Entanglement of quantum fluctuations in the inflationary universe

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(Received 12 May 2008; published 12 August 2008)

We investigate quantum entanglement of a scalar field in the inflationary universe. By introducing a bipartite system using a lattice model of scalar field, we apply the separability criterion based on the partial transpose operation and numerically calculate the bipartite entanglement between separate spatial regions. We find that the initial entangled state becomes separable or disentangled when the size of the spatial regions exceed the Hubble horizon. This is a necessary condition for the appearance of classicality of the quantum fluctuation. We further investigate the condition for the appearance of the classical distribution function and find that the condition is given by the inequality for the symplectic eigenvalue of the covariance matrix of the scalar field.

DOI: 10.1103/PhysRevD.78.044023 PACS numbers: 04.62.+v

I. INTRODUCTION

Inflation provides a mechanism of generation of primordial inhomogeneity, which is needed for the formation of large-scale structures in our present universe. During the qusi-de Sitter expansion stage of the inflationary universe, short wavelength quantum fluctuations of the inflaton field are generated by particle creations and then they are stretched by the cosmic expansion and their wavelength exceeds the Hubble horizon beyond which the physical process proceeds independently. After the wavelength of generated quantum fluctuations become larger than the Hubble horizon, the quantum nature of the fluctuation is expected to be lost and the statistical property of fluctuations is replaced by the classical distribution function. This is the assumption of the quantum to classical transition of the quantum fluctuation generated during the inflation. Once this assumption is adopted, we can use generated "classical" fluctuations as initial perturbations for the large-scale structure formation, which obeys deterministic classical dynamics.

We must explain or justify this assumption of the quantum to classical transition of primordial fluctuation and many investigations have been done on this subject so far [1–12]. A rough outline of the classicalization mechanism proposed in researches [1,5–9] is as follows: In the inflationary universe, particle creations occur due to the accelerated expansion of the Universe and the quantum field becomes a highly squeezed state. For such a highly excited state as a squeezed state, noncommutativity between canonically conjugate variables can be neglected and the operator nature of variables are effectively lost when we evaluate the expectation value of quantum variables. This means we can regard operators as c-numbers. Furthermore, it can be shown that there appears a sharp peak around a line of the Wigner function for the state of quantum fluctuations, and this indicates establishment of classical correlation between canonically conjugate variables on the phase space. At this stage, the Wigner function itself can be interpreted as a classical probability distribution function, and the quantum fluctuations themselves can be treated as classical stochastic variables. Hence, after sufficient squeezing, we can represent the nature of the quantum fluctuations by the classical stochastic variables with an appropriate probability distribution function of which property is determined by the state of the quantum fluctuations. For the classicality of the quantum system, we also need the mechanism of decoherence and the studies along this line have been done with the assumption of an appropriate coupling between the system and the environment [2–4,10–12].

However, these analysis do not pay attention to an important aspect of quantum mechanics, quantum entanglement, which definitely distinguishes a quantum world from a classical one. When we calculate a correlation function of observables between spatially separated two regions, we have a possibility that the quantum correlation function cannot be reproduced using a local classical probability distribution function if these two regions are entangled [13,14] and the classical locality is violated. In other words, we cannot regard the quantum fluctuations as the classical stochastic fluctuations as long as the system is entangled. Therefore, it is important to clarify the relation between the entanglement and the appearance of the classical nature to fully understand the mechanism of the quantum to classical transition of primordial fluctuations.

In this paper, we consider the entanglement of quantum field between two spatially separated regions in a de Sitter universe and aim to understand the mechanism of the quantum to classical transition of fluctuation from the viewpoint of quantum entanglement. This paper is organized as follows: We introduce the concept of entanglement and separability in Sec. II. In Sec. III, we calculate the bipartite entanglement of a scalar field in the de Sitter universe. In Sec. IV, we discuss the relation between the entanglement and the quantum to classical transition in the

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inflationary universe. Sec. V is devoted to summary and conclusion. We use units in which $c = \hbar = 8\pi G = 1$ throughout the paper.

II. SEPARABILITY AND ENTANGLEMENT

In this paper, we focus on a bipartite system composed of two Gaussian modes described by canonical variables $(\hat{q}_1, \hat{p}_1)$ and $(\hat{q}_2, \hat{p}_2)$. This is the simplest case deriving entanglement for continuous variable system. Let Alice be in possession of mode 1 and let Bob be in possession of mode 2. A quantum state $\hat{\rho}$ of the bipartite system is defined to be separable if and only if $\hat{\rho}$ can be expressed in the form [15]

$$\hat{\rho} = \sum_j p_j \hat{\rho}_{jA} \otimes \hat{\rho}_{jB}, \qquad \sum_j p_j = 1, \qquad p_j \geq 0, \quad (1)$$

where $\hat{\rho}_{jA}$ and $\hat{\rho}_{jB}$ are density operators of the modes of Alice and Bob, respectively. If the state of the system cannot be expressed in this form, the quantum state of the system is called entangled. When the state is entangled, the observables associated to the party A and B are correlated and their correlations cannot be reproduced with purely classical means. This leads to the phenomena peculiar to the quantum mechanics such as the EPR correlation [13] and the violation of Bell's inequality [14].

For a bipartite Gaussian two mode system, we have necessary and sufficient conditions for separability [16,17] and we can judge whether the system is entangled or not using these criteria. In this paper, we adopt a criterion proposed by Simon [16], which uses the partial transpose operation for a bipartite system. We define the phase space variables as

$$\hat{\boldsymbol{\xi}} = \begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 \\ \hat{q}_2 \\ \hat{p}_2 \end{pmatrix}. \quad (2)$$

Using these variables, the canonical commutation relations are expressed as

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] = i\Omega_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3, 4,$$
$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{J} & 0 \\ 0 & \boldsymbol{J} \end{pmatrix}, \qquad \boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3)$$

The Gaussian state is completely characterized by the covariance matrix

$$V_{\alpha\beta} = \tfrac{1}{2}\langle \hat{\xi}_\alpha \hat{\xi}_\beta + \hat{\xi}_\beta \hat{\xi}_\alpha \rangle = \tfrac{1}{2}\mathrm{tr}((\hat{\xi}_\alpha \hat{\xi}_\beta + \hat{\xi}_\beta \hat{\xi}_\alpha)\hat{\rho}), \quad (4)$$

where we assume the state with $\langle \xi_\alpha \rangle = 0$. For a physical state, the density matrix must be non-negative and the corresponding covariance matrix must satisfy the inequality [16]

$$\boldsymbol{V} + \frac{i}{2}\boldsymbol{\Omega} \geq 0, \quad (5)$$

which is the generalization of the uncertainty relation between two canonically conjugate variables. The separability of the bipartite Gaussian state is expressed in terms of the partial transpose operation defined by

$$\hat{\boldsymbol{\xi}}' = \boldsymbol{\Lambda}\hat{\boldsymbol{\xi}}, \qquad \boldsymbol{\Lambda} = \mathrm{diag}(1, 1, 1, -1). \quad (6)$$

This operation reverses the sign of Bob's momentum. With this operation, the covariance matrix transforms as

$$\tilde{\boldsymbol{V}} = \boldsymbol{\Lambda}\boldsymbol{V}\boldsymbol{\Lambda}^T. \quad (7)$$

The necessary and sufficient condition of the separability is given by the inequality

$$\tilde{\boldsymbol{V}} + \frac{i}{2}\boldsymbol{\Omega} \geq 0, \quad (8)$$

which represents the physical condition for the partially transposed state. For an entangled state, this inequality is violated and the partially transposed state becomes unphysical. To measure the degrees of entanglement, we introduce the logarithmic negativity via symplectic eigenvalues of the covariance matrix. The covariance matrix can be diagonalized by an appropriate symplectic transformation $\boldsymbol{S} \in \mathrm{Sp}(4, R)$, $\boldsymbol{S}\boldsymbol{\Omega}\boldsymbol{S}^T = \boldsymbol{\Omega}$ as follows: [18,19]

$$\boldsymbol{S}\boldsymbol{V}\boldsymbol{S}^T = \mathrm{diag}(\nu_+, \nu_+, \nu_-, \nu_-), \qquad \nu_+ \geq \nu_- \geq 0, \quad (9)$$

where $\nu_\pm$ are symplectic eigenvalues. In terms of symplectic eigenvalues, the physical condition (5) can be expressed as

$$\nu_- \geq \tfrac{1}{2} \quad (10)$$

and the separability condition (8) can be expressed as

$$\tilde{\nu}_- \geq \tfrac{1}{2}. \quad (11)$$

The logarithmic negativity is defined by

$$E_N = -\min[\log_2(2\tilde{\nu}_-), 0]. \quad (12)$$

For an entangled state, $\tilde{\nu}_- < 1/2$ and we have $E_N > 0$. For a separable state, $\tilde{\nu}_- \geq 1/2$ and we have $E_N = 0$. Practically, the symplectic eigenvalues can be obtained as eigenvalues of the matrix $i\boldsymbol{\Omega}\boldsymbol{V}$ [18,19].

III. ENTANGLEMENT OF QUANTUM FIELD IN THE DE SITTER UNIVERSE

A. One-dimensional lattice model of scalar field

To comprehend the behavior of the entanglement of quantum fields in the inflationary universe, we consider a real massless scalar field ϕ in the de Sitter universe. The metric and the Lagrangian are

$$ds^2 = a(\eta)^2(-d\eta^2 + dx^2), \qquad a = -\frac{1}{H\eta},$$
$$-\infty < \eta < 0, \qquad L = \int d^3x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right), \tag{13}$$

where η is the conformal time and H is the Hubble parameter, which is assumed to be constant in time. By introducing a conformally rescaled variable $q = a\phi$,

$$L = \int d^3x\frac{1}{2}\left[\left(q' - \frac{a'}{a}q\right)^2 - (\partial_i q)^2\right], \tag{14}$$

and the equations of motion of the scalar field is

$$q'' - \frac{a''}{a}q - \partial_i^2 q = 0, \tag{15}$$

where $'$ denotes the derivative with respect to the conformal time η.

To investigate the property of the quantum entanglement of the scalar field, we adopt a discrete lattice model of the scalar field in this paper. This model introduces a cutoff of short wavelength mode of the scalar field, which regularizes the ultraviolet divergence of the vacuum fluctuations. The same model is used to investigate the spatial structure of entanglement in a Minkowski spacetime [20]. To simplify the analysis, we assume that the scalar field depends only on one spatial coordinate and the space is one dimensional. Then the lattice version of the scalar field Lagrangian is

$$L = \frac{\Delta x}{2}\sum_{j=1}^{N}\left[\left(q_j' - \frac{a'}{a}q_j\right)^2(\Delta x)^2 - (q_j - q_{j-1})^2\right], \tag{16}$$

where q_j denotes the scalar field at the j-th lattice site, Δx is a lattice spacing and N is the total number of lattice sites. The equation of motion is

$$q_j'' - \frac{a''}{a}q_j + \frac{1}{(\Delta x)^2}[2q_j - \alpha(q_{j+1} + q_{j-1})] = 0, \qquad j = 1, 2, \cdots, N, \tag{17}$$

$$q_0 = q_N, \qquad q_{N+1} = q_1,$$

where we assume a periodic boundary condition for q_j, and the parameter $\alpha \neq 1$ is introduced to regularize the infrared divergence, which appears in the correlation function of the scalar field. This divergence is peculiar to one-dimensional massless scalar field. The nonunity value of the parameter α corresponds to adding a small mass to the scalar field

$$m^2 = \frac{2(1-\alpha)}{(\Delta x)^2}, \tag{18}$$

and we choose the value of α sufficiently close to unity so that the our result of calculation does not depends on the value of this cutoff parameter. By rescaling the time variable as $\eta \to \eta\Delta x$, the equation of motion can be written as

$$q_j'' - \frac{a''}{a}q_j + 2q_j - \alpha(q_{j+1} + q_{j-1}) = 0. \tag{19}$$

The Hamiltonian is

$$H = \sum_{j=1}^{N}\left[\frac{1}{2}p_j^2 + q_j^2 - \alpha q_j q_{j-1} + \frac{a'}{a}p_j q_j\right]. \tag{20}$$

To quantize this system, we introduce the Fourier expansion of the scalar field on the lattice as follows:

$$q_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\tilde{q}_k e^{i\theta_k j}, \qquad p_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\tilde{p}_k^* e^{i\theta_k j}, \qquad \theta_k = \frac{2\pi k}{N}. \tag{21}$$

The equation of motion for the Fourier mode $\tilde{q}_k$ is

$$\tilde{q}_k'' + \left(\omega_k^2 - \frac{a''}{a}\right)q_k = 0, \qquad \omega_k^2 = 2(1 - \alpha\cos\theta_k). \tag{22}$$

Introducing creation and annihilation operators, the quantized canonical variables are represented as follows:

$$\hat{q}_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}(f_k\hat{a}_k + f_k^*\hat{a}_{N-k}^\dagger)e^{i\theta_k j}, \tag{23}$$

$$\hat{p}_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}(-i)(g_k\hat{a}_k - g_k^*\hat{a}_{N-k}^\dagger)e^{i\theta_k j}, \tag{24}$$

$$[\hat{q}_j, \hat{p}_\ell] = i\delta_{j\ell}, \qquad [\hat{a}_{k_1}, \hat{a}_{k_2}^\dagger] = \delta_{k_1,k_2}, \qquad [\hat{a}_{k_1}, \hat{a}_{k_2}] = [\hat{a}_{k_1}^\dagger, \hat{a}_{k_2}^\dagger] = 0, \tag{25}$$

$$f_k'' + \left(\omega_k^2 - \frac{a''}{a}\right)f_k = 0, \qquad g_k = i\left(f_k' - \frac{a'}{a}f_k\right), \qquad f_k f_k^{*\prime} - f_k' f_k^* = i. \tag{26}$$

As the quantum state of the scalar field, we assume the Bunch-Davis vacuum state, and it corresponds to the following form of the mode functions:

$$f_k = \frac{1}{\sqrt{2\omega_k}}\left(1 + \frac{1}{i\omega_k\eta}\right)e^{-i\omega_k\eta}, \qquad g_k = \sqrt{\frac{\omega_k}{2}}e^{-i\omega_k\eta}. \tag{27}$$

The two point correlation functions between the canonical variables on the lattice sites are given by

$$g_{|j-\ell|} \equiv \frac{1}{2}\langle \hat{q}_j \hat{q}_\ell + \hat{q}_\ell \hat{q}_j \rangle = \frac{1}{N}\sum_{k=0}^{N-1} |f_k|^2 \cos(\theta_k(j-\ell)), \tag{28}$$

$$h_{|j-\ell|} \equiv \frac{1}{2}\langle \hat{p}_j \hat{p}_\ell + \hat{p}_\ell \hat{p}_j \rangle = \frac{1}{N}\sum_{k=0}^{N-1} |g_k|^2 \cos(\theta_k(i-\ell)), \tag{29}$$

$$\begin{aligned} k_{|j-\ell|} &\equiv \frac{1}{2}\langle \hat{q}_j \hat{p}_\ell + \hat{p}_\ell \hat{q}_j \rangle \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \frac{i}{2}(f_k g_k^* - f_k^* g_k)\cos(\theta_k(j-\ell)). \end{aligned} \tag{30}$$

Now, we define a bipartite system using this lattice model. As we are interested in the correlation and the entanglement between different spatial regions, we introduce the following block variables by spatially averaging the variables in given regions A and B (see Fig. 1).

$$\begin{aligned} \hat{q}_A &= \frac{1}{\sqrt{n}}\sum_{j\in A}\hat{q}_j, \qquad \hat{p}_A = \frac{1}{\sqrt{n}}\sum_{j\in A}\hat{p}_j, \\ \hat{q}_B &= \frac{1}{\sqrt{n}}\sum_{j\in B}\hat{q}_j, \qquad \hat{p}_B = \frac{1}{\sqrt{n}}\sum_{j\in B}\hat{p}_j. \end{aligned} \tag{31}$$

The regions A and B contain n lattice sites and the coarse-grained field values are assigned to each regions. The separation between A and B is d. The commutation relations between these coarse-grained variables are

$$[\hat{q}_A, \hat{p}_A] = [\hat{q}_B, \hat{p}_B] = i, \qquad [\hat{q}_A, \hat{p}_B] = 0 \tag{32}$$

and the set of canonical variables $(\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)$ constitutes a bipartite system. The covariance matrix of this bipartite system is given by the following symmetric 4×4 matrix:

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{A} \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}, \qquad \boldsymbol{C} = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix}, \tag{33}$$

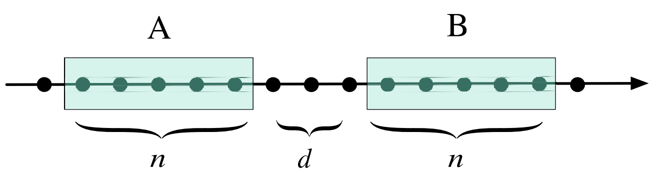

FIG. 1 (color online). A bipartite system in our one-dimensional lattice model.

$$\begin{aligned} a_1 &= \langle \hat{q}_A^2 \rangle = \langle \hat{q}_B^2 \rangle = \frac{1}{n}\sum_{i,j\in A} g_{|i-j|}, \\ a_2 &= \langle \hat{p}_A^2 \rangle = \langle \hat{p}_B^2 \rangle = \frac{1}{n}\sum_{i,j\in A} h_{|i-j|}, \\ a_3 &= \frac{1}{2}\langle \hat{q}_A \hat{p}_A + \hat{p}_A \hat{q}_A \rangle = \frac{1}{n}\sum_{i,j\in A} k_{|i-j|}, \\ c_1 &= \frac{1}{2}\langle \hat{q}_A \hat{q}_B + \hat{q}_B \hat{q}_A \rangle = \frac{1}{n}\sum_{i\in A, j\in B} g_{|i-j|}, \\ c_2 &= \frac{1}{2}\langle \hat{p}_A \hat{p}_B + \hat{p}_B \hat{p}_A \rangle = \frac{1}{n}\sum_{i\in A, j\in B} h_{|i-j|}, \\ c_3 &= \frac{1}{2}\langle \hat{q}_A \hat{p}_B + \hat{p}_B \hat{q}_A \rangle = \frac{1}{n}\sum_{i\in A, j\in B} k_{|i-j|}. \end{aligned}$$

As we do not observe the degrees of freedom of outside the regions A and B, the evolution of this bipartite system is nonunitary. Thus, we take into account the effect of decoherence through our definition of bipartite system. Using these components of the covariance matrix $\boldsymbol{V}$, the symplectic eigenvalues are given by

$$(\nu_-)^2 = a_1 a_2 - a_3^2 + c_1 c_2 - c_3^2 - |a_1 c_2 + a_2 c_1 - 2a_3 c_3|, \tag{34}$$

$$\begin{aligned} (\tilde{\nu}_-)^2 = a_1 a_2 - a_3^2 - c_1 c_2 + c_3^2 - |(a_1 c_2 - a_2 c_1)^2 \\ + 4(a_1 c_3 - a_3 c_1)(a_2 c_3 - a_3 c_2)|^{1/2}, \end{aligned} \tag{35}$$

and we can apply the separability criterion (11) to judge whether the system is separable or entangled.

B. Numerical result

We calculated the logarithmic negativity of this system numerically. The number of lattice sites is $N = 100$ and the value of the infrared cutoff parameter is chosen to be $\alpha = 0.9999$.

Fig. 2 shows the logarithmic negativity E_N as a function of separation d between the regions A and B with the region size $n = 4$. Initially ($\eta = -10$, the left panel), $E_N \neq 0$ for $d = 0$, and $E_N = 0$ for $d \geq 1$. The regions A and B are entangled for $d = 0$ and separable for $d \geq 1$. This implies that the system is intrinsically entangled at this time because the choice of the separation d corresponds to the choice of measurement; how to observe the system. As the system evolves, the logarithmic negativity becomes zero for any d ($\eta = -0.9$, the right panel) and we can say that the system becomes separable at this time. This behavior is not changed for the other value of the region size n. The spatial structure of entanglement for this lattice model is simple and we only pay attention to the behavior of entanglement for $d = 0$.

Then, we consider the evolution of the entanglement. Fig. 3 shows the evolution of symplectic eigenvalues ν_-

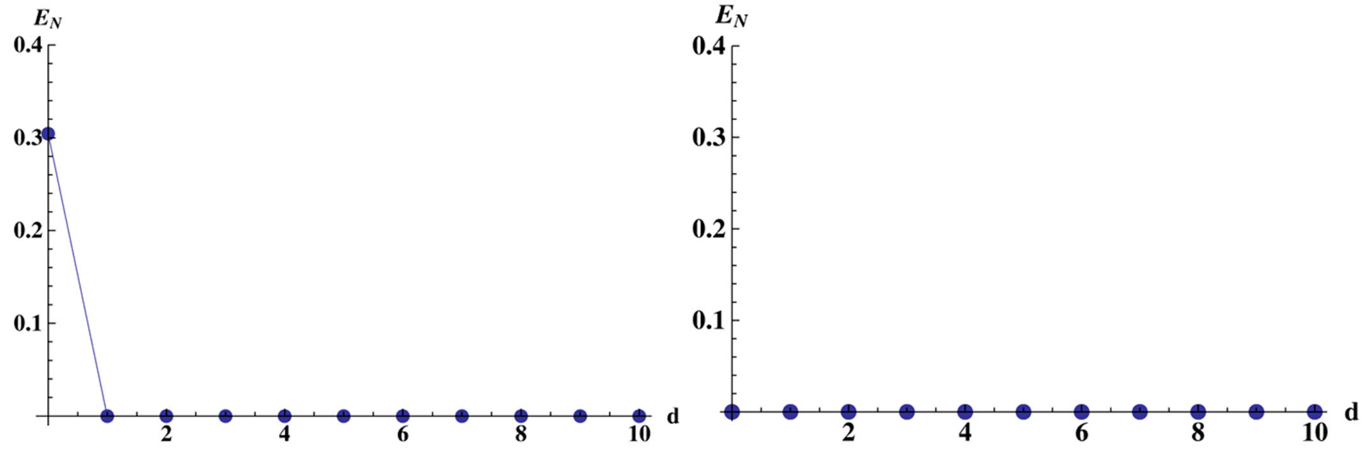

FIG. 2 (color online). The dependence of the separation d on E_N for region size $n = 4$. The left panel is E_N at $\eta = -10$ and the right panel is E_N at $\eta = -0.9$.

and $\tilde{\nu}_-$ for $d = 0$ with $n = 4$. During the entire period of evolution, the value of ν_- is greater than $1/2$ and the physical condition (10) is always satisfied. On the other hand, the value of $\tilde{\nu}_-$ is smaller than $1/2$ initially, then increases and exceeds the value $1/2$. Thus, the initial entangled state changes into the separable state.

We interpret these symplectic eigenvalues behaviors using the logarithmic negativity. Fig. 4 shows the logarithmic negativity for $d = 0$ as a function of conformal time. At some critical time $\eta_c \approx -1$, the logarithmic negativity E_N becomes zero and the initially entangled state changes into a separable state after η_c. As the quantum state, we assume the Bunchi-Davis vacuum, which imposes the Minkowski vacuum state in the short wave length limit. Thus, the entanglement of the scalar field before η_c implies the remnant of the entanglement of the Minkowski vacuum. After $\eta = \eta_c$, the regions A and B do not have quantum correlation and we expect that the correlation between two regions can be mimicked by an appropriate classical distribution function. To understand what time scale determines the critical time η_c, we observed how the critical time η_c varies when we change the region size n.

Fig. 5 shows the relation between the critical time η_c and the region size n. For small value of n, the relation coincides with the line $\eta_c = -n$ (dotted line). By restoring the dimension of the variables, this relation corresponds to

$$a(\eta_c) \times n\Delta x = \frac{1}{H}. \tag{36}$$

Thus, the quantum entanglement between the regions A and B disappears when the physical size of each region exceeds the Hubble horizon length. We must keep in mind that the physical size of each region grows as $a(\eta) \times n\Delta x$ by the cosmic expansion. For larger values of n, the relation deviates from the line $\eta_c = -n$ and we suppose

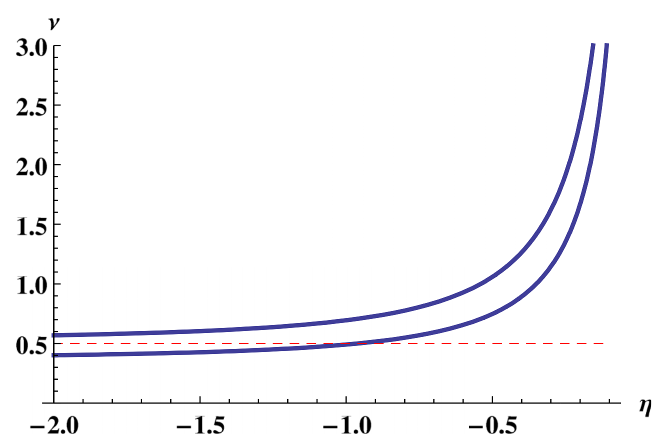

FIG. 3 (color online). Evolution of symplectic eigenvalues for $d = 0$, $n = 4$. The upper line represents ν_- and the lower line represents $\tilde{\nu}_-$. The physical condition $\nu_- > 1/2$ is always satisfied.

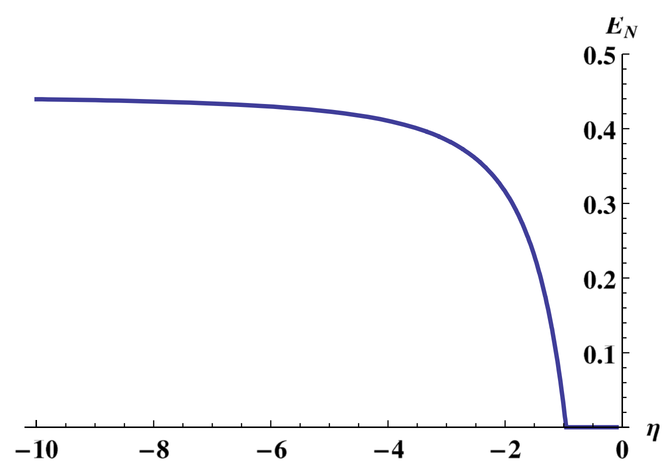

FIG. 4 (color online). $E_N(d = 0)$ as a function of the conformal time η for $n = 4$ case. After $\eta = \eta_c \approx -1$, E_N becomes zero and the system is separable.

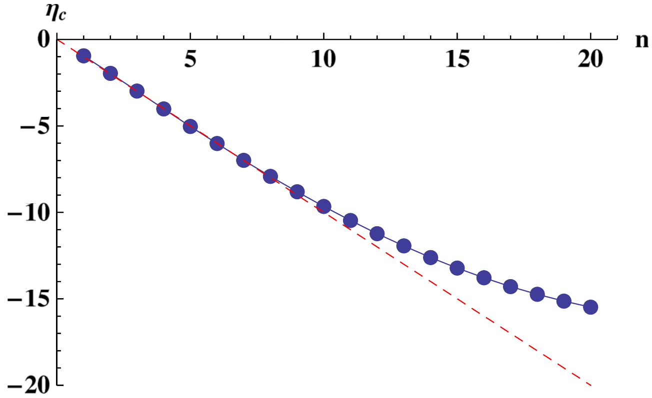

FIG. 5 (color online). The relation between the critical time η_c and the region size n. The deviation from a linear relation represents the effect of boundary condition.

this is due to the effect of the periodic boundary condition we imposed for the lattice model. The appearance of separability or disentanglement at the Hubble horizon scale coincides with our naive classical picture of the separate universe; the Hubble size inflationary domains evolves independently and they can be treated as independent Friedman-Robertson-Walker universes. As is shown here, the quantum entanglement between the Hubble size regions is lost and these regions do not have the quantum correlation. This is quantum version of the separate universe picture. Hence, we can expect that the quantum fluctuations in these regions behave classical.

IV. ENTANGLEMENT AND QUANTUM TO CLASSICAL TRANSITION

To clarify the condition of quantum to classical transition of fluctuations in the inflationary universe, we consider the relation between the disentanglement and the classicalization of quantum fluctuations. We can say that the quantum fluctuations become classical if they are mimicked by appropriate classical stochastic variables and their statistical nature cannot be distinguished from quantum ones. More precisely, the bipartite quantum system under the consideration is defined to be classical if there exists a positive normalizable distribution function $\mathcal{P}$ on the phase space and the following relation holds for any function F of canonical variables:

$$\langle F(\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)\rangle = \int dq_A dp_A dq_B dp_B \mathcal{P}(q_A, p_A, q_B, p_B) F(q_A, p_A, q_B, p_B). \tag{37}$$

The left-hand side of this relation is evaluated with respect to the considering quantum state. If this relation holds, any correlation function of quantum variables in the bipartite system can be imitated by the classical distribution function $\mathcal{P}$ on the phase space and appropriate classical stochastic variables.

As is shown by Simon [16] and Duan *et al.*[17], for a bipartite two-mode Gaussian system, if the system is separable then the state of the system can be written by the following form of P representation: [21]

$$\hat{\rho} = \int d^2\alpha d^2\beta P(\alpha, \beta)|\alpha, \beta\rangle\langle\alpha, \beta|, \tag{38}$$

where $|\alpha, \beta\rangle = |\alpha\rangle|\beta\rangle$ is the product of the coherent state of A and B, and $P(\alpha, \beta)$ is a positive normalizable function called P function. For a Gaussian state with the covariance matrix $\boldsymbol{V}$, the P function is also a Gaussian function and given by

$$\begin{aligned} P(\boldsymbol{\xi}) &= \frac{1}{4\pi^2}\sqrt{\det \boldsymbol{P}}\exp\left(-\frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{P}\boldsymbol{\xi}\right), \\ \boldsymbol{P} &= \left(\boldsymbol{V} + \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{S}^T\boldsymbol{S}\boldsymbol{\Omega}^T\right)^{-1}, \end{aligned} \tag{39}$$

where $\boldsymbol{S} \in \mathrm{Sp}(2, R) \otimes \mathrm{Sp}(2, R)$ is the symplectic transformation that transforms the covariance matrix $\boldsymbol{V}$ to the following standard form [17]:

$$\begin{aligned} \boldsymbol{V}_{II} &= \boldsymbol{S}\boldsymbol{V}\boldsymbol{S}^T = \begin{pmatrix} ar & & cr & \\ & a/r & & c'/r \\ cr & & ar & \\ & c'/r & & a/r \end{pmatrix}, \\ r &= \sqrt{\frac{a-|c'|}{a-|c|}}. \end{aligned} \tag{40}$$

From the definition of P function (39), we have

$$\boldsymbol{P} = \boldsymbol{S}^T\left(\boldsymbol{V}_{II} - \frac{\boldsymbol{I}}{2}\right)^{-1}\boldsymbol{S} \tag{41}$$

and the existence of a positive normalizable P function is guaranteed by the condition $(\boldsymbol{V}_{II} - \boldsymbol{I}/2) \geq 0$. In terms of the components of $\boldsymbol{V}_{II}$, this condition is represented as

$$(a - |c|)(a - |c'|) \geq \tfrac{1}{4}. \tag{42}$$

On the other hand, the symplectic eigenvalues of $\boldsymbol{V}$, which is invariant under symplectic transformations, are

$$\nu^2 = \begin{cases} (a-|c|)(a-|c'|), (a+|c|)(a+|c'|), & cc' \geq 0 \\ (a-|c|)(a+|c'|), (a+|c|)(a-|c'|), & cc' < 0 \end{cases}, \tag{43}$$

$$\tilde{\nu}^2 = \begin{cases} (a-|c|)(a+|c'|), (a+|c|)(a-|c'|), & cc' \geq 0 \\ (a-|c|)(a-|c'|), (a+|c|)(a+|c'|), & cc' < 0. \end{cases} \tag{44}$$

Therefore, the condition of existence of positive normalizable P function is equivalent to the separability condition (11) provided that the physical condition (10) is satisfied. From the definition of P representation (38), if the system is separable, it is possible to calculate the quantum expectation value of the normally ordered product of any operators using the P function as a distribution function

$$\langle :F(\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B):\rangle = \int dq_A dp_A dq_B dp_B P(q_A, p_A, q_B, p_B) F(q_A, p_A, q_B, p_B). \tag{45}$$

However, the existence of P function is not sufficient for the establishment of classicality of the system; it only guarantees the existence of the distribution function for the normally ordered quantities.

To derive the condition of classicality of the quantum field, we introduce a Wigner distribution function on the phase space and its form for a Gaussian state is given by

$$W(\boldsymbol{\xi}) = \frac{1}{4\pi^2\sqrt{\det \boldsymbol{V}}} \exp\left(-\frac{1}{2}\boldsymbol{\xi}^T \boldsymbol{V}^{-1}\boldsymbol{\xi}\right). \tag{46}$$

The normalizable Winger function exists for $\boldsymbol{V} \geq 0$ and this condition is weaker than the physical condition of the state (5). Hence, there exists a normalizable Wigner function, which does not represent the physical state. The Wigner function (46) for $\boldsymbol{V} \geq 0$ is positive normalizable and can be interpreted as a distribution function giving the expectation value for the symmetrically ordered product of operators

$$\langle \{F(\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)\}_{\rm sym}\rangle = \int dq_A dp_A dq_B dp_B W(q_A, p_A, q_B, p_B) F(q_A, p_A, q_B, p_B). \tag{47}$$

If the difference between the P function and the Wigner function is negligible, these distribution functions return the same answer for the expectation value of any operator $\hat{F}$ and we have the relation

$$\langle :\hat{F}:\rangle \approx \langle \{\hat{F}\}_{\rm sym}\rangle \approx \langle \hat{F}\rangle. \tag{48}$$

This means that noncommutativity between operators is negligible and the P function or the Wigner function plays a role of the classical distribution function, which reproduces the quantum expectation value for any operators. Hence, the condition of classicality (37) is established.

We look for the condition for the establishment of the relation (48). For this purpose, it is sufficient to consider the condition for the standard form of the covariance matrix $\boldsymbol{V}_{II}$ because this form of the covariance matrix is related to the original covariance matrix $\boldsymbol{V}$ via a symplectic transformation. In terms of the covariance matrix, the condition for the classicality $P \approx W$ is given by

$$\boldsymbol{V}_{II}^{-1} \approx (\boldsymbol{V}_{II} - \boldsymbol{I}/2)^{-1}. \tag{49}$$

We write down the components of $\boldsymbol{V}_{II}^{-1}$ and $(\boldsymbol{V}_{II} - \boldsymbol{I}/2)^{-1}$ explicitly

$$\boldsymbol{V}_{II}^{-1} = \begin{pmatrix} \frac{a/r}{a^2 - c^2} & & -\frac{c/r}{a^2-c^2} & \\ & \frac{ar}{a^2-c'^2} & & -\frac{c'r}{a^2-c'^2} \\ -\frac{c/r}{a^2-c^2} & & \frac{a/r}{a^2-c^2} & \\ & -\frac{c'r}{a^2-c'^2} & & \frac{ar}{a^2-c'^2} \end{pmatrix},$$

$$(\boldsymbol{V}_{II} - \boldsymbol{I}/2)^{-1} = \begin{pmatrix} \frac{(a-1/2r)/r}{(a-1/2r)^2 - c^2} & & -\frac{c/r}{(a-1/2r)^2-c^2} & \\ & \frac{(a-r/2)r}{(a-r/2)^2-c'^2} & & -\frac{c'r}{(a-r/2)^2-c'^2} \\ -\frac{c/r}{(a-1/2r)^2-c^2} & & \frac{(a-1/2r)/r}{(a-1/2r)^2-c^2} & \\ & -\frac{c'r}{(a-r/2)^2-c'^2} & & \frac{(a-r/2)r}{(a-r/2)^2-c'^2} \end{pmatrix}.$$

Thus, if the condition

$$f_1 \equiv a^2 r^2 = a^2 \frac{a - |c'|}{a - |c|} \gg \frac{1}{4}, \qquad f_2 \equiv \frac{a^2}{r^2} = a^2 \frac{a - |c|}{a - |c'|} \gg \frac{1}{4} \tag{50}$$

is satisfied, $\boldsymbol{V}_{II}^{-1} \approx (\boldsymbol{V}_{II} - \boldsymbol{I}/2)^{-1}$ and the P function equals the Wigner function. We rewrite the condition (50) in terms of the symplectic eigenvalues. For $cc' > 0$, $\nu_-^2 = (a - |c|)(a - |c'|)$ and

$$f_1 = \frac{a^2 \nu_-^2}{(a - |c|)^2} > \nu_-^2, \qquad f_2 = \frac{a^2 \nu_-^2}{(a - |c'|)^2} > \nu_-^2, \tag{51}$$

and the condition (50) is satisfied if $\nu_- \gg 1/2$. For $cc' < 0$,

$$f_1 = \frac{a^2 \tilde{\nu}_-^2}{(a - |c|)^2} > \tilde{\nu}_-^2, \qquad f_2 = \frac{a^2 \tilde{\nu}_-^2}{(a - |c'|)^2} > \tilde{\nu}_-^2, \tag{52}$$

and the condition (50) is satisfied if $\tilde{\nu}_- \gg 1/2$. Combining these two cases, we have the following result:

$$\nu_- \gg \tfrac{1}{2}, \quad \tilde{\nu}_- \gg \tfrac{1}{2} \Rightarrow \boldsymbol{V}_{II}^{-1} \approx (\boldsymbol{V}_{II} - \boldsymbol{I}/2)^{-1} \Longleftrightarrow P \approx W. \tag{53}$$

Therefore, $\nu_-, \tilde{\nu}_- \gg 1/2$ is the sufficient condition for the system can be treated as classical.

We can check whether this condition is satisfied in our lattice model. As is show in Fig. 3, before the critical time $\eta_c \approx -1$, $\nu_- > 1/2$ and $\tilde{\nu}_- < 1/2$ and the system is entangled. After η_c, the value of $\tilde{\nu}_-$ becomes greater than $1/2$ and increases in time. In our lattice model, the relation $\tilde{\nu}_- < \nu_-$ always holds. For $-1 \lesssim \eta < 0$, the behavior of $\tilde{\nu}_-$ is approximately given by

$$(\tilde{\nu}_-)^2 - \frac{1}{4} \approx \frac{0.10\,605}{\eta^2} - 0.115\,144 + O(\eta^2) \tag{54}$$

and after $\eta = \eta_c$, the condition $\tilde{\nu}_- \gg 1/2$ is rapidly realized. Hence, the difference between the P function and the Wigner function becomes negligible in one Hubble time after the system becomes separable at the horizon crossing. As a subset of the separability condition for $\boldsymbol{V}_{II}$, the inequality

$$a^2 \geq \tfrac{1}{4} \tag{55}$$

holds for the physical state and this corresponds to a standard uncertainty relation. The condition of classicality $\nu_-, \tilde{\nu}_- \gg 1/2$ leads to $a^2 \gg 1/4$ and this also implies the noncommutativity between canonical variables can be neglected when we evaluating the expectation values of operators. This is consistent with the result obtained in the paper [1,5–8]; for superhorizon scale quantum fluctuations, the noncommutativity between canonical variables becomes negligible because the growing mode solution is dominant. In other words, we can neglect $\hbar$ in the uncertainty relation. We derived the equivalent condition for the classicality from the condition of the existence of the classical distribution function and the symplectic eigenvalues.

V. SUMMARY AND CONCLUSION

We investigated the appearance of the classical distribution function for the quantum fluctuation in the inflationary model using the lattice model of the scalar field. By following the evolution of entanglement between two spatially separated regions, we found the classicality of the quantum field appears as follows: Initially, when the size of the considering region is smaller than the Hubble horizon, the quantum field is in the entangled state. As the Universe expands, the quantum state becomes separable when the size of the region equals the size of the Hubble horizon. At this stage, the quantum correlation between neighboring regions is lost. Then, within about one Hubble time after the horizon crossing, noncommutativity of operators becomes negligible and the system can be treated as classical. Any quantum expectation values can be evaluated using the P function or the Wigner function. In other words, there appears classical stochastic nature for variables, which mimics the original quantum dynamics. As we have shown, disentanglement is not a sufficient condition for the establishment of classicality of fluctuations. This condition only guarantees the loss of EPR-type nonlocal correlations, which are peculiar to quantum mechanics.

In our analysis, we defined a bipartite system as the subsystem of the entire universe and we discard the unobserved dynamical degrees of freedom outside of the observed region. Thus, our bipartite system evolves in a nonunitary way and this definition of our system effectively takes into account the decoherence mechanism of the considering region.

We comment on the relation of our analysis to the stochastic approach of inflation [22], which treats the quantum dynamics of inflaton fields as the classical stochastic process. By coarse graining the scalar field on the large scale $(\epsilon H)^{-1}$, $\epsilon \ll 1$, it can be shown that the coarse grained field obeys the Langevin equation and the dynamics of the quantum inflaton field is replaced by the classical stochastic process. In the stochastic approach, the classical nature of the inflaton field is guaranteed by the appropriate small value of the coarse graining parameter ϵ. However, in this approach, the connection between the probability distribution and the state of the inflaton field is not clear. The stochastic approach assumes the existence of the classical probability distribution from the first. However, as we have shown in this paper, it is possible to define a probability distribution function only when the system becomes separable. We expect that the condition $\epsilon \ll 1$ corresponds to the condition $\tilde{\nu} \gg 1/2$, which is stronger than the separability. Anyways, we must reconsider the meaning of the probability in the stochastic approach from the view point of entanglement. We will report on this topic in a separate publication.

In this paper, we assumed the Bunch-Davis vacuum state. Previous analysis by J. Lesgourgues *et al.* [7] considered the nonvacuum initial states that are non-Gaussian and concluded that the non-Gaussian nature of the state does not affect the transition to the classicalbehavior. However, from the viewpoint of the entanglement, the condition of the separability for non-Gaussian states is unknown and the determination of a classicality condition for such states is an unsolved problem. Further, we considered the quantum to classical transition based on the bipartite entanglement only. This is because the criterion on the separability for the general N-partite system is unknown [15]. However it is necessary to look for the classicality condition for non-Gaussian states and the N-partite system to fully understand the mechanism of classical to quantum transition of primordial fluctuation. This is a future problem to be tackled.

ACKNOWLEDGMENTS

The author would like to thank Yuji Ohsumi for valuable discussions on this subject. This work was supported in part by a JSPS Grant-In-Aid for Scientific Research [C] (19540279).

[1] A. H. Guth and S.-Y. Pi, Phys. Rev. D **32**, 1899 (1985).
[2] M. Sakagami, Prog. Theor. Phys. **79**, 442 (1988).
[3] R.H. Brandenberger, R. Laflame, and M. Mijic, Mod. Phys. Lett. A **5**, 2311 (1990).
[4] Y. Nambu, Phys. Lett. B **276**, 11 (1992).
[5] A. Albrecht, P. Ferreira, M. Joyce, and T. Prokopec, Phys. Rev. D **50**, 4807 (1994).
[6] D. Polarski and A. A. Starobinsky, Classical Quantum Gravity **13**, 377 (1996).
[7] J. Lesgourgues, D. Polarski, and A. A. Starobinsky, Nucl. Phys. **B497**, 479 (1997).
[8] C. Kiefer, J. Lesgourgues, D. Polarski, and A. A. Starobinsky, Classical Quantum Gravity **15**, L67 (1998).
[9] C. Kiefer, I. Lohmar, D. Polarski, and A. A. Starobinsky, Classical Quantum Gravity **24**, 1699 (2007).
[10] J.W. Sharman and G.D. Moore, J. Cosmol. Astropart. Phys. 11 (2007) 020.
[11] P. Martineau, Classical Quantum Gravity **24**, 5817 (2007).
[12] C. P. Burgess, R. Holman, and D. Hoover, Phys. Rev. D **77**, 063534 (2008).
[13] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[14] J.S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[15] S.L. Braunstein and P. Loock, Rev. Mod. Phys. **77**, 513 (2005).
[16] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[17] L. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[18] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, 087901 (2004).
[19] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
[20] J. Kofler, V. Vedral, M. S. Kim, and Č. Brukner, Phys. Rev. A **73**, 052107 (2006).
[21] C.W. Gardiner and P. Zoller, *Quantum Noise* (Springer, New York, 2004).
[22] A. A. Starobinsky, in *Current Topics in Field Theory, Quantum Gravity and Strings*, edited by H.J. de Vega and N. Sanchez, Lecture Notes in Physics, 107 vol. 206 (Springer, Heidelberg, 1986).