

# Energy of gravitational radiation in plane-symmetric space-times

Sean A. Hayward

*Center for Astrophysics, Shanghai Normal University, 100 Guilin Road, Shanghai 200234, China*

(Received 19 May 2008; published 13 August 2008)

Gravitational radiation in plane-symmetric space-times can be encoded in a complex potential, satisfying a nonlinear wave equation. An effective energy tensor for the radiation is given, taking a scalar-field form in terms of the potential, entering the field equations in the same way as the matter energy tensor. It reduces to the Isaacson energy tensor in the linearized, high-frequency approximation. An energy conservation equation is derived for a quasilocal energy, essentially the Hawking energy. A transverse pressure exerted by interacting low-frequency gravitational radiation is predicted.

DOI: [10.1103/PhysRevD.78.044027](https://doi.org/10.1103/PhysRevD.78.044027)

PACS numbers: 04.30.Nk

## I. INTRODUCTION

Gravitational radiation, as predicted by Einstein gravity, is indirectly observed in such examples as the Hulse-Taylor pulsar, and widely expected to be directly observed in the coming years, offering a new window to understand various astrophysical processes, such as binary inspiral and merger of black holes or neutron stars. However, the textbook theory of gravitational radiation mostly concerns weak radiation, either in the linearized approximation or at infinity in an asymptotically flat space-time [1]. Comparatively little is known about strong-field radiation. Exceptions include plane symmetry, cylindrical symmetry, Robinson-Trautman space-times, and boost-rotation symmetric space-times, as reviewed by Bičák [2]. The simplest example is plane gravitational radiation, where exact solutions describe radiation propagating in one direction. The simplest scenario to study interaction effects is the head-on collision of two such beams, as pioneered by Szekeres [3,4] and reviewed by Griffiths [5]. More generally, one may study plane-symmetric space-times, which in vacuum generally consist of gravitational radiation propagating in opposite directions and interacting [6].

Much is known about such space-times, including that the interaction is nonlinear, that the key dynamical equations can be cast as a complex Ernst equation [7], and that the cross-focusing of the radiation produces a caustic which is generically a curvature singularity, though there are nongeneric exceptions [8–10]. This article introduces an effective energy tensor  $\Theta_{ab}$  for the gravitational radiation, taking a scalar-field form in terms of a complex potential  $\Phi$ . Then  $\Theta_{ab}$  enters the field equations in the same way as the matter energy tensor  $T_{ab}$ , in particular, entering an energy conservation law. The Ernst equation is manifestly a wave equation for  $\Phi$ , generally with a nonlinear source, which vanishes for collinear polarization.

The method involves a conserved time vector  $k^a$ , a conserved energy-momentum density  $j^a$ , a corresponding energy  $E$ , and a first law for  $E$  involving energy-supply and work terms. Surface gravity  $\kappa$  is also defined and takes a quasi-Newtonian form. This is intended to complete the

same programme of identifying physical quantities and equations which has previously been performed in spherical symmetry [11,12], cylindrical symmetry [13], and a quasispherical approximation [14–17]. These references will be assumed for comparison throughout the text without repeated citation, though the treatment here is self-contained.

## II. METRIC VARIABLES AND FIELD EQUATIONS

Cartesian coordinates  $(z, y)$  on the planes of symmetry will be used, to allow easy comparisons with standard coordinates  $(z, \varphi)$  in cylindrical symmetry and  $(\vartheta, \varphi)$  in spherical symmetry and the quasispherical approximation. It is convenient to use null coordinates  $x^\pm$  in the normal space, as they are adapted to gravitational radiation. Then the metric can be written locally as

$$ds^2 = -2e^{2\gamma} dx^+ dx^- + A(e^{2\phi} \sec 2\chi dy^2 + 2 \tan 2\chi dy dz + e^{-2\phi} \sec 2\chi dz^2), \quad (1)$$

where  $(A, \phi, \chi, \gamma)$  are functions of  $(x^+, x^-)$ . Here  $A$  is the specific area, meaning that it is the area of a square coordinate patch  $(0, 1) \times (0, 1)$  in the  $(y, z)$  plane. It is invariant up to constant linear transformations of  $y$  and  $z$ , under which it scales by a constant factor. The remaining freedom in  $(y, z)$  is by rotations, under which  $A$  is invariant. The functions  $(\phi, \chi)$  encode the gravitational radiation, as will be seen below. They are invariant up to the above-mentioned transformations of  $(y, z)$ , which will be treated as fixed henceforth. The remaining function  $\gamma$  is invariant up to functional rescalings  $x^\pm \mapsto \tilde{x}^\pm(x^\pm)$ , under which it transforms by additive functions of  $x^+$  and  $x^-$ . The variables have been chosen so that the induced metric on the planes of symmetry takes a similar form to that used in the quasispherical approximation, with  $(dz, dy)$  replaced by  $(d\vartheta, \sin\vartheta d\varphi)$ , and takes a similar form to that used in cylindrical symmetry. The Szekeres variables  $(P, M, Q, W)$  are related by

$$\begin{aligned} P &= -\log A, & M &= -2\gamma, \\ Q &= -2\phi, & \sinh W &= \tan 2\chi, \end{aligned} \quad (2)$$

or  $\cosh W = \sec 2\chi$ .

The six independent components of the Einstein equation may be found directly, or by comparison with the Szekeres form, as

$$2A\partial_{\pm}\partial_{\pm}A - (\partial_{\pm}A)^2 - 4A\partial_{\pm}A\partial_{\pm}\gamma + 4A^2\sec^2 2\chi((\partial_{\pm}\phi)^2 + (\partial_{\pm}\chi)^2) = -16\pi A^2 T_{\pm\pm}, \quad (3)$$

$$\partial_{+}\partial_{-}A = 8\pi AT_{+-}, \quad (4)$$

$$\begin{aligned} &2A\partial_{+}\partial_{-}\phi + \partial_{+}A\partial_{-}\phi + \partial_{-}A\partial_{+}\phi \\ &+ 4A\tan 2\chi(\partial_{+}\phi\partial_{-}\chi + \partial_{-}\phi\partial_{+}\chi) \\ &= 4\pi Ae^{2\gamma}\cos^2 2\chi(T_y^y - T_z^z), \end{aligned} \quad (5)$$

$$2A\partial_{+}\partial_{-}\chi + \partial_{+}A\partial_{-}\chi + \partial_{-}A\partial_{+}\chi + 4A\tan 2\chi(\partial_{+}\chi\partial_{-}\chi - \partial_{+}\phi\partial_{-}\phi) = 4\pi Ae^{2\gamma}\cos^2 2\chi(e^{2\phi}T_z^z + e^{-2\phi}T_y^y), \quad (6)$$

$$4A^2\partial_{+}\partial_{-}\gamma - \partial_{+}A\partial_{-}A + 4A^2\sec^2 2\chi(\partial_{+}\phi\partial_{-}\phi + \partial_{+}\chi\partial_{-}\chi) = -8\pi A^2(2T_{+-} + e^{2\gamma}(T_y^y + T_z^z)), \quad (7)$$

where  $\partial_{\pm} = \partial/\partial x^{\pm}$ ,  $T_{\pm\pm} = T(\partial_{\pm}, \partial_{\pm})$ ,  $T_{+-} = T(\partial_{+}, \partial_{-})$ , and the units are such that Newton's gravitational constant is unity. Equations (3) can be regarded as constraint equations on initial null hypersurfaces  $\Sigma_{\pm}$  of constant  $x^{\pm}$ , as they are preserved in the  $\partial_{\mp}$  directions due to the Bianchi identities or energy-momentum conservation. The other Eqs. (4)–(7) are then the evolution equations.

In vacuum,  $T_{ab} = 0$ , it is well known that these equations describe the propagation and interaction of gravitational radiation in the opposite  $\partial_{\pm}$  directions, and that the radiation may be encoded in  $(\phi, \chi)$ . The solution to (4) is trivial and can be used to fix the rescaling freedom in  $x^{\pm}$ .

$$\Theta_{ab} = \frac{2\nabla_a\phi\nabla_b\phi + 2\nabla_a\chi\nabla_b\chi - g_{ab}g^{cd}(\nabla_c\phi\nabla_d\phi + \nabla_c\chi\nabla_d\chi)}{8\pi\cos^2 2\chi}. \quad (10)$$

If  $\chi = 0$ , it reduces to the energy tensor of a massless scalar field  $\phi$ , as in cylindrical symmetry, where the corresponding  $\phi$  reduces to the Newtonian gravitational potential in the Newtonian limit. Here there are generally two polarizations of the radiation, as is familiar from the linearized approximation. Inspection of the metric (1) for small  $\Phi$  identifies  $\phi$  as encoding the “plus” polarization and  $\chi$  as encoding the “cross” polarization. These properties justify the numerical factors chosen in the definitions of  $(\phi, \chi)$  and partly motivated the chosen symbols.

The nontrivial components of  $\Theta_{ab}$  follow explicitly as

$$4\pi\Theta_{\pm\pm} = \sec^2 2\chi((\partial_{\pm}\phi)^2 + (\partial_{\pm}\chi)^2), \quad (11)$$

One may give initial data for  $(\phi, \chi)$  on  $\Sigma_{\pm}$ , corresponding to initial radiation profiles, with (3) determining  $\gamma$  on  $\Sigma_{\pm}$ . Then the main task is to solve (5) and (6) simultaneously for  $(\phi, \chi)$ , after which the full solution follows from (7) by quadrature for  $\gamma$ . The main Eqs. (5) and (6) can be written as a complex Ernst equation, corresponding physically to a nonlinear wave equation, as will be verified below.

### III. EFFECTIVE ENERGY TENSOR FOR GRAVITATIONAL RADIATION

The next aim is to find an effective energy tensor  $\Theta_{ab}$  for the gravitational radiation, analogous to those found in cylindrical symmetry and the quasispherical approximation, and consistent with the Isaacson effective energy tensor in the high-frequency linearized approximation [1]. In all cases, the components of the energy tensor are quadratic in first derivatives of the metric, in this case the  $\partial_{\pm}$  derivatives of  $(\phi, \chi)$ , and such terms can be seen in the last term in parentheses on the left-hand side of each of (3) and (5)–(7). The idea is to identify these terms as components of the desired  $\Theta_{ab}$ , corresponding to the components of  $T_{ab}$  on the right-hand sides. The result is that one may introduce a complex potential

$$\Phi = \phi + i\chi \quad (8)$$

and define the effective energy tensor as

$$\Theta_{ab} = \frac{2\nabla_{(a}\Phi\nabla_{b)}\bar{\Phi} - g_{ab}g^{cd}\nabla_c\Phi\nabla_d\bar{\Phi}}{8\pi\cosh^2(\Phi - \bar{\Phi})}, \quad (9)$$

where  $g_{ab}$  is the space-time metric and  $\nabla_a$  its covariant derivative operator. It is manifestly a tensor, taking a scalar-field form in terms of  $\Phi$ , with the same form, including the same denominator, as in the quasispherical approximation. Apart from this denominator, it is the energy tensor of a massless complex scalar field  $\Phi$ . Explicitly in terms of  $(\phi, \chi)$ ,

$$\Theta_{+-} = 0, \quad (12)$$

$$4\pi\perp\Theta = e^{-2\gamma}\sec^2 2\chi(\partial_{+}\phi\partial_{-}\phi + \partial_{+}\chi\partial_{-}\chi)\perp g, \quad (13)$$

where  $\perp$  denotes projection onto the planes of symmetry and the transverse metric is given in  $(y, z)$  coordinates by

$$\perp g = A \begin{pmatrix} e^{2\phi}\sec 2\chi & \tan 2\chi \\ \tan 2\chi & e^{-2\phi}\sec 2\chi \end{pmatrix}. \quad (14)$$

It is then straightforward to verify that adding  $\Theta_{ab}$  to  $T_{ab}$  on the right-hand sides of the Einstein equations (3)–(7) cancels the quadratic terms in  $(\phi, \chi)$  on the left-hand sides. In abstract terms, the Einstein equation  $G_{ab} = 8\pi T_{ab}$  may

be rewritten as  $C_{ab} = 8\pi(T_{ab} + \Theta_{ab})$  in terms of a truncated Einstein tensor  $C_{ab}$ , whose components have a simpler form to those of the Einstein tensor  $G_{ab}$ .

The physical interpretation of  $\Theta_{\pm\pm}/2$  is the energy density of gravitational radiation propagating in the  $\partial_{\mp}$  direction. Apart from the nonlinear modification due to the  $\text{sec}^2 2\chi$  factor in (11), it is the energy density of a complex scalar field  $\Phi$ . The numerical factor also corresponds to the energy density of electromagnetic radiation in Gaussian units, with  $\phi$  corresponding to the electric potential and  $\chi$  vanishing. The vanishing of  $\Theta_{+-}$  (12) is familiar from cylindrical symmetry and the quasispherical approximation, and indicates that the gravitational radiation is workless. Note that this is generally not so for a similar effective energy tensor found in the context of black holes [18,19] and uniformly expanding flows [20,21]. The non-negativity of  $\Theta_{\pm\pm}$  indicates that, as an energy tensor,  $\Theta_{ab}$  satisfies the dominant energy condition, meaning physically that gravitational radiation carries positive energy. The other nonzero terms (13) indicate that interacting gravitational radiation generally exerts transverse pressure and shear, proportional to the transverse metric. These terms vanish for radiation propagating in one direction only, where  $\Phi$  is a function of  $x^+$  (or  $x^-$ ) only. They are commonly known as plane waves, but since this would appear to imply periodicity in some sense, this article uses the more general terminology of radiation.

#### IV. CONSERVATION OF ENERGY

To see how  $\Theta$  (switching to index-free notation) further qualifies as an effective energy tensor, one may proceed by analogy with spherical symmetry, cylindrical symmetry, and the quasispherical approximation. Here the definitions and equations will be stated first in a manifestly invariant way, then verified in coordinates. First introduce the specific area radius

$$r = \sqrt{A/4\pi}. \quad (15)$$

This is defined in order to compare with spherically symmetric space-times or the quasispherical approximation, so that one may easily treat astrophysical gravitational radiation as observed on or near Earth, since distant sources can be treated as points, producing roughly spherical wave fronts which can be treated as planes when observed.

The Hodge operator  $*$  defines the Hodge dual  $*\alpha$  of a normal one-form, up to sign, by

$$g^{-1}(*\alpha, \alpha) = 0, \quad g^{-1}(*\alpha, *\alpha) = -g^{-1}(\alpha, \alpha). \quad (16)$$

Then a preferred time vector is defined by

$$k = g^{-1}(*dr), \quad (17)$$

where the qualification ‘‘specific’’ is omitted here and henceforth. This vector is conserved:

$$\nabla \cdot k = 0. \quad (18)$$

The corresponding energy-momentum density is

$$j = -g^{-1}((T + \Theta) \cdot k). \quad (19)$$

Then  $j$  is also conserved:

$$\nabla \cdot j = 0. \quad (20)$$

Here the standard physical interpretation is conservation of energy, and the role of  $\Theta$  as an effective energy tensor is clear in that it appears additively with  $T$  in  $j$ .

Put another way, both  $k$  and  $j$  are Noether currents, and the corresponding Noether charges are area volume

$$V = \frac{4}{3}\pi r^3 \quad (21)$$

and energy  $E$ , defining the latter. Specifically,

$$Ag(k) = *dV, \quad Ag(j) = *dE. \quad (22)$$

Integrating for  $E$  and requiring it to vanish for flat space-time,

$$E = -\frac{1}{2}rg^{-1}(dr, dr) \quad (23)$$

which has a similar form to the Misner-Sharp energy in spherical symmetry and the modified Thorne energy in cylindrical symmetry. In fact, if the planes of symmetry are toroidally compacted by periodic identifications in  $(y, z)$  at 0 and 1, so that  $A$  is actually the area, then  $E$  coincides with the Hawking energy [22].

Note that  $E > 0$  for trapped surfaces,  $E = 0$  for marginal surfaces and  $E < 0$  for untrapped surfaces. In particular,  $E$  vanishes for radiation propagating in one direction only. Thus it should not be interpreted as the energy of a wave in any sense. Taking the example of two colliding beams, where the surfaces in the interaction region are trapped if the null energy condition holds, one may interpret  $E$  as measuring energy due to cross-focusing of radiation. In particular, it diverges at the caustic formed by such cross-focusing.

Introduce the work density

$$w = -\text{tr}T/2 \quad (24)$$

and the energy flux

$$\psi = (T + \Theta) \cdot g^{-1}(dr) + wdr, \quad (25)$$

where the trace is in the normal space. Then conservation of energy (20) can be written in the form of a first law:

$$dE = A\psi + wdV \quad (26)$$

which has the same form as in spherical symmetry and the quasispherical approximation. Here the two terms can be interpreted as energy supply and work, respectively, as in the first law of thermodynamics. Note again that  $\Theta$  appears additively with  $T$  in  $\psi$  and (in a null sense)  $w$ , playing the role of an effective energy tensor.

The corresponding definition of surface gravity is

$$\kappa = *d*dr/2, \quad (27)$$

where  $d$  is the exterior derivative of the normal space. Then the Einstein equations yield

$$\kappa = \frac{E}{r^2} - 4\pi r w \quad (28)$$

which again has the same form as that in spherical symmetry and the quasispherical approximation. Apart from the matter term, this has the form of Newtonian gravitational acceleration.

In dual-null coordinates (1), the corresponding expressions are

$$\begin{aligned} * \alpha &= -\alpha_+ dx^+ + \alpha_- dx^-, \\ \text{where } \alpha &= \alpha + dx^+ + \alpha_- dx^-, \end{aligned} \quad (29)$$

$$k = e^{-2\gamma}(\partial_+ r \partial_- - \partial_- r \partial_+), \quad (30)$$

$$\begin{aligned} j &= e^{-4\gamma} [((T_{--} + \Theta_{--})\partial_+ r - T_{+-}\partial_- r)\partial_+ \\ &\quad - ((T_{++} + \Theta_{++})\partial_- r - T_{-+}\partial_+ r)\partial_-], \end{aligned} \quad (31)$$

$$E = e^{-2\gamma} r \partial_+ r \partial_- r, \quad (32)$$

$$w = e^{-2\gamma} T_{+-}, \quad (33)$$

$$\psi_{\pm} = -e^{-2\gamma}(T_{\pm\pm} + \Theta_{\pm\pm})\partial_{\mp} r, \quad (34)$$

$$\kappa = -e^{-2\gamma}\partial_+\partial_- r. \quad (35)$$

Writing  $4(4\pi)^{3/2}E = e^{-2\gamma}A^{-1/2}\partial_+A\partial_-A$  and using the Einstein equations (3) and (4), a calculation yields

$$\partial_{\pm}E = Ae^{-2\gamma}(\partial_{\pm}rT_{+-} - \partial_{\mp}r(T_{\pm\pm} + \Theta_{\pm\pm})). \quad (36)$$

Comparison with

$$\begin{aligned} Ag(j) &= Ae^{-2\gamma} [((T_{++} + \Theta_{++})\partial_- r - T_{+-}\partial_+ r)dx^+ \\ &\quad - ((T_{--} + \Theta_{--})\partial_+ r - T_{-+}\partial_- r)dx^-] \\ &= [-\partial_+ E dx^+ + \partial_- E dx^-] = *dE \end{aligned} \quad (37)$$

verifies (22). Similarly, the calculation

$$\begin{aligned} A(\psi_{\pm} + w\partial_{\pm}r) &= Ae^{-2\gamma}(-\partial_{\mp}r(T_{\pm\pm} + \Theta_{\pm\pm}) + \partial_{\pm}rT_{+-}) \\ &= \partial_{\pm}E \end{aligned} \quad (38)$$

verifies (26). The easiest way to verify the conservation equations (18) and (20) is to use (22) and exterior calculus:

$$\nabla \cdot k = A^{-1} * d * (Ag(k)) = A^{-1} * d * *dV = 0, \quad (39)$$

$$\nabla \cdot j = A^{-1} * d * (Ag(j)) = A^{-1} * d * *dE = 0 \quad (40)$$

since  $** = \pm 1$  and  $dd = 0$ . Finally, a calculation using the Einstein equation (4) verifies (28).

## V. GRAVITATIONAL WAVE EQUATION

As is well known, the propagation equations (5) and (6) for  $(\phi, \chi)$  can be written as a single complex Ernst equation, usually given in terms of an Ernst potential  $Z = e^{2\Phi}$  or  $E = \tanh\Phi$  [5]. The corresponding form for  $\Phi$  is

$$\nabla^2\Phi = 2 \tanh(\Phi - \bar{\Phi})g^{-1}(\nabla\Phi, \nabla\Phi), \quad (41)$$

where  $\perp T = 0$  for simplicity. This has the same form as that in the quasispherical approximation. The calculation is straightforward:

$$\nabla^2\Phi = -e^{-2\gamma}(2\partial_+\partial_-\bar{\Phi} + A^{-1}(\partial_+A\partial_-\bar{\Phi} + \partial_-A\partial_+\bar{\Phi})) \quad (42)$$

and

$$\begin{aligned} 2 \tanh(\Phi - \bar{\Phi})g^{-1}(\nabla\Phi, \nabla\Phi) &= -4e^{-2\gamma} \tanh 2i\chi(\partial_+\phi + i\partial_+\chi)(\partial_-\phi + i\partial_-\chi) \\ &= 4e^{-2\gamma} \tan 2\chi((\partial_+\phi\partial_-\chi + \partial_-\phi\partial_+\chi) \\ &\quad + i(\partial_+\chi\partial_-\chi - \partial_+\phi\partial_-\phi)) \end{aligned} \quad (43)$$

then the result follows by comparing with (5) and (6).

Note that (41) is manifestly a wave equation for  $\Phi$ , equating  $\nabla^2\Phi$  to a nonlinear term in  $\Phi$ . This source term is highly nonlinear, being quadratic in  $\nabla\Phi$  and also involving  $\tanh(\Phi - \bar{\Phi})$ . In the special case of collinear polarization  $\chi = 0$ , the source term vanishes and the equation reduces to the wave equation for  $\phi$ ,  $\nabla^2\phi = 0$ . This can be written as a Euler-Poisson-Darboux equation, for which general solutions are available. The full Ernst equation has been studied by various methods both in plane symmetry and in the original context of stationary axisymmetric space-times; see e.g. the reviews of Bičák [2] and Griffiths [5], and references therein.

## VI. LINEARIZED GRAVITATIONAL RADIATION

To compare with the usual description of linearized gravitational radiation [1], it is convenient to switch temporarily to Minkowski coordinates  $(t, x, y, z)$  defined by  $\sqrt{2}x^{\pm} = t \pm x$ . Expanding about the Minkowski metric  $\eta = \text{diag}\{-1, 1, 1, 1\}$  by  $g = \eta + h$  consists of expanding about  $(A, \phi, \chi, \gamma) = (1, 0, 0, 0)$ , so one can write  $A = 1 + \alpha$  and use  $(\alpha, \phi, \chi, \gamma)$  as perturbative fields, each assumed  $\ll 1$ . Linearizing, the metric perturbation  $h$  is given by

$$\begin{aligned} -2\gamma(dt^2 - dx^2) &+ (\alpha + 2\phi)dy^2 + 2(\alpha + 2\chi)dydz \\ &+ (\alpha - 2\phi)dz^2. \end{aligned} \quad (44)$$

Then the trace of  $h$  is  $2\alpha + 4\gamma$  and the trace-reversed metric perturbation  $\bar{h}$  is given by

$$\begin{aligned} \alpha(dt^2 - dx^2) &+ (2\phi - 2\gamma)dy^2 + 2(\alpha + 2\chi)dydz \\ &+ (-2\phi - 2\gamma)dz^2. \end{aligned} \quad (45)$$

Applying the transverse traceless gauge conditions:

$\partial^a \bar{h}_{ab} = 0$  yields constant  $\alpha$ ,  $\bar{h}_{0b} = 0$  yields  $\alpha = 0$ , and  $\bar{h}_a^a = 0$  yields  $\gamma = 0$ . Then  $h = \bar{h}$  is indeed transverse: in  $(y, z)$  coordinates,

$$h = \begin{pmatrix} 2\phi & 2\chi \\ 2\chi & -2\phi \end{pmatrix}. \quad (46)$$

This verifies the appropriateness of the transverse traceless gauge conditions in plane symmetry. Noting that the space-time strain is  $h/2$ , this also confirms that  $\phi$  and  $\chi$  encode the plus and cross polarizations, respectively.

In the high-frequency approximation, the Isaacson effective energy tensor  $\bar{\Theta}$  for gravitational waves is defined by

$$32\pi\bar{\Theta}_{ab} = \langle (\partial_a h_{cd})(\partial_b h^{cd}) \rangle, \quad (47)$$

where the angle brackets denote averaging over several wavelengths [1]. Returning to dual-null coordinates, the explicit expressions are

$$4\pi\bar{\Theta}_{\pm\pm} = \langle (\partial_{\pm}\phi)^2 + (\partial_{\pm}\chi)^2 \rangle, \quad (48)$$

$$4\pi\bar{\Theta}_{+-} = \langle \partial_+\phi\partial_-\phi + \partial_+\chi\partial_-\chi \rangle, \quad (49)$$

$$\perp\bar{\Theta} = 0. \quad (50)$$

Comparing with (11) and (13), one sees that the radiative components  $\bar{\Theta}_{\pm\pm}$  agree with  $\Theta_{\pm\pm}$ , but the other components apparently do not. However, this is due to the averaging, as follows.

First note that the gravitational wave equation (41) linearizes to the flat-space form

$$\partial_+\partial_-\Phi = 0 \quad (51)$$

with general solution

$$\Phi = \Phi_+(x^+) + \Phi_-(x^-) \quad (52)$$

as expected. Considering linear superpositions of Fourier modes in the high-frequency approximation, it suffices to consider solutions of the form

$$\Phi_{\pm} = \phi_{\pm} \sin\sqrt{2}\omega_{\pm}x^{\pm} + i\chi_{\pm} \sin\sqrt{2}\nu_{\pm}x^{\pm} \quad (53)$$

for constant amplitudes  $(\phi_{\pm}, \chi_{\pm})$  and angular frequencies  $(\omega_{\pm}, \nu_{\pm})$ . Then

$$\partial_{\pm}\phi = \sqrt{2}\phi_{\pm}\omega_{\pm} \cos\sqrt{2}\omega_{\pm}x^{\pm}, \quad (54)$$

$$\partial_{\pm}\chi = \sqrt{2}\chi_{\pm}\nu_{\pm} \cos\sqrt{2}\nu_{\pm}x^{\pm} \quad (55)$$

and

$$4\pi\Theta_{\pm\pm} = 2\phi_{\pm}^2\omega_{\pm}^2\cos^2\sqrt{2}\omega_{\pm}x^{\pm} + 2\chi_{\pm}^2\nu_{\pm}^2\cos^2\sqrt{2}\nu_{\pm}x^{\pm} \quad (56)$$

$$4\pi\perp\Theta = 2(\phi_+\phi_-\omega_+\omega_-\cos\sqrt{2}\omega_+x^+\cos\sqrt{2}\omega_-\omega_-\omega_-\omega_-\cos\sqrt{2}\nu_+x^+\cos\sqrt{2}\nu_-\omega_-\omega_-\omega_-\cos\sqrt{2}\nu_+x^+\cos\sqrt{2}\nu_-\omega_-\omega_-\omega_-\omega_-\delta), \quad (57)$$

where  $\delta = \text{diag}\{1, 1\}$ . Since  $\langle \cos^2 \rangle = 1/2$  but  $\langle \cos \rangle = 0$ ,  $\langle \perp\Theta \rangle = 0$  and similarly  $\bar{\Theta}_{+-} = 0$ . Then

$$4\pi\langle \Theta_{\pm\pm} \rangle = 4\pi\bar{\Theta}_{\pm\pm} = \phi_{\pm}^2\omega_{\pm}^2 + \chi_{\pm}^2\nu_{\pm}^2, \quad (58)$$

$$\langle \Theta_{+-} \rangle = \bar{\Theta}_{+-} = 0, \quad (59)$$

$$\langle \perp\Theta \rangle = \perp\bar{\Theta} = 0, \quad (60)$$

or

$$\langle \Theta \rangle = \bar{\Theta} \quad (61)$$

as expected. Note that the energy densities  $\bar{\Theta}_{\pm\pm}/2$  (58) have the expected form of squares of amplitudes times angular frequencies, with the same numerical factor  $1/8\pi$  as for electromagnetic radiation in Gaussian units.

On the other hand, for low-frequency waves, transverse pressure is generally present in  $\perp\Theta$  even in the linearized approximation, for which  $\Theta$  reduces to the energy tensor of a massless complex scalar field in flat space-time:

$$8\pi\Theta_{ab} = 2\partial_{(a}\Phi\partial_{b)}\bar{\Phi} - \eta_{ab}\eta^{cd}\partial_c\Phi\partial_d\bar{\Phi}. \quad (62)$$

The nonzero components (11)–(13) reduce to

$$4\pi\Theta_{\pm\pm} = (\partial_{\pm}\phi)^2 + (\partial_{\pm}\chi)^2, \quad (63)$$

$$4\pi\perp\Theta = (\partial_+\phi\partial_-\phi + \partial_+\chi\partial_-\chi)\delta \quad (64)$$

and, in particular, the transverse shear vanishes, but transverse pressure generally remains. Recall that this is an effect for interacting radiation, vanishing for radiation propagating in one direction only. However, if two beams with similar amplitude and frequency are passing through one another, the transverse pressure is generally of the same order as the energy densities  $\Theta_{\pm\pm}/2$ . Although this has been derived here only for plane-symmetric radiation propagating in opposite directions, one may expect it to generalize to gravitational radiation from any two sources in different directions.

## ACKNOWLEDGMENTS

Research supported by the National Natural Science Foundation of China under Grants No. 10375081, No. 10473007, and No. 10771140, by Shanghai Municipal Education Commission under Grant No. 06DZ1111, and by Shanghai Normal University under Grant No. PL609.



- [1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [2] J. Bičák, *Lect. Notes Phys.* **540**, 1 (2000).
- [3] P. Szekeres, *Nature (London)* **228**, 1183 (1970).
- [4] P. Szekeres, *J. Math. Phys. (N.Y.)* **13**, 286 (1972).
- [5] J. B. Griffiths, *Colliding Waves in General Relativity* (Oxford University, New York, 1991).
- [6] S. A. Hayward, *Classical Quantum Gravity* **7**, 1117 (1990).
- [7] F. J. Ernst, *Phys. Rev.* **167**, 1175 (1968).
- [8] C. J. S. Clarke and S. A. Hayward, *Classical Quantum Gravity* **6**, 615 (1989).
- [9] S. A. Hayward, *Classical Quantum Gravity* **6**, 1021 (1989).
- [10] S. A. Hayward, *Classical Quantum Gravity* **6**, L179 (1989).
- [11] S. A. Hayward, *Phys. Rev. D* **53**, 1938 (1996).
- [12] S. A. Hayward, *Classical Quantum Gravity* **15**, 3147 (1998).
- [13] S. A. Hayward, *Classical Quantum Gravity* **17**, 1749 (2000); **17**, 4159 (2000), corrigendum.
- [14] S. A. Hayward, *Phys. Rev. D* **61**, 101503 (2000).
- [15] H. Shinkai and S. A. Hayward, *Phys. Rev. D* **64**, 044002 (2001).
- [16] S. A. Hayward, *Classical Quantum Gravity* **18**, 5561 (2001).
- [17] S. A. Hayward, *Phys. Lett. A* **294**, 179 (2002).
- [18] S. A. Hayward, *Phys. Rev. Lett.* **93**, 251101 (2004).
- [19] S. A. Hayward, *Phys. Rev. D* **70**, 104027 (2004).
- [20] S. A. Hayward, *Classical Quantum Gravity* **23**, L15 (2006).
- [21] H. Bray, S. A. Hayward, M. Mars, and W. Simon, *Commun. Math. Phys.* **272**, 119 (2007).
- [22] S. W. Hawking, *J. Math. Phys. (N.Y.)* **9**, 598 (1968).