Thin shell dynamics and equations of state

J. P. Krisch and E. N. Glass

Department of Physics, University of Michigan, Ann Arbor, Michigan 48109, USA

(Received 6 June 2008; published 4 August 2008)

A relation between stress-energy and motion is derived for accelerated Israel layers. The relation, for layers between two Schwarzschild manifolds, generalizes the equation of state for geodesic collapse. A set of linked layers is discussed.

DOI: 10.1103/PhysRevD.78.044003 PACS numbers: 04.40.Dg, 04.20.Cv

I. INTRODUCTION

Equations of state and boundary matching are two important tools for developing exact solutions of Einstein's field equations. As models have become more physical, the requirements of strict boundary matching have been relaxed by joining two exact solutions across a boundary layer, matching their metrics on the layer but allowing jumps in the derivative structure. The Israel junction conditions [1] are often used to make the broader matching and are widely applied because they provide a simple dynamic boundary description for a variety of scenarios ranging from thin shell descriptions [2–5], to shell applications like bubbles [6–10], walls [11–13], gravastars [14–18], and extensions of general relativity like dilatons [19] or Gauss-Bonnet gravity [20,21]. Since their introduction, Israel layers [1,22] have played an increasingly important role in gravitational physics. Barrabes and Israel [22] began their paper with a description of the Israel layer as a thermodynamic phase boundary, but the initial applications of Israel layers considered metric matching in dynamic collapse processes involving dust shells, null shells, and cosmic string loops [1,22]. Poisson [23] has summarized some of the early seminal work [1,22,24,25]. As our knowledge of the variety of astrophysical objects and their dynamic processes has expanded, Israel layers have become physically interesting in their own right and the questions that have been investigated for large scale three-dimensional mass distributions are now being asked about Israel layers [26–30].

In this paper we investigate the relation between a layer's equation of state and its motion. An equation of state for a layer dropping from rest at infinity on an exterior Schwarzschild geodesic has been formulated [31], and can be generalized to include accelerated motions. We develop the extension and apply it to static and dynamic examples. The motion input to the geodesic extension is a single function. In the next section we briefly review the thin shell formalism used in the rest of the paper and give the geodesic extension. The applications of the extension are in Sec. III. Many of the applications use a layer with a linear equation of state (L-layer), $P = a\sigma$. The description of an L-layer has been included in Appendix B. The accelerated extension singles out several special a values, $a = (-1/2, -1/4, 0)$. We show that the $a = -1/4$ L-layer is simply related to a geodesic layer. A set of linked L-layers is described and the other two special a values are shown to be boundaries for the linked layers.

II. THIN SHELL DESCRIPTION

We consider an Israel layer as a thin shell Σ between two Schwarzschild manifolds with exterior mass parameter m_0 and interior mass parameter M. The spacetime consists of the two manifolds which join across surface Σ ($+/-$ denotes exterior/interior).

$$g^{\pm}_{ab}dx^a dx^b = -(f_\pm)dt^2 + (1/f_\pm)dr^2 + r^2 d\Omega^2, \tag{1a}$$

$$f_+ = 1 - 2m_0/R, \qquad f_- = 1 - 2M/R, \tag{1b}$$

$$ds^2_\Sigma = -d\tau^2 + R^2(\tau)d\Omega^2. \tag{1c}$$

The layer is tracked by two observers comoving with the layer. The observers use $r = R(\tau)$ and $T_\pm = T_\pm(\tau)$ to describe the layers. With this parametrization and with overdots denoting $d/d\tau$, the observers' velocities and associated normal vectors are

$$U^i_\pm = (\dot{T}_\pm, \dot{R}, 0, 0), \qquad i = t, r, \vartheta, \varphi \tag{2}$$

$$n_{i\pm} = (-\dot{R}, \dot{T}_\pm, 0, 0). \tag{3}$$

Velocity normalization $g^{\pm}_{ij}U^i_\pm U^j_\pm = -1$ implies $(f_\pm)^2\dot{T}^2_\pm = \dot{R}^2 + f_\pm$. The jump in $f_\pm$, $\dot{T}_\pm$, is an important input to layer motion, and we define

$$\Delta = \Delta_+ - \Delta_- \tag{4}$$

$$\Delta_\pm = f_\pm \dot{T}_\pm = \sqrt{\dot{R}^2 + f_\pm}. \tag{5}$$

A. Matter content

The layer has stress-energy content

$$S^i{}_j := \sigma U^i U_j + P(g^i{}_j + U^i U_j) \tag{6}$$

with density σ and stress P ($-P$ is tension). The stress-energy content of layers depends on the bounding metrics and the layer motion. From the Israel conditions [23] we have

$$4\pi\sigma = -\Delta/R, \tag{7}$$

$$8\pi P = \Delta/R + \dot{\Delta}/\dot{R}, \tag{8}$$

with the layer mass m_L defined by

$$m_L = 4\pi R^2 \sigma. \tag{9}$$

B. Layer motion

Motion parameters $\dot{R}^2$ and $\ddot{R}$ are both important inputs to the stress-energy structure of the layer,

$$\dot{R}^2 = \left(\frac{\Delta}{2}\right)^2 + \left(\frac{m_0 - M}{R\Delta}\right)^2 + \frac{m_0 + M}{R} - 1, \tag{10}$$

$$\ddot{R} = \frac{\dot{\Delta}}{\dot{R}}\left[\frac{\Delta}{4} - \frac{(m_0 - M)^2}{\Delta^3 R^2}\right] - \frac{m_0 + M}{2R^2} - \frac{(m_0 - M)^2}{\Delta^2 R^3}. \tag{11}$$

The radial components of the 4-accelerations of observers comoving with the layer (Appendix A) are

$$\dot{U}^r_{\pm} = \ddot{R} + (1 - f_{\pm})/(2R). \tag{12}$$

The radial accelerations are related to a stress-energy sum

$$4\pi(\sigma + 2P) = \frac{\dot{\Delta}}{\dot{R}} = \frac{\dot{U}^r_+}{\sqrt{1 + \dot{R}^2 - 2m_0/R}} - \frac{\dot{U}^r_-}{\sqrt{1 + \dot{R}^2 - 2M/R}}. \tag{13}$$

These relations provide insight about the role of jumps in the observer accelerations and velocities in defining the stress energy. It is clear that the stress-energy structure of the layer and its motion are related so that different assumptions about the motion will produce different equations of state or, inversely, that an imposed equation of state will determine the layer motions.

C. The geodesic generalization

A layer starting from rest at infinity and collapsing along a geodesic in the exterior spacetime has motion parameters

$$\dot{R}_g = -\sqrt{2m_0/R}, \tag{14a}$$

$$\ddot{R}_g = -m_0/R^2, \tag{14b}$$

$$R_g(\tau) = (2m_0)^{1/3}[(3/2)(c_1 - \tau)]^{2/3}. \tag{14c}$$

The equation of state for the geodesic layer [31] is

$$\sigma(1 + 4P/\sigma)^3 = (2P/\sigma)^2 \frac{(1 + 2P/\sigma)}{\pi(m_0 - M)}. \tag{15}$$

This geodesic equation of state is the limit ($\gamma \to 0, A \to 0$) of a layer with exterior motion parameters

$$\gamma = \dot{R}^2 - \frac{2m_0}{R}, \tag{16}$$

$$A = \dot{U}^r_+ = \ddot{R} + \frac{m_0}{R^2}, \tag{17}$$

where A is the radial 4-acceleration of the exterior comoving observer and $1 + \gamma = \Delta_+^2$. A and γ are related by $A = \dot{\gamma}/(2\dot{R})$. With these parameters, Eq. (13) can be written as

$$4\pi(\sigma + 2P) = \frac{A}{\sqrt{1+\gamma}} - \frac{A + (M - m_0)/R}{\sqrt{1 + \gamma + 2(m_0 - M)/R}}. \tag{18}$$

In this equation, the motion parameters R, A, and γ require several separate choices to set integration constants. The stress-energy-motion relation (Appendix C) involving a single motion function $F := AR/(1 + \gamma)$ is

$$\pi\sigma(m_0 - M)(1 + 4P/\sigma)^3 = (1 + \gamma)^{3/2}(F - 2P/\sigma)^2 \times (1 + F + 2P/\sigma). \tag{19}$$

This is the generalization of the geodesic equation of state. For $m_0 = M$ there is no layer and the motion is described by $\dot{R}A = 0$, $\gamma = \text{const}$. In developing applications of this equation, one notes that two key inputs are either the ratio P/σ or the value of F. Choosing a static layer ($\dot{R} = 0, \ddot{R} = 0$) sets the value of F and a general equation of state results. Choosing P/σ selects the motion; for example, the equations of state $P/\sigma = (-1/4, -1/2, 0)$ are all simplifying special values for the relation. In the next section we will examine all three values. We begin with dynamic layers and consider static layers as a second example.

III. APPLICATION TO NONGEODESIC LAYER MOTION

A. Dynamic layers: $P = -(1/4)\sigma$

$P = a\sigma = -(1/4)\sigma$ is singled out by the generalized stress-energy-motion relation. From Eq. (19), for this equation of state, one has

$$F = \frac{AR}{1 + \gamma} = -1/2. \tag{20}$$

This can be integrated for $1 + \gamma$ giving

$$1 + \gamma = \frac{C_0}{R} \tag{21}$$

with C_0 an integration constant. Using the motion function, A is

$$A = -\frac{C_0}{2R^2} = \frac{C_0}{2m_0}\ddot{R}_g. \tag{22}$$

Layers that have a radial 4-acceleration linearly related to the geodesic $\ddot{R}_g$ value will have the $P = -\sigma/4$ equation of state. They are L-layers with tension. In the section on static layers, we will see that $a = -1/4$ is an excluded value for static L-layers. We will also see that $a = -1/4$ is an important boundary point for some of them. Note that there are moving L-layers with tension.

B. Linked dynamic layers

Because of its simplicity, the linear equation of state is often used in discussing thin shells [6,13,32]. For L-layers (Appendix B)

$$\Delta = -c_a R^{-(1+2a)}.$$

For this Δ, the motion of the layer is described by Eq. (10)

$$\dot{R}^2 = \frac{c_a^2 R^{-2(1+2a)}}{4} + \frac{(m_0 - M)^2}{c_a^2 R^{-4a}} + \frac{m_0 + M}{R} - 1. \quad (23)$$

There is an interesting symmetry in this equation. Over the positive a range, $a = a_p$, $0 \le a_p \le 1$, Eq. (23) describes the motion. For $a = -a_n$, $0 \le a_n \le 1$, the motion is described by

$$\dot{R}^2 = \frac{c_a^2 R^{-2(1-2a_n)}}{4} + \frac{(m_0 - M)^2}{c_a^2 R^{4a_n}} + \frac{m_0 + M}{R} - 1. \quad (24)$$

The substitutions

$$a_n = a_p + 1/2, \qquad c_{a_n}^2 = 4(m_0 - M)^2/c_{a_p}^2, \quad (25)$$

map these two motion equations into each other. The a range over which a layer with tension is linked to a layer with pressure is

$$1/2 \le a_n \le 1 \qquad 0 \le a_p \le 1/2. \quad (26)$$

For each negative $a = -a_n$ in this range, there is a layer with tension which has the same motion as a layer with pressure and positive $a = a_p$. The linked layers have the same $\dot{R}$ and $\ddot{R}$. $a_p = 0$ and $a_n = 1/2$ are the lowest a values for linked pressure/tension shells and are two of the values which simplify Eq. (19). For these two values we have

$$a = 0\colon\ \pi(m_0 - M)\sigma_{a=0} = (1+\gamma)^{3/2} F^2 (1+F) \quad (27)$$

$$\begin{aligned} a = -1/2\colon\ &\pi(m_0 - M)\sigma_{a=-1/2} \\ &= -(1+\gamma)^{3/2}(F+1)^2(F). \end{aligned} \quad (28)$$

The motion functions are

$$a = 0\colon\ F = -\frac{c_0^2}{R[c_0^2 - 2R(m_0 - M)]},$$

$$a = -1/2\colon\ F = -\frac{2(m_0 - M)}{R[2(m_0 - M) - Rc_{-1/2}^2]},$$

and are identical under the linkage. The matter content of the two layers is different. From Eqs. (27) and (28) we have

$$\frac{\sigma_{a=0}}{\sigma_{a=-1/2}} = -\frac{F}{1+F}.$$

For $m_0 = M$, we would expect no layer to exist and $a = -1/2$ describes Schwarzschild vacuum.

A missing part of the a range for linked pressure/tension shells is $-1/2 < a < 0$. There are linked L-layers in this region but the linked layers both have tension. This linkage is centered around $a = -1/4$, one of the special values for the geodesic extension, describing layers whose radial 4-accelerations are linearly related to $\ddot{R}_g$. Consider two negative a values, a_{n_1} and a_{n_2} in the ranges

$$1/2 < a_{n_1} \le 1/4 \qquad 1/4 \le a_{n_2} < 0. \quad (29)$$

As might be expected from the pressure/tension linking relations, the relations for these layers are

$$a_{n_1} = -a_{n_2} + 1/2 \quad (30)$$

$$c_{a_{n_1}}^2 = 4(m_0 - M)^2/c_{a_{n_2}}^2. \quad (31)$$

When $a = -1/4$, then a_{n_1} and a_{n_2} coincide.

C. Static layers

Static layers with $\dot{R} = 0$, $\ddot{R} = 0$ have motion parameters

$$\gamma_s = -2m_0/R_s, \qquad A_s = m_0/R_s^2, \quad (32)$$

and are, in some sense, the negatives of the geodesic layer with $\gamma_s = -\dot{R}_g^2$, $A_s = -\ddot{R}_g$. Using Eqs. (8) and (19) we have

$$4P/\sigma = -1 + \frac{1}{1 - 2M/R_s - 4\pi R_s \sigma \sqrt{1 - 2M/R_s}}. \quad (33)$$

For $M = 0$ this is the stress-energy-radius relation for static layers given by Khourami and Mansouri [3], with their stress and density related to P and σ by $P_{\rm km} = 8\pi P$, $\sigma_{\rm km} = 8\pi\sigma$

$$P = \frac{\pi R_s \sigma^2}{1 - 4\pi R_s \sigma}. \quad (34)$$

This relation is particularly interesting when compared with the classical van der Waals form for three-dimensional fluids

$$P = \frac{nRT}{V - nb} - a\left(\frac{n}{V}\right)^2$$

with n the number of moles, volume V, gas constant R, and temperature T. a and b are constants. The denominator of the first term corrects for a minimum volume available to the fluid constituents. This can be attributed to a finite size of the particle constituents or to the existence of a repulsive core in constituent interactions. The second term accounts for an attractive long range attraction between constituents which reduces the stress. For low densities (n/V) the equation of state describes a perfect fluid, $PV = nRT$.

The numerator of Eq. (34) could imply that, for $\pi R_s \sigma \ll 1$, an Israel layer is a first order polytrope. The current relativistic polytrope assumes a linear low density equation of state [33]. The part of the classical van der Waals equation describing long range interaction is missing from Eq. (34) but the denominator suggests, as in the classical equation, there is either a minimum or a zero. The existence of a minimum value could be related to the existence of a repulsive core in the interaction potential between allowed layer constituents. Detailed models of this possibility will be discussed elsewhere. Equation (34) also suggests that there are static shells with tension and pressure with the zero denominator related to the boundary between the two kinds of stress. At the zero value the Israel Δ can be found from the density

$$4\pi R_s \sigma = 1, \qquad \Delta = -1.$$

$\dot{R} = 0$ is one of the static shell conditions. From the definition of Δ, Eq. (4), we have for an $M = 0$ static shell

$$\Delta = \sqrt{1 - \frac{2m_0}{R}} - 1.$$

We see that there are no static shells outside the horizon corresponding to $\Delta = -1$. The boundary between the static linked layers is not static.

The static shell boundaries can be explored using $M = 0$, L-shells as an example. For static L-shells the Israel radius value is (Appendix B)

$$R_s = m_0 \frac{(4a + 1)^2}{4a(1 + 2a)}$$

and the range for physical static radii is $-1 \leq a < 1/2$, $0 < a \leq 1$. There are linked static shells with tension and pressure, just as in the dynamic case. For each R_s in the range $0 < a \leq 1/2$, there is an identical radius in the range $-1 \leq a < -1/2$. For example, $a = 1/2$ and $a = -1$ have the same static radius $R_s = \frac{9}{4} m_0$, as suggested by the linking relation $a_n = a_p + 1/2$. The range for static shells excludes the points $a = -1/2$ and $a = 0$. These points are the moving boundaries of the static layer region. The boundary between the pressure/tension linked $M = 0$ static L-layers is a moving layer that is not the dust layer one might have expected, but the layer linked to dust.

The $M \neq 0$ static relation is less easy to interpret in a van der Waals sense, since the size of the interior mass M becomes an important parameter

$$P = \frac{\pi R_s \sigma^2}{\sqrt{1 - 2M/R_s} - 4\pi R_s \sigma} + \frac{M\sigma/2}{R_s - 2M - 4\pi R_s \sigma \sqrt{R_s - 2M}}. \qquad (35)$$

For $4\pi R_s \sigma \ll \sqrt{1 - 2M/R_s}$ the equation of state becomes approximately linear

$$P \approx \frac{2M}{R_s - 2M} \sigma \qquad (36)$$

and a mixed equation of state results as the effect of M increases.

IV. CONCLUSION

A general stress-energy-motion relation was derived for Israel layers between Schwarzschild manifolds. The equation generalizes the equation of state for layers dropping on exterior Schwarzschild geodesics. It was used to discuss the relation between motion and equation of state. The motion input is a single function of the exterior comoving observer acceleration and velocity. Using the relation, the motion of a layer with equation of state $P = -\sigma/4$ was shown to be linearly related to geodesic motion.

A set of linked L-layers with a common motion was described over the parameter range $-1 \leq a \leq -1/2$, $0 \leq a \leq 1/2$. In the linked range, each layer with pressure has a partner layer with tension. There is also a set of linked shells, both with tension in the range $-1/2 < a < 0$. These layers coincide for $P = -\sigma/4$. There are layers in the range $1/2 < a \leq 1$ but they are not linked to physical layers with tension. Positive a values in this range are linked to negative values larger than 1. Because the motion of the linked shells is the same, their motion functions agree and the new relation can be used to compare densities and pressures.

The stress energy in the Israel formalism is described by two observers comoving with the layer. As pointed out by Ipser and Skivie [11,12], the existence of static layers with tension is related to the accelerations of the comoving observers. For a static layer, the two observers are hovering over the layer but must accelerate in order to remain static with respect to the layer. For L-layers, the radially projected 4-accelerations given in Eq. (13) are related to the size of state parameter a

$$n_{r+} \dot{U}^r_+ - n_{r-} \dot{U}^r_- = 4\pi\sigma(1 + 2a). \qquad (37)$$

This can be used to interpret the relative sizes of $\dot{U}^r$ needed for the two hovering observers. For the $M = 0$ case, the interior observer is not accelerated at all and this equation describes whether the projected 4-acceleration needed by the exterior Schwarzschild observer points inward or outward. For $-1/2 < a$ the observer needs to accelerate away from the layer, counteracting the gravitational attraction of the layer and for $a < -1/2$, the region where there are static shells with tension, the observer has to accelerate toward the layer. The repelling nature of layers with tension has also been discussed by Vilenkin [34].

The applications focused mainly on L-layers because the P/σ structure of the geodesic extension makes this a simple example to develop with clarity. There are many interesting questions yet to be studied. Other equations of state easily could be investigated, for example, a dynamic first order polytrope with $P = K\sigma^2$ would have a density

$$\sigma(1 + 4K\sigma)^3 = (1 + \gamma)^{3/2}(F - 2K\sigma)^2\left[\frac{1 + F + 2K\sigma}{\pi(m_0 - M)}\right].$$

Using this equation, the suggestion that low density static shells are first order polytropes could be explored for dynamic shells. The general stress-energy-motion equation has been developed for Schwarzschild but could be a useful tool in understanding layers bounding other metrics. The layer linkage discussed here depends on the radial structure of Δ in Schwarzschild L-layers. The idea of linkages for L-layers bounding other metrics in a variety of dimensions may have broad applications.

APPENDIX A: OBSERVER ACCELERATION

The layer is tracked by two observers comoving with the layer who agree on the layer metric. The radial 4-accelerations of these observers are computed from the 4-acceleration

$$A^i = U^a \nabla_a U^i = U^a \partial_a U^i + \Gamma^i_{ab} U^a U^b. \qquad (A1)$$

Using

$$\frac{d\dot{R}}{d\tau} = \frac{\partial \dot{R}}{\partial t}\frac{\partial t}{\partial \tau} + \frac{\partial \dot{R}}{\partial r}\frac{\partial r}{\partial \tau} = \frac{\partial \dot{R}}{\partial t}\dot{T} + \frac{\partial \dot{R}}{\partial r}\dot{R}$$

one finds

$$A^r = \frac{d\dot{R}}{d\tau} + \left[\frac{m_0}{R^2} f\right](\dot{T})^2 - \left[\frac{m_0}{R^2}(1/f)\right](\dot{R})^2. \qquad (A2)$$

The velocity normalization is

$$f\dot{T}^2 = \dot{R}^2/f + 1.$$

For the exterior observer we have

$$A^r = \ddot{R} + \frac{m_0}{R^2}. \qquad (A3)$$

Thus, in general

$$\dot{U}^r_{\pm} = \ddot{R} + (1 - f_{\pm})/(2R). \qquad (A4)$$

APPENDIX B: $P = a\sigma$

1. General motions

Because of its simplicity, the linear equation of state is often used with the field equations and it is frequently used in discussing thin shells [6,13,32]. For layers with $P = a\sigma$, we have

$$\frac{\dot{\Delta}}{\Delta} = -(1 + 2a)\frac{\dot{R}}{R}. \qquad (B1)$$

If a is a constant we have

$$\Delta = -c_a R^{-(1+2a)} \qquad 4\pi P_a = a c_a R^{-2(1+a)}$$
$$4\pi\sigma_a = c_a R^{-2(1+a)} \qquad m_{L_a} = c_a R^{-2a}. \qquad (B2)$$

c_a carries an a index because, from a unit standpoint, it will have to vary with the value of a. The motion of the layer is described by

$$\dot{R}^2 = \left(\frac{\Delta}{2}\right)^2 + \left(\frac{m_0 - M}{R\Delta}\right)^2 + \frac{m_0 + M}{R} - 1$$
$$= \frac{c_a^2}{4R^{2(1+2a)}} + \frac{(m_0 - M)^2 R^{4a}}{c_a^2} + \frac{m_0 + M}{R} - 1 \qquad (B3)$$

$$\ddot{R} = -\frac{c_a^2(1 + 2a)}{4R^{3+4a}} + \frac{2a(m_0 - M)^2 R^{4a-1}}{c_a^2} - \frac{m_0 + M}{2R^2}. \qquad (B4)$$

The points $\dot{R} = 0$ provide an equation for c_a

$$2R_0^{1+4a}[R_0 - m_0 - M \pm \sqrt{R_0^2 - 2R_0(m_0 + M) + 4m_0 M}] = c_a^2. \qquad (B5)$$

Not all values of a will correspond to a static layer with both $\dot{R} = 0$ and $\ddot{R} = 0$. Where there is a static layer, the constant c_a can be evaluated in terms of the equation of state parameter a.

2. Static layer

The static points R_s follow from $\ddot{R}_s = 0$, $\dot{R}_s = 0$. This identifies R_0 with R_s. Using Eqs. (B4) and (B5), one finds

$$(1 + 2a)c_a^4 R_s^{-4(1+2a)} - \frac{8a(m_0 - M)^2}{R_s^2} + \frac{2(m_0 + M)c_a^2 R_s^{-2(1+2a)}}{R_s} = 0. \qquad (B6)$$

The parameter values $a = 0$, $a = -1/2$ have no solutions. The static radius in terms of c_a is

$$R_s^{1+4a} = \frac{-(1 + 2a)c_a^2}{(m_0 + M) \pm \sqrt{(m_0 + M)^2 + 8a(1 + 2a)(m_0 - M)^2}}. \qquad (B7)$$

Using Eq. (B5) for c_a, the static layer radius is

$$R_s = \frac{(m_0 + M)(1+4a)^2 \pm \sqrt{(m_0+M)^2(1+4a)^4 - 32am_0M(1+2a)(1+4a)^2}}{8a(1+2a)}. \quad \text{(B8)}$$

For $M = 0$ the layer description is especially simple and we have

$$R_s = m_0\frac{(4a+1)^2}{4a(1+2a)} \qquad c_a^2 = R_s^{1+4a}\frac{4am_0}{1+2a} \quad \text{(B9)}$$

$$4\pi\sigma_a = \frac{1}{m_0}\frac{(4a)^2(1+2a)}{(4a+1)^3}$$
$$4\pi P_a = \frac{1}{m_0}\frac{16a^3(1+2a)}{(4a+1)^3} \quad \text{(B10)}$$
$$m_{L_a} = m_0\frac{1+4a}{(1+2a)}$$

over the a range

$$-1 \le a < -1/2 \qquad 0 < a \le 1.$$

It is clear that there are static shells with tension as well as with pressure.

APPENDIX C: DERIVATION OF GEODESIC EXTENSION

The extension is developed in terms of A and γ, which are zero for exterior geodesic motion

$$A = \ddot{R} + m_0/R^2 \quad \text{(C1)}$$

$$\gamma = \dot{R}^2 - 2m_0/R. \quad \text{(C2)}$$

The Israel formalism gives the stress and density in terms of a function Δ,

$$8\pi P = \Delta/R + \dot{\Delta}/\dot{R}, \qquad 4\pi\sigma = -\Delta/R,$$

with

$$\Delta_+ = \sqrt{1+\dot{R}^2 - 2m_0/R} = \sqrt{1+\gamma} \quad \text{(C3a)}$$
$$\Delta_- = \sqrt{1+\dot{R}^2 - 2M/R} = \sqrt{1+\gamma+2\frac{m_0-M}{R}} \quad \text{(C3b)}$$
$$\Delta = \Delta_+ - \Delta_-. \quad \text{(C3c)}$$

Calculating the derivatives with respect to τ we have

$$\dot{\Delta}_+ = \frac{\dot{R}(\ddot{R}+m_0/R^2)}{\sqrt{1+\dot{R}^2-2m_0/R}} = \frac{\dot{R}A}{\Delta_+} \quad \text{(C4)}$$

$$\dot{\Delta}_- = \frac{\dot{R}(\ddot{R}+M/R^2)}{\sqrt{1+\dot{R}^2-2M/R}} = \frac{\dot{R}(A+M/R^2-m_0/R^2)}{\Delta_-}. \quad \text{(C5)}$$

There are several useful relations. Using Eq. (C4) and (C5) one finds

$$\Delta_+^2 - \Delta_-^2 = 2\frac{M-m_0}{R} \quad \text{(C6)}$$

and using this, the density of the layer is

$$4\pi\sigma = \frac{(\Delta_+ - \Delta_-)(\Delta_+^2 - \Delta_-^2)}{2(m_0 - M)}. \quad \text{(C7)}$$

A pressure-density relation can also be found

$$8\pi PR = -4\pi\sigma R + \frac{RA}{\Delta_+} - \left(A + \frac{M-m_0}{R^2}\right)\frac{R}{\Delta_-},$$
$$8\pi PR + 4\pi\sigma R = \frac{RA(\Delta_- - \Delta_+)}{\Delta_-\Delta_+} - \frac{M-m_0}{R\Delta_-},$$
$$8\pi PR + 4\pi\sigma R = \frac{RA(\Delta_- - \Delta_+)}{\Delta_-\Delta_+} - \frac{\Delta_+^2 - \Delta_-^2}{2\Delta_-},$$
$$8\pi PR + 4\pi\sigma R = \frac{RA(4\pi\sigma R)}{\Delta_-\Delta_+} + \frac{(\Delta_+ + \Delta_-)4\pi\sigma R}{2\Delta_-},$$
$$2P/\sigma + 1/2 = \frac{AR}{\Delta_-\Delta_+} + \frac{\Delta_+}{2\Delta_-},$$
$$(4P/\sigma + 1)\Delta_- = \frac{2AR}{\Delta_+} + \Delta_+. \quad \text{(C8)}$$

A useful relation is

$$\left(4\frac{P}{\sigma} + 1\right)\frac{\Delta_-}{\Delta_+} = \frac{2AR}{1+\gamma} + 1, \quad \text{(C9)}$$

with

$$\frac{\Delta_-}{\Delta_+} + 1 = \frac{2AR/(1+\gamma) + 2 + 4P/\sigma}{(4P/\sigma + 1)}, \quad \text{(C10a)}$$
$$\frac{\Delta_-}{\Delta_+} - 1 = \frac{2AR/(1+\gamma) - 4P/\sigma}{(4P/\sigma + 1)}. \quad \text{(C10b)}$$

Using these, substituting into Eq. (C8), the general extension of the geodesic equation of state follows

$$\pi(m_0 - M)\sigma\left(4\frac{P}{\sigma} + 1\right)^3 = (1+\gamma)^{3/2}[F - 2P/\sigma]^2 \times [F + 1 + 2P/\sigma]. \quad \text{(C11)}$$

For $A = \gamma = 0$, this becomes the geodesic equation of state.

[1] W. Israel, Nuovo Cimento B **44**, 1 (1966); **48**, 463 (1967).
[2] K. Lake, Phys. Rev. D **19**, 2847 (1979).
[3] M. Khorrami and R. Mansouri, Phys. Rev. D **44**, 557 (1991).
[4] R. Mansouri and M. Khorrami, J. Math. Phys. (N.Y.) **37**, 5672 (1996).
[5] S. M. C. V. Gonçalves, Phys. Rev. D **66**, 084021 (2002).
[6] K. Lake and R. Wevrick, Can. J. Phys. **64**, 165 (1986).
[7] P. Laguna-Castillo and R. A. Matzner, Phys. Rev. D **34**, 2913 (1986).
[8] V. A. Berezin, V. A. Kuzmin, and I. I. Trachev, Phys. Rev. D **36**, 2919 (1987).
[9] J. R. Ipser, Phys. Rev. D **36**, 1933 (1987).
[10] J. Kijowski, G. Magli, and D. Malafarina, Gen. Relativ. Gravit. **38**, 1697 (2006).
[11] J. R. Ipser and P. Sikivie, Phys. Rev. D **30**, 712 (1984).
[12] J. R. Ipser, Phys. Rev. D **30**, 2452 (1984).
[13] D. Garfinkle and C. Vuille, Classical Quantum Gravity **6**, 1819 (1989).
[14] P. O. Mazur and E. Mottola, arXiv:gr-qc/0109035.
[15] M. Visser and D. L. Wiltshsire, Classical Quantum Gravity **21**, 1135 (2004).
[16] B. M. N. Carter, Classical Quantum Gravity **22**, 4551 (2005).
[17] C. Cattoen, T. Faber, and M. Visser, Classical Quantum Gravity **22**, 4189 (2005).
[18] A. deBenedictis, D. Horvat, S. Ilijic, S. Kloster, and K. S. Viswanathan, Classical Quantum Gravity **23**, 2303 (2006).
[19] E. F. Eiroa and C. Simeone, Phys. Rev. D **71**, 127501 (2005).
[20] C. Barrabes and W. Israel, Phys. Rev. D **71**, 064008 (2005).
[21] E. Gravanis and S. Willison, Phys. Rev. D **75** 084025, (2007).
[22] C. Barrabes and W. Israel, Phys. Rev. D **43**, 1129 (1991).
[23] E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, England, 2004).
[24] V. de la Cruz and W. Israel, Phys. Rev. **170**, 1187 (1968).
[25] P. Musgrave and K. Lake, Classical Quantum Gravity **13**, 1885 (1996).
[26] S. Ansoldi, Classical Quantum Gravity **19**, 6321 (2002).
[27] P. R. Brady, J. Louko, and E. Poisson, Phys. Rev. D **44**, 1891 (1991).
[28] R. Vera, Classical Quantum Gravity **19**, 5249 (2002).
[29] F. S. N. Lobo and P. Crawford, Classical Quantum Gravity **22**, 4869 (2005).
[30] M. Mars, Classical Quantum Gravity **22**, 3325 (2005).
[31] J. P. Krisch and E. N. Glass, Phys. Rev. D **76**, 104006 (2007).
[32] U. S. Nilsson and C. Uggla, Ann. Phys. (N.Y.) **286**, 278 (2000).
[33] F. S. N. Lobo, Phys. Rev. D **75**, 024023 (2007).
[34] A. Vilenkin, Phys. Rev. D **23**, 852 (1981).