# Gravitational field of a spinning sigma-model cosmic string

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(Received 28 March 2008; published 15 August 2008)

We study the effect of internal space rotation on the gravitational properties of infinite straight and stationary cosmic strings. From the approximate solution of Einstein equations for the spinning $Q$-lump string, we obtain long-range gravitational acceleration resembling that of a rotating massive cylindrical shell. We also compute the angular velocity of the inertial frame dragging and the angle of light deflection by the $Q$-lump string. Matter accretion onto spinning strings can play a role in galaxy formation when the angular velocity times the string width is comparable to the speed of light.

DOI: 10.1103/PhysRevD.78.045014 PACS numbers: 11.27.+d, 12.39.Dc, 98.62.Ai, 98.80.Cq

## I. INTRODUCTION

Cosmic strings are one of a family of topological defects generated by phase transitions in the early universe [1,2]. The gravitational field of strings is of particular interest in view of their possible role in the galaxy formation [2]. Vilenkin [3] studied an infinite straight static string and found that its gravitational field is very different from that of a massive rod. First of all, there is no net active gravitational mass associated with this string. This is because the tension in the string leads to a negative active gravitational mass which balances the positive mass of the energy density. Thus a test particle in the neighborhood would experience no gravitational force. Consequently, straight static strings would not initiate the gravitational instability necessary for galaxy formation. Nevertheless, owing to the unusual geometry of their exterior metric, they are of considerable astronomical importance. Vilenkin [3] found that the metric exhibits a geometry of a conical space with a wedge removed, and concluded that the string acts as a gravitational lens.

The absence of the active gravitational mass for a static string, parallel to the $x^3$ axis, is linked to independence of string fields of $x^0$ and $x^3$. This leads to the invariance of the stress-energy tensor $T_\mu^\nu$ with respect to the Lorentz boost in the $x^3$ direction. It follows that $T_0^0 = T_3^3$, an equality which makes the active mass density vanish. However, if the string solution is time dependent, the Lorentz-boost invariance is lost and a finite mass density ensues. For the Abelian Higgs model, the string solutions are fully characterized by their position and there is no rotational degrees of freedom. Hence, unless the string moves, the Abelian Higgs field remains time independent. On the other hand, a non-Abelian model provides a degree of freedom corresponding to rotations in the internal space. One particularly elegant non-Abelian model allowing a classical internal spin is the sigma-model lump due to Leese [4]. The solutions of this model were termed $Q$ lumps where $Q$ is the charge of the model. Owing to the spinning in the internal phase, the $Q$-lump solutions are time dependent but their position as well as energy remains static. Thus, a stationary cosmic string built from the $Q$-lump solution may provide a source of an active gravitational mass. This contrasts with the absence of the active mass for static sigma-model string [5].

The purpose of the present work is to investigate the gravitational properties of the $Q$-lump string in the weak field approximation. In Secs. II and III, we derive the long-range gravitational acceleration and show that, for distances well beyond the string width, the string acts as a massive rod. Section IV is devoted to the dragging of the inertial frame caused by the spinning. This effect has been investigated for rotating spherical mass shells since 1918 and it carries the name "Lense-Thirring" effect [6–8]. In Sec. V, we study the deflection of light ray propagating in the plane perpendicular to the string axis. Finally, in Sec. VI we discuss some cosmological consequences of the present work.

## II. LAGRANGIAN AND STRESS-ENERGY TENSOR

We consider an infinite, straight, and stationary string parallel to the $x^3$ axis. The Lagrangian density is obtained by trivially extending the 2 + 1 theory [4] of the $Q$ lump to 3 + 1 dimensions. Since the fields of our model are independent of $x^3$, the added dimension does not change the original form of the Lagrangian density of the $Q$ lump [4]. The basic field is a triplet $\vec{\phi} \equiv (\phi_1, \phi_2, \phi_3)$ of real scalar fields satisfying the constraint $\vec{\phi} \cdot \vec{\phi} = 1$. We write the Lagrangian density in the form ($\hbar = c = 1$)

$$\mathcal{L} = \eta^2\left[\frac{1}{4}g^{\mu\nu}\partial_\mu\vec{\phi} \cdot \partial_\nu\vec{\phi} - \frac{\alpha^2}{4}(1 - \phi_3^2)\right]. \quad (1)$$

The constant $\eta^2$ is proportional to the energy per unit length of the pure sigma-model string [5] whose Lagragian follows from (1) by putting $\alpha = 0$.

The second term in Eq. (1) is reminiscent of the uniaxial anisotropy energy of a nonrelativistic Heisenberg ferromagnet where it is responsible for a Larmor precession

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of the magnetization vector $\vec{M}$ about the $x^3$ axis. In that case, the equations of motion for $\vec{M}$ have localized solutions in 2 + 1 dimensions called precessional solitons [9]. In the absence of an external magnetic field, the precessional frequency of these solitons is given by the ferromagnetic resonance frequency proportional to the anisotropy field [9] (given by $\alpha^2$ in the present notation). The precessional soliton resulting from Eq. (1) is, however, of a different nature. Solving the equations of motion, Leese [4] finds precessional frequency equal to $\alpha$. This stems from the fact that the dynamics of the Lagrangian (1) is second order in the time derivatives of $\vec{\phi}$, whereas it is first order for the Lagrangian for the nonrelativistic ferromagnet [9]. Note that the sign of the potential energy in Eq. (1) is opposite to that seen in Eq. (2.1) of Ref. [4]. This is due to our choice of the space-time metric which in the flat space limit has signature $(1, -1, -1, -1)$ [see also Eq. (6.70) in Manton and Sutcliffe [10]].

Anticipating nonzero gravitational density for the spinning string, we write the metric in the form

$$ds^2 = g_{00}(r)(dx^0)^2 - \Omega^2(r)[(dx^1)^2 + (dx^2)^2] + 2[g_{01}(\vec{x})dx^0dx^1 + g_{02}(\vec{x})dx^0dx^2] + g_{33}(r)(dx^3)^2, \tag{2}$$

where $\vec{x}$ is a vector in the $x^1$, $x^2$ space, and $r = [(x^1)^2 + (x^2)^2]^{1/2}$.

The metric for the static string of the pure ($\alpha = 0$) sigma model [5] is recovered from Eq. (2) by letting $g_{00} = 1$, $g_{01} = g_{02} = 0$, and $g_{33} = -1$.

The components of our metric tensor ($g_{00}$, $g_{11} = g_{22} = -\Omega^2$, $g_{33}$, $g_{01}$, and $g_{02}$) are obtained by solving the corresponding components of the Einstein equation [11]:

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T), \tag{3}$$

where $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the stress-energy tensor, and $T = T^{\mu}_{\mu}$. Using Eq. (1), we have

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{|g|}\mathcal{L}) = \frac{1}{2}\eta^2\partial_\mu\vec{\phi}\cdot\partial_\nu\vec{\phi} - g_{\mu\nu}\mathcal{L}, \tag{4}$$

where $g$ is the determinant of the metric. In what follows, we treat Eq. (3) in the weak field limit which amounts to replacing $g_{\mu\nu}$ in Eq. (4) by $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. It is convenient to calculate $T_{\mu\nu}$ by discarding the field $\vec{\phi}$ in favor of a complex scalar field $u(t, z)$, where $z = x^1 + ix^2$, and

$$u = \frac{\phi_1 + i\phi_2}{1 + \phi_3}. \tag{5}$$

This leads to the $CP^1$ representation in which the Lagrangian density of Eq. (1) becomes [4,10]

$$\mathcal{L} = \eta^2\left[\frac{1}{2}g^{\mu\nu}\frac{\partial_\mu u\partial_\nu\bar{u} + \partial_\mu\bar{u}\partial_\nu u}{(1+|u|^2)^2} - \frac{\alpha^2 u\bar{u}}{(1+|u|^2)^2}\right]. \tag{6}$$

For the stress tensor (4), we obtain in this representation

$$T_{\mu\nu} = \eta^2\frac{\partial_\mu u\partial_\nu\bar{u} + \partial_\mu\bar{u}\partial_\nu u}{(1+|u|^2)^2} - g_{\mu\nu}\mathcal{L}. \tag{7}$$

## III. ACTIVE GRAVITATIONAL MASS

To calculate the active gravitational mass, we take the 00-component of Eq. (3). We write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. In the weak field limit, we have

$$R_{00} = -\tfrac{1}{2}(\partial_1^2 + \partial_2^2)h_{00} = -\tfrac{1}{2}\nabla^2 h_{00}. \tag{8}$$

On the right-hand side (rhs) of Eq. (3) we need the expression $T_{00} - \frac{1}{2}T$. Using Eqs. (6) and (7), we obtain

$$T_{00} = \frac{\eta^2}{(1+|u|^2)^2}(\partial_t u\partial_t\bar{u} + \partial_i u\partial_i\bar{u} + \alpha^2 u\bar{u}) \tag{9}$$

and

$$T_{33} = \frac{\eta^2}{(1+|u|^2)^2}(\partial_t u\partial_t\bar{u} - \partial_i u\partial_i\bar{u} - \alpha^2 u\bar{u}), \quad i = 1, 2. \tag{10}$$

The axial symmetry of the string fields implies $\partial_1 u\partial_1\bar{u} = \partial_2 u\partial_2\bar{u}$. With the use of this relation, we have from Eqs. (6) and (7)

$$T_{11} = T_{22} = \frac{\eta^2}{(1+|u|^2)^2}(\partial_t u\partial_t\bar{u} - \alpha^2 u\bar{u}). \tag{11}$$

Using Eqs. (9)–(11) we obtain

$$T_{00} - \frac{1}{2}T = \frac{\eta^2}{(1+|u|^2)^2}(2\partial_t u\partial_t\bar{u} - \alpha^2 u\bar{u}). \tag{12}$$

Introducing this result and Eq. (8) in the 00-component of Eq. (3), we obtain

$$\nabla^2 h_{00} = \frac{16\pi G\eta^2\alpha^2 u\bar{u}}{(1+|u|^2)^2}. \tag{13}$$

We have used the identity $\partial_t u\partial_t\bar{u} = \alpha^2 u\bar{u}$ which follows from the solution [4] (see the Appendix for derivation)

$$u(t, z) = \exp(-i\alpha t)u(z), \tag{14}$$

where $u(z)$ is a degree $N$ rational map in $z = x^1 + ix^2$

$$u(z) = \left(\frac{\lambda}{z}\right)^N. \tag{15}$$

Leese [4] finds that in order to have a lump of finite energy, the integer $N$ must be equal or larger than 2. Incidentally, the same condition applies to the nonrelativistic spinning soliton [9]. This can be understood by considering the energy per unit string length

$$\epsilon = \int d^2x T_{00} = 2\pi\eta^2 N + \frac{2\pi^2\eta^2\alpha^2\lambda^2}{N^2 \sin\frac{\pi}{N}}, \tag{16}$$

where the second equality follows by integrating Eq. (9) with $u(t, z)$ given by Eqs. (14) and (15). As shown in the Appendix, $N$ is the topological charge.

We see that the second term in Eq. (16) diverges for $N = 1$. Since this term originates from the anisotropy energy, it is of no surprise to have the same condition for the nonrelativistic spinning soliton [9].

For the $Q$ lump, this term can be expressed by means of the Noether charge $Q$ [4]

$$Q = i \int d^2x \frac{\bar{u}\partial_t u - u\partial_t \bar{u}}{(1 + |u|^2)^2} = 2\alpha \int d^2x \frac{u\bar{u}}{(1 + |u|^2)^2}. \tag{17}$$

With the use of this result, the energy given in Eq. (16) becomes [10]

$$\epsilon = \eta^2(2\pi|N| + |\alpha Q|), \tag{18}$$

which is valid for $N$ and $Q$ positive or negative (see the Appendix). Starting from Eq. (13), we now proceed to the evaluation of the gravitational acceleration $\vec{g}(r)$. Noting that $h_{00} = 2\phi_g$, where $\phi_g$ is the gravitational potential related to the acceleration by $\vec{g}(r) = -\vec{\nabla}\phi_g$, Eq. (13) can be cast into the form of a Poisson equation

$$\vec{\nabla} \cdot \vec{g}(r) = -4\pi G\rho(r), \tag{19}$$

where $\rho(r)$ is the active gravitational mass density (per unit volume)

$$\rho(r) = \frac{2\eta^2\alpha^2\lambda^{2N}r^{2N}}{(r^{2N} + \lambda^{2N})^2}. \tag{20}$$

Using the Gauss theorem, Eq. (19) yields $g(r)$ (not to be confused with the determinant of the metric)

$$g(r) = -\frac{4\pi G}{r}\int_0^r dr' r' \rho(r'). \tag{21}$$

For $N = 2$, we obtain from Eqs. (20) and (21)

$$g(r) = 2\pi G\eta^2\alpha^2\lambda^4\left(\frac{r}{r^4 + \lambda^4} - \frac{1}{\lambda^2 r}\tan^{-1}\frac{r^2}{\lambda^2}\right). \tag{22}$$

This function has a maximum at $r \simeq \frac{1}{2}\lambda$. Since $\lambda$ is the width of the sigma-model string, it is useful to have asymptotic results deep in the interior region, $r \ll \lambda$, and in the exterior region, $r \gg \lambda$. From Eq. (22), we obtain

$$g(r) \to -\frac{4\pi G\eta^2\alpha^2 r^5}{3\lambda^4} \tag{23}$$

for $r/\lambda \to 0$ and

$$g(r) \to -\frac{\pi^2 G\eta^2\alpha^2\lambda^2}{r} \tag{24}$$

for $r/\lambda \to \infty$. From Eq. (23), we see that $g(r)$ is a nonsingular function of $r$ as we approach the string axis. Introducing the linear active mass density $m$ [obtained for $N = 2$ from Eq. (20)]

$$m = 2\pi\int_0^\infty dr r\rho(r) = \frac{\pi^2\eta^2\alpha^2\lambda^2}{2}, \tag{25}$$

we can express Eq. (24) as follows:

$$g(r) \to -\frac{2Gm}{r} = -\frac{G\eta^2|\alpha Q|}{r}. \tag{26}$$

This result shows that, for $r$ well beyond the soliton width $\lambda$, the string acts as a massive rod of linear active mass density $m$. Recalling Eq. (18), we see that $m = \eta^2|\alpha Q|$.

## IV. INDUCED FRAME DRAGGING

The dragging of inertial frames relative to the asymptotic frame inside a rotating mass shell has been first investigated by Thirring and Lense [6]. We expect that this effect is also present for a spinning string. Calculation of the mixed space-time components of the metric tensor (2), described below, confirms this expectation.

First consider the component $g_{01}(\vec{x})$. Taking the 01-component of Eq. (3) and using the weak field result $R_{01} = -\frac{1}{2}\nabla^2 g_{01}$, we have

$$\nabla^2 g_{01}(\vec{x}) = 16\pi G T_{01}(\vec{x}). \tag{27}$$

From Eq. (7), we obtain in the weak field approximation

$$T_{01} = \frac{\eta^2}{(1 + |u|^2)^2}(\partial_0 u\partial_1\bar{u} + \partial_0\bar{u}\partial_1 u). \tag{28}$$

Substituting for $u(t, z)$ the solution (14), Eq. (28) yields for $N = 2$

$$T_{01}(\vec{x}) = -\frac{4\eta^2\alpha\lambda^4 r^3}{(r^4 + \lambda^4)^2}\sin\theta, \tag{29}$$

where we introduced polar coordinates $x^1 = r\cos\theta$ and $x^2 = r\sin\theta$. To solve Eq. (27), we use the two-dimensional Green's function

$$G(\vec{x} - \vec{x}') = \frac{1}{2\pi}\left\{\log r - \sum_{n=1}^\infty \frac{1}{n}\left(\frac{r'}{r}\right)^n \cos[n(\theta - \theta')]\right\} \times S(r - r') + \frac{1}{2\pi}\left\{\log r' - \sum_{n=1}^\infty \frac{1}{n}\left(\frac{r}{r'}\right)^n \cos[n(\theta - \theta')]\right\} \times S(r' - r), \tag{30}$$

where $S(r)$ is the unit step function. Using Eqs. (29) and (30), the solution of Eq. (27) is written as follows:

$$g_{01}(\vec{x}) = \frac{\Gamma}{2\pi}\int_0^\infty dr' r' F(r')\int_0^{2\pi} d\theta' \sin\theta'\left[\frac{r'}{r}\cos(\theta - \theta') \times S(r - r') + \frac{r}{r'}\cos(\theta - \theta')S(r' - r)\right], \tag{31}$$

where $\Gamma = 64\pi G\eta^2\alpha\lambda^4$, and $F(r) = r^3(r^4 + \lambda^4)^{-2}$. Performing the angular integrations, Eq. (31) yields

$$g_{01}(\vec{x}) = \frac{1}{2}\Gamma \sin\theta\left[\frac{1}{r}\int_0^r dr' F(r') r'^2 + r\int_r^\infty dr' F(r')\right] = 8\pi G\eta^2\alpha\lambda^2 \frac{\sin\theta}{r}\tan^{-1}\frac{r^2}{\lambda^2}. \tag{32}$$

In a similar way, we calculate $g_{02}(\vec{x})$. Solving the 02-component of Eq. (3), we have

$$g_{02}(\vec{x}) = -8\pi G\eta^2\alpha\lambda^2 \frac{\cos\theta}{r}\tan^{-1}\frac{r^2}{\lambda^2}. \tag{33}$$

Using Eqs. (32) and (33), we calculate the local angular velocity, $\omega(r)$, of the frame rotation induced by the spinning of the $Q$-lump string. This is done by expressing the third term of Eq. (2) in polar coordinates. Using Eqs. (32) and (33), we have

$$2[g_{01}(\vec{x})dx^1 + g_{02}(\vec{x})dx^2]dx^0 = -16\pi G\eta^2\alpha\lambda^2 d\theta dx^0 \tan^{-1}\frac{r^2}{\lambda^2}. \tag{34}$$

By comparing the rhs of this equation with the cross term of the rotating metric component $r^2[d\theta - \omega(r)dx^0]^2$, we obtain

$$\omega(r) = 8\pi G\eta^2\alpha\left(\frac{\lambda}{r}\right)^2 \tan^{-1}\frac{r^2}{\lambda^2}. \tag{35}$$

Interesting conclusions emerge from the asymptotic expansions of (35). For $r/\lambda \to 0$, we have

$$\omega(r) \simeq 8\pi G\eta^2\alpha = 2\epsilon_0 G\omega_s, \tag{36}$$

where $\epsilon_0 = 4\pi\eta^2$ is the linear mass density of the pure sigma model [5] given by the first term of Eq. (18). We have also set $\alpha = \omega_s$. Now, the rhs of Eq. (36) allows us to make a comparison with the well-known Thirring result [7] for the rotation rate $\Omega(r)$ of the inertial frame induced by a thin spherical shell of radius $r_0$ and mass $M$, rotating with the angular velocity $\omega_s$. In the interior region, $r < r_0$, $\Omega(r) = \Omega_{\rm int}(r)$ is given by [7]

$$\Omega_{\rm int}(r) = \frac{4MG\omega_s}{3r_0}. \tag{37}$$

Equation (37) exhibits a similarity with the rhs of Eq. (36), except for the geometric factor $2/3r_0$ characteristic of the spherical symmetry. For $r/\lambda \to \infty$, we have from Eq. (35)

$$\omega(r) \simeq \pi\epsilon_0 G\omega_s \frac{\lambda^2}{r^2}. \tag{38}$$

This is to be compared with the Thirring formula [7] for the exterior region, $r > r_0$,

$$\Omega_{\rm ext}(r) = \frac{4MG\omega_s}{3}\frac{r_0^2}{r^3}. \tag{39}$$

This time, the geometric factor $4/3\pi r$ enters in Eq. (39). Moreover, the comparison suggests that $\lambda$ in Eq. (38) plays a role of the radius of the rotating mass shell. This view is substantiated by looking at the $r$ dependence of the mass density $T_{00}(r)$ of the pure sigma model. According to Eq. (9), we have for $N = 2$

$$T_{00}(r) = \frac{\eta^2\partial_i u \partial_i \bar{u}}{(1 + |u|^2)^2} = \frac{8\eta^2\lambda^4 r^2}{(r^4 + \lambda^4)^2}. \tag{40}$$

From this expression we see that $T_{00}(r)$ goes as $r^2$, and $r^{-6}$ for $r/\lambda \to 0$, and $r/\lambda \to \infty$, respectively. The maximum of $T_{00}(r)$ is at $r \simeq 0.75\lambda$. In view of this comparison, we conjecture that the inertial frame dragging effect for the $Q$-lump string is similar to that for the cylindrical shell of radius of order $\lambda$, rotating at the angular velocity $\omega_s = \alpha$.

This conjecture can be substantiated quantitatively by computing the $x^3$ component, $J_3$, of the angular momentum per unit length of the string. Using the expression $\epsilon_{jk}x_j T_{0k}$ for the angular momentum density, we obtain in the weak field approximation $J_3 \simeq \pi^2\alpha\eta^2\lambda^2$. This should be compared with the angular momentum, $J = 4\pi\alpha\eta^2 r_0^2$, of a thin cylindrical shell of radius $r_0$ with linear mass density, $\epsilon_0$, rotating at angular velocity $\alpha$. From this comparison, we have $r_0^2 \simeq \pi\lambda^2/4$ confirming the above conjecture. Note that the rhs of Eq. (38) can be expressed in terms of $J$ yielding $\omega(r) \simeq 4JG/r^2$. This is a string analog of the Thirring [7] formula (39) expressed via the angular momentum of the spherical shell, $J_{\rm sph} = 2\omega_s M r_0^2/3$, as $\Omega_{\rm ext}(r) = 2J_{\rm sph}G/r^3$.

## V. DEFLECTION OF LIGHT

We assume that the light path is in the $(x^1, x^2)$ plane. First we neglect the contribution to the deflection caused by the frame dragging. Then the relevant metric is

$$ds^2 = g_{00}(dx^0)^2 + g_{11}[(dx^1)^2 + (dx^2)^2], \tag{41}$$

where

$$g_{00} = 1 + 2\phi, \qquad g_{11} = -1 + 2\bar{\phi}. \tag{42}$$

The gravitational potentials $\phi = \frac{1}{2}h_{00}$ and $\bar{\phi} = \frac{1}{2}h_{11}$ are obtained by solving the 00- and 11-components of the Einstein equation (3), respectively.

To calculate the deflection angle, we consider the geodesic equation of motion for the four momentum, $p_i$, of the photon [11]

$$\frac{dp_i}{dt} = \frac{1}{2}g_{jk,i}\frac{p^j p^k}{p^0}. \tag{43}$$

Assuming that the light ray is approaching the string in the $y$ direction, the deflection is produced by the rate of increase of the $x$ component of the coordinate velocity $u^x = p^x/p^0$. Thus, we consider Eq. (43) for $i = x$, and obtain with use of Eqs. (41) and (42)

$$-\frac{d}{dt}[(1-2\bar{\phi})p^x] = p^0\left[\partial_x\phi + \partial_x\bar{\phi}\frac{(p^x)^2+(p^y)^2}{(p^0)^2}\right]. \tag{44}$$

Taking $i = 0$, and noting that the metric tensor $g_{jk}$ is independent of time, Eq. (43) yields

$$\frac{d}{dt}[(1+2\phi)p^0] = 0. \tag{45}$$

Using this result, it follows that to first order in $\phi$ and $\bar{\phi}$, the following identity holds:

$$\frac{1}{p^0}\frac{d}{dt}[(1-2\bar{\phi})p^x] \simeq \frac{d}{dt}\left[(1-2\bar{\phi}-2\phi)\frac{p^x}{p^0}\right]. \tag{46}$$

Further simplification of Eq. (44) follows by noting that $p^i$ is a null vector. With the metric (41), this implies

$$\frac{(p^x)^2+(p^y)^2}{(p^0)^2} = \frac{1+2\phi}{1-2\bar{\phi}} \simeq 1. \tag{47}$$

Dividing Eq. (44) by $p^0$, we obtain with use of Eqs. (46) and (47) to order $\phi$, $\bar{\phi}$

$$\frac{du^x}{dt} \simeq -\partial_x[\phi(r)+\bar{\phi}(r)] = -\frac{x}{r}\partial_r[\phi(r)+\bar{\phi}(r)]. \tag{48}$$

From Eq. (22), we have $\partial_r\phi = \partial_r\phi_g = -g(r)$. To obtain the quantity $\partial_r\bar{\phi}$, we consider the Einstein equation (3) for $h_{11}$

$$\nabla^2 h_{11} = 16\pi G\left(T_{11}+\frac{1}{2}T\right) = \frac{16\pi G\eta^2\lambda^4}{(r^4+\lambda^4)^2}(8r^2+\alpha^2 r^4), \tag{49}$$

where the second equality follows from Eqs. (9) and (11) using the ansatz (15) with $N = 2$. Similar to the derivation of Eq. (22), we use the Gauss theorem to solve Eq. (49) for $\partial_r h_{11}$. Combining this result with Eq. (22), we have

$$\partial_r(\phi+\bar{\phi}) = 4\pi G\eta^2\left[\frac{4r^3}{r^4+\lambda^4} + \alpha^2\lambda^2\left(\frac{1}{r}\tan^{-1}\frac{r^2}{\lambda^2} - \frac{\lambda^2 r}{r^4+\lambda^4}\right)\right]. \tag{50}$$

The differential of the deflection angle $\delta\Phi$, acquired during time interval $dt$, is $d(\delta\Phi) = (du^x/dt)dy$. Using Eq. (48), the net deflection angle $\delta\Phi$ becomes

$$\delta\Phi = -x\int_{-\infty}^{\infty} dy\frac{1}{r}\partial_r(\phi+\bar{\phi}), \tag{51}$$

where $r = [x^2+y^2]^{1/2}$ and $x$ is the impact parameter. In what follows, we assume $x \gg \lambda$ implying that also $r \gg \lambda$. In this limit, Eq. (50) yields

$$\partial_r(\phi+\bar{\phi}) \approx \frac{\epsilon_0 G}{r}\left(4+\frac{\pi}{2}\alpha^2\lambda^2\right). \tag{52}$$

Using this result in Eq. (51), the deflection angle becomes

$$\delta\Phi \approx -x\epsilon_0 G\left(4+\frac{\pi}{2}\alpha^2\lambda^2\right)\int_{-\infty}^{\infty}\frac{dy}{(x^2+y^2)} = -\pi\epsilon_0 G\left(4+\frac{\pi}{2}\alpha^2\lambda^2\right). \tag{53}$$

We see that in the limit $x/\lambda \to \infty$, the deflection angle is independent of $x$. Moreover, the rhs of Eq. (53) can be written as $\delta\Phi \approx -4\pi G\epsilon$, where $\epsilon$ is the net energy per unit length given in Eq. (16). These results are reminiscent of the deflection angle obtained from the wedge angle deficit of the conical space [3,5]. In fact, for $\alpha = 0$, the angle $2\delta\Phi \approx -8\pi G\epsilon_0$ coincides with the deficit angle $\delta$ obtained in Ref. [5] using the Gauss-Bonnet formula.

We now consider the additional contribution to light deflection due to the frame dragging. Denoting the corresponding deflection angle as $\delta\Phi_d$, we obtain with use of Eq. (35)

$$\delta\Phi_d = \int_{-\infty}^{\infty} dy\,\omega(r) = 8\pi G\eta^2\alpha\lambda^2\int_{-\infty}^{\infty} dy\frac{\tan^{-1}(\frac{r^2}{\lambda^2})}{r^2}. \tag{54}$$

An approximate evaluation of $\delta\Phi_d(x)$ for $0 < x < \infty$ can be made by replacing the integrand of Eq. (54) by $(\pi/2)\times(r^2+\pi\lambda^2/2)^{-1}$. In this way, we get

$$\delta\Phi_d(x) \sim \frac{\pi^2\epsilon_0 G\alpha\lambda^2}{\sqrt{x^2+\frac{\pi\lambda^2}{2}}}. \tag{55}$$

## VI. DISCUSSION

According to Eq. (25), an infinite stationary $Q$-lump spinning string acquires an active linear mass density $m = \frac{\pi}{8}\epsilon_0(\alpha\lambda)^2$, where $\epsilon_0$ is the linear mass density of pure sigma-model string. The fact that matter can be attracted onto these strings prompts us to examine their role in the formation of galaxies.

The main concern is the magnitude of the density inhomogeneity due to infinite strings, $\delta\rho/\rho = \rho_{\rm inf}/\rho$. According to Ref. [2], $\rho_{\rm inf}$ at cosmic time $t$ is given by $m/t^2$. In the radiation dominated era we have $\rho = 3/32\pi G t^2$. Thus

$$\frac{\rho_{\rm inf}}{\rho} = \frac{32\pi}{3}Gm. \tag{56}$$

Galaxy formation scenarios require that $Gm \sim 10^{-6}$. Let us first consider the pure sigma-model string with linear mass density $\epsilon_0 = 4\pi\eta^2$. These strings are formed when the Universe cools to temperature $T \sim \eta$. We note that at this temperature the Higgs field acquires a nonzero vacuum expectation value owing to the O(3) invariant Higgs potential. With $m$ replaced by $\epsilon_0$, the condition $G\epsilon_0 \sim 10^{-6}$ implies $\eta \sim 3\times 10^{15}$ Gev which falls into the grand unification (GU) region.

For the stationary $Q$-lump spinning string we have $Gm = G\epsilon_0 \pi(\alpha\lambda)^2/8$. Since the formation of galaxies requires $Gm \sim 10^{-6}$, we get a condition $\alpha\lambda \sim 1$ if $G\epsilon_0 \sim 10^{-6}$ is assumed.

There are observational constraints implying limits on the cosmic string tension $\epsilon_0$. Recent attention has focused on the power spectrum of the cosmic microwave background (CMB). Albrecht, Battye, and Robinson [12] report $G\epsilon_0 \leq 10^{-6}$ based on normalization of the calculated CMB temperature fluctuations to the Cosmic Background Explorer (COBE) satellite data [13]. They calculate these fluctuations using a standard scaling model in which the string network is represented as a collection of uncorrelated string segments with random uncorrelated velocities. Consequently, the resulting two-point correlation function for the stress-energy tensor exhibits a temporal decoherence and the calculated angular power spectrum is in disagreement with the observed CMB anisotropies [13,14]. In particular, it does not show the acoustic (Doppler) peaks seen in the spectrum, a feature that is due to a high degree of temporal coherence of the pressure perturbations [15]. On the other hand, adiabatic inflationary models exhibit coherence leading to phase focusing on the subhorizon pressure waves and producing a secondary Doppler peak in the angular power spectrum. For this reason, it is believed that topological defect models cannot provide the dominant source of structure formation.

More detailed measurements of the CMB have been made by the Wilkinson microwave anisotropy probe (WMAP) [16,17]. Jeong and Smoot [18] have searched the first year WMAP $W$-band CMB anisotropy map and deduced a limit on the string tension $G\epsilon_0 < 6 \times 10^{-7}$. Bevis, Hindmarsh, and Kunz [19] have used the first year WMAP data to produce an upper bound, 0.13, on the fraction of global defects contributing to cosmic structure formation.

Standard defect networks are not frozen and coherent as inflationary models. It should be noted that these networks are modeled as an ensemble of Nambu-Goto strings [20] or their modifications due to wiggliness [21]. The tension of wiggly strings is smaller than the effective mass per unit length. Thus they represent another example of the breakdown of the Lorentz-boost invariance that also motivated the present work. Similar to the $Q$-lump string, there is a long-range gravitational field of the wiggly string related to the difference between the tension and the effective mass [21]. Representing the network of wiggly strings as a collection of uncorrelated segments and performing the COBE normalization of the calculated CMB temperature fluctuations, Pogosian and Vachaspati [21] predict $G \leq 1.9 \times 10^{-6}$.

Let us now consider possible effects of the long-range gravitational field on the modeling of $Q$-lump string network. It can be shown [22] that between two parallel strings (spinning in the same sense) there is an attractive force due to graviton exchange which (for $\alpha\lambda \sim 1$) goes asymptotically as $F \sim \eta^4 G/d$, where $d \gg \eta$ is the distance between strings. This long-range interaction may be responsible for correlations between string segments in the network which in turn could produce temporal coherence. Initially, the stationary pair begins to move together under the influence of the graviton-induced attraction. However, when their spacing $d$ is of order $\lambda$ they do not necessarily annihilate since their interaction is dominated by short-range repulsive force due to boson exchange [22].

The potential energy associated with these forces may contribute to localization of strings. This suggests the possibility of a freezing of the network into a three-dimensinal structure in a local minimum of the potential energy. There is an analogy with the formation of a vortex lattice in a type II superconductor [23]. Collective modes in a frozen network of strings would then act as a source of coherent fluctuations that could contribute to the generation of acoustic peaks in the angular spectrum of CMB. This, of course, implies a drastic departure from the standard modeling of string networks as a collection of segments with uncorrelated velocities. In the absence of quantitative treatment of a correlated or frozen network, an estimate of the limit of $Q$-lump string tension using CMB data is presently uncertain.

The idea of the formation of frozen networks of strings is not entirely new; it has been considered previously for the case when strings of different types collide [1,24]. A renewed interest in this possibility comes from the presence of both $F$- and $D$-type strings in superstring networks [25–27]. These strings are produced at the end of the inflationary epoch in the brane world. In this scenario the inflationary expansion proceeds by the attractive interaction of $D$-branes and anti-$D$-branes and the strings are produced during brane collision. Sarangi and Tye [28] used the inflationary, brane scenario to estimate the string tension. Comparing the combined effect of the inflaton and the "uncorrelated" cosmic string fluctuations with the COBE data [13], they find that $G\epsilon_0 \sim 10^{-7}$.

If non-Abelian symmetry is broken to a discrete subgroup, multiple types of cosmic strings are produced. When strings of different types collide they do not intercommute. Instead, they form frustrated networks with trilinear vertices [27]. Bucher and Spergel [29] proposed a solid dark matter component created from frustrated networks of non-Abelian cosmic strings and domain walls. There is a significant negative pressure associated with this mechanism. Vachaspati and Vilenkin [24] calculated the average pressure $P_s$ starting from the energy-momentum tensor of a string moving with the velocity $v_s$ and obtained $P_s/\rho_s = -1/3(1 - \langle v_s^2 \rangle)$, where $\rho_s$ is the energy density of the string network. For stationary strings, $\langle v_s^2 \rangle = 0$, and the ratio $P_s/\rho_s = -1/3$ implying a universe that is neither accelerating nor deccelerating. As pointed out by Bucher and Spergel [29], the recent observations suggesting that the Universe is presently accelerating requires that the ratio $w = P/\rho < -1/3$. Of course, other dark matter components may supply the needed negative pressure contribution. In this context, the model of frustrated networks formed from sigma-model spinning strings may be of interest. As shown in the present paper, these strings represent a source of active gravitational mass even when stationary. This allows the ratio $P_s/\rho_s$ to reach the most negative value of $-1/3$. Moreover, as suggested by the virial equation of state, an additional contribution to the negative pressure may come from the attractive interaction between the strings [22].

Another observational bound on the cosmic string tension comes from pulsar timing. When gravity waves sweep over the pulsar, they affect the ticking rate of the pulsar clock. Vibrations of closed loops of cosmic strings generate stochastic gravitational wave background characterized by the quantity $\Omega_{\rm GW}$ defined as the energy density per logarithmic frequency range [2]. The limits on $\Omega_{\rm GW}$, inferred from the pulsar timing, range from [30] $\Omega_{\rm GW} < 1.2 \times 10^{-7}$ to a value that is about 30 times weaker [31]. The stochastic background produced by a quadrupole radiation from a network of strings has been estimated by Vilenkin [2] to yield $\Omega_{\rm GW} \simeq 0.04 G\epsilon_0$. By comparing this result with the above pulsar timing limits [30,31], we obtain $G\epsilon_0 < 10^{-5.5}$ and $G\epsilon_0 < 10^{-7}$. A critical analysis of these results is presented by Polchinski [27]. It should be pointed out that the theoretical estimate [2] of $\Omega_{\rm GW}$ is based on the standard model of the network which does not involve any correlation effects that may be present in a network of $Q$-lump strings. This may introduce further uncertainties in the estimate of the tension of $Q$-lump strings based on pulsar timing.

The $Q$-lump string can be detected through gravitational deflection of light. When the impact parameter is large so that $x \gg \lambda$, Eq. (53) shows that the deflection angle is independent of $x$ and its magnitude $\delta\Phi \simeq 4\pi G\epsilon$, where $\epsilon$ is the net energy per unit length of the string. In the limit $\alpha\lambda \rightarrow 0$, these features agree with the deflection angle for the vacuum string obtained in Ref. [3]. For $\alpha\lambda \sim 1$, there is an enhancement (about 40%) of the deflection angle due to spinning.

Equation (55) shows that the drag-induced light deflection angle depends not only on the product $\alpha\lambda$ but also on the string radius $\lambda$ itself. However, in the $Q$-lump model, the radius is not fixed (being determined by the Noether charge $Q$ which is a free parameter). An orientational estimate of $\lambda$ can be made with the use of a related string model that is based on a spinning baby Skyrmion model of Piette, Schroers, and Zakrzewski [32]. In this model, the soliton has a preferred size determined by the competition between the potential and Skyrme terms in the Lagrangian. Recently, we studied the gravitational field of a string obtained by trivially extending the baby Skyrmion model from $2+1$ to $3+1$ dimensions [22]. We note that the spinning frequency, $\omega$, of this model is a free parameter. However, by requiring that the active gravitational mass be a relevant factor in galaxy formation, the product $\omega\lambda$ must be of the order of 1. Also, to ensure exponential localization of the soliton, $\omega$ must be smaller than the magnitude of the potential term. From these conditions, we deduce that $\lambda < 1/\eta$. Thus, the radius of a GU string is comparable to a Higgs Compton wavelength $\lambda \sim 10^{-31}$ m. We note that this result is similar to the transverse dimensions of strings studied in Ref. [3]. With this value of $\lambda$, we now proceed to make an estimate of $\delta\Phi_d$ from Eq. (55). Denoting by $R$ the proper distance of the observer to the string axis, the impact parameter $x \sim R\delta\Phi_d$. Owing to the extremely small value of $\lambda$, the second term in the denominator of Eq. (55) can be neglected yielding $\delta\Phi_d \sim (\pi^2\epsilon_0 G\alpha\lambda^2/R)^{1/2}$. Taking $\epsilon_0 G \sim 10^{-6}$, $\alpha\lambda \sim 1$, $\lambda \sim 10^{-31}$ m, and $R \sim 10^4$ m, we get $\delta\Phi_d \sim 10^{-20}$ rad which is negligible compared with the deflection angle $\delta\Phi \sim 10^{-5}$ resulting from Eq. (53).

Sazhin *et al.* [33] have considered an interpretation of the extragalactic double source CSL-1 (Capodimonte-Sternberg-Lens candidate, No. 1) as a gravitational lensing by a cosmic string. The pair of images exhibits a separation of 2 arc s which correspond to $G\epsilon_0$ equal to $4 \times 10^{-7}$ times a geometric factor of order 1. However, this interpretation has now been ruled out by high quality imaging data from the Hubble Space Telescope (HTS) [34]. Rather the HTS data show that the galaxy image pair CSL-1 is not a lens but a pair of galaxies.

Let us turn to some problems of the dynamics of spinning strings. First note that a string can remain static only if it is straight. Curved strings oscillate under their own tension. Of particular importance for the galaxy formation are oscillating closed loops [2]. In this case, the oscillation of the string axis is responsible for the generation of an active gravitational mass that bears qualitative similarities with the mechanism outlined in Sec. III. This may be seen by recalling Eq. (25), which shows that the active mass induced in a unit length of a spinning string is of the order of $\eta^2$ times the square of the velocity at the radius $\lambda$ of the cylindrical shell. This should be compared with the mass $M \sim \eta^2 v_{\rm rms}^2$ derived by Turok [35] for a loop oscillating with rms velocity $v_{\rm rms}$.

Now, the question that needs to be addressed is if closed loops of spinning strings could also serve as seeds for galaxy formation. The scenario of galaxy formation proposed by Vilenkin [2] requires that the main energy loss mechanism of large loops be gravitational radiation. For gauge-symmetry nonspinning strings Vachaspati, Everett, and Vilenkin [36] have shown that electromagnetic radiation and the radiation of massive particles fail to yield significant energy loss in comparison to the gravitational radiation. For the $Q$-lump string, considered in this paper, there is no gauge field in the Lagrangian density of Eq. (1). Thus, there is no coupling of the photon field to the oscillations of the string as a whole. However, there is nonzero coupling of these oscillations to the field corresponding to small oscillations, $\delta\vec{\phi}$, of the vector $\vec{\phi}$ about the vacuum configuration $\vec{\phi}_0 = (0, 0, 1)$. This coupling is of a similar nature to that considered by Davis [37] for global nonspinning string. Specifically, the $\delta\vec{\phi}$ field is rigidly carried along with the string as it moves. Nevertheless, the power radiated due to this coupling is zero as long as the frequency of the string oscillation $\omega$ is less than the spinning frequency $\alpha$. This is because $\delta\vec{\phi}$ satisfies a Klein-Gordon equation describing massive bosons with the mass $\alpha$. For a loop of size $R$, $\omega \sim 1/R$, whereas $\alpha$ is of order $\eta \sim 10^{15}$ GeV. Thus, $\alpha \gg \omega$, implying that the decay into the $\delta\vec{\phi}$ bosons is ruled out.

The global properties of the 2 + 1-dimensional spacetimes generated by massive point particle with angular momentum $J$ have been thoroughly investigated in Ref. [38]. It is of interest to note the relation of the line element given in Eq. (4.17) of this reference to Eq. (34) of the present paper. As $r/\lambda \to \infty$, the rhs of this equation goes to $-8GJ_3 d\theta dx^0$, where $J_3$ is the $x^3$ component of the angular momentum per unit length of the $Q$-lump string. This result should be compared with the term $2A dt d\theta = -8GJ dt d\theta$ in Eq. (4.17). This agreement is not surprising, since we show that the spinning string behaves as a rotating cylindrical mass shell (see Sec. IV). In Refs. [39,40], the 2 + 1-dimensional metric of Ref. [38] has been extended to study the gravitational effects of straight spinning string. The physical consequences of the "time-helical" structure of the locally flat coordinates derived in Ref. [38] have been thoroughly studied in Ref. [41] which presents solutions of the Klein-Gordon and Dirac equations in the presence of massive point particle with arbitrary angular momentum.

Closer to our present work appears to be a more recent paper by Verbin and Larsen [42] who study the $Q$-lump string with a general oscillatory behavior of the fields given by $\exp(iqz - i\omega t)$. The case $q = 0$ corresponds to the present model, but since Ref. [42] goes beyond the weak coupling approximation, the solutions for the metric are obtained numerically. From Fig. 6 of Ref. [42] we see that the metric $N = (1 + h_{00})^{1/2}$ is consistent with our result showing that $h_{00}$ goes as $(r/\lambda)^6$ for $r/\lambda \to 0$ and as $\log(r/\lambda)$ for $r/\lambda \to \infty$. Thus the metric $g_{00}$ is not asymptotically flat owing to finite frequency of spinning. This general trend is also exhibited by other examples studied numerically by Verbin and Larsen [42]. This confirms predictions made by these authors following a deep analysis based on the Kaluza-Klein reduction from $D$-dimensional global strings to the $D - 1$-dimensional gauged strings. On the other hand, the numerically obtained quantity $L_\varphi$ is asymptotically flat and negative. This is consistent with our weak coupling result (34), according to which $L_\varphi \to -(8/\pi)GJ_3(r/\lambda)^2$ for $r/\lambda \to 0$, and $L_\varphi \to -4GJ_3$ for $r/\lambda \to \infty$.

*Note added.*—When I wrote this paper I was not aware of several related earlier papers. Thanks to correspondence from Professor S. Deser, Professor R. Jackiw, and Professor Y. Verbin I have been enlightened and thus some of the most relevant papers are now included in Refs. [38–42].

## APPENDIX: $Q$-LUMP SOLUTION TO BOGOMOL'NYI EQUATIONS

We now show that Eqs. (14) and (15) can be obtained as solutions to first order Bogomol'nyi equations for the $Q$-lump string. We start with the expression for the net energy $\epsilon$ per unit length of the string. Using Eq. (9) we have

$$\epsilon = \eta^2 \int d^2x (1 + |u|^2)^{-2} (\partial_t u \partial_t \bar{u} + \partial_i u \partial_i \bar{u} + \alpha^2 u \bar{u}). \tag{A1}$$

Motivated by this formula, the Bogomol'nyi inequality is written as [4,10]

$$\eta^2 \int d^2x (1 + |u|^2)^{-2} [(\partial_i u \pm i\epsilon_{ij}\partial_j u)(\partial_i \bar{u} \mp i\epsilon_{ik}\partial_k \bar{u}) + 2(\partial_t u \pm i\alpha u)(\partial_t \bar{u} \mp i\alpha \bar{u})] \geq 0. \tag{A2}$$

Note the factor of 2 multiplying the last product. Using Eq. (A1), Eq. (A2) simplifies to

$$\epsilon \geq \pm 2\pi\eta^2 N \pm \eta^2 \alpha Q, \tag{A3}$$

where

$$N = \frac{i}{2\pi} \int d^2x \frac{\epsilon_{ij}\partial_i u \partial_j \bar{u}}{(1 + |u|^2)^2} \tag{A4}$$

is the topological charge. Using Eq. (5), it can be shown that Eq. (A4) agrees with the well-known definition [10]

$$N = \frac{1}{4\pi} \int d^2x \vec{\phi} \cdot (\partial_1 \vec{\phi} \times \partial_2 \vec{\phi}). \tag{A5}$$

The quantity $Q$ which appears on the rhs of Eq. (A3), happens to coincide with the Noether charge defined by the first equality in Eq. (17). We note that $Q$ is conserved owing to the gauge invariance of the Lagrangian density (6).

According to Eq. (A2), the bound (A3) is attained when the following first order equations hold:

$$\partial_i u \pm i\epsilon_{ij}\partial_j u = 0, \tag{A6}$$

$$\partial_t u \pm i\alpha u = 0. \tag{A7}$$

When $u(z, t)$ is a solution of these equations, equality occurs in Eq. (A3). Taken with the upper sign, Eq. (A6) shows that $u$ satisfies the Cauchy-Riemann conditions for a holomorphic function of $z = x^1 + ix^2$. For this choice of sign, Eqs. (17) and (A4) yield positive values for $N$ and $Q$. On the other hand, choosing a lower sign in Eq. (A6) makes

$u(z)$ antiholomorphic and $N$ and $Q$ are both negative. Using these results in conjunction with Eq. (A3), we obtain Eq. (18) that holds for both sign choices. [Eq. (16) holds only for $N$ positive]. The general solution of Eq. (A6) is given by the rational map $u(z) = p(z)/q(z)$ where $p$ and $q$ are polynomials. The degree $N$ rational map $u(z) = \frac{\lambda^N}{z^N}$ corresponds to $N$ lumps coincident at the origin. It also describes a radially symmetric $Q$ lump with topological charge $N$. This can be verified by substituting this map into Eq. (A4). Our choice of a radially symmetric map made in Eq. (15) greatly simplifies the solution of the Einstein equation (3). Using this solution on the rhs of Eq. (17) and substituting the result in Eq. (18), we obtain, for $N > 0$, the expression (16). For $N = 1$, the integral in Eq. (17) diverges (as cosec$\pi$) owing to slow power law decay of $|u(z)| = |\lambda/z|$. The lowest degree yielding finite energy $\epsilon$ is $N = 2$ as used in Eq. (15) and throughout the following computations. Equation (A7), taken with upper sign, implies

$$u(t, z) = \exp(-i\alpha t)u(z) \tag{A8}$$

as written in Eq. (14).

It should be pointed out, that, in contrast to the present model, the spinning baby Skyrmion model [32] yields solutions for the fields $\vec{\phi}$ that are exponentially localized as long as the spinning frequency does not exceed the meson mass threshold. Consequently, finite energy solutions are obtained in this model even for $N = 1$. For the $Q$-lump model, the exponential localization is absent due to the fact that the spinning frequency is exactly equal to the meson mass threshold. In the case of the pure sigma model [5], the rational map of degree 1, $u(z) = \lambda/z$, also yields finite energy per unit length of the string since the anisotropic potential is absent.


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