PHYSICAL REVIEW D **78,** 041502(R) (2008)

Black hole thermodynamics from simulations of lattice Yang-Mills theory

Simon Catterall[1] and Toby Wiseman[2]

[1]*Department of Physics, Syracuse University, Syracuse, NY13244, USA*
[2]*Theoretical Physics, Blackett Laboratory, Imperial College London, London, SW7 2AZ, United Kingdom*
(Received 2 April 2008; published 13 August 2008)

We report on lattice simulations of 16 supercharge $SU(N)$ Yang-Mills quantum mechanics in the 't Hooft limit. Maldacena duality conjectures that in this limit the theory is dual to IIA string theory, and, in particular, that the behavior of the thermal theory at low temperature is equivalent to that of certain black holes in IIA supergravity. Our simulations probe the low temperature regime for $N \leq 5$ and the intermediate and high temperature regimes for $N \leq 12$. We observe 't Hooft scaling, and at low temperatures our results are consistent with the dual black hole prediction. The intermediate temperature range is dual to the Horowitz-Polchinski correspondence region, and our results are consistent with continuous behavior there. We include the Pfaffian phase arising from the fermions in our calculations where appropriate.

DOI: 10.1103/PhysRevD.78.041502 PACS numbers: 04.60.Cf, 04.70.Dy, 11.15.Ha, 11.25.Tq

I. INTRODUCTION

String theory has provided remarkable insight into the quantum physics underlying black holes. Much recent progress stems from conjectured dualities, which, in an appropriate limit, relate the finite temperature low energy supergravity limit of the string theory to strongly coupled thermal field theory. The entropy of the black holes that arise in these supergravity theories can then be computed, in principle, by counting microstates in their dual field theories. The pioneering calculations of black hole entropy in [1,2] are examples where the dual field theory is a 2-d conformal field theory which allows computation of the entropy despite the strong coupling.

For a large number N of coincident Dp-branes in the "decoupling" limit [3,4], the dual field theory is $(1+p)$-dimensional, strongly coupled, maximally supersymmetric, $SU(N)$ Yang-Mills theory, taken in the 't Hooft limit. The case of D3-branes yields the original AdS-CFT correspondence. Analytic calculation of the corresponding black hole entropy of these theories has proven elusive despite interesting attempts [5].

Here we use lattice methods to study the thermal gauge theory and hence test these conjectured dualities. The simplest case for lattice work corresponds to D0-branes [6], where the dual is thermal 16 supercharge Yang-Mills quantum mechanics (the "BFSS model" [7]). This theory has recently been numerically studied using a nonlattice formulation [8,9]. Earlier analytic approaches used a variational method [10,11]. Related zero temperature numerical works are [12–14].

In this paper we simulate the super quantum mechanics in the 't Hooft limit over a range of temperature and present preliminary results. We obtain intermediate temperature results for $N \leq 12$ and low temperature results for $N \leq 5$. We pay particular attention to the continuum limit and the behavior of the important Pfaffian phase arising from the fermions. More details of the method and results will be given in [15].

II. DUALITY AND BLACK HOLES

The type IIA string theory reduces to a supergravity theory for low energies compared to the string scale $(\alpha')^{-1/2}$. In this limit the thermal theory contains black holes with N units of D0-charge. Their energy E is a function of their Hawking temperature T. Defining $\lambda = N g_s \alpha'^{-3/2}$ where g_s is the string coupling, we may write a dimensionless energy and temperature $\epsilon = \frac{1}{N^2} E \lambda^{-1/3}$ and $t = T\lambda^{-1/3}$. One finds that, provided we take N large and $t \ll 1$, the black hole is weakly curved on string scales and the quantum string corrections are suppressed. The energy of this black hole can be precisely computed by standard methods [4,16], giving

$$\epsilon = c t^{14/5}, \qquad c = \left(\frac{2^{21} 3^{12} 5^2}{7^{19}} \pi^{14}\right)^{1/5} \simeq 7.41. \tag{1}$$

Duality posits that the thermodynamics of this black hole should be reproduced by the dual Yang-Mills quantum mechanics at the same temperature with $g_s \alpha'^{-3/2} = g_{\mathrm{YM}}^2$ so λ is identified with the 't Hooft coupling.

In the large N limit, at high temperatures $t > 1$, the bound state of D0-branes is of order the size of the string scale, and hence all α' corrections are important. One should best think of the configuration dominating the partition function as a hot gas of D0-branes bound by strings. Horowitz and Polchinski have argued that the low temperature black hole and high temperature gas are the asymptotic descriptions, and intermediate temperatures continuously interpolate between these [17].

III. LATTICE IMPLEMENTATION

The 16 supercharge $SU(N)$ Yang-Mills quantum mechanics arises from dimensional reduction of $\mathcal{N} = 1$ super Yang-Mills theory in 10-d. The 10-d gauge field reduces to the 1-d gauge field A and 9 scalars, X^i, $i = 1, \ldots, 9$, and the 10-d Majorana-Weyl fermion to 16 single component fermions, Ψ_α, $\alpha = 1, \ldots, 16$. All fields transform in the adjoint of the gauge group. In order to simulate the theory we must integrate out the fermions giving rise to a Pfaffian. The continuum Euclidean path integral, $Z = \int dAdX \mathrm{Pf}(\mathcal{O}) e^{-S_{\mathrm{bos}}}$, is then given by

$$\begin{aligned} S_{\mathrm{bos}} &= \frac{N}{\lambda} \mathrm{Tr} \oint^R d\tau \left\{ \frac{1}{2}(D_\tau X_i)^2 - \frac{1}{4}[X_i, X_j]^2 \right\}, \\ \mathcal{O} &= \gamma^\tau D_\tau - \gamma^i [X_i, \cdot]. \end{aligned} \tag{2}$$

The γ^τ, γ^i are the Euclidean Majorana-Weyl gamma matrices, and we choose a representation where

$$\gamma^\tau = \begin{pmatrix} 0 & \mathrm{Id}_8 \\ \mathrm{Id}_8 & 0 \end{pmatrix}.$$

We take Euclidean time to have period R.

We have a choice of fermion boundary conditions. Thermal boundary conditions correspond to taking the fermions antiperiodic on the Euclidean time circle and correspond to a temperature $t = \lambda^{-1/3}/R$. We will also employ periodic fermions, and the continuum partition function is the Witten index, with t the inverse volume. The Pfaffian is, in general, complex [18]. It is important, in principle, to include the phase of the Pfaffian in the Monte-Carlo simulation, and we discuss this later.

We discretize this continuum model as

$$\begin{aligned} S_{\mathrm{bos}} &= \frac{NL^3}{\lambda R^3} \sum_{a=0}^{L-1} \mathrm{Tr} \left[\frac{1}{2}(D_+ X_i)_a^2 - \frac{1}{4}[X_{i,a}, X_{j,a}]^2 \right], \\ \mathcal{O}_{ab} &= \begin{pmatrix} 0 & (D_+)_{ab} \\ (D_-)_{ab} & 0 \end{pmatrix} - \gamma^i [X_{i,a}] \delta_{ab}, \end{aligned} \tag{3}$$

where we have rescaled the fields $X_{i,a}$ and $\Psi_{i,\alpha}$ by powers of the lattice spacing $a = R/L$ where L is the number of lattice points to render them dimensionless. We have introduced a Wilson gauge link field U_a, and taken covariant difference operators $(D_- W)_a = W_a - U_a^\dagger W_{a-1} U_a$, $(D_+ W)_a = U_a W_{a+1} U_a^\dagger - W_a$. Notice that the fermionic operator is free of doublers and is manifestly antisymmetric. This lattice action is finite in lattice perturbation theory and hence will flow *without fine-tuning* to the correct supersymmetric continuum theory as the lattice spacing is reduced [6,15].

We use the rational hybrid Monte Carlo (RHMC) algorithm [19,20] to sample configurations using the absolute value of the Pfaffian. The phase may be reincorporated in the expectation value of an observable $\mathcal{A}$ by reweighting as $\langle \mathcal{A} \rangle = \frac{\sum_m (\mathcal{A} e^{i\phi})}{\sum_m (e^{i\phi})}$. Here $e^{i\phi(\mathcal{O})}$ is the phase of the Pfaffian and the sum runs over all members of our phase quenched ensemble.

We find in practice that the RHMC simulation of the thermal theory at low temperature, $t \lesssim 1$, exhibits an instability corresponding to the scalar fields moving out along the flat directions of the classical potential. Hence the algorithm never thermalizes and cannot be used to approximate the path integral. This has been observed before [9]. We believe this divergence may be a lattice artifact that is related to the discretization of the fermion operator. In previous work [6] we have simulated the 4 supercharge quantum mechanics over a range of t, using a Weyl representation for the fermions where one obtains a real positive determinant. However, we have also tried using a Majorana representation where one obtains a Pfaffian which we have discretized in analogy with the 16 supercharge case discussed here. While no divergence of the scalars was observed in the Weyl simulations over a large range of t [6], the Majorana implementation has the same instability we observe in the 16 supercharge case for $t \lesssim 1$. Since both representations are equivalent in the continuum limit, this implies that the instability may not be a property of the continuum theory as is claimed in [9], but merely an artifact of finite lattice spacing. More details will be given in [15].

We find no such problem simulating the periodic theory at small t. At low temperature we expect the thermal and periodic theories to be similar, and the configurations that dominate the path integral will be similar. Hence in order to simulate the thermal theory at low temperature, $t \lesssim 1$, we have employed a reweighting of the periodic theory. We can expect to get good results for the thermal theory at low temperature by computing expectation values using the periodic theory, and reweighting as

$$\langle \mathcal{A} \rangle_T = \frac{\sum_m^{(P)} (\mathcal{A} \mathrm{Pf}(\mathcal{O}_T)/|\mathrm{Pf}(\mathcal{O}_P)|)}{\sum_m^{(P)} (\mathrm{Pf}(\mathcal{O}_T)/|\mathrm{Pf}(\mathcal{O}_P)|)} \tag{4}$$

where $\sum_m^{(P)}$ is a sum over the phase quenched ensemble generated for the periodic theory, $\mathcal{O}_P$ and $\mathcal{O}_T$ are the periodic and thermal fermion operators, respectively, and $\langle \ldots \rangle_T$ is the expectation value for the thermal theory.

IV. RESULTS

We have simulated the thermal and periodic theories concentrating on the range $0.3 < t < 5$. We have focused on two observables—the mean energy ϵ and absolute value of the trace of the Polyakov loop, P. In the Yang-Mills theory these are given by [6,15]

$$\begin{aligned} \langle \epsilon / t \rangle &= \frac{3}{N^2} \left(\frac{9}{2} L(N^2 - 1) - \langle S_{\mathrm{bos}} \rangle \right), \\ P &= \frac{1}{N} \left\langle \left| \mathrm{Tr} \prod_{a=0}^{L-1} U_a \right| \right\rangle. \end{aligned} \tag{5}$$

The inclusion of $1/N^2$, $1/N$ in these definitions is to ensure these quantities are finite in the 't Hooft limit for a deconfined phase. In the periodic case, since Z is an index, it should not depend continuously on the inverse volume t, and hence in the continuum $\epsilon = 0$. The data we present required approximately 50 000 processor hours to obtain.

To check for a restoration of supersymmetry we have computed ϵ/t in the periodic theory for a variety of lattice sizes $L = 5$, 10, and 20. The upper plot of Fig. 1 shows ϵ/t for $SU(2)$. For large t the index ϵ is already consistent with zero for $L = 5$, while at small t it appears to approach zero as L increases. Notice that, while this quantity is a sensitive test of the restoration of supersymmetry in the continuum limit $L \longrightarrow \infty$, other observables such as P shown in the lower plot are relatively insensitive to the number of lattice sites for $L \geq 5$.

We have also examined the continuum limit of the thermal theory. In Fig. 2 we show $L = 5$ and 10 data for the thermal energy for $SU(5)$ (in the phase quenched approximation—which we discuss shortly). As noted above, we find a lattice instability for the thermal theory with $t \lesssim 1$ (with some dependence on N and L). However, for larger t this does not occur. As argued above we believe this is an artifact of our lattice formulation and has nothing to do with continuum physics. The points plotted in the figure are taken only from simulations where the scalar distribution remained bounded for hundreds of physical RHMC times (the observed instability sets in very quickly in RHMC time, so the change in behavior is easy to identify). The plot shows that these lattice spacing effects are small, and hence for the remainder of our results we show only data from $L = 5$ point lattices.

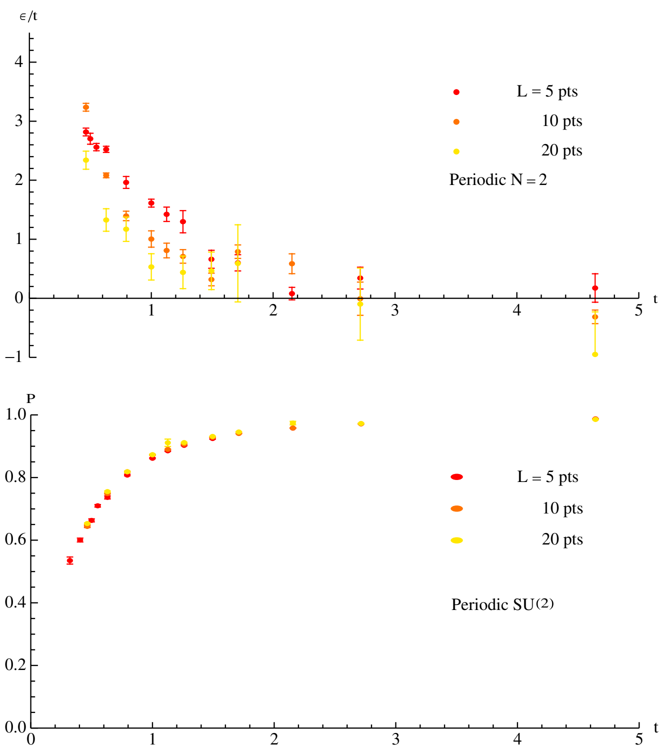

FIG. 1 (color online). Top panel: Plot showing ϵ/t versus dimensionless temperature t for the periodic $SU(2)$ theory for various numbers of lattice points. Bottom panel: Plot of the Polyakov loop against temperature for the same theory.

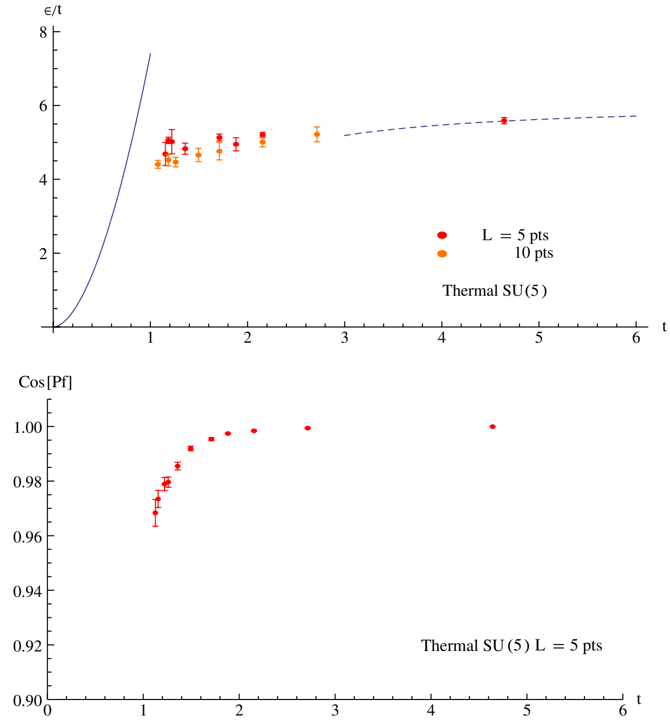

FIG. 2 (color online). Top panel: Plot of dimensionless energy ϵ/t versus dimensionless temperature t for direct simulation of the $SU(5)$ theory with thermal fermion boundary conditions using 5 and 10 lattice points. Bottom panel: Plot of the cosine of the Pfaffian argument for the thermal $SU(5)$ theory with 5 points.

In the lower plot of Fig. 2 we show the mean cosine of the Pfaffian phase for the thermal $SU(5)$ theory with $L = 5$ lattice sites. As expected, this phase becomes more important at lower temperatures but the actual value is close to 1 over the range of temperatures where we can directly simulate the thermal theory. Indeed, the effects of reweighting are negligible in this temperature regime. Hence for the data we present later for direct simulation of the thermal theory, we use the phase quenched approximation. Since the Pfaffian is very costly to compute, this allows us to work at larger N.

We now turn to the main results of this paper. In Fig. 3 we plot the energy and Polyakov loop for various N and $L = 5$ point lattices. For high temperatures we have used direct simulation of the thermal theory (phase quenched as discussed above), and we are able to obtain results up to $N = 12$. At low temperatures we obtain results by a reweighting of the periodic simulations as discussed above. The results from both methods agree in the regime where they overlap $t \sim 1$.

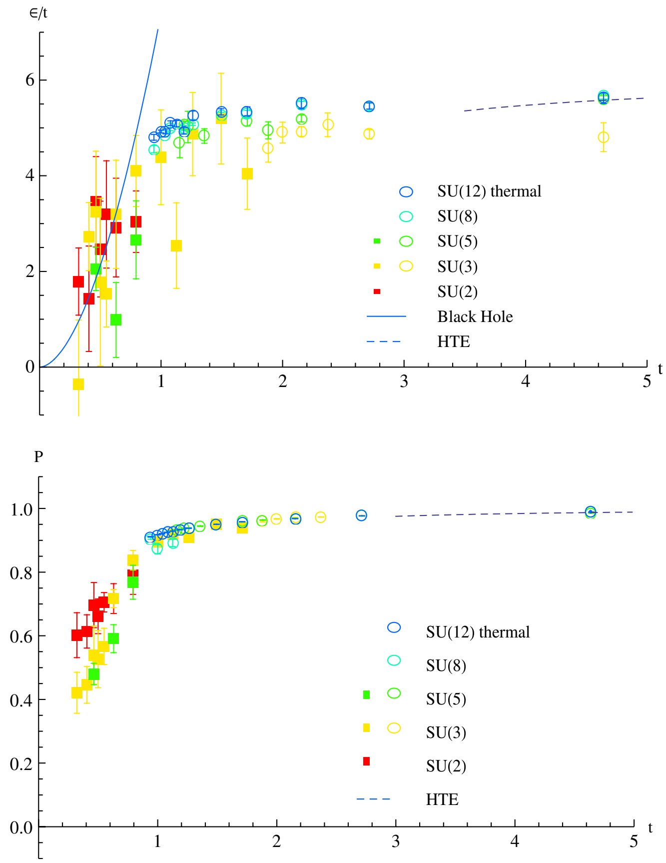

FIG. 3 (color online). Top panel: A plot of the dimensionless energy ϵ/t versus dimensionless temperature t. Data shown are generated in two ways. For temperatures larger than $t \sim 1$ we simulate the thermal theory for $N = 2, 3, 5, 8, 12$ with 5 points. The low temperature results are computed for $N = 2, 3, 5$ for 5 points by simulating the periodic theory, and reweighting with the appropriate combination of the thermal and periodic Pfaffians, as described in the main text. The low temperature black hole prediction is shown. Bottom panel: A plot of the Polyakov loop observable P for the same cases.

At very high temperatures the curves approach a constant corresponding to the result from classical equipartition assuming N^2 deconfined gluonic states (the fermions are lifted out of the dynamics by their thermal mass in this limit). In contrast, for low temperatures the energy approaches zero, signaling the presence of a supersymmetric vacuum at vanishing temperature.

As seen before [6], we see that 't Hooft scaling sets in for small N, with $N = 3$ already giving results close to an extrapolated large N result. The high temperature asymptotics computed in [21] are also plotted and agree with our data. Our results also appear consistent with those found recently using nonlattice methods [9].

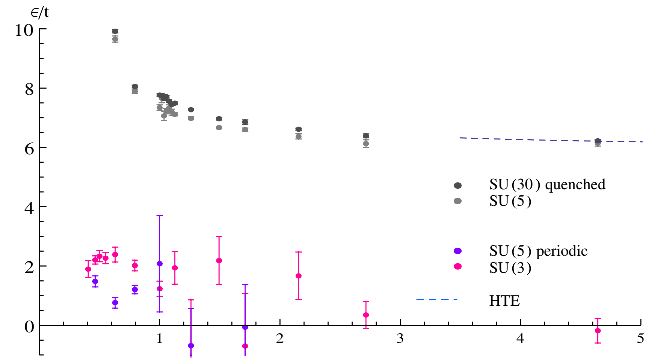

FIG. 4 (color online). For comparison with Fig. 3, ϵ/t versus t is shown for the quenched theory for $N = 5$, 30 and periodic theory with Pfaffian reweighting for $N = 3$, 5, using 5 point lattices.

There are two important physical observations. First, the curves appear to interpolate from high to low temperatures continuously, and apparently smoothly. This is to be contrasted with the quenched version of this theory which has a large N confinement/deconfinement phase transition at $t \simeq 0.9$ [22,23], and a discontinuous Polyakov loop. Since the intermediate temperature range $t \sim 1$ is dual to the regime where the thermal D0-branes have a radius comparable to the string scale, we are probing the Horowitz-Polchinski correspondence regime, and seeing apparently smooth behavior there. Of course, there may be a nonanalyticity here that would be difficult to resolve at the order of these numerical results, and work on the AdS/CFT correspondence suggests a possible Gross-Witten-Wadia phase transition [24,25].

Second, the low temperature behavior of the theory appears consistent with the prediction from supergravity, also shown in the plots. This is to be contrasted with the quenched energy curve shown for comparison in Fig. 4 which departs strongly from the black hole prediction at low temperature. In this figure we also show the periodic theory which shows that the degree of supersymmetry breaking for these lattices is small.

It would be very interesting to extendthese calculations to 2- and 3-dimensional Yang-Mills systems which are thought to be dual to D1- and D2-brane systems using recent lattice formulations retaining exact supersymmetry [26].

ACKNOWLEDGMENTS

S. C. is supported in part by DOE Grant No. DE-FG02-85ER40237. T. W. is supported by PPARC. Simulations were performed using USQCD resources at Fermilab.

[1] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[2] A. Strominger, J. High Energy Phys. 02 (1998) 009.
[3] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[4] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, Phys. Rev. D **58**, 046004 (1998).
[5] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, Commun. Math. Phys. **275**, 209 (2007).
[6] S. Catterall and T. Wiseman, J. High Energy Phys. 12 (2007) 104.
[7] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[8] M. Hanada, J. Nishimura, and S. Takeuchi, Phys. Rev. Lett. **99**, 161602 (2007).
[9] K. N. Anagnostopoulos, M. Hanada, J. Nishimura, and S. Takeuchi, Phys. Rev. Lett. **100**, 021601 (2008).
[10] D. Kabat and G. Lifschytz, Nucl. Phys. B **571**, 419 (2000).
[11] D. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D **64**, 124015 (2001).
[12] M. Campostrini and J. Wosiek, Nucl. Phys. B **703**, 454 (2004).
[13] J. R. Hiller, S. S. Pinsky, N. Salwen, and U. Trittmann, Phys. Lett. B **624**, 105 (2005).
[14] J. R. Hiller, O. Lunin, S. Pinsky, and U. Trittmann, Phys. Lett. B **482**, 409 (2000).
[15] S. Catterall and T. Wiseman (unpublished).
[16] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B **475**, 164 (1996).
[17] G. T. Horowitz and J. Polchinski, Phys. Rev. D **55**, 6189 (1997).
[18] W. Krauth, H. Nicolai, and M. Staudacher, Phys. Lett. B **431**, 31 (1998).
[19] M. A. Clark and A. D. Kennedy, Nucl. Phys. B, Proc. Suppl. **129**, 850 (2004).
[20] M. A. Clark, A. D. Kennedy, and Z. Sroczynski, Nucl. Phys. B, Proc. Suppl. **140**, 835 (2005).
[21] N. Kawahara, J. Nishimura, and S. Takeuchi, J. High Energy Phys. 12 (2007) 103.
[22] O. Aharony, J. Marsano, S. Minwalla, and T. Wiseman, Classical Quantum Gravity **21**, 5169 (2004).
[23] O. Aharony *et al.*, J. High Energy Phys. 01 (2006) 140.
[24] L. Alvarez-Gaume, P. Basu, M. Marino, and S. R. Wadia, Eur. Phys. J. C **48**, 647 (2006).
[25] T. Azuma, P. Basu, and S. R. Wadia, Phys. Lett. B **659**, 676 (2008).
[26] S. Catterall, J. High Energy Phys. 01 (2008) 048.