Hadamard renormalization of the stress-energy tensor for a quantized scalar field in a general spacetime of arbitrary dimension

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We develop the Hadamard renormalization of the stress-energy tensor for a massive scalar field theory defined on a general spacetime of arbitrary dimension. Our formalism could be helpful in treating some aspects of the quantum physics of extra spatial dimensions. More precisely, for spacetime dimensions up to six, we explicitly describe the Hadamard renormalization procedure and for spacetime dimensions from 7 to 11, we provide the framework permitting the interested reader to perform this procedure explicitly in a given spacetime. We complete our study (i) by considering the ambiguities of the Hadamard renormalization of the stress-energy tensor and the corresponding ambiguities for the trace anomaly, (ii) by providing the expressions of the gravitational counterterms involved in the renormalization process, and (iii) by discussing the connections between Hadamard renormalization and renormalization in the effective action. All our results are expanded on standard bases for Riemann polynomials constructed from group theoretical considerations and thus given on irreducible forms.

I. INTRODUCTION

In semiclassical gravity, spacetime is considered from a classical point of view, i.e., its metric $g_{\mu\nu}$ is treated classically, while all the other fields propagating on this background (from matter fields to the graviton field at one-loop order) are assumed to be quantized. In the last 30 years, this approximation of quantum gravity, usually called quantum field theory in curved spacetime, has permitted us to obtain very interesting results concerning more particularly (i) quantum black hole physics in connection with Hawking radiation, (ii) early universe cosmology, (iii) the Casimir effect, and (iv) quantum violations of classical energy conditions in connection with both the singularity theorems of Hawking and Penrose and the existence of traversable wormholes and time-machines. We refer to the monographs of Birrell and Davies [1], Fulling [2], and Wald [3] as well as to references therein for various aspects of semiclassical gravity. We also refer to a recent review by Ford [4] which is a short but rather up to date introduction to semiclassical gravity and to its applications. We finally refer to Sec. II.B of Ref. [5] for a very interesting critical account about the status and the domain of applicability of semiclassical gravity and to Refs. [6,7] for an extension of semiclassical gravity, the so-called semiclassical stochastic gravity, which also permits us to discuss and investigate its validity.

For a quantum field in some normalized state $|\psi\rangle$, the expectation value with respect to $|\psi\rangle$ of its associated stress-energy-tensor operator $T_{\mu\nu}$, denoted $\langle\psi|T_{\mu\nu}|\psi\rangle$, plays a central role in semiclassical gravity. Indeed:

(i) In curved spacetime, the particle concept is in general very nebulous. Here, we adhere completely to the point of view developed by Davies in Ref. [8]. It is then nonsense to speak about the particle content of the quantum state $|\psi\rangle$. From the physical point of view, it is more objectively described by a quantity such as the expectation value $\langle\psi|T_{\mu\nu}|\psi\rangle$.

(ii) It is rather natural to conjecture that the classical metric $g_{\mu\nu}$ is coupled to the quantum field according to the semiclassical Einstein equations

$$G_{\mu\nu} = 8\pi G\langle\psi|T_{\mu\nu}|\psi\rangle \qquad (1)$$

where $G_{\mu\nu}$ is the Einstein tensor $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}$ (here Λ and G denote, respectively, the cosmological constant and Newton's gravitational constant) or some higher-order generalization of this geometrical tensor. The expectation value $\langle\psi|T_{\mu\nu}|\psi\rangle$ which acts as a source in Eq. (1) then governs the backreaction of the quantum field on the spacetime geometry.

As a consequence, in semiclassical gravity, it is fundamental to be able to obtain an expression of the expectation value $\langle\psi|T_{\mu\nu}|\psi\rangle$ showing in detail the influence of the background geometry but also of the quantum state $|\psi\rangle$. But it is well-known that this is not really obvious [1–3].

The stress-energy tensor $T_{\mu\nu}$ is an operator quadratic in the quantum field which is, from the mathematical point of view, an operator-valued distribution. As a consequence, the operator $T_{\mu\nu}$ is ill defined and the associated expectation value $\langle\psi|T_{\mu\nu}|\psi\rangle$ is formally infinite. To deal with such a difficulty, renormalization is required. Much work has been done since the mid-1970s in order to renormalize the

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stress-energy tensor and/or to extract from the expectation value $\langle\psi|T_{\mu\nu}|\psi\rangle$ a finite and physically acceptable contribution which could act as the source in the semiclassical Einstein equations (1) (see, Ref. [1] for the state of affairs of the literature concerning this subject before 1982). Among all the methods employed, the axiomatic approach introduced by Wald [9] is certainly the most general and the most powerful. It is an extension of the "point-splitting method" [10–12] and it has been developed in connection with the Hadamard representation of the Green functions by Wald [9,13], Adler, Lieberman, and Ng [14,15]; Brown and Ottewill [16]; and Castagnino and Harari [17]. We refer to the monographs of Fulling [2] and Wald [3] for rigorous presentations of this approach which is usually called Hadamard renormalization. It permitted us to obtain, in the most general context, the explicit expressions of the renormalized expectation value of the stress-energy tensor for the scalar field theory [18–20] but also for some gauge theories such as (i) electromagnetism [18], (ii) quantum gravity at one-loop order [21] (here the theories described by the standard effective action as well as by the reparametrization-invariant effective action of Vilkovsky and DeWitt were both considered), and (iii) two- and three-form field theories [22] (in this context, the Hadamard formalism allowed us to treat carefully the phenomenon of ghosts for ghosts).

Hadamard renormalization has been exclusively considered for field theories defined on four-dimensional curved spacetimes. (However, it should be noted that a recent work has been achieved in a two-dimensional framework [23] but it is incorrect due to a wrong expression for the Hadamard representation of the Green functions.) According to the "recent" physical theories such as supergravity theories, string theories, and M theory, which were developed in order to understand gravity in a quantum framework and to provide a unified description of all the fundamental interactions, we should live in a spacetime with more dimensions than the four we observe, a scenario which is a resurgence of the old Kaluza-Klein theory [24,25]. Because all the previously mentioned theories are still at an early stage of development and are far from being well understood, it is rather difficult to make predictions by using them directly. In fact, people studying the consequences of supergravity and string theories in cosmology or in black hole physics often develop analysis based on semiclassical approximations or more precisely use the methods of quantum field theory in curved spacetime taking into account the extra dimensions. In this context, it seems to us crucial to extend the powerful Hadamard renormalization procedure to be able to deal, as generally as possible, with quantum fluctuations and with their backreaction effects. In this paper, we shall take some steps in this direction.

It is important to note that many recent articles have already been devoted to the role as well as to the calculation of the expectation value of the stress-energy tensor in the presence of extra spatial dimensions. For example:

(i) In the context of the Randall-Sundrum braneworld models [26,27] introduced in order to solve the hierarchy problem [28–30], i.e., to eliminate the large hierarchy between the electroweak scale and the gravity scale. The vacuum expectation value of the stress-energy tensor and the associated vacuum energy have been called upon to stabilize the size of the extra dimensions. There is extensive literature on the subject. We refer more particularly to Ref. [31] where backreaction effects are in addition considered and to Ref. [32] where cosmological considerations in connection with the inflationary scenario are in addition discussed (see, also, Refs. [33–37] and references therein).

(ii) In the context of the vacuum polarization induced by topological defects such as monopoles [38–40] or cosmic strings (see, Ref. [37] and references therein).

(iii) In the context of the AdS/CFT correspondence [41–43] which asserts the existence of a duality between a theory of gravity in the $(D + 1)$-dimensional anti-de Sitter (AdS) space and a conformal field theory living on its D-dimensional boundary (for a review, see, Ref. [44]) and which could provide a concrete realization of the holographic principle [45,46]. A new renormalization procedure, the so-called holographic renormalization, has been developed. More precisely, it has been shown that the regularized expectation value of the stress-energy tensor corresponding to the conformal field theory living on the boundary can be obtained from the "regularized" action of the gravitational field living in the bulk [47,48] (see, also for a review, Ref. [49] as well as references therein for complements and Refs. [50–60] for related approaches as well as extensions). The counterterm subtraction technique developed in this context permits us to obtain the stress-energy tensor, at large distance, for higher-dimensional black holes such as Kerr-AdS$_5$, Kerr-AdS$_6$, and Kerr-AdS$_7$ [61,62].

(iv) In the context of the validity of semiclassical gravity but also of the avoidance of the singularities predicted by the singularity theorems of Hawking and Penrose [63]. Fluctuations of the stress-energy tensor induce Ricci curvature fluctuations (see, for example, Ref. [64]) or in other words fluctuations of the gravitational field itself. The existence of these fluctuations places limits on the validity of semiclassical gravity but also could lead to important effects on the focusing of a bundle of timelike or null geodesics. The study of such fluctuations in the presence of compact extra spatial dimensions has been discussed more particularly in Ref. [65].

All these works have however been carried out under very strong hypotheses: flat (or conformally flat) spacetimes with extra dimensions or maximally (or asymptotically maximally) symmetric spacetimes as well as massless or conformally invariant field theories. Of course, it is necessary, from a physical point of view, to be able to deal with situations presenting a lower degree of symmetry. With this aim in view, the Hadamard renormalization procedure could be very helpful.

Finally, it should be noted that some mathematical aspects of the Hadamard renormalization procedure for a scalar field in a general ''spacetime'' of arbitrary dimension have been already considered by Moretti in a series of recent articles [66–70]. He has provided a rigorous proof of the symmetry of the off diagonal Hadamard coefficients, i.e., of the coefficients corresponding to the short-distance divergent part of the Hadamard representation of the Green functions for the Euclidean and Lorentzian scalar field theories [67,69]. He has also established a connection between the zeta- and Hadamard-regularization procedures in the Euclidean framework [66,68] and he has finally discussed the possible elimination of the ambiguities plaguing the Hadamard renormalization procedure by using microlocal analysis in the context of the algebraic approach to quantum field theory [70]. In fact, the results we present in this article are very different from those of Moretti. We do not focus our attention on the mathematical aspects of Hadamard renormalization as he did but on its practical aspects: from our results, the interested reader should be able to obtain explicitly the renormalized expression of the expectation value with respect to a given state $|\psi\rangle$ of the stress-energy-tensor operator associated with the scalar field theory if he/she knows (exactly or asymptotically in a sense defined below) the Feynman propagator corresponding to $|\psi\rangle$. With this aim in view, we have provided in Sec. III a step-by-step guide for the reader who simply wishes to calculate this regularized expectation value and is not specially interested in following the derivation of all our results.

Our article is organized as follows. In Sec. II, we develop as generally as possible the Hadamard renormalization of the stress-energy tensor associated with a massive scalar field theory defined on a general spacetime of arbitrary dimension. In Sec. III, we explicitly describe this procedure for arbitrary spacetimes of dimension from 2 to 6. This is done by using recent results we obtained in Ref. [71] and which concern the covariant Taylor series expansions of the Hadamard coefficients. For spacetime dimension from 7 to 11, we provide the framework permitting the interested reader to perform this regularization procedure explicitly in a given spacetime. In Sec. IV, we complete our study (i) by considering the ambiguities of the Hadamard renormalization of the stress-energy tensor and the corresponding ambiguities for the trace anomaly, (ii) by providing the expressions of the gravitational coun-

terterms involved in the renormalization process, and (iii) by discussing the connections between Hadamard renormalization and renormalization in the effective action. Finally, in Sec. V, we briefly discuss possible extensions of our work as well as possible applications. In a short appendix, we provide the traces of various conserved local tensors of rank 2 and orders 4 and 6. These results are more particularly helpful in order to discuss the ambiguity problem for the trace anomaly considered in Sec. IV.

In this paper, we use units with $\hbar = c = 1$ and the geometrical conventions of Hawking and Ellis [72] concerning the definitions of the scalar curvature R, the Ricci tensor $R_{\mu\nu}$, and the Riemann tensor $R_{\mu\nu\rho\sigma}$. We also extensively use the commutation of covariant derivatives in the form

$$T^{\rho\cdots}{}_{\sigma\ldots;\nu\mu} - T^{\rho\cdots}{}_{\sigma\ldots;\mu\nu} = +R^{\rho}{}_{\tau\mu\nu}T^{\tau\cdots}{}_{\sigma\ldots} + \ldots$$
$$- R^{\tau}{}_{\sigma\mu\nu}T^{\rho\cdots}{}_{\tau\ldots} - \ldots. \quad (2)$$

It is furthermore important to note that all the results we provide in Secs. III and IV are given on irreducible forms: indeed, by using some of the geometrical identities displayed in our recent unpublished report [73], our results have been systematically expanded on the standard bases constructed from group theoretical considerations which have been proposed by Fulling, King, Wybourne, and Cummings (FKWC) in Ref. [74]. A reader who would like to follow or to check our calculations is invited to have in hand these two papers and more particularly Ref. [73] which displays, in addition to a list of useful geometrical identities, the slightly modified version of the FKWC bases we used in the present article.

II. HADAMARD RENORMALIZED STRESS-ENERGY TENSOR: GENERAL CONSIDERATIONS

In this section, we shall describe from a general point of view the renormalization of the stress-energy tensor associated with a massive scalar field theory defined on a general spacetime of arbitrary dimension $D \geq 2$. We shall assume that the scalar field is in a normalized quantum state of Hadamard type and we shall consider that the Wald's axiomatic approach (see, Refs. [3,9,13]) developed in the four-dimensional framework remains valid in the D-dimensional one. We shall in fact extend various considerations previously developed in the four-dimensional framework (see, Refs. [9,13–22]).

A. Some aspects of the classical theory

We begin by reviewing the classical theory of a ''free'' massive scalar field Φ propagating on a D-dimensional curved spacetime $(\mathcal{M}, g_{\mu\nu})$ in order to emphasize some results which shall play a crucial role at the quantum level. We first recall that the associated action is given by

$$S = -\frac{1}{2}\int_{\mathcal{M}} d^D x \sqrt{-g}(g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu} + m^2\Phi^2 + \xi R\Phi^2) \tag{3}$$

where m is the mass of the scalar field and ξ is a dimensionless factor which accounts for the possible coupling between the scalar field and the gravitational background. We furthermore assume that spacetime has no boundary, i.e., that $\partial M = \varnothing$. S is a functional of the scalar field Φ and of the gravitational field $g_{\mu\nu}$, i.e., $S = S[\Phi, g_{\mu\nu}]$. The functional derivative of S with respect to Φ is given by

$$\frac{\delta S}{\delta \Phi} = \sqrt{-g}(\Box - m^2 - \xi R)\Phi \tag{4}$$

and its extremization provides the wave (Klein-Gordon) equation

$$(\Box - m^2 - \xi R)\Phi = 0. \tag{5}$$

The functional derivative of S with respect to $g_{\mu\nu}$ permits us to define the stress-energy tensor $T_{\mu\nu}$ associated with the scalar field Φ (see, for example, Ref. [72]). Indeed, we have

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}} S[\Phi, g_{\mu\nu}] \tag{6}$$

and by using that in the variation

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \tag{7}$$

of the metric tensor we have (see, for example, Ref. [75])

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu} \tag{8a}$$
$$\sqrt{-g} \to \sqrt{-g} + \delta\sqrt{-g} \tag{8b}$$
$$R \to R + \delta R \tag{8c}$$

with

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} \tag{8d}$$

$$\delta\sqrt{-g} = \tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu} \tag{8e}$$

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + (\delta g_{\mu\nu})^{;\mu\nu} - (g^{\mu\nu}\delta g_{\mu\nu})^{;\rho}{}_{;\rho}, \tag{8f}$$

we can explicitly find that

$$T_{\mu\nu} = (1 - 2\xi)\Phi_{;\mu}\Phi_{;\nu} + (2\xi - \tfrac{1}{2})g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma}$$
$$- 2\xi\Phi\Phi_{;\mu\nu} + 2\xi g_{\mu\nu}\Phi\Box\Phi$$
$$+ \xi(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R)\Phi^2 - \tfrac{1}{2}g_{\mu\nu}m^2\Phi^2. \tag{9}$$

It is well-known that the stress-energy tensor is conserved, i.e., it satisfies

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{10}$$

This result could be obtained directly from Eq. (5) by using the expression (9). However, it is more instructive from the physical point of view to derive it from the invariance of

the action (3) under spacetime diffeomorphisms and therefore under the infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \epsilon^\mu \quad \text{with} \quad |\epsilon^\mu| \ll 1. \tag{11}$$

Indeed, under this transformation, the scalar field and the background metric transform as

$$\Phi \to \Phi + \delta\Phi \tag{12a}$$
$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \tag{12b}$$

with

$$\delta\Phi = \mathcal{L}_{-\epsilon}\Phi = -\epsilon^\mu\Phi_{;\mu} \tag{12c}$$

$$\delta g_{\mu\nu} = \mathcal{L}_{-\epsilon}g_{\mu\nu} = -\epsilon_{\mu;\nu} - \epsilon_{\nu;\mu} \tag{12d}$$

where $\mathcal{L}_{-\epsilon}$ denotes the Lie derivative with respect to the vector $-\epsilon$. The invariance of the action (3) leads to

$$\int_{\mathcal{M}} d^D x \left[\left(\frac{\delta S}{\delta\Phi}\right)\delta\Phi + \left(\frac{\delta S}{\delta g_{\mu\nu}}\right)\delta g_{\mu\nu}\right] = 0 \tag{13}$$

which implies

$$T^{\mu\nu}{}_{;\nu} = \Phi^{;\mu}[\Box - m^2 - \xi R]\Phi \tag{14}$$

by using (12). Then, from (5) we obtain immediately (10).

It is also well-known that for

$$m^2 = 0 \quad \text{and} \quad \xi = \xi_c(D) \tag{15}$$

with

$$\xi_c(D) = \frac{1}{4}\left(\frac{D-2}{D-1}\right) \tag{16}$$

the stress-energy tensor is traceless, i.e., it satisfies

$$T^\mu{}_\mu = 0. \tag{17}$$

This result could be obtained directly from Eq. (5) by using the expression (9). In fact, from the physical point of view, it is more instructive to derive it by noting that for the values of the parameters m^2 and ξ given by (15) the scalar field theory is conformally invariant (see, for example, Appendix D of Ref. [76]). As a consequence, the action (3) is invariant under the so-called conformal transformation

$$\Phi \to \hat\Phi = \Omega^{(2-D)/2}\Phi \tag{18a}$$
$$g_{\mu\nu} \to \hat g_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{18b}$$

and therefore under the infinitesimal conformal transformation

$$\Phi \to \hat\Phi = \Phi + \delta\Phi \tag{19a}$$
$$g_{\mu\nu} \to \hat g_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \tag{19b}$$

with

$$\delta\Phi = \frac{2-D}{2}\epsilon\Phi \tag{19c}$$

$$\delta g_{\mu\nu} = 2\epsilon g_{\mu\nu} \qquad (19\text{d})$$

which corresponds to $\Omega = 1 + \epsilon$ with $|\epsilon| \ll 1$. The invariance of the action (3) leads to (13) which now implies

$$T^\mu{}_\mu = \frac{D-2}{2} \Phi[\Box - \xi_c(D)R]\Phi \qquad (20)$$

by using (19). Then, from (5) with (15), we obtain immediately (17).

B. Hadamard quantum states and Feynman propagator

From now on, we shall assume that the scalar field theory previously described has been quantized and that the scalar field Φ is in a normalized quantum state $|\psi\rangle$ of Hadamard type. The associated Feynman propagator

$$G^{\mathrm{F}}(x, x') = i\langle\psi|T\Phi(x)\Phi(x')|\psi\rangle \qquad (21)$$

(here T denotes time ordering) is, by definition, a solution of

$$(\Box_x - m^2 - \xi R)G^{\mathrm{F}}(x, x') = -\delta^D(x, x') \qquad (22)$$

with $\delta^D(x, x') = [-g(x)]^{-1/2}(x)\delta^D(x - x')$. It is symmetric in the exchange of x and x' and its short-distance behavior is of Hadamard type. Its precise form for x' near x depends on whether the dimension D of spacetime is even or odd (see, Refs. [77–79] or the articles by Moretti [66–70] as well as our recent article [71] for more details). It involves the geodetic interval $\sigma(x, x')$ and the biscalar form $\Delta(x, x')$ of the Van Vleck-Morette determinant [80]. Here we recall that $2\sigma(x, x')$ is a biscalar function which is defined as the square of the geodesic distance between x and x' and which satisfies

$$2\sigma = \sigma^{;\mu}\sigma_{;\mu}. \qquad (23)$$

We have $\sigma(x, x') < 0$ if x and x' are timelike related, $\sigma(x, x') = 0$ if x and x' are null related, and $\sigma(x, x') > 0$ if x and x' are spacelike related. We furthermore recall that $\Delta(x, x')$ is given by

$$\Delta(x, x') = -[-g(x)]^{-1/2} \det(-\sigma_{;\mu\nu'}(x, x'))[-g(x')]^{-1/2} \qquad (24)$$

and satisfies the partial differential equation

$$\Box_x \sigma = D - 2\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \qquad (25\text{a})$$

as well as the boundary condition

$$\lim_{x' \to x} \Delta(x, x') = 1. \qquad (25\text{b})$$

For $D = 2$, the Hadamard expansion of the Feynman propagator is given by

$$G^{\mathrm{F}}(x, x') = \frac{i\alpha_2}{2}(V(x, x') \ln[\sigma(x, x') + i\epsilon] + W(x, x')) \qquad (26)$$

where $V(x, x')$ and $W(x, x')$ are symmetric biscalars, regu-

lar for $x' \to x$ and which possess expansions of the form

$$V(x, x') = \sum_{n=0}^{+\infty} V_n(x, x')\sigma^n(x, x'), \qquad (27\text{a})$$

$$W(x, x') = \sum_{n=0}^{+\infty} W_n(x, x')\sigma^n(x, x'). \qquad (27\text{b})$$

For D even with $D \neq 2$, the Hadamard expansion of the Feynman propagator is given by

$$G^{\mathrm{F}}(x, x') = \frac{i\alpha_D}{2}\left(\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{D/2-1}} + V(x, x')\right.$$
$$\left. \times \ln[\sigma(x, x') + i\epsilon] + W(x, x')\right) \qquad (28)$$

where $U(x, x')$, $V(x, x')$, and $W(x, x')$ are symmetric biscalars, regular for $x' \to x$ and which possess expansions of the form

$$U(x, x') = \sum_{n=0}^{D/2-2} U_n(x, x')\sigma^n(x, x'), \qquad (29\text{a})$$

$$V(x, x') = \sum_{n=0}^{+\infty} V_n(x, x')\sigma^n(x, x'), \qquad (29\text{b})$$

$$W(x, x') = \sum_{n=0}^{+\infty} W_n(x, x')\sigma^n(x, x'). \qquad (29\text{c})$$

For D odd, the Hadamard expansion of the Feynman propagator is given by

$$G^{\mathrm{F}}(x, x') = \frac{i\alpha_D}{2}\left(\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{D/2-1}} + W(x, x')\right) \qquad (30)$$

where $U(x, x')$ and $W(x, x')$ are again symmetric and regular biscalar functions which now possess expansions of the form

$$U(x, x') = \sum_{n=0}^{+\infty} U_n(x, x')\sigma^n(x, x'), \qquad (31\text{a})$$

$$W(x, x') = \sum_{n=0}^{+\infty} W_n(x, x')\sigma^n(x, x'). \qquad (31\text{b})$$

In Eqs. (26), (28), and (30), the coefficient α_D is given by

$$\alpha_D = \begin{cases} 1/(2\pi) & \text{for } D = 2, \\ \Gamma(D/2 - 1)/(2\pi)^{D/2} & \text{for } D \neq 2, \end{cases} \qquad (32)$$

while the factor $i\epsilon$ with $\epsilon \to 0_+$ is introduced to give to $G^{\mathrm{F}}(x, x')$ a singularity structure that is consistent with the definition of the Feynman propagator as a time-ordered product [see Eq. (21)].

For $D = 2$, the Hadamard coefficients $V_n(x, x')$ and $W_n(x, x')$ are symmetric and regular biscalar functions.

The coefficients $V_n(x, x')$ satisfy the recursion relations

$$2(n + 1)^2 V_{n+1} + 2(n + 1)V_{n+1;\mu}\sigma^{;\mu}$$
$$- 2(n + 1)V_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (\Box_x - m^2 - \xi R)V_n = 0 \quad \text{for } n \in \mathbb{N} \qquad (33a)$$

with the boundary condition

$$V_0 = -\Delta^{1/2}. \qquad (33b)$$

The coefficients $W_n(x, x')$ satisfy the recursion relations

$$2(n + 1)^2 W_{n+1} + 2(n + 1)W_{n+1;\mu}\sigma^{;\mu}$$
$$- 2(n + 1)W_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ 4(n + 1)V_{n+1} + 2V_{n+1;\mu}\sigma^{;\mu}$$
$$- 2V_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (\Box_x - m^2 - \xi R)W_n = 0 \quad \text{for } n \in \mathbb{N}. \qquad (34)$$

From the recursion relations (33a) and (34), the boundary condition (33b), and the relations (23) and (25) it is possible to prove that $G^F(x, x')$ given by (26) and (27) solves the wave equation (22). This can be done easily by noting that we have

$$(\Box_x - m^2 - \xi R)V = 0 \qquad (35)$$

as a consequence of (33) and

$$\sigma(\Box_x - m^2 - \xi R)W = -2V_{;\mu}\sigma^{;\mu} + 2V\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu} \qquad (36)$$

as a consequence of (33b) and (34).

For D even with $D \neq 2$, the Hadamard coefficients $U_n(x, x')$, $V_n(x, x')$, and $W_n(x, x')$ are symmetric and regular biscalar functions. The coefficients $U_n(x, x')$ satisfy the recursion relations

$$(n + 1)(2n + 4 - D)U_{n+1} + (2n + 4 - D)U_{n+1;\mu}\sigma^{;\mu}$$
$$- (2n + 4 - D)U_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (\Box_x - m^2 - \xi R)U_n = 0$$

for $n = 0, 1, \ldots, D/2 - 3$ $\qquad (37a)$

with the boundary condition

$$U_0 = \Delta^{1/2}. \qquad (37b)$$

The coefficients $V_n(x, x')$ satisfy the recursion relations

$$(n + 1)(2n + D)V_{n+1} + 2(n + 1)V_{n+1;\mu}\sigma^{;\mu}$$
$$- 2(n + 1)V_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (\Box_x - m^2 - \xi R)V_n = 0 \quad \text{for } n \in \mathbb{N} \qquad (38a)$$

with the boundary condition

$$(D - 2)V_0 + 2V_{0;\mu}\sigma^{;\mu} - 2V_0\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (\Box_x - m^2 - \xi R)U_{D/2-2} = 0. \qquad (38b)$$

The coefficients $W_n(x, x')$ satisfy the recursion relations

$$(n + 1)(2n + D)W_{n+1} + 2(n + 1)W_{n+1;\mu}\sigma^{;\mu}$$
$$- 2(n + 1)W_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (4n + 2 + D)V_{n+1} + 2V_{n+1;\mu}\sigma^{;\mu}$$
$$- 2V_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (\Box_x - m^2 - \xi R)W_n = 0 \quad \text{for } n \in \mathbb{N}. \qquad (39)$$

From the recursion relations (37a), (38a), and (39), the boundary conditions (37b) and (38b), and the relations (23) and (25) it is possible to prove that $G^F(x, x')$ given by (28) and (29) solves the wave equation (22). This can be done easily by noting that we have

$$(\Box_x - m^2 - \xi R)V = 0 \qquad (40)$$

as a consequence of (38a) and

$$\sigma(\Box_x - m^2 - \xi R)W = -(\Box_x - m^2 - \xi R)U_{D/2-2}$$
$$- (D - 2)V - 2V_{;\mu}\sigma^{;\mu}$$
$$+ 2V\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu} \qquad (41)$$

as a consequence of (38b) and (39).

For D odd, the Hadamard coefficients $U_n(x, x')$ and $W_n(x, x')$ are symmetric and regular biscalar functions. The coefficients $U_n(x, x')$ satisfy the recursion relations

$$(n + 1)(2n + 4 - D)U_{n+1} + (2n + 4 - D)U_{n+1;\mu}\sigma^{;\mu}$$
$$- (2n + 4 - D)U_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (\Box_x - m^2 - \xi R)U_n = 0 \quad \text{for } n \in \mathbb{N} \qquad (42a)$$

with the boundary condition

$$U_0 = \Delta^{1/2}. \qquad (42b)$$

The coefficients $W_n(x, x')$ satisfy the recursion relations

$$(n + 1)(2n + D)W_{n+1} + 2(n + 1)W_{n+1;\mu}\sigma^{;\mu}$$
$$- 2(n + 1)W_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;\mu}\sigma^{;\mu}$$
$$+ (\Box_x - m^2 - \xi R)W_n = 0 \quad \text{for } n \in \mathbb{N}. \qquad (43)$$

From the recursion relations (42a) and (43), the boundary conditions (42b) and the relations (23) and (25) it is possible to prove that $G^F(x, x')$ given by (30) and (31) solves the wave equation (22). This can be done easily from

$$(\Box_x - m^2 - \xi R)W = 0 \qquad (44)$$

which is a consequence of (43).

For $D = 2$, the Hadamard coefficients $V_n(x, x')$ can be formally obtained by integrating the recursion relations (33a) along the unique geodesic joining x to x' (it is unique for x' near x or more generally for x' in a convex normal neighborhood of x). Similarly, for D even with $D \neq 2$, the Hadamard coefficients $U_n(x, x')$ and $V_n(x, x')$ can be ob-

tained by integrating the recursion relations (37a) and (38a) along the unique geodesic joining x to x' while, for D odd, the Hadamard coefficients $U_n(x, x')$ can be obtained by integrating the recursion relations (42a) along the unique geodesic joining x to x'. As a consequence, all these Hadamard coefficients are determined uniquely and are purely geometrical objects, i.e., they only depend on the geometry along the geodesic joining x to x'. By contrast, the Hadamard coefficients $W_n(x, x')$ with $n \in \mathbb{N}$ are neither uniquely defined nor purely geometrical. Indeed, the first coefficient of this sequence, i.e., $W_0(x, x')$, is unrestrained by the recursion relations (34) for $D = 2$, (39) for D even with $D \neq 2$, and (43) for D odd. As a consequence, this is also true for all the $W_n(x, x')$ with $n \geq 1$. This arbitrariness is in fact very interesting and it can be used to encode the quantum state dependence in the biscalar $W(x, x')$ by specifying the Hadamard coefficient $W_0(x, x')$. Once it has been specified, the recursion relations (34) or (39) or (43) uniquely determine the coefficients $W_n(x, x')$ with $n \geq 1$ and therefore the biscalar $W(x, x')$. In other words, the Hadamard expansions (26)–(31) comprise a purely geometrical part, divergent for $x' \to x$ and given by

$$G^{\mathrm{F}}_{\mathrm{sing}}(x, x') = \frac{i\alpha_2}{2}(V(x, x') \ln[\sigma(x, x') + i\epsilon]) \quad (45)$$

for $D = 2$, by

$$G^{\mathrm{F}}_{\mathrm{sing}}(x, x') = \frac{i\alpha_D}{2}\left(\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{D/2-1}} + V(x, x')\right.$$
$$\left. \times \ln[\sigma(x, x') + i\epsilon]\right) \quad (46)$$

for D even with $D \neq 2$ and by

$$G^{\mathrm{F}}_{\mathrm{sing}}(x, x') = \frac{i\alpha_D}{2}\left(\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{D/2-1}}\right) \quad (47)$$

for D odd as well as a regular state-dependent part given by

$$G^{\mathrm{F}}_{\mathrm{reg}}(x, x') = \frac{i\alpha_D}{2}W(x, x'). \quad (48)$$

It should be noted that, bearing in mind practical applications, it is very interesting to replace the Hadamard coefficients by their covariant Taylor series expansions. Here, we shall provide some associated results which will be helpful afterwards. As far as the geometrical Hadamard coefficients $U_n(x, x')$ and $V_n(x, x')$ which determine the singular part of the Feynman propagator are concerned, they are usually obtained by looking for the solutions of the recursion relations defining them as covariant Taylor series expansions for x' near x given by

$$U_n(x, x') = u_n(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!}u_{n(p)}(x, x') \quad (49a)$$

$$V_n(x, x') = v_n(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!}v_{n(p)}(x, x') \quad (49b)$$

where the $u_{n(p)}(x, x')$ and $v_{n(p)}(x, x')$ with $p = 1, 2, \ldots$ are all biscalars in x and x' which are of the form

$$u_{n(p)}(x, x') = u_{na_1 \ldots a_p}(x)\sigma^{;a_1}(x, x') \ldots \sigma^{;a_p}(x, x') \quad (49c)$$

$$v_{n(p)}(x, x') = v_{na_1 \ldots a_p}(x)\sigma^{;a_1}(x, x') \ldots \sigma^{;a_p}(x, x'). \quad (49d)$$

This method, due to DeWitt [80,81], has been used in the four-dimensional framework to construct the covariant Taylor series expansions of $U_0(x, x')$, $V_0(x, x')$, and $V_1(x, x')$ (see, for example, Ref. [18] and references therein for the scalar field). In Ref. [71], we have recently discussed the construction of the expansions of the geometrical Hadamard coefficients $U_n(x, x')$ and $V_n(x, x')$ of lowest orders in the D-dimensional framework (with $D \geq 3$) and we intend to use these results later. The case $D = 2$ has not been explicitly treated in Ref. [71] but a comparison of Eq. (23) of Ref. [71] with (33) permits us to express the geometrical Hadamard coefficients $V_n(x, x')$ in terms of the mass-dependent DeWitt coefficients $\tilde{A}_n(m^2; x, x')$ [71]. We have $V_n(x, x') = -((-1)^n/(2^n n!))\tilde{A}_n(m^2; x, x')$ and this relation together with the covariant Taylor series expansions of the mass-dependent DeWitt coefficients obtained in Ref. [71] provide the covariant Taylor series expansions of the geometrical Hadamard coefficients $V_n(x, x')$ of lowest orders for $D = 2$.

As far as the biscalar $W(x, x')$ which encodes the state dependence of the Feynman propagator is concerned, its covariant Taylor series expansion is written as

$$W(x, x') = w(x) + \sum_{p=1}^{+\infty} \frac{(-1)^p}{p!}w_{(p)}(x, x') \quad (50a)$$

where the $w_{(p)}(x, x')$ with $p = 1, 2, \ldots$ are all biscalars in x and x' which are of the form

$$w_{(p)}(x, x') = w_{a_1 \ldots a_p}(x)\sigma^{;a_1}(x, x') \ldots \sigma^{;a_p}(x, x'). \quad (50b)$$

The coefficients $w(x)$ and $w_{a_1 \ldots a_p}(x)$ with $p = 1, 2, \ldots$ are constrained by the symmetry of $W(x, x')$ in the exchange of x and x' as well as by the wave equations (36) or (41) or (44) according to the values of D. The symmetry of $W(x, x')$ permits us to express the odd coefficients of the covariant Taylor series expansion of $W(x, x')$ in terms of the even ones. We have for the odd coefficients of lowest orders (see, for example, Refs. [18,19] or Ref. [71])

$$w_{a_1} = (1/2)w_{;a_1} \quad (51a)$$
$$w_{a_1 a_2 a_3} = (3/2)w_{(a_1 a_2;a_3)} - (1/4)w_{;(a_1 a_2 a_3)}. \quad (51b)$$

The wave equation satisfied by $W(x, x')$ for D even permits us to write

$$(\Box_x - m^2 - \xi R)W = -(D + 2)V_1 - 2V_{1;\mu}\sigma^{;\mu} + O(\sigma). \quad (52)$$

This relation is valid for $D = 2$ as well as for D even with $D \neq 2$. It is obtained from (36) or (41) by using (27a) and

(33b) or (29b) and (38b) as well as the following two expansions (see, for example, Refs. [11,12] or Ref. [71])

$$\Delta^{1/2} = 1 + (1/12)R_{a_1 a_2}\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}) \qquad (53)$$

and

$$\sigma_{;\mu\nu} = g_{\mu\nu} - (1/3)R_{\mu a_1 \nu a_2}\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}). \qquad (54)$$

Then, by inserting the expansion of $V_1(x, x')$ given by (49b) and (49d) and by using (54), we have

$$(\Box_x - m^2 - \xi R)W = -(D + 2)v_1 + (D/2)v_{1;\mu}\sigma^{;\mu} \\ + O(\sigma). \qquad (55)$$

By inserting the expansion (50a) and (50b) of $W(x, x')$ up to order $\sigma^{3/2}$ into the left-hand side of (55) and by using (51) as well as (54) we find that

$$w^\rho_{\ \rho} = (m^2 + \xi R)w - (D + 2)v_1 \qquad (56a)$$

$$w^\rho_{\ a;\rho} = (1/4)(\Box w)_{;a} + (1/2)w^\rho_{\ \rho;a} + (1/2)R^\rho_{\ a}w_{;\rho} \\ - (1/2)(m^2 + \xi R)w_{;a} + (D/2)v_{1;a} \qquad (56b)$$

and by combining (56a) and (56b) we establish another relation

$$w^\rho_{\ a;\rho} = (1/4)(\Box w)_{;a} + (1/2)R^\rho_{\ a}w_{;\rho} \\ + (1/2)\xi R_{;a}w - v_{1;a} \qquad (57)$$

which will be helpful in the next subsection. The wave equation (44) satisfied by $W(x, x')$ for D odd can be worked in the same manner. It leads to

$$w^\rho_{\ \rho} = (m^2 + \xi R)w \qquad (58a)$$

$$w^\rho_{\ a;\rho} = (1/4)(\Box w)_{;a} + (1/2)w^\rho_{\ \rho;a} + (1/2)R^\rho_{\ a}w_{;\rho} \\ - (1/2)(m^2 + \xi R)w_{;a} \qquad (58b)$$

and to

$$w^\rho_{\ a;\rho} = (1/4)(\Box w)_{;a} + (1/2)R^\rho_{\ a}w_{;\rho} + (1/2)\xi R_{;a}w. \qquad (59)$$

C. Hadamard renormalization of the stress-energy tensor

The expectation value with respect to the Hadamard quantum state $|\psi\rangle$ of the stress-energy-tensor operator is formally given as the limit

$$\langle\psi|T_{\mu\nu}(x)|\psi\rangle = \lim_{x'\to x}\mathcal{T}_{\mu\nu}(x, x')[-iG^{\mathrm{F}}(x, x')] \qquad (60)$$

where $G^{\mathrm{F}}(x, x')$ is the Feynman propagator (21) which is assumed to possess one of the Hadamard form displayed in the previous subsection. In Eq. (60), $\mathcal{T}_{\mu\nu}(x, x')$ is a differential operator which is constructed by point splitting from the classical expression (9) of the stress tensor. It is a tensor of type (0, 2) in x and a scalar in x'. It is given by

$$\mathcal{T}_{\mu\nu} = (1 - 2\xi)g_\nu^{\ \nu'}\nabla_\mu\nabla_{\nu'} + (2\xi - \tfrac{1}{2})g_{\mu\nu}g^{\rho\sigma'}\nabla_\rho\nabla_{\sigma'} \\ - 2\xi g_\mu^{\ \mu'}g_\nu^{\ \nu'}\nabla_{\mu'}\nabla_{\nu'} + 2\xi g_{\mu\nu}\nabla_\rho\nabla^\rho \\ + \xi(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) - \tfrac{1}{2}g_{\mu\nu}m^2 \qquad (61)$$

where $g_{\mu\nu'}$ denotes the bivector of parallel transport from x to x' (see Refs. [80,81]) which is defined by the partial differential equation

$$g_{\mu\nu';\rho}\sigma^{;\rho} = 0 \qquad (62a)$$

and the boundary condition

$$\lim_{x'\to x}g_{\mu\nu'} = g_{\mu\nu}. \qquad (62b)$$

Of course, because of the short-distance behavior of the Feynman propagator, the expression (60) of the expectation value of the stress-energy-tensor operator in the Hadamard state $|\psi\rangle$ is divergent and therefore meaningless. This pathological behavior comes from the purely geometrical part of the Hadamard expansion given by (45) for $D = 2$ or (46) for D even with $D \neq 2$ or by (47) for D odd. More precisely, for $D = 2$ the terms in $\ln\sigma$ and $\sigma\ln\sigma$ which are present in (45) induce divergences in $1/\sigma$ and $\ln\sigma$ in the expression (60) of $\langle\psi|T_{\mu\nu}|\psi\rangle$. For D even with $D \neq 2$, the terms in $1/\sigma^{D/2-1}, \ldots, 1/\sigma, \ln\sigma$, and $\sigma\ln\sigma$ which are present in (46) induce divergences in $1/\sigma^{D/2}, \ldots, 1/\sigma^2, 1/\sigma$, and $\ln\sigma$ in the expression (60) of $\langle\psi|T_{\mu\nu}|\psi\rangle$ while, for D odd, the terms in $1/\sigma^{D/2-1}, \ldots, 1/\sigma^{1/2}$ and $\sigma^{1/2}$ which are present in (47) induce divergences in $1/\sigma^{D/2}, \ldots, 1/\sigma^{1/2}$ in this expression.

With Wald [3,9,13] it is possible to cure the pathological behavior of $\langle\psi|T_{\mu\nu}|\psi\rangle$ given by (60) and to construct from it a meaningful expression which can act as a source in the semiclassical Einstein equations (1) and which can be considered as the renormalized expectation value with respect to the Hadamard quantum state $|\psi\rangle$ of the stress-energy tensor operator. The Hadamard regularization prescription permits us to accomplish this in the following manner: we first discard in the right-hand side of (60) the purely geometrical part (45) or (46) or (47) of G^F, i.e., we make the replacement

$$\lim_{x'\to x}\mathcal{T}_{\mu\nu}(x, x')[-iG^{\mathrm{F}}(x, x')] \\ \to \frac{\alpha_D}{2}\lim_{x'\to x}\mathcal{T}_{\mu\nu}(x, x')W(x, x'). \qquad (63)$$

We then add to the right-hand side of (63) a state-independent tensor $\tilde{\Theta}_{\mu\nu}$ which only depends on the parameters m^2 and ξ of the theory and on the local geometry and which ensures the conservation of the resulting expression. The renormalized expectation value of the stress-energy tensor operator in the Hadamard state $|\psi\rangle$ is therefore given by

$$\langle\psi|T_{\mu\nu}(x)|\psi\rangle_{\text{ren}} = \frac{\alpha_D}{2} \lim_{x'\to x} \mathcal{T}_{\mu\nu}(x, x')W(x, x') + \tilde{\Theta}_{\mu\nu}(x). \tag{64}$$

Bearing in mind practical applications, it is also interesting to reexpress the previous result in terms of the lowest order coefficients of the covariant Taylor series expansion of the biscalar $W(x, x')$. By inserting (50a) and (50b) into (64) and by using the expansions (54) and (see, for example, Refs. [11,12] or Ref. [71])

$$g_{\nu}{}^{\nu'}\sigma_{;\mu\nu'} = -g_{\mu\nu} - (1/6)R_{\mu a_1 \nu a_2}\sigma^{;a_1}\sigma^{;a_2} + O(\sigma^{3/2}) \tag{65}$$

as well as the relations (see, for example, Refs. [11,12])

$$g_{\mu}{}^{\rho'}g_{\nu\rho'} = g_{\mu\nu} \tag{66a}$$

$$g_{\nu}{}^{\nu'}g_{\mu\nu';\rho} = -(1/2)R_{\mu\nu\rho a}\sigma^{;a} + O(\sigma) \tag{66b}$$

$$g_{\nu}{}^{\nu'}g_{\rho}{}^{\rho'}g_{\mu\nu';\rho'} = -(1/2)R_{\mu\nu\rho a}\sigma^{;a} + O(\sigma) \tag{66c}$$

we obtain

$$\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{\alpha_D}{2}\left[-\left(w_{\mu\nu} - \frac{1}{2}g_{\mu\nu}w^{\rho}{}_{\rho}\right)\right.$$
$$+ \frac{1}{2}(1 - 2\xi)w_{;\mu\nu} + \frac{1}{2}\left(2\xi - \frac{1}{2}\right)g_{\mu\nu}\Box w$$
$$\left. + \xi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)w - \frac{1}{2}g_{\mu\nu}m^2 w\right]$$
$$+ \tilde{\Theta}_{\mu\nu}. \tag{67}$$

Now, by requiring the conservation of $\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{ren}}$ given by (67), we find that $\tilde{\Theta}_{\mu\nu}$ must satisfy

$$[\tilde{\Theta}^{\mu\nu} - (D/4)\alpha_D g^{\mu\nu}v_1]_{;\nu} = 0 \tag{68}$$

when D is even and

$$\tilde{\Theta}^{\mu\nu}{}_{;\nu} = 0 \tag{69}$$

when D is odd. Equations (68) and (69) are derived by using (56a) and (57) for the former and (58a) and (59) for the latter.

It is now possible to provide a definitive expression for the renormalized expectation value of the stress-energy tensor operator in the Hadamard state $|\psi\rangle$. From (64) and by taking into account (68), we have for D even

$$\langle\psi|T_{\mu\nu}(x)|\psi\rangle_{\text{ren}} = \frac{\alpha_D}{2}\left[\lim_{x'\to x}\mathcal{T}_{\mu\nu}(x, x')W(x, x')\right.$$
$$\left. + \frac{D}{2}g_{\mu\nu}v_1\right] + \Theta_{\mu\nu}(x). \tag{70}$$

This result can be also written in the form

$$\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{\alpha_D}{2}\left[-w_{\mu\nu} + \frac{1}{2}(1 - 2\xi)w_{;\mu\nu}\right.$$
$$+ \frac{1}{2}\left(2\xi - \frac{1}{2}\right)g_{\mu\nu}\Box w + \xi R_{\mu\nu}w$$
$$\left. - g_{\mu\nu}v_1\right] + \Theta_{\mu\nu} \tag{71}$$

which is obtained by inserting (56a) into (67) and by taking into account (68). From (64) and by taking into account (69), we have for D odd

$$\langle\psi|T_{\mu\nu}(x)|\psi\rangle_{\text{ren}} = \frac{\alpha_D}{2}\lim_{x'\to x}\mathcal{T}_{\mu\nu}(x, x')W(x, x') + \Theta_{\mu\nu}(x). \tag{72}$$

This result can be also written in the form

$$\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{\alpha_D}{2}\left[-w_{\mu\nu} + \frac{1}{2}(1 - 2\xi)w_{;\mu\nu}\right.$$
$$\left. + \frac{1}{2}\left(2\xi - \frac{1}{2}\right)g_{\mu\nu}\Box w + \xi R_{\mu\nu}w\right] + \Theta_{\mu\nu} \tag{73}$$

which is obtained by inserting (58a) into (67) and by taking into account (69). In Eqs. (70)–(73), the tensor $\Theta_{\mu\nu}$ only depends on the parameters m^2 and ξ of the theory and on the local geometry and it is now conserved, i.e., it satisfies

$$\Theta^{\mu\nu}{}_{;\nu} = 0. \tag{74}$$

To conclude this subsection, we think it is interesting to recall to the reader that the two coefficients $w(x)$ and $w_{\mu\nu}(x)$ which appear in the final expressions (71) and (73) and which encode the state dependence are obtained as Taylor coefficients of the expansion of the biscalar $W(x, x')$ but also more directly by the following two formulas

$$w(x) = \lim_{x'\to x}W(x, x') \tag{75a}$$

$$w_{\mu\nu}(x) = \lim_{x'\to x}W(x, x')_{;\mu\nu} \tag{75b}$$

which can be derived easily from (50a) and (50b) by using (51a) and (54). They are useful to treat practical applications.

D. Ambiguities in the renormalized expectation value of the stress-energy tensor

As we have previously noted, the renormalized expectation value $\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{ren}}$ is unique up to the addition of a local conserved tensor $\Theta_{\mu\nu}$. This problem plagues the Hadamard renormalization procedure since its invention (see, Sec. III of Ref. [13]). It has been recurrently discussed in the four-dimensional context: we refer to the monographs of Fulling [2] and Wald [3] and to references therein as well as to more recent considerations developed in Refs. [70,82–87]. In our opinion, this problem cannot be

solved in the lack of a complete quantum theory of gravity. As a consequence, it induces a serious difficulty with regard to the study of backreaction effects, the right-hand side of the semiclassical Einstein equation (1) being ambiguously defined.

In the present subsection, we shall not consider the ambiguity problem from a general point of view. We shall only discuss the standard ambiguity associated with the choice of a mass scale M—the so-called renormalization mass—introduced in order to make the argument of the logarithm in Eq. (28) dimensionless. We intend to provide a more general (but still incomplete) discussion in Sec. IV. The ambiguity associated with the renormalization mass only exists when the dimension D of spacetime is even. It corresponds to the replacement of the term $V(x, x') \times \ln[\sigma(x, x') + i\epsilon]$ by the term $V(x, x') \ln[M^2(\sigma(x, x') + i\epsilon)]$ and therefore to an indeterminacy in the function $W(x, x')$ previously considered which corresponds to the replacement

$$W(x, x') \rightarrow W(x, x') - V(x, x') \ln M^2 \quad (76)$$

for which the theory developed in Sec. II C remains valid. This indeterminacy is therefore associated with the term

$$\Theta_{\mu\nu}^{M^2}(x) = -\frac{\alpha_D}{2} \lim_{x' \to x} \mathcal{T}_{\mu\nu}(x, x') V(x, x') \ln M^2. \quad (77)$$

By using Eqs. (29b), (49b), and (49d), we can see also that the transformation (76) leads to the replacement

$$w \rightarrow w - v_0 \ln M^2 \quad (78a)$$

$$w_{\mu\nu} \rightarrow w_{\mu\nu} - (v_{0\mu\nu} + g_{\mu\nu} v_1) \ln M^2 \quad (78b)$$

into Eq. (71) and thus we have

$$\Theta_{\mu\nu}^{M^2} = -\frac{\alpha_D}{2}\left[-(v_{0\mu\nu} + g_{\mu\nu} v_1) + \frac{1}{2}(1 - 2\xi) v_{0;\mu\nu} \right.$$
$$\left. + \frac{1}{2}\left(2\xi - \frac{1}{2}\right) g_{\mu\nu} \Box v_0 + \xi R_{\mu\nu} v_0\right] \ln M^2. \quad (79)$$

As a consequence, the knowledge of the first Taylor coefficients of the purely geometrical Hadamard coefficients $V_0(x, x')$ and $V_1(x, x')$ permits us to treat partially the ambiguity problem. It should be finally recalled that the renormalization mass can be fixed by imposing additional physical conditions on the renormalized expectation value of the stress-energy tensor, these conditions being appropriate to the problem treated.

E. Trace anomaly

Here, we shall assume that the renormalized expectation value of the stress-energy tensor $\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{ren}}$ is given by (71) for D even with the geometrical tensor $\Theta_{\mu\nu}$ which reduces to $\Theta_{\mu\nu}^{M^2}$ given by (79) and by (73) for D odd with the geometrical tensor $\Theta_{\mu\nu}$ which vanishes. We neglect all the other possible contributions (see however Sec. IV A for a more general discussion).

By using (56a), we can show that the trace of $\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{ren}}$ is then given by

$$\langle\psi|T^{\mu}{}_{\mu}|\psi\rangle_{\text{ren}} = \frac{\alpha_D}{2}[-m^2 w + (D - 1)(\xi - \xi_c(D))\Box w$$
$$+ 2v_1] + g^{\mu\nu}\Theta_{\mu\nu}^{M^2} \quad (80)$$

for D even and by using (58a) that it reduces to

$$\langle\psi|T^{\mu}{}_{\mu}|\psi\rangle_{\text{ren}} = \frac{\alpha_D}{2}[-m^2 w + (D - 1)(\xi - \xi_c(D))\Box w] \quad (81)$$

for D odd. Furthermore, we have

$$g^{\mu\nu}\Theta_{\mu\nu}^{M^2} = -\frac{\alpha_D}{2}[-m^2 v_0 + (D - 1)(\xi - \xi_c(D))\Box v_0]$$
$$\times \ln M^2 \quad (82)$$

which is obtained from (79) by using $v_0{}^{\rho}{}_{\rho} = -Dv_1 + (m^2 + \xi R)v_0$, this last relation being easily derived from (35) or (40).

For $m^2 = 0$ and $\xi = \xi_c(D)$, i.e,. when the scalar field theory is conformally invariant, the trace $g^{\mu\nu}\Theta_{\mu\nu}^{M^2}$ vanishes and Eq. (80) yields

$$\langle\psi|T^{\mu}{}_{\mu}|\psi\rangle_{\text{ren}} = \alpha_D v_1 \quad (83)$$

for D even. After renormalization, the expectation value of the stress-energy tensor has acquired a nonvanishing or "anomalous" trace even though the classical stress-energy tensor is traceless [see, Eq. (17)]. We refer to the monographs of Birrell and Davies [1], Fulling [2], and Wald [3] as well as to references therein for various discussions and considerations concerning trace anomalies in quantum field theory in curved spacetime. For D odd, $m^2 = 0$, and $\xi = \xi_c(D)$, Eq. (81) yields

$$\langle\psi|T^{\mu}{}_{\mu}|\psi\rangle_{\text{ren}} = 0 \quad (84)$$

and it appears that the trace anomaly does not exist when the dimension of spacetime is odd.

III. HADAMARD RENORMALIZED STRESS-ENERGY TENSOR: EXPLICIT CONSTRUCTION

In this section, we shall mainly discuss the practical aspects of the Hadamard renormalization of the expectation value of the stress-energy tensor. This section is written for the reader who simply wishes to calculate this renormalized expectation value in a particular case and is not specially interested in the derivation of all the previous general results.

We assume that we know the explicit expression of the Feynman propagator $G^{\text{F}}(x, x')$ associated with a given Hadamard quantum state $|\psi\rangle$. We first obtain the state-dependent Hadamard biscalar $W(x, x')$ from the relation

$$W(x, x') = \frac{2}{i\alpha_D}[G^{\text{F}}(x, x') - G^{\text{F}}_{\text{sing}}(x, x')] \quad (85)$$

where $G_{\text{sing}}^{\text{F}}(x, x')$ is given by (45) or (46) or (47) according to the dimension D of spacetime. Of course, we need only the covariant Taylor series expansion of $W(x, x')$ up to order σ and therefore we do not need to know the terms of the expansion of $G_{\text{sing}}^{\text{F}}(x, x')$ which vanish faster than $\sigma(x, x')$ for x' near x. For the same reason, the Feynman propagator $G^{\text{F}}(x, x')$ does not need to be known exactly: we need only its asymptotic expansion for x' near x and we do not need to know the terms of this expansion which vanish faster than $\sigma(x, x')$ for x' near x. From the expansion up to order σ of the biscalar $W(x, x')$ we then obtain the Taylor coefficients $w(x)$ and $w_{\mu\nu}(x)$ either directly or by using the relations (75). This permits us to finally construct the renormalized expectation value in the Hadamard quantum state $|\psi\rangle$ of the stress-energy tensor by using (71) and (79) or (73) according to the parity of D. Of course, for D even, we must in addition construct the geometrical tensor $\Theta_{\mu\nu}^{M^2}$ from the Taylor coefficients v_0, $v_{0\mu\nu}$, and v_1 in order to do this last step.

In the subsections below, we shall provide for spacetime dimension from $D = 2$ to $D = 6$ the explicit expansion of $G_{\text{sing}}^{\text{F}}(x, x')$ and for $D = 2$, 4, and 6 we shall in addition give the explicit expression of the geometrical tensor $\Theta_{\mu\nu}^{M^2}$ as well as of the trace anomaly. We shall use some of the results we obtained in Ref. [71]. We have simplified them from the geometrical identities displayed in our unpublished report [73]. These geometrical identities are helpful to expand the Riemann polynomials encountered in our calculations on the FKWC bases constructed from group theoretical considerations in Ref. [74]. They have permitted us to provide irreducible expressions for all our results. For spacetime dimension from 7 to 11, we shall describe the method permitting the interested reader to construct explicitly $G_{\text{sing}}^{\text{F}}(x, x')$ (as well as $\Theta_{\mu\nu}^{M^2}$ when it is necessary) in a given spacetime by using the results obtained in Ref. [71].

A. $D = 2$

For $D = 2$, the expansion of the singular part

$$G_{\text{sing}}^{\text{F}}(x, x') = \frac{i}{4\pi}(V(x, x') \ln[\sigma(x, x') + i\epsilon]) \quad (86)$$

of the Feynman propagator is obtained, up the required order, for

$$V = V_0 + V_1\sigma + O(\sigma^{3/2}) \quad (87)$$

with

$$V_0 = v_0 - v_{0a}\sigma^{;a} + \frac{1}{2!}v_{0ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}) \quad (88)$$

$$V_1 = v_1 + O(\sigma^{1/2}). \quad (89)$$

The Taylor coefficients appearing in Eqs. (88) and (89) are given by

$$v_0 = -1 \quad (90a)$$
$$v_{0a} = 0 \quad (90b)$$
$$v_{0ab} = -(1/12)Rg_{ab} \quad (90c)$$

and

$$v_1 = -(1/2)m^2 - (1/2)(\xi - 1/6)R. \quad (91)$$

The geometrical tensor $\Theta_{\mu\nu}^{M^2}$ which is associated with the renormalization mass is obtained from (79) by using (90a), (90c), and (91) and is given by

$$\Theta_{\mu\nu}^{M^2} = \frac{\ln M^2}{4\pi}[-(1/2)m^2 g_{\mu\nu}]. \quad (92)$$

The trace anomaly (83) is obtained by using $m^2 = 0$ and $\xi = \xi_c(2) = 0$ into (91). It reduces to [1,88]

$$\langle\psi|T^\mu{}_\mu|\psi\rangle_{\text{ren}} = \frac{R}{24\pi}. \quad (93)$$

B. $D = 3$

For $D = 3$, the expansion of the singular part

$$G_{\text{sing}}^{\text{F}}(x, x') = \frac{i}{4\sqrt{2}\pi}\left(\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{1/2}}\right) \quad (94)$$

of the Feynman propagator is obtained, up the required order, for

$$U = U_0 + U_1\sigma + O(\sigma^2) \quad (95)$$

with

$$U_0 = u_0 - u_{0a}\sigma^{;a} + \frac{1}{2!}u_{0ab}\sigma^{;a}\sigma^{;b}$$
$$- \frac{1}{3!}u_{0abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} + O(\sigma^2) \quad (96)$$

$$U_1 = u_1 - u_{1a}\sigma^{;a} + O(\sigma). \quad (97)$$

The Taylor coefficients appearing in Eqs. (96) and (97) are given by

$$u_0 = 1 \quad (98a)$$
$$u_{0a} = 0 \quad (98b)$$
$$u_{0ab} = (1/6)R_{ab} \quad (98c)$$
$$u_{0abc} = (1/4)R_{(ab;c)} \quad (98d)$$

and

$$u_1 = m^2 + (\xi - 1/6)R \quad (99)$$
$$u_{1a} = (1/2)(\xi - 1/6)R_{;a}. \quad (100)$$

C. $D = 4$

For $D = 4$, the expansion of the singular part

$$G_{\text{sing}}^{\text{F}}(x, x') = \frac{i}{8\pi^2}\left(\frac{U(x, x')}{\sigma(x, x') + i\epsilon}\right.$$
$$\left. + V(x, x') \ln[\sigma(x, x') + i\epsilon]\right) \quad (101)$$

of the Feynman propagator is obtained, up the required order, for

$$U = U_0 \tag{102}$$

$$V = V_0 + V_1\sigma + O(\sigma^{3/2}) \tag{103}$$

with

$$U_0 = u_0 - u_{0a}\sigma^{;a} + \frac{1}{2!}u_{0ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}u_{0abc}\sigma^{;a}\sigma^{;b}\sigma^{;c}$$
$$+ \frac{1}{4!}u_{0abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O(\sigma^{5/2}) \tag{104}$$

$$V_0 = v_0 - v_{0a}\sigma^{;a} + \frac{1}{2!}v_{0ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}) \tag{105}$$

$$V_1 = v_1 + O(\sigma^{1/2}). \tag{106}$$

The Taylor coefficients appearing in Eqs. (104)–(106) are given by

$$u_0 = 1 \tag{107a}$$

$$u_{0a} = 0 \tag{107b}$$

$$u_{0ab} = (1/6)R_{ab} \tag{107c}$$

$$u_{0abc} = (1/4)R_{(ab;c)} \tag{107d}$$

$$u_{0abcd} = (3/10)R_{(ab;cd)} + (1/12)R_{(ab}R_{cd)}$$
$$+ (1/15)R_{p(a|q|b}R^p{}_c{}^q{}_{d)} \tag{107e}$$

and

$$v_0 = (1/2)m^2 + (1/2)(\xi - 1/6)R \tag{108a}$$

$$v_{0a} = (1/4)(\xi - 1/6)R_{;a} \tag{108b}$$

$$v_{0ab} = (1/12)m^2R_{ab} + (1/6)(\xi - 3/20)R_{;ab}$$
$$- (1/120)\Box R_{ab} + (1/12)(\xi - 1/6)RR_{ab}$$
$$+ (1/90)R^p{}_aR_{pb} - (1/180)R^{pq}R_{paqb}$$
$$- (1/180)R^{pqr}{}_aR_{pqrb} \tag{108c}$$

and

$$v_1 = (1/8)m^4 + (1/4)(\xi - 1/6)m^2R - (1/24)(\xi - 1/5)\Box R$$
$$+ (1/8)(\xi - 1/6)^2R^2 - (1/720)R_{pq}R^{pq}$$
$$+ (1/720)R_{pqrs}R^{pqrs}. \tag{109}$$

The geometrical tensor $\Theta_{\mu\nu}^{M^2}$ which is associated with the renormalization mass is obtained from (79) by using (108a), (108c), and (109) and is given by

$$\Theta_{\mu\nu}^{M^2} = \frac{\ln M^2}{2(2\pi)^2}[-(1/2)(\xi - 1/6)m^2R_{\mu\nu} + (1/2)[\xi^2 - (1/3)\xi + 1/30]R_{;\mu\nu} - (1/120)\Box R_{\mu\nu} - (1/2)(\xi - 1/6)^2RR_{\mu\nu}$$
$$+ (1/90)R^p{}_\mu R_{p\nu} - (1/180)R^{pq}R_{p\mu q\nu} - (1/180)R^{pqr}{}_\mu R_{pqr\nu} + g_{\mu\nu}((1/8)m^4 + (1/4)(\xi - 1/6)m^2R$$
$$- (1/2)[\xi^2 - (1/3)\xi + 1/40]\Box R + (1/8)(\xi - 1/6)^2R^2 - (1/720)R_{pq}R^{pq} + (1/720)R_{pqrs}R^{pqrs})]. \tag{110}$$

The trace anomaly (83) is obtained by using $m^2 = 0$ and $\xi = \xi_c(4) = 1/6$ into (109). It reduces to [1,88]

$$\langle\psi|T^\mu{}_\mu|\psi\rangle_{\text{ren}} = \frac{1}{(2\pi)^2}[(1/720)\Box R - (1/720)R_{pq}R^{pq}$$
$$+ (1/720)R_{pqrs}R^{pqrs}]. \tag{111}$$

D. $D = 5$

For $D = 5$, the expansion of the singular part

$$G_{\text{sing}}^F(x, x') = \frac{i}{16\sqrt{2}\pi^2}\left(\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^{3/2}}\right) \tag{112}$$

of the Feynman propagator is obtained, up the required order, for

$$U = U_0 + U_1\sigma + U_2\sigma^2 + O(\sigma^3) \tag{113}$$

with

$$U_0 = u_0 - u_{0a}\sigma^{;a} + \frac{1}{2!}u_{0ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}u_{0abc}\sigma^{;a}\sigma^{;b}\sigma^{;c}$$
$$+ \frac{1}{4!}u_{0abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}$$
$$- \frac{1}{5!}u_{0abcde}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e} + O(\sigma^3) \tag{114}$$

$$U_1 = u_1 - u_{1a}\sigma^{;a} + \frac{1}{2!}u_{1ab}\sigma^{;a}\sigma^{;b}$$
$$- \frac{1}{3!}u_{1abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} + O(\sigma^2) \tag{115}$$

$$U_2 = u_2 - u_{2a}\sigma^{;a} + O(\sigma). \tag{116}$$

The Taylor coefficients appearing in Eqs. (114)–(116) are given by

$$u_0 = 1 \tag{117a}$$

$$u_{0a} = 0 \tag{117b}$$

$$u_{0ab} = (1/6)R_{ab} \tag{117c}$$

$$u_{0abc} = (1/4)R_{(ab;c)} \tag{117d}$$

$$u_{0abcd} = (3/10)R_{(ab;cd)} + (1/12)R_{(ab}R_{cd)} + (1/15)R_{p(a|q|b}R^p{}_c{}^q{}_{d)} \tag{117e}$$

$$u_{0abcde} = (1/3)R_{(ab;cde)} + (5/12)R_{(ab}R_{cd;e)} + (1/3)R_{p(a|q|b}R^p{}_c{}^q{}_{d;e)} \tag{117f}$$

and

$$u_1 = -m^2 - (\xi - 1/6)R \tag{118a}$$

$$u_{1a} = -(1/2)(\xi - 1/6)R_{;a} \tag{118b}$$

$$u_{1ab} = -(1/6)m^2R_{ab} - (1/3)(\xi - 3/20)R_{;ab} + (1/60)\Box R_{ab} - (1/6)(\xi - 1/6)RR_{ab} - (1/45)R^p{}_aR_{pb}$$
$$+ (1/90)R^{pq}R_{paqb} + (1/90)R^{pqr}{}_aR_{pqrb} \tag{118c}$$

$$u_{1abc} = -(1/4)m^2R_{(ab;c)} - (1/4)(\xi - 2/15)R_{;(abc)} + (1/40)(\Box R_{(ab)};c) - (1/4)(\xi - 1/6)R_{;(a}R_{bc)} - (1/4)(\xi - 1/6)RR_{(ab;c)}$$
$$- (1/15)R_{p(a}R^p{}_{b;c)} + (1/60)R_{pq;(a}R^p{}_b{}^q{}_{c)} + (1/60)R_{pq}R^p{}_{(a}{}^q{}_{b;c)} + (1/30)R_{pqr(a}R^{pqr}{}_{b;c)} \tag{118d}$$

and

$$u_2 = -(1/2)m^4 - (\xi - 1/6)m^2R + (1/6)(\xi - 1/5)\Box R - (1/2)(\xi - 1/6)^2R^2 + (1/180)R_{pq}R^{pq}$$
$$- (1/180)R_{pqrs}R^{pqrs} \tag{119a}$$

$$u_{2a} = -(1/2)(\xi - 1/6)m^2R_{;a} + (1/12)(\xi - 1/5)(\Box R)_{;a} - (1/2)(\xi - 1/6)^2RR_{;a} + (1/180)R_{pq}R^{pq}{}_{;a}$$
$$- (1/180)R_{pqrs}R^{pqrs}{}_{;a}. \tag{119b}$$

E. $D = 6$

For $D = 6$, the expansion of the singular part

$$G_{\text{sing}}^F(x, x') = \frac{i}{16\pi^3}\left(\frac{U(x, x')}{[\sigma(x, x') + i\epsilon]^2}\right.$$
$$\left. + V(x, x')\ln[\sigma(x, x') + i\epsilon]\right) \tag{120}$$

of the Feynman propagator is obtained, up the required order, for

$$U = U_0 + U_1\sigma \tag{121}$$

$$V = V_0 + V_1\sigma + O(\sigma^{3/2}) \tag{122}$$

with

$$U_0 = u_0 - u_{0a}\sigma^{;a} + \frac{1}{2!}u_{0ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}u_{0abc}\sigma^{;a}\sigma^{;b}\sigma^{;c}$$
$$+ \frac{1}{4!}u_{0abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}$$
$$- \frac{1}{5!}u_{0abcde}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e}$$
$$+ \frac{1}{6!}u_{0abcdef}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e}\sigma^{;f} + O(\sigma^{7/2}) \tag{123}$$

$$U_1 = u_1 - u_{1a}\sigma^{;a} + \frac{1}{2!}u_{1ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}u_{1abc}\sigma^{;a}\sigma^{;b}\sigma^{;c}$$
$$+ \frac{1}{4!}u_{1abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O(\sigma^{5/2}) \tag{124}$$

$$V_0 = v_0 - v_{0a}\sigma^{;a} + \frac{1}{2!}v_{0ab}\sigma^{;a}\sigma^{;b} + O(\sigma^{3/2}) \tag{125}$$

$$V_1 = v_1 + O(\sigma^{1/2}). \tag{126}$$

The Taylor coefficients appearing in Eqs. (123)–(126) are given by

$$u_0 = 1 \tag{127a}$$

$$u_{0a} = 0 \tag{127b}$$

$$u_{0ab} = (1/6)R_{ab} \tag{127c}$$

and

$$u_{0abc} = (1/4)R_{(ab;c)} \tag{127d}$$

$$u_{0abcd} = (3/10)R_{(ab;cd)} + (1/12)R_{(ab}R_{cd)} + (1/15)R_{p(a|q|b}R^p{}_c{}^q{}_{d)} \tag{127e}$$

$$u_{0abcde} = (1/3)R_{(ab;cde)} + (5/12)R_{(ab}R_{cd;e)} + (1/3)R_{p(a|q|b}R^p{}_c{}^q{}_{d;e)} \tag{127f}$$

$$u_{0abcdef} = (5/14)R_{(ab;cdef)} + (3/4)R_{(ab}R_{cd;ef)} + (4/7)R_{p(a|q|b}R^p{}_c{}^q{}_{d;ef)} + (5/8)R_{(ab;c}R_{de;f)} + (15/28)R_{p(a|q|b;c}R^p{}_d{}^q{}_{e;f)}$$
$$+ (5/72)R_{(ab}R_{cd}R_{ef)} + (1/6)R_{(ab}R^p{}_c{}^q{}_d R_{|p|e|q|f)} + (8/63)R^p{}_{(a|q|b}R^q{}_{c|r|d}R^r{}_{e|p|f)} \tag{127g}$$

and

$$u_1 = -(1/2)m^2 - (1/2)(\xi - 1/6)R \tag{128a}$$
$$u_{1a} = -(1/4)(\xi - 1/6)R_{;a} \tag{128b}$$
$$u_{1ab} = -(1/12)m^2 R_{ab} - (1/6)(\xi - 3/20)R_{;ab} + (1/120)\Box R_{ab} - (1/12)(\xi - 1/6)RR_{ab} - (1/90)R^p{}_a R_{pb}$$
$$+ (1/180)R^{pq}R_{paqb} + (1/180)R^{pqr}{}_a R_{pqrb} \tag{128c}$$
$$u_{1abc} = -(1/8)m^2 R_{(ab;c)} - (1/8)(\xi - 2/15)R_{;(abc)} + (1/80)(\Box R_{(ab)};c) - (1/8)(\xi - 1/6)R_{;(a}R_{bc)}$$
$$- (1/8)(\xi - 1/6)RR_{(ab;c)} - (1/30)R_{p(a}R^p{}_{b;c)} + (1/120)R_{pq;(a}R^p{}_b{}^q{}_c) + (1/120)R_{pq}R^p{}_{(a}{}^q{}_{b;c)}$$
$$+ (1/60)R_{pqr(a}R^{pqr}{}_{b;c)} \tag{128d}$$
$$u_{1abcd} = -(3/20)m^2 R_{(ab;cd)} - (1/24)m^2 R_{(ab}R_{cd)} - (1/30)m^2 R_{p(a|q|b}R^p{}_c{}^q{}_d) - (1/10)(\xi - 5/42)R_{;(abcd)}$$
$$+ (1/70)(\Box R_{(ab)};cd) - (1/6)(\xi - 3/20)R_{;(ab}R_{cd)} - (3/20)(\xi - 1/6)RR_{(ab;cd)} + (1/120)R_{(ab}\Box R_{cd)}$$
$$- (3/70)R^p{}_{(a}R_{|p|b;cd)} + (1/210)R^p{}_{(a}R_{bc;d)p} + (1/70)R_{pq;(ab}R^p{}_c{}^q{}_d) - (2/105)R_{p(a;b|q|}R^p{}_c{}^q{}_d)$$
$$+ (1/70)R_{(ab}{}^{;pq}R_{|p|c|q|d)} + (1/105)R_{pq}R^p{}_{(a}{}^q{}_{b;cd)} + (2/105)R^{pqr}{}_{(a}R_{|pqr|b;cd)} - (1/4)(\xi - 1/6)R_{;(a}R_{bc;d)}$$
$$- (11/420)R^p{}_{(a;b}R_{|p|c;d)} - (3/140)R^p{}_{(a;b}R_{cd);p} + (17/1680)R_{(ab}{}^{;p}R_{cd);p} + (1/60)R_{pq;(a}R^p{}_b{}^q{}_{c;d)}$$
$$+ (1/210)R_{p(a;|q|}R^p{}_b{}^q{}_{c;d)} + (1/56)R^{pqr}{}_{(a;b}R_{|pqr|c;d)} + (1/280)R^p{}_{(a}{}^q{}_b{}^{;r}R_{|p|c|q|d);r} - (1/24)(\xi - 1/6)RR_{(ab}R_{cd)}$$
$$- (1/90)R_{(ab}R^p{}_c R_{|p|d)} + (1/180)R_{(ab}R^{pq}R_{|p|c|q|d)} + (1/90)R^p{}_{(a}R^q{}_b R_{|p|c|q|d)} - (1/30)(\xi - 1/6)RR^p{}_{(a}{}^q{}_b R_{|p|c|q|d)}$$
$$+ (1/180)R_{(ab}R^{pqr}{}_c R_{|pqr|d)} + (1/315)R_{p(a}R^{rps}{}_b R_{|r|c|s|d)} - (1/315)R_{pq}R^r{}_{(a}{}^p{}_b R_{|r|c}{}^q{}_d)$$
$$- (2/315)R_{prqs}R^p{}_{(a}{}^q{}_b R^r{}_c{}^s{}_d) - (1/315)R^p{}_{(a}{}^q{}_b R^{rs}{}_{|p|c}R_{|rsq|d)} + (4/315)R^p{}_{(a}{}^q{}_b R_{|p|}{}^{rs}{}_c R_{|qrs|d)} \tag{128e}$$

and

$$v_0 = -(1/8)m^4 - (1/4)(\xi - 1/6)m^2 R + (1/24)(\xi - 1/5)\Box R - (1/8)(\xi - 1/6)^2 R^2 + (1/720)R_{pq}R^{pq}$$
$$- (1/720)R_{pqrs}R^{pqrs} \tag{129a}$$
$$v_{0a} = -(1/8)(\xi - 1/6)m^2 R_{;a} + (1/48)(\xi - 1/5)(\Box R)_{;a} - (1/8)(\xi - 1/6)^2 RR_{;a} + (1/720)R_{pq}R^{pq}{}_{;a}$$
$$- (1/720)R_{pqrs}R^{pqrs}{}_{;a} \tag{129b}$$

and

$$v_{0ab} = -(1/48)m^4 R_{ab} - (1/12)(\xi - 3/20)m^2 R_{;ab} + (1/240)m^2 \Box R_{ab} - (1/24)(\xi - 1/6)m^2 R R_{ab} - (1/180)m^2 R_{pa}R^p{}_b$$
$$+ (1/360)m^2 R^{pq}R_{paqb} + (1/360)m^2 R^{pqr}{}_a R_{pqrb} + (1/80)(\xi - 4/21)(\Box R)_{;ab} - (1/3360)\Box\Box R_{ab}$$
$$- (1/12)(\xi - 1/6)(\xi - 3/20)R R_{;ab} + (1/144)(\xi - 1/5)(\Box R)R_{ab} + (1/360)(\xi - 1/7)R_{;p(a}R^p{}_{b)}$$
$$+ (1/240)(\xi - 1/6)R\Box R_{ab} + (1/1008)R_{p(a}\Box R^p{}_{b)} + (1/1680)R^{pq}R_{pq;(ab)} + (1/1260)R^{pq}R_{p(a;b)q}$$
$$- (1/1680)R^{pq}R_{ab;pq} + (1/180)(\xi - 3/14)R^{;pq}R_{paqb} - (1/2520)(\Box R^{pq})R_{paqb} + (1/630)R^{pq;r}{}_{(a}R_{|rqp|b)}$$
$$+ (1/420)R^p{}_{(a}{}^{;qr}R_{|pqr|b)} - (1/1260)R^{pqrs}R_{pqrs;(ab)} - (1/16)(\xi - 1/6)^2 R_{;a}R_{;b} - (1/120)(\xi - 3/14)R_{;p}R^p{}_{(a;b)}$$
$$+ (1/120)(\xi - 17/84)R_{;p}R_{ab}{}^{;p} + (1/1440)R^{pq}{}_{;a}R_{pq;b} - (1/5040)R^p{}_{a;q}R_{pb}{}^{;q} + (1/1008)R^p{}_{a;q}R^q{}_{b;p}$$
$$- (1/2520)R^{pq;r}R_{rqp(a;b)} - (1/1680)R^{pq;r}R_{paqb;r} - (1/1344)R^{pqrs}{}_{;a}R_{pqrs;b} - (1/1680)R^{pqr}{}_{a;s}R_{pqrb}{}^{;s}$$
$$- (1/48)(\xi - 1/6)^2 R^2 R_{ab} - (1/180)(\xi - 1/6)R R_{pa}R^p{}_b + (1/4320)R^{pq}R_{pq}R_{ab} - (1/3780)R^{pq}R_{pa}R_{qb}$$
$$+ (1/360)(\xi - 1/6)R R^{pq}R_{paqb} + (1/7560)R^{pr}R^q{}_r R_{paqb} + (1/7560)R^{pq}R^r{}_{(a}R_{|rqp|b)}$$
$$+ (1/360)(\xi - 1/6)R R^{pqr}{}_a R_{pqrb} - (1/4320)R_{ab}R^{pqrs}R_{pqrs} - (1/1890)R^p{}_{(a}R^{qrs}{}_{|p}R_{qrs|b)}$$
$$- (1/3780)R^{pq}R^{rs}{}_{pa}R_{rsqb} + (1/1890)R_{pq}R^{prqs}R_{rasb} - (1/7560)R_{pq}R^{prs}{}_a R^q{}_{rsb}$$
$$+ (1/3780)R^{pqrs}R_{pqta}R_{rs}{}^t{}_b + (1/378)R^{prqs}R^t{}_{pqa}R_{trsb} - (1/3780)R^{pqr}{}_s R_{pqrt}R^s{}_a{}^t{}_b \tag{129c}$$

and

$$v_1 = -(1/48)m^6 - (1/16)(\xi - 1/6)m^4 R + (1/48)(\xi - 1/5)m^2 \Box R - (1/16)(\xi - 1/6)^2 m^2 R^2 + (1/1440)m^2 R_{pq}R^{pq}$$
$$- (1/1440)m^2 R_{pqrs}R^{pqrs} - (1/480)(\xi - 3/14)\Box\Box R + (1/48)(\xi - 1/6)(\xi - 1/5)R\Box R$$
$$- (1/720)(\xi - 3/14)R_{;pq}R^{pq} - (1/5040)R_{pq}\Box R^{pq} + (1/840)R_{pq;rs}R^{prqs} + (1/96)[\xi^2 - (2/5)\xi + 17/420]R_{;p}R^{;p}$$
$$- (1/20160)R_{pq;r}R^{pq;r} - (1/10080)R_{pq;r}R^{pr;q} + (1/4480)R_{pqrs;t}R^{pqrs;t} - (1/48)(\xi - 1/6)^3 R^3$$
$$+ (1/1440)(\xi - 1/6)R R_{pq}R^{pq} + (1/45360)R_{pq}R^p{}_r R^{qr} - (1/15120)R_{pq}R_{rs}R^{prqs} - (1/1440)(\xi - 1/6)R R_{pqrs}R^{pqrs}$$
$$+ (1/2160)R_{pq}R^p{}_{rst}R^{qrst} - (1/5670)R_{pqrs}R^{pquv}R^{rs}{}_{uv} - (11/11340)R_{prqs}R^p{}_u{}^q{}_v R^{rusv}. \tag{130}$$

The geometrical tensor $\Theta^{M^2}_{\mu\nu}$ which is associated with the renormalization mass is obtained from (79) by using (129a), (129c), and (130) and is given by

$$\Theta^{M^2}_{\mu\nu} = \frac{\ln M^2}{2(2\pi)^3}[(1/8)(\xi - 1/6)m^4 R_{\mu\nu} - (1/4)[\xi^2 - (1/3)\xi + 1/30]m^2 R_{;\mu\nu} + (1/240)m^2 \Box R_{\mu\nu}$$
$$+ (1/4)(\xi - 1/6)^2 m^2 R R_{\mu\nu} - (1/180)m^2 R_{p\mu}R^p{}_\nu + (1/360)m^2 R^{pq}R_{p\mu q\nu} + (1/360)m^2 R^{pqr}{}_\mu R_{pqr\nu}$$
$$+ (1/24)[\xi^2 - (2/5)\xi + 3/70](\Box R)_{;\mu\nu} - (1/3360)\Box\Box R_{\mu\nu} - (1/4)(\xi - 1/6)[\xi^2 - (1/3)\xi + 1/30]R R_{;\mu\nu}$$
$$- (1/24)(\xi - 1/6)(\xi - 1/5)(\Box R)R_{\mu\nu} + (1/360)(\xi - 1/7)R_{;p(\mu}R^p{}_{\nu)} + (1/240)(\xi - 1/6)R\Box R_{\mu\nu}$$
$$+ (1/1008)R_{p(\mu}\Box R^p{}_{\nu)} + (1/360)(\xi - 2/7)R^{pq}R_{pq;(\mu\nu)} + (1/1260)R^{pq}R_{p(\mu;\nu)q} - (1/1680)R^{pq}R_{\mu\nu;pq}$$
$$+ (1/180)(\xi - 3/14)R^{;pq}R_{p\mu q\nu} - (1/2520)(\Box R^{pq})R_{p\mu q\nu} + (1/630)R^{pq;r}{}_{(\mu}R_{|rqp|\nu)} + (1/420)R^p{}_{(\mu}{}^{;qr}R_{|pqr|\nu)}$$
$$- (1/360)(\xi - 3/14)R^{pqrs}R_{pqrs;(\mu\nu)} - (1/4)(\xi - 1/6)^2(\xi - 1/4)R_{;\mu}R_{;\nu} - (1/120)(\xi - 3/14)R_{;p}R^p{}_{(\mu;\nu)}$$
$$+ (1/120)(\xi - 17/84)R_{;p}R_{\mu\nu}{}^{;p} + (1/360)(\xi - 1/4)R^{pq}{}_{;\mu}R_{pq;\nu} - (1/5040)R^p{}_{\mu;q}R_{p\nu}{}^{;q} + (1/1008)R^p{}_{\mu;q}R^q{}_{\nu;p}$$
$$- (1/2520)R^{pq;r}R_{rqp(\mu;\nu)} - (1/1680)R^{pq;r}R_{p\mu q\nu;r} - (1/360)(\xi - 13/56)R^{pqrs}{}_{;\mu}R_{pqrs;\nu} - (1/1680)R^{pqr}{}_{\mu;s}R_{pqr\nu}{}^{;s}$$
$$+ (1/8)(\xi - 1/6)^3 R^2 R_{\mu\nu} - (1/180)(\xi - 1/6)R R_{p\mu}R^p{}_\nu - (1/720)(\xi - 1/6)R^{pq}R_{pq}R_{\mu\nu} - (1/3780)R^{pq}R_{p\mu}R_{q\nu}$$

$$+ (1/360)(\xi - 1/6)RR^{pq}R_{p\mu q\nu} + (1/7560)R^{pr}R^q{}_r R_{p\mu q\nu} + (1/7560)R^{pq}R^r{}_{(\mu}R_{|rqp|\nu)} + (1/360)(\xi - 1/6)RR^{pqr}{}_\mu R_{pqr\nu}$$

$$+ (1/720)(\xi - 1/6)R_{\mu\nu}R^{pqrs}R_{pqrs} - (1/1890)R^p{}_{(\mu}R^{qrs}{}_{|p}R_{qrs|\nu)} - (1/3780)R^{pq}R^{rs}{}_{p\mu}R_{rsq\nu} + (1/1890)R_{pq}R^{pqrs}R_{r\mu s\nu}$$

$$- (1/7560)R_{pq}R^{prs}{}_\mu R^q{}_{rs\nu} + (1/3780)R^{pqrs}R_{pqt\mu}R_{rs}{}^t{}_\nu + (1/378)R^{prqs}R^t{}_{pq\mu}R_{trs\nu} - (1/3780)R^{pqr}{}_s R_{pqrt}R^s{}_\mu{}^t{}_\nu$$

$$+ g_{\mu\nu}(-(1/48)m^6 - (1/16)(\xi - 1/6)m^4 R + (1/4)[\xi^2 - (1/3)\xi + 1/40]m^2 \Box R - (1/16)(\xi - 1/6)^2 m^2 R^2$$

$$+ (1/1440)m^2 R_{pq}R^{pq} - (1/1440)m^2 R_{pqrs}R^{pqrs} - (1/24)[\xi^2 - (2/5)\xi + 11/280]\Box\Box R + (1/4)(\xi - 1/6)$$

$$\times [\xi^2 - (1/3)\xi + 1/40]R\Box R - (1/720)(\xi - 3/14)R_{;pq}R^{pq} - (1/360)(\xi - 5/28)R_{pq}\Box R^{pq}$$

$$+ (1/90)(\xi - 1/7)R_{pq;rs}R^{pqrs} + (1/4)[\xi^3 - (13/24)\xi^2 + (17/180)\xi - 53/10080]R_{;p}R^{;p}$$

$$- (1/360)(\xi - 13/56)R_{pq;r}R^{pq;r} - (1/10080)R_{pq;r}R^{pr;q} + (1/360)(\xi - 19/112)R_{pqrs;t}R^{pqrs;t} - (1/48)(\xi - 1/6)^3 R^3$$

$$+ (1/1440)(\xi - 1/6)RR_{pq}R^{pq} + (1/45360)R_{pq}R^p{}_r R^{qr} - (1/15120)R_{pq}R_{rs}R^{prqs} - (1/1440)(\xi - 1/6)RR_{pqrs}R^{pqrs}$$

$$+ (1/180)(\xi - 1/6)R_{pq}R^p{}_{rst}R^{qrst} - (1/360)(\xi - 47/252)R_{pqrs}R^{pquv}R^{rs}{}_{uv} - (1/90)(\xi - 41/252)R_{prqs}R^p{}_u{}^q{}_v R^{rusv})].$$

$$(131)$$

The trace anomaly (83) is obtained by using $m^2 = 0$ and $\xi = \xi_c(6) = 1/5$ into (130). It reduces to

$$\langle\psi|T^\mu{}_\mu|\psi\rangle_{\text{ren}} = \frac{1}{(2\pi)^3}[(1/33600)\Box\Box R + (1/50400)R_{;pq}R^{pq} - (1/5040)R_{pq}\Box R^{pq} + (1/840)R_{pq;rs}R^{prqs}$$

$$+ (1/201600)R_{;p}R^{;p} - (1/20160)R_{pq;r}R^{pq;r} - (1/10080)R_{pq;r}R^{pr;q} + (1/4480)R_{pqrs;t}R^{pqrs;t}$$

$$- (1/1296000)R^3 + (1/43200)RR_{pq}R^{pq} + (1/45360)R_{pq}R^p{}_r R^{qr} - (1/15120)R_{pq}R_{rs}R^{prqs}$$

$$- (1/43200)RR_{pqrs}R^{pqrs} + (1/2160)R_{pq}R^p{}_{rst}R^{qrst} - (1/5670)R_{pqrs}R^{pquv}R^{rs}{}_{uv}$$

$$- (11/11340)R_{prqs}R^p{}_u{}^q{}_v R^{rusv}].$$

$$(132)$$

F. $D = 7, 8, 9, 10, 11$

The complexity of the explicit expressions of $G^{\text{F}}_{\text{sing}}(x, x')$ and of the geometrical tensor $\Theta^{M^2}_{\mu\nu}$ greatly increases with the dimension D of spacetime. That clearly appears in the previous subsections. For this reason, we cannot write them explicitly for spacetime dimension from $D = 7$ to $D = 11$ even though we have at our disposal all the tools permitting us to carry out all the necessary calculations. Indeed, in the appendices of Ref. [71], we have obtained the covariant Taylor series expansions of the Van Vleck-Morette determinant $U_0(x, x') = \Delta^{1/2}(x, x')$ up to order $\sigma^{11/2}$ and of the bitensor $\sigma^{\mu\nu}(x, x')$ up to order $\sigma^{9/2}$. We have also developed the general theory permitting us to construct the covariant derivative and the d'Alembertian of an arbitrary biscalar $F(x, x')$ symmetric in the exchange of x and x'. From a theoretical point of view, all these results could permit us to solve the recursion relations (37) and (38) for D even and the recursion relations (42) for D odd and therefore to obtain the explicit expressions of $G^{\text{F}}_{\text{sing}}(x, x')$ up to the required order and of the geometrical tensor $\Theta^{M^2}_{\mu\nu}$ when necessary. Of course, this could be realized but at the cost of odious calculations in a general spacetime.

By contrast, in a given spacetime, i.e., if we know explicitly the Riemann tensor $R_{\mu\nu\rho\sigma}$ and therefore the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R, interesting simplifications may occur, the construction of $G^{\text{F}}_{\text{sing}}(x, x')$ and of $\Theta^{M^2}_{\mu\nu}$ done explicitly and the renormalization of the expectation value of the stress-energy tensor "easily" achieved. For example, in D-dimensional Schwarzschild black hole spacetimes where we have $R = 0$, $R_{\mu\nu} = 0$, and more generally in Ricci-flat spacetimes, considerable simplifications could permit us to obtain explicitly $G^{\text{F}}_{\text{sing}}(x, x')$ and $\Theta^{M^2}_{\mu\nu}$ even for $D > 6$. This certainly also happens in D-dimensional spacetimes such as $\text{AdS}_p \times \text{S}_q$ with $p + q = D$ where the covariant derivative of the Riemann tensor vanishes $(R_{\mu\nu\rho\sigma;\tau} = 0)$ as well as in D-dimensional de Sitter and Anti-de Sitter spacetimes, i.e., in maximally symmetric spacetimes, where $R_{\mu\nu\rho\sigma} = [R/D(D-1)](g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ with $R = \text{Cte}$.

IV. IMPORTANT REMARKS AND COMPLEMENTS

In this section, we shall complete our study by discussing some aspects of the Hadamard renormalization of the stress-energy tensor which are more or less directly related to the explicit calculations described in Secs. II and III. They are helpful in order to simplify some of the results displayed above. Furthermore, they permit us to discuss more generally the ambiguity problem and the trace anomaly as well as to clarify the links existing between the Hadamard formalism and the more popular method based

on regularization and renormalization in the effective action.

A. Ambiguities and trace anomaly

As already noted in Sec. II, the renormalized expectation value $\langle\psi|T_{\mu\nu}|\psi\rangle_{\mathrm{ren}}$ is unique up to the addition of a local conserved tensor $\Theta_{\mu\nu}$. For D even, we have been able to construct the standard ambiguity associated with the choice of the renormalization mass M [see, Eq. (79)] and, in Sec. III, we have explicitly obtained its expression for $D = 2$, 4, and 6 [see, Eqs. (92), (110), and (131)]. In the present subsection, following Wald's arguments of Ref. [13], we shall push further our discussion and provide for $D = 2$, 3, 4, 5, and 6, the bases (i.e., all the independent conserved local tensors) permitting us to constructed the most general expression for the tensor $\Theta_{\mu\nu}$. Here, we adhere to a conventional point of view [13] by discarding ambiguities diverging as $m^2 \to 0$. It should be however noted that a less conventional point of view has been considered by Tichy and Flanagan in Ref. [82].

In order to extend Wald's arguments, it is important to keep in mind that $\Theta_{\mu\nu}$ is a local conserved tensor of dimension $(\mathrm{mass})^D$ and that it can be obtained by functional derivation with respect to the metric tensor from a geometrical Lagrangian of dimension $(\mathrm{mass})^D$. We note also that $g_{\mu\nu}$ is dimensionless while R, $R_{\mu\nu}$, and $R_{\mu\nu\rho\sigma}$ have dimension $(\mathrm{mass})^2$.

1. D = 2

For $D = 2$, there are only two "independent" geometrical Lagrangians of dimension $(\mathrm{mass})^2$ which remain finite in the massless limit: $\mathcal{L} = m^2$ and $\mathcal{L} = R$. However, by functional derivation, the latter does not provide any contribution to $\Theta_{\mu\nu}$ because, in two dimensions, the Euler number

$$\int_{\mathcal{M}} d^2x\sqrt{-g}R \tag{133}$$

is a topological invariant. $\Theta_{\mu\nu}$ is then necessarily proportional to the functional derivative of $\mathcal{L} = m^2$ and therefore of the form

$$\Theta_{\mu\nu} = Am^2 g_{\mu\nu} \tag{134}$$

where A is a dimensionless constant.

It is interesting to note that $\Theta_{\mu\nu}$ given by (134) vanishes for $m^2 = 0$ and therefore does not modify the trace anomaly (93).

2. D = 3

For $D = 3$, there are only two independent geometrical Lagrangians of dimension $(\mathrm{mass})^3$ which remain finite in the massless limit: $\mathcal{L} = m^3$ and $\mathcal{L} = mR$. So, it is natural to consider that $\Theta_{\mu\nu}$ is necessarily a linear combination of their functional derivatives $m^3 g_{\mu\nu}/2$ and $m[(1/2)Rg_{\mu\nu} -$

$R_{\mu\nu}]$, i.e., that

$$\Theta_{\mu\nu} = Am^3 g_{\mu\nu} + Bm[R_{\mu\nu} - (1/2)Rg_{\mu\nu}] \tag{135}$$

where A and B are dimensionless constants.

It should be noted that $\Theta_{\mu\nu}$ given by (135) vanishes for $m = 0$. Thus, it cannot be used in order to modified (84). In other words, the trace anomaly does not exist for $D = 3$ even if we take into account the possible ambiguities of the Hadamard renormalization process.

3. D = 4

For $D = 4$, there are five "independent" geometrical Lagrangians of dimension $(\mathrm{mass})^4$ which remain finite in the massless limit: $\mathcal{L} = m^4$, $\mathcal{L} = m^2R$, $\mathcal{L} = R^2$, $\mathcal{L} = R_{pq}R^{pq}$, and $\mathcal{L} = R_{pqrs}R^{pqrs}$. By functional derivation with respect to the metric tensor, they define the conserved tensors $m^4 g_{\mu\nu}/2$, $m^2[(1/2)Rg_{\mu\nu} - R_{\mu\nu}]$ as well as the three conserved tensors of rank 2 and order 4

$$H^{(4,2)(1)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int_{\mathcal{M}} d^Dx\sqrt{-g}R^2 \tag{136a}$$

$$= 2R_{;\mu\nu} - 2RR_{\mu\nu} + g_{\mu\nu}[-2\Box R + (1/2)R^2], \tag{136b}$$

$$H^{(4,2)(2)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int_{\mathcal{M}} d^Dx\sqrt{-g}R_{pq}R^{pq} \tag{137a}$$

$$= R_{;\mu\nu} - \Box R_{\mu\nu} - 2R^{pq}R_{p\mu q\nu}$$
$$+ g_{\mu\nu}[-(1/2)\Box R + (1/2)R_{pq}R^{pq}], \tag{137b}$$

$$H^{(4,2)(3)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int_{\mathcal{M}} d^Dx\sqrt{-g}R_{pqrs}R^{pqrs} \tag{138a}$$

$$= 2R_{;\mu\nu} - 4\Box R_{\mu\nu} + 4R^p{}_\mu R_{p\nu} - 4R^{pq}R_{p\mu q\nu}$$
$$- 2R^{pqr}{}_\mu R_{pqr\nu} + g_{\mu\nu}[(1/2)R_{pqrs}R^{pqrs}]. \tag{138b}$$

$\Theta_{\mu\nu}$ is therefore necessarily of the form

$$\Theta_{\mu\nu} = Am^4 g_{\mu\nu} + Bm^2[R_{\mu\nu} - (1/2)Rg_{\mu\nu}] + C_1 H^{(4,2)(1)}_{\mu\nu}$$
$$+ C_2 H^{(4,2)(2)}_{\mu\nu} + C_3 H^{(4,2)(3)}_{\mu\nu} \tag{139}$$

where A, B, C_1, C_2, and C_3 are dimensionless constants. Here, it should be also noted that it is possible to simplify the previous expression because, in a four-dimensional background, the Euler number

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\mathcal{L}_{(2)}, \tag{140}$$

where $\mathcal{L}_{(2)}$ is the quadratic Gauss-Bonnet Lagrangian given by

$$\mathcal{L}_{(2)} = R^2 - 4R_{pq}R^{pq} + R_{pqrs}R^{pqrs}, \tag{141}$$

is a topological invariant. By functional derivation of (140) we obtain

$$H_{\mu\nu}^{(4,2)(1)} - 4H_{\mu\nu}^{(4,2)(2)} + H_{\mu\nu}^{(4,2)(3)} = 0 \qquad (142)$$

which could be helpful in order to eliminate one of the three conserved tensors of rank 2 and order 4 into (139). In other words, without loss of generality it is possible to use $C_1 = 0$ or $C_2 = 0$ or $C_3 = 0$ into (139).

It should be noted that the "basis" exhibited above which has permitted us to provide the general form for the tensor $\Theta_{\mu\nu}$ can be used to simplify considerably the expression (110) obtained for $\Theta_{\mu\nu}^{M^2}$. Indeed, from Eqs. (136)–(138), we can write

$$\begin{aligned}\Theta_{\mu\nu}^{M^2} = \frac{\ln M^2}{(4\pi)^2} &\times (+(1/2)(\xi - 1/6)^2 H_{\mu\nu}^{(4,2)(1)} \\ &- (1/180)H_{\mu\nu}^{(4,2)(2)} + (1/180)H_{\mu\nu}^{(4,2)(3)} \\ &- (\xi - 1/6)m^2[R_{\mu\nu} - (1/2)Rg_{\mu\nu}] + (1/4)m^4 g_{\mu\nu}).\end{aligned} \qquad (143)$$

Finally, it is interesting to note that $\Theta_{\mu\nu}$ given by (139) can be used in order to modify the trace anomaly (111). Indeed, for $m^2 = 0$ and by using Eqs. (A1)–(A3) with $D = 4$, we obtain

$$g^{\mu\nu}\Theta_{\mu\nu} = [-6C_1 - 2C_2 - 2C_3]\square R. \qquad (144)$$

For example, by taking $C_1 = 1/4320(2\pi)^2$ and $C_2 = C_3 = 0$, we can remove the $\square R$ term from (111). This elimination can be achieved by adding a finite R^2 term to the gravitational Lagrangian [see, Eq. (136)] and is in accordance with the discussion we shall develop in Sec. IV B. On the contrary, the $R_{pq}R^{pq}$ term and the $R_{pqrs}R^{pqrs}$ term cannot be modified. We refer to Sec. 6.3 of Ref. [1] for various physical comments concerning the possible modifications of the trace anomaly in a four-dimensional gravitational background.

4. D = 5

For $D = 5$, there are five independent geometrical Lagrangians of dimension (mass)^5 which remain finite in the massless limit: $\mathcal{L} = m^5$, $\mathcal{L} = m^3 R$, $\mathcal{L} = mR^2$, $\mathcal{L} = mR_{pq}R^{pq}$, and $\mathcal{L} = mR_{pqrs}R^{pqrs}$. By functional derivation, they define the conserved tensors $m^5 g_{\mu\nu}/2$, $m^3[(1/2)Rg_{\mu\nu} - R_{\mu\nu}]$ as well as the three conserved tensors of rank 2 and order 4 $mH_{\mu\nu}^{(4,2)(1)}$, $mH_{\mu\nu}^{(4,2)(2)}$, and $mH_{\mu\nu}^{(4,2)(3)}$. $\Theta_{\mu\nu}$ is therefore necessarily of the form

$$\begin{aligned}\Theta_{\mu\nu} = &Am^5 g_{\mu\nu} + Bm^3[R_{\mu\nu} - (1/2)Rg_{\mu\nu}] \\ &+ C_1 mH_{\mu\nu}^{(4,2)(1)} + C_2 mH_{\mu\nu}^{(4,2)(2)} + C_3 mH_{\mu\nu}^{(4,2)(3)}\end{aligned} \qquad (145)$$

where A, B, C_1, C_2, and C_3 are dimensionless constants.

Here, the conserved tensors $H_{\mu\nu}^{(4,2)(1)}$, $H_{\mu\nu}^{(4,2)(2)}$, and $H_{\mu\nu}^{(4,2)(3)}$ are still, respectively, defined by Eqs. (136a), (137a), and (138a) but now, in these equations, $D = 4$ must be replaced by $D = 5$. Their explicit expressions (136b), (137b), and (138b) remain unchanged. For $D = 5$, it is not possible to simplify Eq. (145) by using (142). Indeed, for $D > 4$ this topological constraint is not valid because the Euler number (140) does not remain a topological invariant.

It should be noted that $\Theta_{\mu\nu}$ given by (145) vanishes for $m = 0$. Thus, it cannot be used in order to modify (84). In other words, the trace anomaly does not exist for $D = 5$ even if we take into account the possible ambiguities of the Hadamard renormalization process.

5. D = 6

For $D = 6$, there are 15 "independent" geometrical Lagrangians of dimension (mass)^6 which remain finite in the massless limit: $\mathcal{L} = m^6$, $\mathcal{L} = m^4 R$ and the three Riemann polynomials of rank 0 and order 4 $\mathcal{L} = m^2 R^2$, $\mathcal{L} = m^2 R_{pq}R^{pq}$, $\mathcal{L} = m^2 R_{pqrs}R^{pqrs}$ as well as the ten Riemann monomials of rank 0 and order 6 (see Refs. [73,74]) $\mathcal{L} = R\square R$, $\mathcal{L} = R_{pq}\square R^{pq}$, $\mathcal{L} = R^3$, $\mathcal{L} = RR_{pq}R^{pq}$, $\mathcal{L} = R_{pq}R^p{}_r R^{qr}$, $\mathcal{L} = R_{pq}R_{rs}R^{prqs}$, $\mathcal{L} = RR_{pqrs}R^{pqrs}$, $\mathcal{L} = R_{pq}R^p{}_{rst}R^{qrst}$, $\mathcal{L} = R_{pqrs}R^{pquv}R^{rs}{}_{uv}$, $\mathcal{L} = R_{prqs}R^p{}_u{}^q{}_v R^{rusv}$. By functional derivation, they define the conserved tensors $m^6 g_{\mu\nu}/2$, $m^4[(1/2)Rg_{\mu\nu} - R_{\mu\nu}]$ and the three conserved tensors of rank 2 and order 4 $m^2 H_{\mu\nu}^{(4,2)(1)}$, $m^2 H_{\mu\nu}^{(4,2)(2)}$, and $m^2 H_{\mu\nu}^{(4,2)(3)}$ as well as the ten conserved tensors of rank 2 and order 6

$$H_{\mu\nu}^{\{2,0\}(1)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} R\square R \qquad (146)$$

$$H_{\mu\nu}^{\{2,0\}(3)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pq}\square R^{pq} \qquad (147)$$

$$H_{\mu\nu}^{(6,3)(1)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} R^3 \qquad (148)$$

$$H_{\mu\nu}^{(6,3)(2)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} RR_{pq}R^{pq} \qquad (149)$$

$$H_{\mu\nu}^{(6,3)(3)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pq}R^p{}_r R^{qr} \qquad (150)$$

$$H_{\mu\nu}^{(6,3)(4)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pq}R_{rs}R^{prqs} \qquad (151)$$

$$H_{\mu\nu}^{(6,3)(5)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} RR_{pqrs}R^{pqrs} \qquad (152)$$

$$H_{\mu\nu}^{(6,3)(6)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pq}R^p{}_{rst}R^{qrst} \qquad (153)$$

$$H_{\mu\nu}^{(6,3)(7)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pqrs} R^{pquv} R^{rs}{}_{uv} \quad (154)$$

$$H_{\mu\nu}^{(6,3)(8)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{prqs} R^p{}_u{}^q{}_v R^{rusv}. \quad (155)$$

Here we do not provide the explicit expressions of these ten tensors. They are very complicated ones and can be found in Ref. [89] [see, Eqs. (2.22)–(2.31) of this article]. As far as the conserved tensors $H_{\mu\nu}^{(4,2)(1)}$, $H_{\mu\nu}^{(4,2)(2)}$, and $H_{\mu\nu}^{(4,2)(3)}$ are concerned, they are still, respectively, defined by Eqs. (136a), (137a), and (138a) but now, in these equations, $D = 4$ must be replaced by $D = 6$. Their explicit expressions (136b), (137b), and (138b) remain unchanged. $\Theta_{\mu\nu}$ is therefore necessarily of the form

$$\begin{aligned}
\Theta_{\mu\nu} = &\, Am^6 g_{\mu\nu} + Bm^4[(1/2)R g_{\mu\nu} - R_{\mu\nu}] \\
&+ C_1 m^2 H_{\mu\nu}^{(4,2)(1)} + C_2 m^2 H_{\mu\nu}^{(4,2)(2)} + C_3 m^2 H_{\mu\nu}^{(4,2)(3)} \\
&+ D_1 H_{\mu\nu}^{\{2,0\}(1)} + D_2 H_{\mu\nu}^{\{2,0\}(3)} + D_3 H_{\mu\nu}^{(6,3)(1)} \\
&+ D_4 H_{\mu\nu}^{(6,3)(2)} + D_5 H_{\mu\nu}^{(6,3)(3)} + D_6 H_{\mu\nu}^{(6,3)(4)} \\
&+ D_7 H_{\mu\nu}^{(6,3)(5)} + D_8 H_{\mu\nu}^{(6,3)(6)} + D_9 H_{\mu\nu}^{(6,3)(7)} \\
&+ D_{10} H_{\mu\nu}^{(6,3)(8)}
\end{aligned} \quad (156)$$

where A, B, C_1, C_2, and C_3 as well as $D_1, \ldots D_9$ and D_{10} are dimensionless constants. Finally, it should be noted that it is possible to simplify the previous expression for $\Theta_{\mu\nu}$

because, in a six-dimensional background, the Euler number

$$\int_{\mathcal{M}} d^6 x \sqrt{-g} \mathcal{L}_{(3)}, \quad (157)$$

where $\mathcal{L}_{(3)}$ is the cubic Lovelock Lagrangian explicitly given by

$$\begin{aligned}
\mathcal{L}_{(3)} = &\, R^3 - 12 R R_{pq} R^{pq} + 16 R_{pq} R^p{}_r R^{qr} \\
&+ 24 R_{pq} R_{rs} R^{prqs} + 3 R R_{pqrs} R^{pqrs} \\
&- 24 R_{pq} R^p{}_{rst} R^{qrst} + 4 R_{pqrs} R^{pquv} R^{rs}{}_{uv} \\
&- 8 R_{prqs} R^p{}_u{}^q{}_v R^{rusv},
\end{aligned} \quad (158)$$

is a topological invariant. By functional derivation of (157) we obtain the relation

$$\begin{aligned}
&H_{\mu\nu}^{(6,3)(1)} - 12 H_{\mu\nu}^{(6,3)(2)} + 16 H_{\mu\nu}^{(6,3)(3)} + 24 H_{\mu\nu}^{(6,3)(4)} \\
&+ 3 H_{\mu\nu}^{(6,3)(5)} - 24 H_{\mu\nu}^{(6,3)(6)} + 4 H_{\mu\nu}^{(6,3)(7)} - 8 H_{\mu\nu}^{(6,3)(8)} = 0.
\end{aligned} \quad (159)$$

Equation (159) could be helpful in order to eliminate into (156) one of the conserved tensors of rank 2 and order 6.

Of course, the "basis" exhibited above and which has permitted us to provide the general form for the tensor $\Theta_{\mu\nu}$ can be used to simplify considerably the expression (131) obtained for $\Theta_{\mu\nu}^{M^2}$. By using Eqs. (2.22)–(2.31) of Ref. [89] and after a tedious calculation, we obtain the compact expression

$$\begin{aligned}
\Theta_{\mu\nu}^{M^2} = &\, \frac{\ln M^2}{(4\pi)^3} \times \big([(1/12)\xi^2 - (1/30)\xi + 1/336] H_{\mu\nu}^{\{2,0\}(1)} + (1/840) H_{\mu\nu}^{\{2,0\}(3)} - (1/6)(\xi - 1/6)^3 H_{\mu\nu}^{(6,3)(1)} \\
&+ (1/180)(\xi - 1/6) H_{\mu\nu}^{(6,3)(2)} - (4/2835) H_{\mu\nu}^{(6,3)(3)} + (1/945) H_{\mu\nu}^{(6,3)(4)} - (1/180)(\xi - 1/6) H_{\mu\nu}^{(6,3)(5)} \\
&+ (1/7560) H_{\mu\nu}^{(6,3)(6)} + (17/45360) H_{\mu\nu}^{(6,3)(7)} - (1/1620) H_{\mu\nu}^{(6,3)(8)} - (1/2)(\xi - 1/6)^2 m^2 H_{\mu\nu}^{(4,2)(1)} \\
&+ (1/180) m^2 H_{\mu\nu}^{(4,2)(2)} - (1/180) m^2 H_{\mu\nu}^{(4,2)(3)} + (1/2)(\xi - 1/6) m^4 [R_{\mu\nu} - (1/2) R g_{\mu\nu}] - (1/12) m^6 g_{\mu\nu} \big). \quad (160)
\end{aligned}$$

Finally, it is interesting to note that $\Theta_{\mu\nu}$ given by (156) could permit us to modify the trace anomaly (132). Indeed, for $m^2 = 0$ and by using Eqs. (A4)–(A7) and (A7)–(A13) with $D = 6$, we obtain

$$\begin{aligned}
g^{\mu\nu} \Theta_{\mu\nu} = &\, [-10 D_1 - 3 D_2] \Box\Box R + [-2 D_1 - 30 D_3 - 4 D_4 - D_6/2 - 2 D_7] R \Box R + [-8 D_2 - 4 D_4 - 6 D_5 + 2 D_6 - 4 D_7 \\
&- D_8 + 3 D_{10}/2] R_{;pq} R^{pq} + [2 D_2 - 10 D_4 - 3 D_5 - 6 D_6 - 2 D_8 - 3 D_{10}] R_{pq} \Box R^{pq} \\
&+ [8 D_2 - 4 D_6 - 40 D_7 - 14 D_8 - 24 D_9 + 3 D_{10}] R_{pq;rs} R^{prqs} \\
&+ [-2 D_1 - 3 D_2/2 - 30 D_3 - 6 D_4 - 3 D_5/2 - 3 D_6/4 - 4 D_7 - D_8/2] R_{;p} R^{;p} + [10 D_2 - 10 D_4 - 3 D_5 \\
&- 10 D_6 - 8 D_8 - 12 D_9 - 3 D_{10}] R_{pq;r} R^{pq;r} + [-18 D_2 - 6 D_5 + 9 D_6 + 6 D_8 + 12 D_9 + 3 D_{10}] R_{pq;r} R^{pr;q} \\
&+ [-18 D_2 - 6 D_5 + 9 D_6 + 6 D_8 + 12 D_9 + 3 D_{10}] R_{pq;r} R^{pr;q} + [-10 D_7 - 2 D_8 - 3 D_9 + 3 D_{10}/4] R_{pqrs;t} R^{pqrs;t} \\
&+ [-10 D_2 - 6 D_5 + 5 D_6 + 3 D_{10}] (R_{pq} R^p{}_r R^{qr} - R_{pq} R_{rs} R^{prqs}) + [-10 D_7 - 2 D_8 - 3 D_9 + 3 D_{10}/4] \\
&\times (2 R_{pq} R^p{}_{rst} R^{qrst} - R_{pqrs} R^{pquv} R^{rs}{}_{uv} - 4 R_{prqs} R^p{}_u{}^q{}_v R^{rusv}). \quad (161)
\end{aligned}$$

It should be noted that three of the ten scalar Riemann monomials of order 6, namely R^3, $RR_{pq}R^{pq}$, and $RR_{pqrs}R^{pqrs}$, do not appear in (161). As a consequence, it is impossible to remove such terms from the trace anomaly (132). On the contrary, by choosing correctly the coefficients D_i, it is possible to remove any other term from (132).

B. Infinities and gravitational actions

In the previous sections, we have constructed the renormalized expectation value of the stress-energy operator for a massive scalar field in a general spacetime of arbitrary dimension by assuming that the Wald's axiomatic approach (see, Refs. [3,9,13]) remains valid for all dimensions. In the Wald's axiomatic approach, the treatment of the divergences present in the formal expression (60) does not necessitate a particular study, i.e., absorption into renormalized gravitational parameters. These divergences are simply discarded and the cosmological constant Λ and the Newton's gravitational constant G (as well as the other coupling constants associated with higher-order gravitational terms if we need to consider such terms) appearing in the semiclassical Einstein equations (1) are directly the physical gravitational parameters while the expectation value $\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{ren}}$ constructed from the Hadamard biscalar $W(x, x')$ is automatically the physically meaningful source.

In the present subsection, we shall depart from the path marked out by Wald. We shall briefly describe one way to deal with the divergent part of (60) by extending the approach developed by Christensen in Refs. [11,12] (see, also, Adler, *et. al.* in Refs. [14,15] for a related but slightly different approach). We intend to discuss at more length this very technical aspect of our work in a paper in preparation [90]. However, in order to be as complete as possible, we shall here provide partial results related to the present work.

For a given spacetime dimension, we can formally evaluate the divergent part of (60) and express the result of our calculation as a power series in $\sigma^{;a}(x, x')$. By consistently averaging this power series over all the angular directions joining x' and x and by adding to it, for D even, the opposite of (79) as well as $-(D/4)\alpha_D g_{\mu\nu}v_1$, we find a final divergent expression constructed from "simple" conserved geometrical tensors which can be absorbed into a bare gravitational Lagrangian. It is important to note that the averaging process of the direction-dependant terms adopted in Refs. [11,12,14,15]) must be modified in order to take into account the spacetime dimension.

For $D = 2$, we obtain for the averaged divergent part of (60) an expression of the form

$$\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{sing}} \sim A\frac{g_{\mu\nu}}{\sigma} + [B_1m^2g_{\mu\nu} + B_2Rg_{\mu\nu}]\ln(M^2\sigma)$$
$$+ \text{ finite terms in } m^2g_{\mu\nu} \text{ and } Rg_{\mu\nu}. \quad (162)$$

Here A, B_1, and B_2 are dimensionless constants. This singular tensor cannot be absorbed into a bare gravitational Lagrangian of Einstein-Hilbert type because, in two dimensions, the Euler number (133) being a topological invariant, the tensor $R_{\mu\nu} - (1/2)Rg_{\mu\nu}$ vanishes identically. However, this tensor can be absorbed into the Polyakov nonlocal bare gravitational action

$$S_{\text{grav}} = \int_{\mathcal{M}} d^2x\sqrt{-g}\left(a_BR\frac{1}{\Box}R - 2\Lambda_B\right). \quad (163)$$

It should be noted that the nonlocal Lagrangian $\mathcal{L} = R\frac{1}{\Box}R$ provides, by functional derivation with respect to the metric tensor, a contribution in $2Rg_{\mu\nu}$ but also a nonlocal contribution proportional to

$$-2\left(\frac{1}{\Box}R\right)_{;\mu\nu} + \left(\frac{1}{\Box}R\right)_{;\mu}\left(\frac{1}{\Box}R\right)_{;\nu} - \frac{1}{2}g_{\mu\nu}\left(\frac{1}{\Box}R\right)_{;p}\left(\frac{1}{\Box}R\right)^{;p}. \quad (164)$$

We think that these two contributions must be added to (134). In the particular case of a two-dimensional background, it is not natural to follow Wald's prescription and to construct the conserved tensor $\Theta_{\mu\nu}$ from a purely local Lagrangian.

For $D = 3$, we obtain for the averaged divergent part of (60) an expression of the form

$$\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{sing}} \sim A\frac{g_{\mu\nu}}{\sigma^{3/2}} + B\frac{[R_{\mu\nu} - (1/2)Rg_{\mu\nu}]}{\sigma^{1/2}}$$
$$+ \text{ finite terms in } m^3g_{\mu\nu} \quad \text{and}$$
$$m[R_{\mu\nu} - (1/2)Rg_{\mu\nu}]. \quad (165)$$

Here A and B are dimensionless constants. This singular tensor can be absorbed into a bare gravitational action given by

$$S_{\text{grav}} = -\frac{1}{16\pi G_B}\int_{\mathcal{M}} d^3x\sqrt{-g}(R - 2\Lambda_B). \quad (166)$$

For $D = 4$, we obtain for the averaged divergent part of (60) an expression of the form

$$\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{sing}} \sim A\frac{g_{\mu\nu}}{\sigma^2} + B\frac{[R_{\mu\nu} - (1/2)Rg_{\mu\nu}]}{\sigma} + (C_1H_{\mu\nu}^{(4,2)(1)}$$
$$+ C_2H_{\mu\nu}^{(4,2)(2)} + C_3H_{\mu\nu}^{(4,2)(3)})\ln(M^2\sigma)$$
$$+ \text{ finite terms in } m^4g_{\mu\nu}, m^2$$
$$\times [R_{\mu\nu} - (1/2)Rg_{\mu\nu}], H_{\mu\nu}^{(4,2)(1)},$$
$$H_{\mu\nu}^{(4,2)(2)} \text{ and } H_{\mu\nu}^{(4,2)(3)}. \quad (167)$$

Here A, B, C_1, C_2, and C_3 are dimensionless constants. This singular tensor can be absorbed into a bare gravitational action given by

$$S_{\text{grav}} = -\frac{1}{16\pi G_B}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda_B + \alpha_B^{(1)}R^2$$
$$+ \alpha_B^{(2)}R_{pq}R^{pq} + \alpha_B^{(3)}R_{pqrs}R^{pqrs}). \qquad (168)$$

In this bare gravitational action, the term in $\alpha_B^{(3)}R_{pqrs}R^{pqrs}$ could be removed because, as we have already noted, the

Euler number (140) is a topological invariant in four dimensions. Similarly, it would have been possible to remove the $H_{\mu\nu}^{(4,2)(3)}$ term from (167).

For $D = 5$, we obtain for the averaged divergent part of (60) an expression of the form

$$\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{sing}} \sim A\frac{g_{\mu\nu}}{\sigma^{5/2}} + B\frac{[R_{\mu\nu} - (1/2)Rg_{\mu\nu}]}{\sigma^{3/2}} + \frac{(C_1H_{\mu\nu}^{(4,2)(1)} + C_2H_{\mu\nu}^{(4,2)(2)} + C_3H_{\mu\nu}^{(4,2)(3)})}{\sigma^{1/2}}$$
$$+ \text{ finite terms in } m^5g_{\mu\nu}, m^3[R_{\mu\nu} - (1/2)Rg_{\mu\nu}], mH_{\mu\nu}^{(4,2)(1)}, mH_{\mu\nu}^{(4,2)(2)} \quad \text{and} \quad mH_{\mu\nu}^{(4,2)(3)}. \quad (169)$$

Here A, B, C_1, C_2, and C_3 are dimensionless constants. This singular tensor can be absorbed into a bare gravitational action given by

$$S_{\text{grav}} = -\frac{1}{16\pi G_B}\int_{\mathcal{M}} d^5x\sqrt{-g}(R - 2\Lambda_B + \alpha_B^{(1)}R^2 + \alpha_B^{(2)}R_{pq}R^{pq} + \alpha_B^{(3)}R_{pqrs}R^{pqrs}). \qquad (170)$$

Of course, this bare gravitational action cannot be simplified because, for $D > 4$, the Euler number (140) does not remain a topological invariant. Similarly, the $H_{\mu\nu}^{(4,2)(3)}$ term cannot be removed from (169).

Finally, for $D = 6$, we obtain for the averaged divergent part of (60) an expression of the form

$$\langle\psi|T_{\mu\nu}|\psi\rangle_{\text{sing}} \sim A\frac{g_{\mu\nu}}{\sigma^3} + B\frac{[R_{\mu\nu} - (1/2)Rg_{\mu\nu}]}{\sigma^2} + \frac{(C_1H_{\mu\nu}^{(4,2)(1)} + C_2H_{\mu\nu}^{(4,2)(2)} + C_3H_{\mu\nu}^{(4,2)(3)})}{\sigma} + (D_1H_{\mu\nu}^{\{2,0\}(1)} + D_2H_{\mu\nu}^{\{2,0\}(3)}$$
$$+ D_3H_{\mu\nu}^{(6,3)(1)} + D_4H_{\mu\nu}^{(6,3)(2)} + D_5H_{\mu\nu}^{(6,3)(3)} + D_6H_{\mu\nu}^{(6,3)(4)} + D_7H_{\mu\nu}^{(6,3)(5)} + D_8H_{\mu\nu}^{(6,3)(6)} + D_9H_{\mu\nu}^{(6,3)(7)}$$
$$+ D_{10}H_{\mu\nu}^{(6,3)(8)})\ln(M^2\sigma) + \text{ finite terms in } m^6g_{\mu\nu}, m^4[R_{\mu\nu} - (1/2)Rg_{\mu\nu}], m^2H_{\mu\nu}^{(4,2)(1)}, m^2H_{\mu\nu}^{(4,2)(2)},$$
$$m^2H_{\mu\nu}^{(4,2)(3)}, H_{\mu\nu}^{\{2,0\}(1)}, H_{\mu\nu}^{\{2,0\}(3)}, H_{\mu\nu}^{(6,3)(1)}, H_{\mu\nu}^{(6,3)(2)}, H_{\mu\nu}^{(6,3)(3)}, H_{\mu\nu}^{(6,3)(4)}, H_{\mu\nu}^{(6,3)(5)}, H_{\mu\nu}^{(6,3)(6)}, H_{\mu\nu}^{(6,3)(7)} \quad \text{and} \quad H_{\mu\nu}^{(6,3)(8)}.$$
$$(171)$$

Here A, B, C_1, C_2, C_3, D_1, ..., D_9, and D_{10} are dimensionless constants. This singular tensor can be absorbed into a bare gravitational action given by

$$S_{\text{grav}} = -\frac{1}{16\pi G_B}\int_{\mathcal{M}} d^6x\sqrt{-g}(R - 2\Lambda_B + \alpha_B^{(1)}R^2 + \alpha_B^{(2)}R_{pq}R^{pq} + \alpha_B^{(3)}R_{pqrs}R^{pqrs} + \beta_B^{(1)}R^3 + \beta_B^{(2)}RR_{pq}R^{pq}$$
$$+ \beta_B^{(3)}R_{pq}R^p{}_rR^{qr} + \beta_B^{(4)}R_{pq}R_{rs}R^{prqs} + \beta_B^{(5)}RR_{pqrs}R^{pqrs} + \beta_B^{(6)}R_{pq}R^p{}_{rst}R^{qrst} + \beta_B^{(7)}R_{pqrs}R^{pquv}R^{rs}{}_{uv}$$
$$+ \beta_B^{(8)}R_{prqs}R^p{}_u{}^q{}_vR^{rusv}). \qquad (172)$$

This bare gravitational action could be simplified by using the fact that the Euler number (157) is a topological invariant. This result could be used to remove from the bare gravitational action a term such as $\beta_B^{(8)}R_{prqs}R^p{}_u{}^q{}_vR^{rusv}$. Similarly, it would have been possible to remove the $H_{\mu\nu}^{(6,2)(8)}$ term from (171).

To conclude this subsection, it is important to note that the previous results must be taken with a grain of salt. Indeed, Eqs. (162), (165), (167), (169), and (171) are formal relations: they display on a very condensed form the true behavior of the averaged divergent part of (60) in the limit of small σ (for more details, we refer to Ref. [90]).

C. Hadamard renormalization versus renormalization in the effective action

Field quantization in curved spacetime can be addressed very efficiently by using the effective action [80,88,91]. This basic object contains, in principle, all the information about a given quantum field theory but, unfortunately, it is not usually possible to express it explicitly. Even in the very simple case of the scalar field theory considered in the present article, we have only an approximation for the associated effective action, the so-called DeWitt-Schwinger (DS) approximation [1,80,88,91–93], which may be represented by the asymptotic series [88]

$$W^{DS} = \int_{\mathcal{M}} d^D x \sqrt{-g(x)} \times \left[\frac{1}{2(4\pi)^{D/2}} \right.$$

$$\left. \times \int_0^{+\infty} \frac{d(is)}{(is)^{D/2+1}} e^{-m^2 is} \Lambda(x;s) \right] \qquad (173)$$

where $\Lambda(x;s)$ is a purely geometrical object (see, Ref. [88]) which satisfies

$$\lim_{s \to +\infty} e^{-m^2 is} \Lambda(x;s) = 0 \qquad (174)$$

and which can be formally written for $s \to 0$ on the form

$$\Lambda(x;s) \approx \sum_{k=0}^{+\infty} a_k(x)(is)^k. \qquad (175)$$

Here, $a_k(x)$ are the diagonal DeWitt coefficients. The four first ones can be found in Refs. [88,94,95] and, because we

have previously assumed that spacetime has no boundary, we have for their global (or integrated) expressions

$$\int_{\mathcal{M}} d^D x \sqrt{-g} a_0 = \int_{\mathcal{M}} d^D x \sqrt{-g}, \qquad (176)$$

$$\int_{\mathcal{M}} d^D x \sqrt{-g} a_1 = \int_{\mathcal{M}} d^D x \sqrt{-g}[-(\xi - 1/6)R], \qquad (177)$$

$$\int_{\mathcal{M}} d^D x \sqrt{-g} a_2 = \int_{\mathcal{M}} d^D x \sqrt{-g}[(1/2)(\xi - 1/6)^2 R^2$$
$$- (1/180)R_{pq}R^{pq}$$
$$+ (1/180)R_{pqrs}R^{pqrs}], \qquad (178)$$

and

$$\int_{\mathcal{M}} d^D x \sqrt{-g} a_3 = \int_{\mathcal{M}} d^D x \sqrt{-g}([(1/12)\xi^2 - (1/30)\xi + 1/336]R\Box R + (1/840)R_{pq}\Box R^{pq} - (1/6)(\xi - 1/6)^3 R^3$$
$$+ (1/180)(\xi - 1/6)RR_{pq}R^{pq} - (4/2835)R_{pq}R^p{}_r R^{qr} + (1/945)R_{pq}R_{rs}R^{prqs}$$
$$- (1/180)(\xi - 1/6)RR_{pqrs}R^{pqrs} + (1/7560)R_{pq}R^p{}_{rst}R^{qrst}$$
$$+ (17/45360)R_{pqrs}R^{pquv}R^{rs}{}_{uv} - (1/1620)R_{prqs}R^p{}_u{}^q{}_v R^{rusv}). \qquad (179)$$

The DeWitt-Schwinger representation (173) of the effective action is a purely local geometrical object which contains all the information on the ultraviolet behavior of the quantum theory of the scalar field obeying the wave equation (5) but which does not take into account its state dependence. By functional derivation of (173) with respect to the metric tensor, we can construct the formal stress-energy tensor

$$\langle T^{DS}_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W^{DS}}{\delta g^{\mu\nu}}. \qquad (180)$$

Of course, it is also a purely local geometrical object which is furthermore state independent. However, in spite of this last drawback, it has been extensively used, in the four-dimensional context, in order (i) to understand the regularization and renormalization of the true (i.e., state-dependent) stress-energy tensor (see, for example, Ref. [1]) or (ii) to provide approximations valid in the large mass limit for this true stress-energy tensor (see, for example, Refs. [89,91,96,97]). In the present subsection, we shall briefly discuss some aspects of the renormalization of the formal stress-energy tensor (180) directly linked to our propose in order to shed light, from a different point of view, on the results obtained above but also to advocate, with in mind practical applications, the use of the Hadamard method we have developed.

First, it is important to note that the effective action W^{DS} is divergent at the lower limit of the integral over s for all the positive values of the dimension D. For $D = 2$ and

$D = 3$, this divergent behavior is associated with the integrated DeWitt coefficients (176) and (177); for $D = 4$ and $D = 5$, it is associated with the integrated DeWitt coefficients (176)–(178); and for $D = 6$, it is associated with the integrated DeWitt coefficients (176)–(179). As a consequence, from (180) and by using (136)–(138) and (146)–(155), we can very easily obtain results analogous to those described in Sec. IV B concerning the formal expression of the divergent part of the stress-energy tensor.

The treatment of the divergent behavior of (180) can be achieved by first regularizing the effective action (173), then by absorbing its divergent part into a bare gravitational action and finally by functionally deriving the renormalized effective action so obtained. By considering the dimensionality D of spacetime as a complex number, the effective action W^{DS} can be regularized by analytic continuation and its divergent part can be extracted coherently and naturally absorbed into a bare gravitational action [1,88]. The resulting renormalized effective action can be written in the form

$$W^{DS}_{ren} = \int_{\mathcal{M}} d^D x \sqrt{-g(x)} \times \left[-\frac{1}{2[(D/2)!](4\pi)^{D/2}} \right.$$

$$\times \int_0^{+\infty} d(is) \ln(4\pi M^2 is)\left(\frac{\partial}{\partial(is)}\right)^{D/2+1}$$

$$\left. \times [e^{-m^2 is} \Lambda(x;s)] \right] \qquad (181)$$

for D even and in the form

$$W_{\text{ren}}^{\text{DS}} = \int_{\mathcal{M}} d^D x \sqrt{-g(x)} \times \left[\frac{1}{2[\Gamma(D/2)/\sqrt{\pi}](4\pi)^{D/2}} \right.$$
$$\left. \times \int_0^{+\infty} \frac{d(is)}{(is)^{1/2}} \left(\frac{\partial}{\partial(is)} \right)^{D/2+1/2} [e^{-m^2 is} \Lambda(x; s)] \right]$$

$$(182)$$

for D odd. Formulas (181) and (182) generalize results displayed in Ref. [88] for $D = 2, 3, 4$. In Eq. (181), M is an arbitrary mass scale parameter (the renormalization mass) which is necessary in dimensional regularization because only dimensionless quantities can be analytically continued. M remains in the renormalized effective action for D even. Now, by inserting (181) or (182) into (180), we can obtain a renormalized stress-energy tensor for D even or D odd. Of course, the object calculated in that way is only a state-independent approximation of the true expectation value of the stress-energy operator. Furthermore, because in order to obtain (181) and (182) we have discarded not only infinite terms involving the integrated DeWitt coefficients but also finite ones which have been absorbed by finite renormalization, this object is also ambiguously defined. The corresponding ambiguities are obtained by functional derivation of the integrated DeWitt coefficients and are those displayed in Sec. IVA.

Let us now consider the part of the renormalized effective action (181) associated with the renormalization mass M. By using (174) we obtain for its expression

$$W_{M^2}^{\text{DS}} = \int_{\mathcal{M}} d^D x \sqrt{-g(x)} \left[\frac{\ln M^2}{2[(D/2)!](4\pi)^{D/2}} \left(\frac{\partial}{\partial(is)} \right)^{D/2} \right.$$
$$\left. \times [e^{-m^2 is} \Lambda(x; s)] \right]_{s=0}$$

$$(183)$$

and, from (175) and (180), it provides a geometrical ambiguity associated with the stress-energy tensor given by

$$\Theta_{\mu\nu}^{M^2} = \frac{\ln M^2}{2(4\pi)^{D/2}} \times \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(\int_{\mathcal{M}} d^D x \sqrt{-g(x)} \right.$$
$$\left. \times \sum_{k=0}^{D/2} \frac{(-1)^k}{k!} (m^2)^k a_{D/2-k}(x) \right).$$

$$(184)$$

This ambiguity is in fact equivalent to that obtained from the Hadamard formalism in Sec. II [see, Eq. (79)]. Indeed, for $D = 2$ it reduces to

$$\Theta_{\mu\nu}^{M^2} = \frac{\ln M^2}{2(4\pi)} \times \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}}$$
$$\times \left(\int_{\mathcal{M}} d^2 x \sqrt{-g(x)} [a_1(x) - m^2 a_0(x)] \right) \quad (185)$$

which permits us to recover (92) from (176) and (177). For $D = 4$, it reduces to

$$\Theta_{\mu\nu}^{M^2} = \frac{\ln M^2}{2(4\pi)^2} \times \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(\int_{\mathcal{M}} d^4 x \sqrt{-g(x)} [a_2(x) \right.$$
$$\left. - m^2 a_1(x) + (m^4/2) a_0(x)] \right)$$

$$(186)$$

which permits us to recover (143) from (176)–(178) by using (136)–(138). For $D = 6$, it reduces to

$$\Theta_{\mu\nu}^{M^2} = \frac{\ln M^2}{2(4\pi)^3} \times \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(\int_{\mathcal{M}} d^6 x \sqrt{-g(x)} [a_3(x) \right.$$
$$\left. - m^2 a_2(x) + (m^4/2) a_1(x) - (m^6/6) a_0(x)] \right)$$

$$(187)$$

which permits us to recover (160) from (176)–(179) by using (136)–(138) and (146)–(155).

We shall now conclude the present subsection by comparing the respective merits of the Hadamard formalism developed in this article and of the approach based on renormalization in the effective action. Renormalization in the effective action is a powerful tool which permits us to understand the structure of the ultraviolet divergences contained in the unrenormalized expression of the stress-energy tensor and to discuss the ambiguity problem. Because it uses functional derivation with respect to the metric instead of the point-splitting method, it permits us to obtain very easily the results mentioned above with a formalism which is rather independent of the dimension of spacetime. Hadamard formalism, if we depart from the axiomatic point of view advocated by Wald, does not seem so interesting. Unfortunately, calculations based on renormalization in the effective action cannot permit us to take into account the state dependence of the considered quantum theory and therefore to obtain, in a general framework, the full renormalized expectation value of the stress-energy operator. In fact, bearing in mind practical calculations, Hadamard formalism is much more efficient than the method based on renormalization in the effective action even if, at first sight and because of its use of the point-splitting method, it seems rather heavier. It is also important to note that, in the present article, we have achieved the major part of the boring job. The reader who simply wishes to calculate the renormalized expectation value of the stress-energy tensor in a particular case must only extract from the available Feynman propagator the first two coefficients of the biscalar $W(x, x')$ by using the formulas displayed in Sec. III. If he/she wants furthermore to discuss the ambiguities of the renormalized stress-energy tensor obtained, he/she can used the expressions displayed in Sec. IVA. He/she has nothing else to do.

V. CONCLUSION AND PERSPECTIVES

In this article, we have developed the Hadamard renormalization of the stress-energy tensor for a massive scalar field theory defined on a general spacetime of arbitrary

dimension. For spacetime dimensions up to 6, we have explicitly described the renormalization procedure while for spacetime dimensions from 7 to 11, we have provided the framework permitting the interested reader to perform this procedure explicitly in a given spacetime.

Our formalism is very general: we do not assume any symmetry for spacetime and we do not limit our study to the massless or the conformally invariant scalar fields. As a consequence, we have provided a powerful formalism which could permit us to deal with some particular aspects of the quantum physics of extra spatial dimensions in a rather general way or, more precisely, in a more general way than usual (see, references in Sec. I). We think that this formalism could be immediately used to discuss, from a more general point of view, the consequence of the presence of extra spatial dimensions upon:

  (i) The stabilization of Randall-Sundrum braneworld models of cosmological interest (in connection with the inflationary scenario and the dark energy problem).
 (ii) The quantum violations of the classical energy conditions (in connection with the singularity theorems of Hawking and Penrose) as well as of the averaged null energy condition (in connection with the existence of traversable wormholes and time machines).
(iii) The fluctuations of the stress-energy tensor (in connection with the validity of semiclassical gravity and again with the singularity theorems of Hawking and Penrose).

Furthermore, we think it would be very interesting to revisit holographic renormalization from the point of view of the Hadamard formalism and, above all, to use the Hadamard renormalization procedure developed in this article to perform calculations of stress-energy tensors for higher-dimensional black holes. Indeed, even though such a subject has been a central topic of four-dimensional semiclassical gravity, very little has been realized in the higher-dimensional framework. This is rather incomprehensible since string theory (or more precisely the so-called TeV-scale quantum gravity [28–30]) predicts the possibility of production of such black holes at CERN's Large Hadron Collider [98–100] with a production rate around 1 Hz [101,102]. In this context, the semiclassical Einstein equations (1) could permit us to partially describe the black hole evaporation and to test TeV-scale quantum gravity.

Of course, with the various applications previously mentioned in mind, it is necessary to extend our present work to more general field theories and more particularly to the graviton field. In order to perform such a generalization, it is first of all necessary to carry out the program described at the end of the conclusion of Ref. [71], i.e., to construct the covariant Taylor series expansions for the off diagonal Hadamard coefficients for these field theories by going beyond the existing results.

APPENDIX: TRACES FOR THE ANOMALOUS TRACE OF THE STRESS TENSOR

In this appendix, we provide the expressions for the traces of the three conserved tensors of rank 2 and order 4 $H_{\mu\nu}^{(4,2)(1)}$, $H_{\mu\nu}^{(4,2)(2)}$, and $H_{\mu\nu}^{(4,2)(3)}$ and of the ten conserved tensors of rank 2 and order 6 $H_{\mu\nu}^{\{2,0\}(1)}$, $H_{\mu\nu}^{\{2,0\}(3)}$, $H_{\mu\nu}^{(6,3)(1)}$, $H_{\mu\nu}^{(6,3)(2)}$, $H_{\mu\nu}^{(6,3)(3)}$, $H_{\mu\nu}^{(6,3)(4)}$, $H_{\mu\nu}^{(6,3)(5)}$, $H_{\mu\nu}^{(6,3)(6)}$, $H_{\mu\nu}^{(6,3)(7)}$, and $H_{\mu\nu}^{(6,3)(8)}$.

From Eqs. (136a), (137a), and (138a), we easily obtain

$$g^{\mu\nu}H_{\mu\nu}^{(4,2)(1)} = -(2D-2)\Box R + (D/2-2)R^2, \quad \text{(A1)}$$

$$g^{\mu\nu}H_{\mu\nu}^{(4,2)(2)} = -(D/2)\Box R + (D/2-2)R_{pq}R^{pq}, \quad \text{(A2)}$$

$$g^{\mu\nu}H_{\mu\nu}^{(4,2)(3)} = -2\Box R + (D/2-2)R_{pqrs}R^{pqrs}. \quad \text{(A3)}$$

From Eqs. (2.22)–(2.31) of Ref. [89], we obtain after tedious calculations using results and geometrical identities of Ref. [73]

$$g^{\mu\nu}H_{\mu\nu}^{\{2,0\}(1)} = -(2D-2)\Box\Box R - 2R\Box R$$
$$- (D/2-1)R_{;p}R^{;p} \quad \text{(A4)}$$

$$g^{\mu\nu}H_{\mu\nu}^{\{2,0\}(3)} = -(D/2)\Box\Box R + (4-2D)R_{;pq}R^{pq}$$
$$+ (D-4)R_{pq}\Box R^{pq} + (2D-4)R_{pq;rs}R^{prqs}$$
$$- (D/2-3/2)R_{;p}R^{;p} + (5D/2-5)R_{pq;r}R^{pq;r}$$
$$- (4D-6)R_{pq;r}R^{pr;q} - (2D-2)R_{pq}R^p{}_rR^{qr}$$
$$+ (2D-2)R_{pq}R_{rs}R^{prqs} \quad \text{(A5)}$$

$$g^{\mu\nu}H_{\mu\nu}^{(6,3)(1)} = -(6D-6)R\Box R - (6D-6)R_{;p}R^{;p}$$
$$+ (D/2-3)R^3 \quad \text{(A6)}$$

$$g^{\mu\nu}H_{\mu\nu}^{(6,3)(2)} = -(D/2+1)R\Box R - (D-2)R_{;pq}R^{pq}$$
$$- (2D-2)R_{pq}\Box R^{pq} - DR_{;p}R^{;p}$$
$$- (2D-2)R_{pq;r}R^{pq;r}$$
$$+ (D/2-3)RR_{pq}R^{pq} \quad \text{(A7)}$$

$$g^{\mu\nu}H^{(6,3)(3)}_{\mu\nu} = -(3D/2-3)R_{;pq}R^{pq} - 3R_{pq}\Box R^{pq}$$
$$- (3D/8 - 3/4)R_{;p}R^{;p} - 3R_{pq;r}R^{pq;r}$$
$$- (3D/2 - 3)R_{pq;r}R^{pr;q} - DR_{pq}R^{p}{}_{r}R^{qr}$$
$$+ (3D/2 - 3)R_{pq}R_{rs}R^{prqs} \qquad (A8)$$

$$g^{\mu\nu}H^{(6,3)(4)}_{\mu\nu} = -(1/2)R\Box R + (D/2 - 1)R_{;pq}R^{pq}$$
$$- DR_{pq}\Box R^{pq} - (D-2)R_{pq;rs}R^{prqs}$$
$$- (3/4)R_{;p}R^{;p} - (2D-2)R_{pq;r}R^{pq;r}$$
$$+ (2D-3)R_{pq;r}R^{pr;q} + (D-1)R_{pq}R^{p}{}_{r}R^{qr}$$
$$- (D/2+2)R_{pq}R_{rs}R^{prqs} \qquad (A9)$$

$$g^{\mu\nu}H^{(6,3)(5)}_{\mu\nu} = -2R\Box R - 4R_{;pq}R^{pq}$$
$$- (8D-8)R_{pq;rs}R^{prqs} - 4R_{;p}R^{;p}$$
$$- (2D-2)R_{pqrs;t}R^{pqrs;t}$$
$$+ (D/2-3)RR_{pqrs}R^{pqrs}$$
$$- (4D-4)R_{pq}R^{p}{}_{rst}R^{qrst}$$
$$+ (2D-2)R_{pqrs}R^{pquv}R^{rs}{}_{uv}$$
$$+ (8D-8)R_{prqs}R^{p}{}_{u}{}^{q}{}_{v}R^{rusv} \qquad (A10)$$

$$g^{\mu\nu}H^{(6,3)(6)}_{\mu\nu} = -R_{;pq}R^{pq} - 2R_{pq}\Box R^{pq}$$
$$- (2D+2)R_{pq;rs}R^{prqs} - (1/2)R_{;p}R^{;p}$$
$$- (D+2)R_{pq;r}R^{pq;r} + DR_{pq;r}R^{pr;q}$$
$$- (D/4+1/2)R_{pqrs;t}R^{pqrs;t} - 4R_{pq}R^{p}{}_{rst}R^{qrst}$$
$$+ (D/4+1/2)R_{pqrs}R^{pquv}R^{rs}{}_{uv}$$
$$+ (D+2)R_{prqs}R^{p}{}_{u}{}^{q}{}_{v}R^{rusv} \qquad (A11)$$

$$g^{\mu\nu}H^{(6,3)(7)}_{\mu\nu} = -24R_{pq;rs}R^{prqs} - 12R_{pq;r}R^{pq;r}$$
$$+ 12R_{pq;r}R^{pr;q} - 3R_{pqrs;t}R^{pqrs;t}$$
$$- 6R_{pq}R^{p}{}_{rst}R^{qrst} + (D/2)R_{pqrs}R^{pquv}R^{rs}{}_{uv}$$
$$+ 12R_{prqs}R^{p}{}_{u}{}^{q}{}_{v}R^{rusv} \qquad (A12)$$

$$g^{\mu\nu}H^{(6,3)(8)}_{\mu\nu} = (3/2)R_{;pq}R^{pq} - 3R_{pq}\Box R^{pq} + 3R_{pq;rs}R^{prqs}$$
$$- 3R_{pq;r}R^{pq;r} + 3R_{pq;r}R^{pr;q}$$
$$+ (3/4)R_{pqrs;t}R^{pqrs;t} + 3R_{pq}R^{p}{}_{r}R^{qr}$$
$$- 3R_{pq}R_{rs}R^{prqs} + (3/2)R_{pq}R^{p}{}_{rst}R^{qrst}$$
$$- (3/4)R_{pqrs}R^{pquv}R^{rs}{}_{uv}$$
$$+ (D/2-6)R_{prqs}R^{p}{}_{u}{}^{q}{}_{v}R^{rusv}. \qquad (A13)$$

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