}{\mathrm{K}(q)}. \tag{10}$$

This description of the low-energy dynamics is formally valid in the complex u plane outside a region given by the curve of marginal stability $\mathrm{Im}\frac{a_D(u)}{a(u)} = 0$ where the spectrum of the theory changes: particles become unstable or may disappear from the spectrum completely. In fact, the effective description should only be trusted in the region where the degrees of freedom integrated out are heavy compared to the degrees of freedom kept in the effective action. This condition defines a much larger region where the effective description can be trusted.

In order to find a numerical solution to the BPS equations it is necessary to make some simplifying assumptions. The 't Hooft-Polyakov monopole [13,14] can be found when we impose symmetry under the diagonal $SO(3)$ subgroup of the product of rotations and global gauge transformations $SO(3)\times SO(3)_G$,[1] i.e. invariance under generators $\vec{K} = \vec{J} + \vec{T}$ where $\vec{J}$ generates rotations and $\vec{T}$ gauge transformations. By imposing further a $\mathbb{Z}_2$ symmetry which consists of parity plus a sign change of ϕ we are left with the ansatz

[1] $SO(3)_G$ is the homomorphic image of the gauge group $SU(2)$.

$$\phi^A = \mathbf{e}^A \phi(r), \qquad A_i^A = \epsilon_{ij}^A \mathbf{e}^j \left(\frac{1 - L(r)}{r}\right), \qquad A_0^A = \mathbf{e}^A b(r), \tag{11}$$

where $r = \sqrt{x^i x^i}$ is the usual distance from the origin and $\mathbf{e}^A = x^A / r$ is a unit radial vector. The electric and magnetic fields are then

$$B_i^A = \mathbf{e}_i \mathbf{e}^A \frac{L^2 - 1}{r^2} + \mathcal{P}_i^A \frac{L_r}{r}, \qquad E_i^A = -\mathbf{e}_i \mathbf{e}^A b_r - \mathcal{P}_i^A \frac{bL}{r}, \tag{12}$$

with the projector $\mathcal{P}_i^A = \delta_i^A - \mathbf{e}_i \mathbf{e}^A$. The components proportional to $\mathbf{e}_i \mathbf{e}^A$ and $\mathcal{P}_i^A$ we will call Abelian and non-Abelian respectively, since the Abelian part is related to the Abelian $U(1)$ symmetry which survives symmetry breaking.

Inserting in the BPS equations (3) one finds

$$\sqrt{2} e^{i\alpha} \phi_r = \frac{1 - L^2}{r^2} + i b_r, \qquad \sqrt{2} e^{i\alpha} \phi = -\frac{d}{dr} \ln L + i b. \tag{13}$$

From these we can obtain the differential equation

$$\frac{d^2}{dr^2} (\ln L) = \frac{L^2 - 1}{r^2}, \tag{14}$$

which has the solution

$$L = \frac{\kappa r}{\sinh[\kappa(r + \delta)]}, \tag{15}$$

with constants δ and κ. The constant κ is given by considering the $r \to \infty$ limit: Taking the limit of the real part of the second BPS equation (3) one sees that

$$\sqrt{2} \, \mathrm{Re}\left\{ i \frac{n_m \bar{a}_D + n_e \bar{a}}{|n_m a_D + n_e a|} a \right\} = -\lim_{r \to \infty} \frac{d}{dr} \ln L; \tag{16}$$

the right-hand side of this is

$$\kappa = \lim_{r \to \infty} -\frac{1}{r} + \kappa \coth[\kappa(r + \delta)].$$

Altogether κ depends only on $u_0 = \lim_{r \to \infty}$ (for given quantum numbers)

$$\kappa = \sqrt{2} \frac{n_m a \bar{a}}{|n_m a_D + n_e a|} \, \mathrm{Im} \frac{a_D}{a}. \tag{17}$$

For the classical 't Hooft-Polyakov monopole we require the potentials A_i^A to be finite at $r = 0$ which implies the condition $L \to 1$ for $r \to 0$. Then the parameter δ must be chosen to be zero. However we do not need to impose any such requirement on the quantum-corrected monopole since $r = 0$ always lies in the region where the theory becomes strongly coupled and we cannot trust the low-energy description anymore. We therefore leave δ arbitrary. This does not affect the $r \to \infty$ behavior, so the magnetic quantum number is just as in the classical case $n_m = 1$.

Defining

$$X = \mathrm{Re}(e^{i\alpha} \phi), \qquad X_D = \mathrm{Re}(e^{i\alpha} \phi_D), \tag{18}$$

it was shown in [1] that

$$n_m X_D(r) + n_e X(r) = 0, \tag{19}$$

i.e. the local central charge $Z = n_m \phi_D + n_e \phi$ has a constant phase

$$\mathrm{Re}[e^{i\alpha} Z(r)] = 0. \tag{20}$$

The same result was also derived from a string theory perspective in [15–17]

The imaginary part of $\frac{\phi_D}{\phi}$ can be written using the central charge Z and the field X as

$$\mathrm{Im} \frac{\phi_D}{\phi}(r) = \frac{1}{n_m |\phi|^2} |Z(r)| X(r). \tag{21}$$

Thus if $\mathrm{Im} \frac{\phi_D}{\phi} = 0$ at a critical radius r_0, this corresponds to two possibilities: either $|Z(r_0)| = 0$ (a solution called Z pole) or $X(r_0) = 0$ (a solution called X pole).

III. MODULI SPACE DEPENDENCE

In this section we shall derive the spatial dependence of the moduli. We know both the spatial dependence of the scalar field and the dependence on the moduli. This will enable us to find a differential equation for the moduli. As was mentioned before, the prepotential $\mathcal{F}$ can depend only on $\phi^2 = \Sigma_{A=1}^3 \phi^A \phi^A$. Taking the derivatives we find that the second derivative (the coupling) can be written as

$$\mathcal{F}_{AB} = \frac{\phi_D}{\phi} \mathcal{P}_{AB} + \tau \mathbf{e}_A \mathbf{e}_B, \tag{22}$$

with $\tau = \mathcal{F}''$ given by (10).

Inserting the radial ansatz in Eq. (5) and using the fact that $\mathcal{P}_{AB}$ and $\mathbf{e}_i \mathrm{e}^B$ are orthogonal projection operators we have

$$\Pi_{iA} = -\frac{1}{4\pi} \mathrm{Re}\left[\mathcal{P}_{Ai} \left(\frac{L_r}{r} - i \frac{bL}{r} \right) \frac{\phi_D}{\phi} + \tau \mathbf{e}_i \mathbf{e}_A \left(\frac{L^2 - 1}{r^2} - i b_r \right) \right].$$

The functions L, b are real, so we can write everything in terms of real and imaginary parts and obtain

$$\Pi_{iA} = -\frac{1}{4\pi} \left[\mathcal{P}_{Ai} \left(\frac{L_r}{r} \mathrm{Re} \frac{\phi_D}{\phi} + \frac{bL}{r} \mathrm{Im} \frac{\phi_D}{\phi} \right) + \mathbf{e}_A \mathbf{e}_i \left(\frac{L^2 - 1}{r^2} \mathrm{Re}\tau + b_r \mathrm{Im}\tau \right) \right]. \tag{23}$$

On the other hand we have from the BPS equation (3)

$$B_i^A = -\sqrt{2}\,\mathrm{Re}\{e^{i\alpha}\nabla_i\phi^A\},$$
$$\Pi_{iA} = -\frac{1}{4\pi}\,\mathrm{Re}\{\mathcal{F}_{AB}(B_i^B + iE_i^B)\}$$
$$= \frac{1}{4\pi}\sqrt{2}\,\mathrm{Re}\{\nabla_i(e^{i\alpha}\phi_A^D)\},$$

where we have used that $\mathcal{F}_{AB}\nabla_i\phi^B = \nabla_i\mathcal{F}_A = \nabla_i\phi_A^D$.

In terms of the definitions (18) we may now write

$$B_i^A = -\sqrt{2}\nabla_i(\mathbf{e}^A X), \tag{24}$$

$$\Pi_{iA} = \frac{1}{4\pi}\sqrt{2}\nabla_i(\mathbf{e}_A X_D). \tag{25}$$

Then multiplying Eq. (19) by $\sqrt{2}\mathbf{e}_A$ and letting ∇_i operate on it, we get

$$n_m\sqrt{2}\nabla_i(\mathbf{e}^A X_D) + n_e\sqrt{2}\nabla_i(\mathbf{e}^A X) = 0,$$

which gives us another relation between the conjugate momentum and the magnetic field

$$\Pi_{iC} = \frac{n_e}{n_m}\frac{1}{4\pi}B_i^A\delta_{AC}. \tag{26}$$

Comparing this with (23) (after inserting the radial ansatz) we are left with

$$\frac{n_e}{n_m}\frac{1}{4\pi}\left(\mathbf{e}_i\mathbf{e}_A\frac{L^2-1}{r^2} + \mathcal{P}_{iA}\frac{L_r}{r}\right)$$
$$= -\frac{1}{4\pi}\left(\mathcal{P}_{iA}\left(\frac{Lr}{r}\,\mathrm{Re}\frac{\phi_D}{\phi} + \frac{bL}{r}\,\mathrm{Im}\frac{\phi_D}{\phi}\right)\right.$$
$$\left. + \mathbf{e}_A\mathbf{e}_i\left(\frac{L^2-1}{r^2}\,\mathrm{Re}\tau + b_r\,\mathrm{Im}\tau\right)\right).$$

Splitting this in the real and imaginary parts we have the following relations:

$$\frac{n_e}{n_m}\frac{L^2-1}{r^2} = -\left(\frac{L^2-1}{r^2}\,\mathrm{Re}\tau + b_r\,\mathrm{Im}\tau\right), \tag{27}$$

$$\frac{n_e}{n_m}\frac{L_r}{r} = -\left(\frac{L_r}{r}\,\mathrm{Re}\frac{\phi_D}{\phi} + \frac{bL}{r}\,\mathrm{Im}\frac{\phi_D}{\phi}\right). \tag{28}$$

Thus although we do not have the explicit dependence of b on r or u we know that

$$b = -\frac{\frac{n_e}{n_m} + \mathrm{Re}\frac{\phi_D}{\phi}}{\mathrm{Im}\frac{\phi_D}{\phi}}\frac{L_r}{L}, \qquad b_r = -\frac{L^2-1}{r^2}\frac{\frac{n_e}{n_m} + \mathrm{Re}\tau}{\mathrm{Im}\tau}. \tag{29}$$

When we insert this in the BPS equation (3) we find

$$\sqrt{2}e^{i\alpha}\phi_r = \frac{1-L^2}{r^2}\frac{i}{\mathrm{Im}\tau}\left(\frac{n_e}{n_m} + \bar{\tau}\right).$$

But since $\phi(r) = \phi(u(r))$, we can use the chain rule $\phi_r = \phi_u u_r$ and the dependence $\phi(u)$ in (9), which gives

$$\phi_u = \frac{q}{\pi}\mathrm{K}(q).$$

Inserting everything in the BPS equation we get

$$\sqrt{2}e^{i\alpha}\sqrt{\frac{2}{1+u}}\frac{\mathrm{K}(q)}{\pi}u_r = \frac{1-L^2}{r^2}\frac{i}{\mathrm{Im}\tau}\left(\frac{n_e}{n_m} + \bar{\tau}\right),$$

and we find the differential equation

$$u_r = \frac{\pi}{2}\sqrt{1+u}\,\frac{1-L^2}{r^2}\frac{e^{-i\alpha}}{\mathrm{K}(q)}\frac{i}{\mathrm{Im}\tau}\left(\frac{n_e}{n_m} + \bar{\tau}\right). \tag{30}$$

This is a first-order differential equation for the dependence of the moduli space parameter on the distance r to the center of the monopole. At $r = \infty$ the moduli space parameter u will take the vacuum value u_0 of the theory. Moving towards the center of the monopole the parameter u will change according to the above differential equation. The solutions have one integration constant—we will choose it to be the parameter which labels the vacua, i.e. the value of u at infinity $u(r \to \infty) = u_0$. From u_0 the constants α and κ [which figures in the function $L(r)$] are determined.

The dependence of the solution $u(r)$ on δ is hidden only in the function L and can be removed by changing the parameter from r to X since from (13)

$$\sqrt{2}\frac{dX}{dr} = \frac{1-L^2}{r^2}.$$

This changes the differential equation to

$$u_X = \frac{\pi}{2}\sqrt{\frac{1+u}{2}}\frac{\mathrm{e}^{-i\alpha}}{\mathrm{K}(q)}\frac{i}{\mathrm{Im}\tau}\left(\frac{n_e}{n_m} + \bar{\tau}\right). \tag{31}$$

So δ does not affect the shape of the curve $u(r)$, only its parametrization.

A. Spatial dependence

Since the shape of the $u(r)$ curve can be separated from its parametrization we begin by studying the parametrization given by $X(r)$. From (13) we find that the spatial dependence is given by

$$X = \frac{1}{\sqrt{2}}\left(-\frac{1}{r} + \frac{\kappa}{\tanh(\kappa r + \kappa\delta)}\right). \tag{32}$$

We see from Fig. 1 that there are essentially three different cases depending on whether δ is positive, negative, or zero. When $\delta > 0$, X changes monotonically from $X = \kappa/\sqrt{2}$ at $r = \infty$ to $X \to -\infty$ at $r = 0$. For $\delta = 0$, X also changes monotonically but ends at $X = 0$ for $r = 0$. The $\delta < 0$ case is quite different. For $r = \infty$ it starts at $X = \kappa/\sqrt{2}$ and decreases. For finite r there is a minimum and then X starts to increase and goes to ∞ at $r = -\delta > 0$. The maximum X is at an r (see Fig. 1) which is a solution of the condition

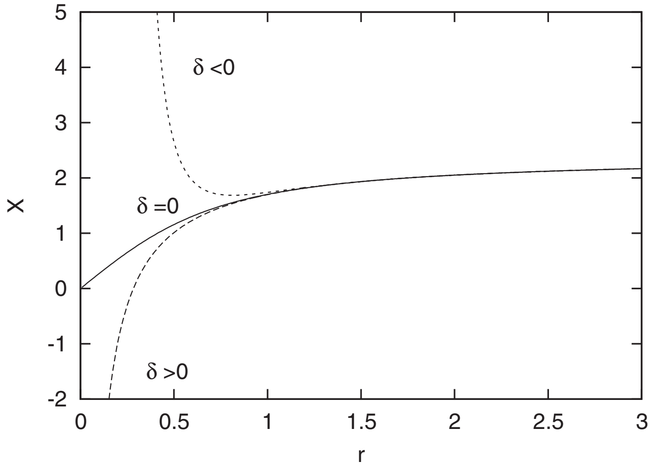

FIG. 1. The dependence $X(r)$.

$$\kappa r = \sinh[\kappa(r + \delta)]. \tag{33}$$

In the limit $\delta \to \infty$ the r dependence of X is very simple: $X = 1/\sqrt{2}(\kappa - 1/r)$. As we will see later, in this limit all non-Abelian parts of the fields are suppressed.

B. Attractor equation

Using X as a parameter and using the Seiberg-Witten metric $ds^2 = \mathrm{Im}\tau d\phi d\bar{\phi}$, the equation for u becomes

$$u_X = \frac{ie^{-i\alpha}}{2n_m} g^{u\bar{u}} \partial_{\bar{u}} \bar{Z} = -\frac{1}{n_m} g^{u\bar{u}} \partial_{\bar{u}} |Z|, \tag{34}$$

where we also have used the local central charge $Z(u) = n_m \phi_D(u) + n_e \phi(u)$ to rewrite the equation in a suggestive form. In fact, since there is a one-to-one map between u and Z we may use Z as a coordinate instead of u. This leads us to the equation

$$\frac{dZ}{dX} = \frac{ie^{-i\alpha}}{2n_m} g^{u\bar{u}} \partial_u Z \partial_{\bar{u}} \bar{Z} = \frac{ie^{-i\alpha}}{2n_m} g^{Z\bar{Z}}, \tag{35}$$

where $g_{Z\bar{Z}}$ is the Seiberg-Witten metric in Z coordinates and, using the fact that the phase of Z is constant for each solution, to the equation

$$\frac{d|Z|}{dX} = \frac{1}{2n_m} g^{u\bar{u}} \partial_u |Z| \partial_{\bar{u}} |Z|. \tag{36}$$

This is an attractor equation as first discovered in [3] and it can alternatively be derived taking the zero gravity limit of the ordinary attractor equations. Using this form of the equation and the fact that the Seiberg-Witten metric is positive definite we see that $\frac{d|Z|}{dX} > 0$. This means that when X is decreasing (which is the usual situation for decreasing r), $|Z|$ will decrease and $|Z| = 0$ is an attractor point.

C. General properties of the solutions

Using the above relations we may write

$$\frac{d|Z|}{dr} = \frac{1}{2n_m} g^{u\bar{u}} \partial_u |Z| \partial_{\bar{u}} |Z| \frac{dX}{dr}, \tag{37}$$

as well as

$$\arg \frac{dZ}{dX} = -\frac{\pi}{2} - \alpha. \tag{38}$$

From (7) it is clear that the phase α and the phase of the central charge sum up to $\frac{\pi}{2}$

$$\alpha = \frac{\pi}{2} - \arg Z. \tag{39}$$

From this we can see that the curve $Z(r)$ is a straight line in the Z plane going from $Z_0 = Z(\infty)$ to $Z = 0$. Since the derivative $\frac{dX}{dr}$ is in general positive, $|Z|$ will decrease when we decrease r. However, if the sign of the derivative $\frac{dX}{dr}$ changes (which is the case for $\delta < 0$), the phase of the derivative jumps by π and $|Z|$ starts to increase for decreasing r, ending up at $X = \infty$ for $r = -\delta$. This behavior, that $|Z|$ "bounces" at some value of r and starts to increase, leads us to call this class of solutions "bouncing solutions."

The point at which the bouncing solution turns around is given by (33). Whether the solution first hits $Z = 0$ or the curve of marginal stability distinguishes the X and Z poles. As can be seen from Fig. 2, if $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the solution is a Z pole; if $\alpha = \pm\frac{\pi}{2}$ it is both an X and a Z pole; otherwise it is an X pole. It follows from the explicit form of X that the X pole will reach the curve of marginal stability at a radius r_* given by a solution of

$$\tanh[\kappa(r_* + \delta)] = \kappa r_*. \tag{40}$$

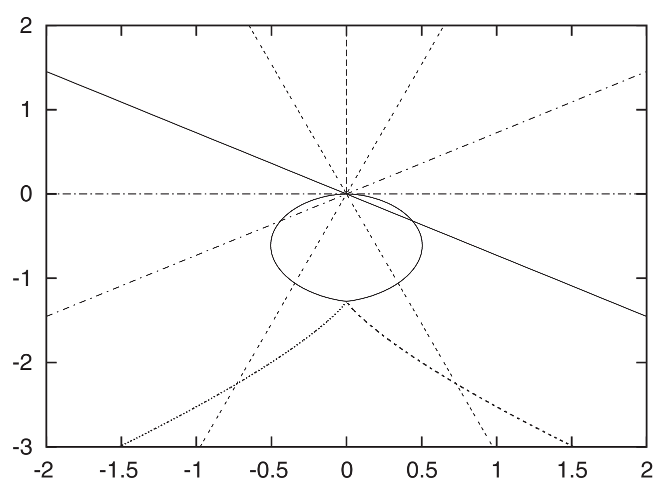

FIG. 2. Solutions, the curve of marginal stability and the branch cut in the complex plane of the central charge. Solutions with phase $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ hit the curve of marginal stability $\mathcal{C}$ at $Z = 0$ and are Z poles. The others are X poles or XZ poles (if $\alpha = \pm\frac{\pi}{2}$).

Solutions which reach the curve of marginal stability at $r > 0$ occur only for positive δ. Thus we see that in the case of X poles, the type of solution (bouncing or not) is given uniquely by the choice of the sign of δ.

We now turn to the Z poles. To simplify things we concentrate on the case with quantum numbers $n_m = 1$, $n_e = 0$ (here and for the rest of the paper). In this case the central charge is given by the dual scalar field. Also the solutions to the differential equation are symmetric with respect to the real u axis, since $\frac{du}{dr}|_u = \overline{\frac{du}{dr}}|_{\bar{u}}$.

The dual scalar field is zero at $u = 1$ so the Z poles end at this point in the u plane. We would like to see what value the parameter r acquires at this point. At this point $\phi = 4/\pi$ which corresponds to $X = 4/\pi \cos\alpha$. In terms of r this corresponds to the value r_c, which solves

$$-\frac{1}{r_c} + \kappa \coth[\kappa(r_c + \delta)] = \sqrt{2}\cos\alpha \frac{4}{\pi}. \quad (41)$$

For X poles the factor $\cos\alpha$ is negative and thus X becomes zero before this point is reached. For Z poles (i.e. $\alpha \in (-\pi/2, \pi/2)$) the factor $\cos\alpha$ is always positive and thus $u = 1$ is reached before $X = 0$.

For positive $\cos\alpha$ and $\delta < 0$ there are in principle two possibilities: either the solution bounces back at some point or it reaches $u = 1$. This is governed by the value of δ: for a value of δ greater than a certain δ_0 the solution will reach $Z = 0$ before it reaches the point of the bounce. If δ is smaller than δ_0 the solution will be a bouncing solution. This particular value $\delta_0 < 0$ solves the equation

$$\frac{8\sqrt{2}}{\pi} \frac{\kappa \cos\alpha}{\kappa^2 - \frac{32}{\pi^2}\cos^2\alpha} = \sinh\left[\frac{8\sqrt{2}\cos\alpha}{\pi} \frac{\kappa}{\kappa^2 - \frac{32}{\pi^2}\cos^2\alpha} + \kappa\delta_0\right]. \quad (42)$$

IV. ELECTRIC AND MAGNETIC FIELDS

As already mentioned the Abelian and non-Abelian parts of the electric and magnetic fields are given by

$$B_{\text{abel}} = -\sqrt{2}\frac{dX}{dr}, \qquad E_{\text{abel}} = B_{\text{abel}} \frac{\text{Re}\tau + \frac{n_e}{n_m}}{\text{Im}\tau}, \quad (43)$$

$$B_{\text{nab}} = -\sqrt{2}\frac{X}{r}\sqrt{1 - \sqrt{2}r^2 \frac{dX}{dr}}, \qquad E_{\text{nab}} = B_{\text{nab}} \frac{\text{Re}\frac{\phi_D}{\phi} + \frac{n_e}{n_m}}{\text{Im}\frac{\phi_D}{\phi}}. \quad (44)$$

In the classical case, for the 't Hooft-Polyakov monopole, the dual field is just a multiple of the scalar $\phi_D = \tau\phi$. So if $n_e = 0$, the factors relating the Abelian and the non-Abelian fields are equal. Furthermore, the coupling, and thus the proportion between electric and magnetic fields, is constant. In terms of the coupling constant g and the theta angle the electric fields are related to the magnetic fields as

$$E = \frac{\theta g^2}{8\pi^2} B. \quad (45)$$

Classically, the non-Abelian magnetic field is always nonzero. In the quantum case the non-Abelian magnetic field can become zero only if either $X = 0$ or $dX/dr = 1/(\sqrt{2}r^2)$. The first case is the X pole and the other case corresponds to the $\delta \to \infty$ limit. Then $X = (\kappa - 1/r)/\sqrt{2}$ and the Abelian magnetic field is $B_{\text{abel}} = -1/r^2$. The Abelian electric field has a more complicated dependence, due to the running coupling τ. Thus we can identify the $\delta \to \infty$ as the Abelian limit, where there are only Abelian fields [3].

The asymptotic behavior of the electric/magnetic fields is the same for all types of solutions. The Abelian fields behave for large r as $1/r^2$; the non-Abelian fields vanish exponentially

$$B_{\text{abel}} \approx -\frac{1}{r^2}, \qquad E_{\text{abel}} \approx -\frac{1}{r^2} \frac{\text{Re}\tau(u_0) + \frac{n_e}{n_m}}{\text{Im}\tau(u_0)}, \quad (46)$$

$$B_{\text{nab}} \approx \frac{\kappa}{r} e^{-\kappa(r+\delta)}, \qquad E_{\text{nab}} \approx \frac{\kappa}{r} e^{-\kappa(r+\delta)} \frac{\text{Re}\frac{\phi_D}{\phi}(u_0) + \frac{n_e}{n_m}}{\text{Im}\frac{\phi_D}{\phi}(u_0)}. \quad (47)$$

V. ENERGY DENSITY

The energy of a configuration is given by the Hamiltonian

$$H = \frac{1}{8\pi} \text{Im} \int d^4x \mathcal{F}_{AB}(E_i^A E_i^B + B_i^A B_i^B + 2\nabla_i \phi^A \nabla_i \bar{\phi}^B), \quad (48)$$

so the energy density is

$$\mathcal{E} = \frac{1}{8\pi} \text{Im}\mathcal{F}_{AB}(E_i^A E_i^B + B_i^A B_i^B + 2\nabla_i \phi^A \nabla_i \bar{\phi}^B).$$

For a BPS solution we see that the electromagnetic field and the Higgs field carry each one-half of the total energy. We can use the BPS equation to substitute for the Higgs field and consider twice the electromagnetic part of the energy. In the radial ansatz the coupling $\mathcal{F}_{AB}$, the electric and magnetic fields split in Abelian and non-Abelian components. The energy density splits in an Abelian and a non-Abelian part as well, with τ being the Abelian coupling and $\frac{\phi_D}{\phi}$ the non-Abelian coupling

$$\mathcal{E} = \frac{1}{4\pi} \text{Im}\tau(B_{\text{abel}}^2 + E_{\text{abel}}^2) + \frac{1}{2\pi} \text{Im}\frac{\phi_D}{\phi}(B_{\text{nab}}^2 + E_{\text{nab}}^2). \quad (49)$$

The Abelian and non-Abelian fields are given in Eq. (43) and (44).

The 't Hooft-Polyakov monopole is the classical case with $\mathcal{F} = \frac{1}{2}\tau\phi^A\phi^A$ (and $\delta = 0$). The Abelian and non-Abelian coupling are the same,; furthermore this coupling is fixed by its asymptotic value at infinity $\tau(u_0)$. Thus the classical energy is

$$\mathcal{E}_{cl} = \frac{\mathrm{Im}\tau}{4\pi}(B^2_{\mathrm{abel}} + E^2_{\mathrm{abel}} + 2B^2_{\mathrm{nab}} + 2E^2_{\mathrm{nab}}). \tag{50}$$

The Hamiltonian can be written as the term including the BPS equation $\mathcal{H}_0$ and a total derivative term, which can be rewritten as a surface term

$$\begin{aligned} H &= H_0 - \sqrt{2}\,\mathrm{Im}\Big[e^{i\alpha}\int d^3x\Big(\frac{1}{4\pi}\nabla_i(B^i_A\phi^A_D) + \nabla_i(\Pi^i_A\phi^A)\Big)\Big] \\ &= H_0 - \sqrt{2}\,\mathrm{Im}\Big[e^{i\alpha}\int_{S^2_\infty} d^2S_i\Big(\frac{1}{4\pi}B^i_A\phi^A_D + \Pi^i_A\phi^A\Big)\Big] \\ &\quad + \sqrt{2}\,\mathrm{Im}\Big[e^{i\alpha}\int_{S^2_{r_0}} d^2S_i\Big(\frac{1}{4\pi}B^i_A\phi^A_D + \Pi^i_A\phi^A\Big)\Big]. \end{aligned}$$

According to the definition of the electric and magnetic quantum numbers, the surface term at infinity is equal to $-(n_m a_D + n_e a)$. We can use the relation between the magnetic field and the conjugate momentum

$$4\pi n_m \Pi^A_i - n_e B^A_i = 0,$$

to write the third term only in terms of the magnetic field

$$\begin{aligned} H &= H_0 + \sqrt{2}\,\mathrm{Im}[e^{i\alpha}(n_m a_D + n_e a)] \\ &\quad + \sqrt{2}\,\mathrm{Im}\Big[e^{i\alpha}\int_{S^2_{r_0}} d^2S_i B^i_A\frac{1}{4\pi}\Big(\phi^A_D + \frac{n_e}{n_m}\phi^A\Big)\Big]. \end{aligned}$$

According to our ansatz, the magnetic field splits in an Abelian and a non-Abelian part $B^A_i = \mathbf{e}^i\mathbf{e}^A(L^2 - 1)/r^2 + (\delta^A_i - \mathbf{e}^i\mathbf{e}^A)L'/r$ and the scalar fields are pure Abelian $\phi^A = \phi\mathbf{e}^A$. Multiplying these by $d^2S^i = d\Omega r^2\mathbf{e}^i$ only the Abelian terms are left

$$\begin{aligned} H &= H_0 + \sqrt{2}\,\mathrm{Im}[e^{i\alpha}(n_m a_D + n_e a)] \\ &\quad + \sqrt{2}\,\mathrm{Im}\Big[e^{i\alpha}\int_{S^2_{r_0}} d\Omega\frac{L^2-1}{4\pi n_m}(n_m\phi_D + n_e\phi)\Big] \\ &= H_0 + \sqrt{2}\,\mathrm{Im}[e^{i\alpha}(n_m a_D + n_e a)] \\ &\quad + \sqrt{2}\,\mathrm{Im}\Big[e^{i\alpha}\frac{1}{n_m}(L^2 - 1)(n_m\phi_D + n_e\phi)\Big]\Big|_{r=r_0}. \end{aligned}$$

The second term includes the asymptotic value of the central charge $Z_0 = Z(r = \infty) = n_m a_D + n_e a$. The phase α was chosen in terms of this central charge as

$$e^{i\alpha} = i\frac{\bar{Z}_0}{|Z_0|},$$

so that the second term is equal to $\sqrt{2}|Z_0|$ as it should be for a BPS state. The third term represents the contribution from the inner boundary. Since the phase of the central charge is constant, we can rewrite the third term in much the same way as the second term and we get for a BPS state ($H_0 = 0$)

$$H = \sqrt{2}|Z(r = \infty)| + \sqrt{2}\frac{1}{n_m}(L^2(r_0) - 1)|Z(r_0)|.$$

The term $L^2 - 1$ is up to a factor the reparametrization term dX/dr. It is negative for $\delta \geq 0$ with $L \to 1$ for $\delta = 0$ and $r \to 0$. For negative delta, however, it can change sign: from negative (at large r) to positive (at small r). This shows that the contribution from the inner shell lowers the total energy of the configuration. For Z poles the energy of the configuration is lowered for any r larger than the critical value r_{cr}, at which the point $u = 1$ (and thus $Z = 0$) is reached. For X poles it is lowered for all r larger than the value at which the curve of marginal stability is crossed and the BPS equations do not necessarily have to hold any more. For bouncing solutions the energy is lowered for r larger than the bouncing point, however, it is increased for smaller values and tends to infinity for $r \to -\delta$.

This behavior gives us a possibility to construct a completely weakly coupled monopole solution by utilizing the properties of the bouncing solution. If we choose δ in such a way that the value of u for which the solution turnaround is in the region where we may trust the low-energy effective description, we may cut off the solution there. The discussion above shows that there is no contribution to the energy from the inner boundary and we may glue in a massless bubble of Higgs vacuum in the center. That is a bubble where the Higgs field is constant radial with the value it has at the cutoff point, while the gauge field is pure gauge $A = g^{-1}dg$ with g being the gauge transformation that takes one from the Higgs field being constant and pointing in, say, the z direction to the radial gauge where the Higgs field points radially outwards. Although there is no conserved charge carried by the non-Abelian fields, they have nonzero energy density. This means that there is a shell-like discontinuity at the cutoff radius.

The question of how to choose the parameter δ does not have a unique answer. We would like to choose it so that the solution stays in the weak coupling region for all values of r. Then a natural boundary is the Wilsonian core defined by the W bosons and the monopoles having equal mass there. Inside the monopoles will be lighter than the W bosons and should thus be used as the degrees of freedom of the effective action. Outside the W bosons are lighter and the standard effective action can be used. Numerically the boundary of this region has the topology of a circle and lies outside the curve of marginal stability but touches it at one point as in Fig. 3. There will of course be many other boundaries where other solitons become lighter than the W bosons but they will not interest us here.

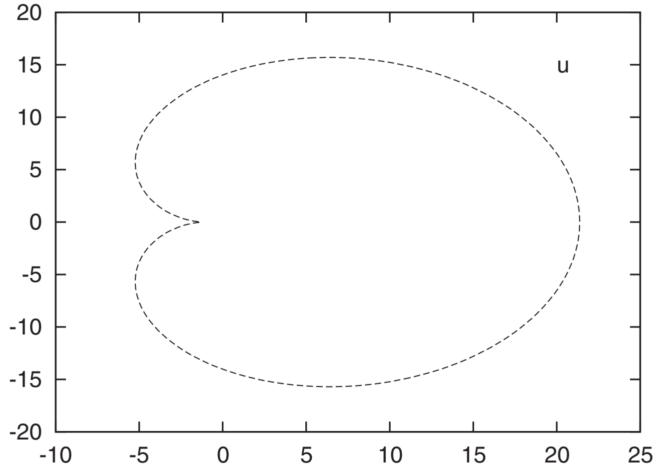

FIG. 3. The Wilsonian core.

VI. EXAMPLES

In this section we will focus on the case with quantum numbers $n_m = 1$, $n_e = 0$.

Solving the equation for the spatial dependence of the moduli numerically we may investigate the behavior of solutions for various choices of parameters. To illustrate this, in Fig. 4 we give a graph which shows how the magnetic and electrical charge of the dyon changes when one approaches the core of the monopole for various values of the phase of the central charge. One can see in the figure that all solutions have unit magnetic charge but the electric charge gets induced by the Witten effect and increases for solutions with increasing $|\alpha|$.

We would now like to investigate the behavior of the solutions on the parameter δ and, in particular, the behavior of bouncing versus nonbouncing solutions. In Figs. 5–7,

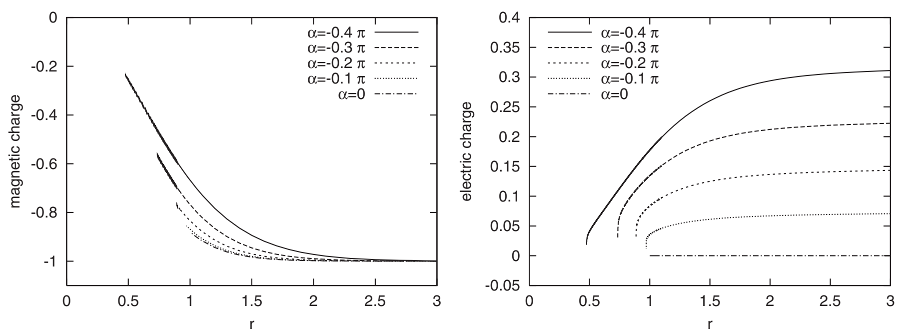

FIG. 4. Magnetic and electric charge.

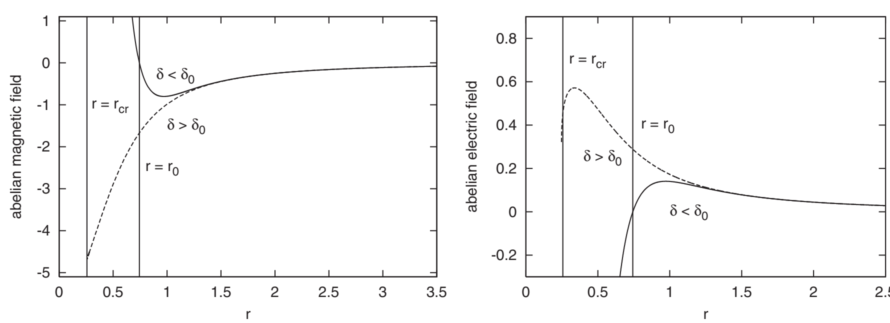

FIG. 5. The Abelian part of the magnetic and electric fields.

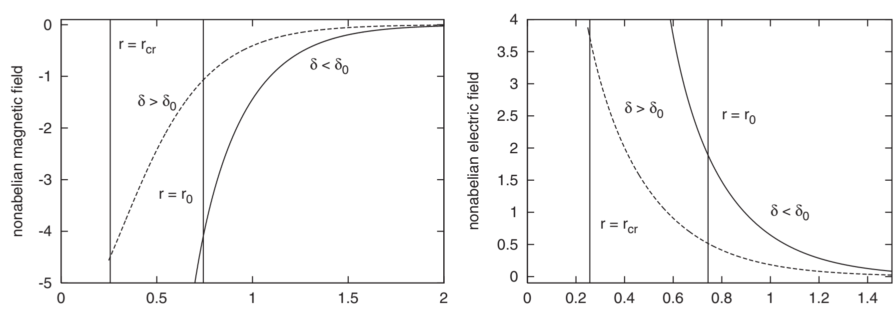

FIG. 6. The non-Abelian part of the magnetic and electric fields.

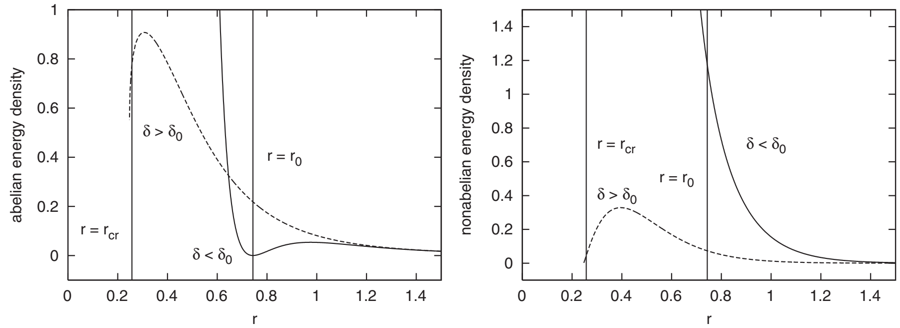

FIG. 7. The energy density carried by the Abelian and non-Abelian fields.

we therefore choose a typical Z pole and plot the fields for two values of δ. For $\delta > \delta_0$ the solution will hit the strong coupling singularity at $u = 1$ at finite r and thus not be a bouncing solution, while for $\delta < \delta_0$ it will.

In all of these figures, the vertical line at larger r represents the point of the bounce while the inner vertical line represents the r for which the nonbouncing solution hits the strong coupling singularity $u = 1$. Notice that all the Abelian fields are zero at the bouncing point while the non-Abelian fields do not seem to take any notice of the fact that the absolute value of the moduli in fact starts to increase again.

Finally, in Fig. 7 we display the energy density of the same two choices for δ for the Z pole. Notice that the energy density of the Abelian components is zero at the bouncing point while the energy density of the non-Abelian fields is not.

VII. SUMMARY AND CONCLUSIONS

We have investigated properties of monopole solutions of the full low-energy effective action of $N = 2$ $SU(2)$ super Yang-Mills theory. We have shown that the solutions of the quantum-corrected BPS equations are such that the local central charge always has constant phase as one approaches the center of the monopole. We further showed that this equation can be rewritten in a form analogous to the attractor equation and concluded that we are studying the attractor mechanism with gravity turned off as was previously concluded in [3,4].

The general solution of the BPS equation is then such that it starts out far away from the center at weak coupling. As one approaches the center the moduli, u changes toward stronger coupling in such a way that the local central charge of the theory has constant phase. The generic solution will at some finite radius enter the strong coupling region and may even hit the point where the monopoles become massless. However, by choosing the integration constants appropriately one may arrange things so that the solution is cut off while the moduli is still in the weak coupling region and the strongly coupled center gets replaced by the weakly coupled Higgs vacuum. Thus we have managed to find a solution to the quantum-corrected BPS equation which is everywhere weakly coupled such that the effective action description can be trusted.

The integration constant δ could in principle be considered as a new moduli for quantum BPS monopoles. This would mean a very drastic modification to the theory of monopoles as we know it. To get rid of this potential moduli one would need a mechanism to fix it. We have not found such a mechanism that would uniquely fix δ in each case but we now discuss various more or less natural choices. The first natural choice is $\delta \to \infty$ which is also considered in [3]. This means that all non-Abelian fields are turned off. In this scenario, since all non-Abelian fields are turned off, the monopoles look more like Dirac monopoles than 't Hooft-Polyakov monopoles.

The second natural choice is to choose δ such that the solution is a bouncing solution that is cut off at the Wilsonian core. This is the choice we have advocated in this article. It has the advantage that the solution lies entirely in the weakly coupled region. However, there is no argument why we must choose exactly the Wilsonian core and not, for instance, a point which lies slightly outside the Wilsonian core. Therefore, in this scenario a mechanism to fix δ uniquely is missing.

A third natural choice of δ would be to try to choose it so that the solution hits the strong coupling singularity at $r = 0$. It is interesting to observe that this is not always possible. While for an X pole if we choose $\delta = 0$ we hit the curve of marginal stability (at $X = 0$) when $r = 0$; for a Z pole, any choice of δ will give a solution that hits the strong coupling singularity $u = 1$ for $r > 0$.

Another interesting question to ask is if in the gravitational attractor mechanism there exists the equivalent of our bouncing solutions. In [3] it seemed that one is forced to take $\delta \to \infty$ for the comparison with the gravitational case to work. However, this is possibly a consequence of the fact that the gravitational side of the problem was being purely classical.

ACKNOWLEDGMENTS

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APPENDIX: EXPANSIONS AROUND $u = 1$

The scalar field ϕ and its dual ϕ_D are written in terms of elliptic integrals. The arguments of these go respectively to 0 and 1 for $u = 1$. The elliptic integral $\mathrm{K}(q)$ diverges for $q \to 1$, so we must use expansions.

The general formulas for expansions of the elliptic integrals around $k = 0$ and $k = 1$ respectively are for $k \to 0$

$$\mathrm{K}(k) = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \cdots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n} + \cdots\right\}, \tag{A1}$$

$$\mathrm{E}(k) = \frac{\pi}{2}\left\{1 - \frac{1}{2^2}k^2 - \cdots - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1} - \cdots\right\}; \tag{A2}$$

for $k \to 1$

$$\begin{aligned}\mathrm{K}(k) = {}& \ln\frac{4}{k'} + \left(\frac{1}{2}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2}\right)(k')^2 \\ & + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4}\right)(k')^4 \\ & + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{2}{5\cdot 6}\right)(k')6 \\ & + \cdots \end{aligned} \tag{A3}$$

$$\begin{aligned}\mathrm{E}(k) = {}& 1 + \frac{1}{2}\left(\ln\frac{4}{k'} - \frac{1}{1\cdot 2}\right)(k')^2 \\ & + \frac{1^2\cdot 3}{2^2\cdot 4}\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{1}{3\cdot 4}\right)(k')^4 \\ & + \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6}\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4} - \frac{1}{5\cdot 6}\right)(k')^6 \\ & + \cdots. \end{aligned} \tag{A4}$$

We shall expand u in the form $u = 1 + \varepsilon e^{i\varphi}$. For the field ϕ and its dual ϕ_D we find the following expansions

$$\begin{aligned}\phi = {}& \frac{4}{\pi}\left(1 - \frac{1}{8}\varepsilon \ln\varepsilon e^{i\varphi} + \left(\frac{5}{8}\ln 2 + \frac{1}{8}\right)\varepsilon e^{i\varphi} - \frac{1}{8}i\varphi\varepsilon e^{i\varphi} \right.\\ & \left. + \frac{1}{4}\varepsilon e^{i\varphi} + O(\varepsilon^2)\right), \end{aligned} \tag{A5}$$

$$\phi_D = i\frac{1}{2}\varepsilon e^{i\varphi}\left(1 - \frac{5}{32}\varepsilon e^{i\varphi}\right). \tag{A6}$$

We can write the expansion of ϕ_D in the form $\phi_D \approx \frac{i}{2}\varepsilon e^{i\varphi} e \exp[-(\frac{5}{32})\varepsilon(\cos\varphi + i\sin\varphi)]$, so its phase is $\arg\phi_D = \frac{\pi}{2} + \varphi - \frac{5}{32}\varepsilon\sin\varphi$. This phase is constant along the solution of the differential Eq. (30) and equal to $\frac{\pi}{2} - \alpha$. Thus we get a relation between ε and φ close to $u = 1$ for curves of constant Z phase

$$\alpha + \varphi \approx \frac{5}{32}\varepsilon\sin\varphi. \tag{A7}$$

From this we see that φ goes to $-\alpha$ as we get closer to $u = 1$.

We can find the differential equation for ε and solve it approximately to lowest order. Inserting $u = 1 + \varepsilon e^{-i\alpha}$ in (31) we find the differential equation

$$\varepsilon_t = \frac{\frac{\pi}{2}\sqrt{2}}{-\frac{1}{2}\ln\varepsilon}, \tag{A8}$$

where we used $\mathrm{K}(q) \approx -\frac{1}{2}\ln\varepsilon$ and $i\bar{\tau}/\mathrm{Im}\tau \approx 1$. The solution of this equation is

$$\varepsilon \approx \frac{-\pi\sqrt{2}(t - t_0)}{\ln[-\pi\sqrt{2}(t - t_0)]}, \tag{A9}$$

where the constant t_0 is chosen so that $\varepsilon(t_0) = 0$, i.e. for Z poles and critical Z poles $t_0 = t(r_c)$, for XZ poles and bouncing XZ poles $t_0 = 0$. Thus close to $u = 1$ the solution goes as

$$u = 1 + \frac{-\pi\sqrt{2}(t - t_0)}{\ln[-\pi\sqrt{2}(t - t_0)]}e^{-i\alpha}. \tag{A10}$$

In terms of the parameter r this can be written (except for critical Z poles)

$$u = 1 - \pi\sqrt{2}t_r(r_a)\frac{r - r_a}{\ln(r - r_a)}e^{-i\alpha},$$

where r_a is the point at which $u = 1$. For critical Z poles $t_r(r_a) = 0$, so we have to take a higher term and get $u = 1 - \pi\sqrt{2}\frac{1}{4}t_{rr}(r_a)(r - r_a)^2/\ln(r - r_a)$.

For the calculation of the electric fields we need the expansions of τ and $\frac{\phi_D}{\phi}$; these are

$$\tau = -\frac{i\pi}{\ln\varepsilon + O(\varepsilon^0)}\left(1 + \frac{1}{8}\varepsilon e^{i\varphi} + O(\varepsilon^2)\right), \tag{A11}$$

$$\frac{\phi_D}{\phi} = \frac{i\pi}{8}\varepsilon e^{i\varphi} + O(\varepsilon^2). \tag{A12}$$

Further we need the following expansions:

$$\frac{\mathrm{Re}\tau}{\mathrm{Im}\tau} = -\frac{-\frac{1}{8}\varepsilon\sin\alpha + O(\varepsilon^2)}{1 + \frac{1}{8}\varepsilon\cos\alpha + O(\varepsilon^2)}, \tag{A13}$$

$$\frac{\mathrm{Re}\frac{\phi_D}{\phi}}{\mathrm{Im}\frac{\phi_D}{\phi}} = \tan\alpha + O(\varepsilon), \tag{A14}$$

$$\frac{\frac{\bar{\phi}_D}{\phi}}{\mathrm{Im}\frac{\phi_D}{\phi}} = -i(1 + i\tan\alpha) + O(\varepsilon). \tag{A15}$$

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