Quantum behavior near a spacelike boundary in the $c = 1$ matrix model

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Certain time-dependent configurations in the $c = 1$ matrix model correspond to string theory backgrounds which have spacelike boundaries and appear geodesically incomplete. We investigate quantum mechanical properties of a class of such configurations in the matrix model, in terms of fermionic eigenvalues. We describe Hamiltonian evolution of the eigenvalue density using several different time variables, some of which are infinite and some of which are finite in extent. We derive unitary transformations relating these different descriptions, and use those to calculate fermion correlators in the time-dependent background. Using the chiral formalism, we write the time-dependent configurations as a state in the original matrix model Hilbert space.

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I. INTRODUCTION

It is well known that the $c = 1$ matrix model is equivalent to two dimensional Liouville string theory [1]. This equivalence is an example of open-closed duality: the density of matrix eigenvalues (representing the tachyonic mode of open strings attached to D0-branes) is directly related to the closed string "tachyon" field. Since the $c = 1$ matrix model is solvable, it provides us with an exact quantum mechanical solution to string theory in two dimensions.

This framework is a nice toy model for the study of time-dependent backgrounds in string theory. Time-dependent solutions can easily be constructed in the matrix model, as there are no conceptual difficulties associated with time dependence in quantum mechanics. These solutions correspond to particular time-dependent string theory backgrounds. Correlators of small fluctuations can be studied in the matrix model to probe the spacetime structure of string backgrounds. Any conceptual difficulties associated with presence of time dependence in string theory can be resolved by going back to the unambiguous description in terms of matrix quantum mechanics.

One of the essential features of the matrix model solution to Liouville string is that space (i.e., the Liouville direction) is emergent: it is constructed from the collective motion of matrix eigenvalues. Time-dependent backgrounds for string theory are constructed in the matrix model by considering large deviations from the static eigenvalue distribution. An outstanding issue of this approach has been that these deviations might be too large to live in the Hilbert space of the original matrix quantum mechanics, which would complicate their interpretation. We address this issue here.

For static Liouville backgrounds, the time variable in string theory is inherited from matrix quantum mechanics; in time-dependent solutions the original quantum mechanical time is mixed with the emergent space dimension. The emergent nature of space and the mixing with the time dimension make these models particularly interesting, potentially leading to insights into the question of emergent *time* in string theory.

In [8], certain time-dependent solutions in the $c = 1$ matrix model were proposed, presenting a variety of physical scenarios which were further studied in [9–12]. Some of the most promising scenarios correspond to spacetimes with spacelike boundaries $\mathcal{I}^+$ and/or $\mathcal{I}^-$ [13–15]. The appearance of spacelike $\mathcal{I}^{\pm}$ is associated with the existence of cosmological horizons, and is reminiscent of de Sitter spacetimes. Some properties of such solutions were studied in [13,15], from the point of view of the classical effective theory. In the present paper, a foray is made into quantum mechanical description of those solutions.

Here we explore, at the full quantum level, the relationship between different solutions of matrix quantum mechanics. One of the results of this paper is that our time-dependent solutions do live in the same Hilbert space as the static ones, and therefore should be thought of as fluctuations in the original theory, a point which has not been made clear before.

The main thrust of the paper is that the same quantum mechanical evolution can be described as taking either a finite or an infinite amount of time, depending on the choice of the time variable. The appearance of a finite time variable is what leads to a spacelike future boundary $\mathcal{I}^+$ in string theory.

The existence of these drastically different and yet equivalent descriptions is interesting in its own right. It is often stressed that one of the difficulties with quantum gravity is that, while quantum mechanics assumes the presence of an *a priori* time, general relativity has no preferred time direction. Our simple example illustrates that the requirement of an *a priori* time in quantum mechanics might not be as rigid a constraint as it is thought to be. Here, quantum mechanical evolution is written in terms of one of two different time variables, which have different topologies: one is infinite in extent, and the other only semi-infinite. This behavior is quite generic in quantum mechanics; here we simply find a specific instance of it which gives us insight into the quantum mechanical properties of particular time-dependent solutions of the $c = 1$ matrix model.

Our quantum correspondence between different solutions allows us to relate the exact quantum correlators in the time-dependent solution to those in the static solution. While beyond the scope of this paper, further exploration of quantum correlators near the boundary might lead to a calculation of the conformal factor of the spacetime metric (which is not computable from classical information), and eventually shine light on the nature of spacelike singularities in string theory.

The paper is organized as follows. In the next section, we briefly review the solutions of interest from [13], and introduce some useful notation. In Sec. III, we write down the correspondence between different solutions in quantum mechanical language, and explain why they all live in the same Hilbert space. In Sec. IV, using the chiral formalism, we write down explicit linear transformations between the wave functions describing different solutions. In Sec. V, we study the fermion correlators, and compare our results to predictions from classical collective field theory. Finally, in Sec. VI, we discuss a few interesting consequences and suggest possible extensions of our work. Our discussion is limited to matrix quantum mechanics side of the duality, except for some comments in the last section. A variety of useful formulas is collected in the Appendix.

II. TIME-DEPENDENT SOLUTIONS IN CLASSICAL EFFECTIVE THEORY

The $c = 1$ matrix model quantum mechanics has as its fundamental degrees of freedom noninteracting fermions in upside down harmonic oscillator potential, with the Hamiltonian

$$H = \tfrac{1}{2}p^2 - \tfrac{1}{2}x^2. \tag{1}$$

The curvature at the top of the potential is fixed by taking $\alpha' = 1$ in the corresponding Liouville string theory. The effective (or bosonized) picture for this system is that of a Fermi fluid moving in phase space (x, p). Its dynamics can be described in terms of the density of this fluid. In the classical limit, the density takes on values of either 1 or 0, since the Fermi fluid is incompressible. Therefore, it is sufficient to specify the region where eigenvalues are present, which is the Fermi sea in phase space, bounded by a Fermi surface. In the simplest case, this surface can be presented as its upper and lower branches at each point x, $p_\pm(x, t)$. The local density of fermions in x space is then given by the distance between the two branches of p:

$$\varphi(x, t) \equiv \tfrac{1}{2}(p_+(x, t) - p_-(x, t)). \tag{2}$$

Static Fermi surfaces are hyperbolas given by the equation

$$x^2 - p^2 = 2\mu, \quad \text{or} \quad \varphi_0 = \sqrt{x^2 - 2\mu}. \tag{3}$$

Any small fluctuation around this static background moves along one branch of the hyperbola from $x = \infty$ towards finite x and back out to $x = \infty$ along the other branch. This is captured by the effective action for small fluctuations about the static solution which is given by

$$S = \int d\tau d\sigma\Big\{\frac{1}{2}((\partial_\tau\eta)^2 - (\partial_\sigma\eta)^2) - \frac{\sqrt{\pi}}{6\varphi_0^2}(3(\partial_\tau\eta)^2(\partial_\sigma\eta) + (\partial_\sigma\eta)^3) + \frac{(\partial_\tau\eta)^2}{2}\sum_{n=2}^{\infty}\Big(-\frac{\sqrt{\pi}(\partial_\sigma\eta)}{\varphi_0^2}\Big)^n\Big\}. \tag{4}$$

η here is the fluctuation of fermion density about its static configuration

$$\varphi(x, t) = \varphi_0(x) + \sqrt{\pi}\partial_x\eta(x, t), \tag{5}$$

and the coordinates σ and τ are related to x and t via

$$x = \sqrt{2\mu}\cosh\sigma, \qquad t = \tau. \tag{6}$$

Note that σ is defined on the interval $[0, \infty)$ and there is a Dirichlet boundary condition for η at $\sigma = 0$. Effectively, the fluctuations live on patch in (σ, τ) space pictured in a Penrose diagram in Fig. 1(a). The σ and τ variables are related to the spacetime coordinates in Liouville string

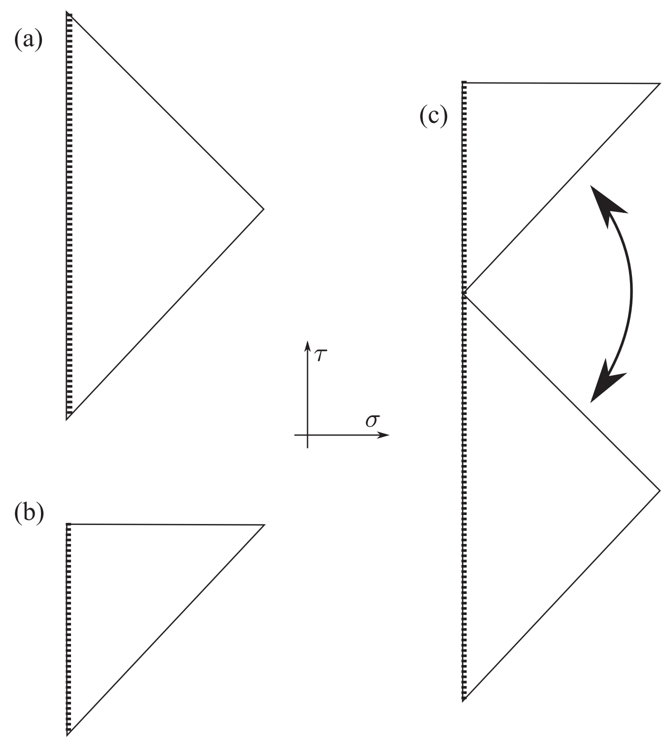

FIG. 1. The Penrose diagram of the causal structure of: (a) static Liouville string; (b) spacetime resulting from the closing hyperbola solution (7); (c) spacetime resulting from the opening hyperbola solution (9). In all three cases, the dashed line on the left-hand side represents the Liouville wall.

theory by a nonlocal transformation, whose exact form is known only at null asymptotia. The characteristic length scale of the nonlocality is string length, and therefore the Penrose diagram in Fig. 1(a), while strictly correct for the fermionic excitations of the matrix model, gives a good representation of the spacetime of Liouville string on length scales longer than string length.

Having reviewed these basic facts, we will now start with simple time-dependent solutions from [13] which exhibit a spacelike $\mathcal{I}^+$ and a null $\mathcal{I}^-$.

At the classical level, the first solution is given as a moving Fermi surface

$$(x + p - e^{2t}(x - p))(x - p) = 2\mu. \tag{7}$$

Geometrically, this is a hyperbola which closes on itself [see Fig. 1(b)]. Surprisingly, the following change of variables

$$x = \frac{\sqrt{2\mu}\cosh\sigma}{\sqrt{1 - e^{2\tau}}}, \qquad (1 + e^{2t})(1 - e^{2\tau}) = 1 \tag{8}$$

brings the action for small fluctuations around this surface exactly to the static action in (4).

The entire evolution of the Fermi surface is described by a coordinate patch $\sigma \geq 0$ and $\tau \leq 0$. The corresponding Penrose diagram is shown in Fig. 1(b). Time dependence of the solution is now hidden in the presence of this boundary, since the effective action, (4), does not depend on τ at all. Even though nothing interesting happens to the action at $\tau = 0$, there is no reason to extend past the spacelike boundary, as the evolution of the original system is fully captured by just this incomplete patch. Extending past $\tau = 0$ has no meaning in the matrix model.

Another interesting solution we will encounter in the later section is given by

$$(x - p)(x + p + e^{2t}(x - p)) = 2\mu. \tag{9}$$

This describes a hyperbola which opens up to a straight line [see Fig. 1(c)]. Again, the effective action can be brought into the static form (4), only this time the change of variables is more involved:

$$t < 0, \qquad x = \pm\frac{\sqrt{2\mu}\cosh\sigma}{\sqrt{1 + e^{2\tau}}}, \qquad (1 - e^{2t})(1 + e^{2\tau}) = 1; \tag{10}$$

$$t > 0, \qquad x = \frac{\sqrt{2\mu}\sinh\sigma}{\sqrt{e^{-2\tau} - 1}}, \qquad (e^{2t} - 1)(e^{-2\tau} - 1) = 1. \tag{11}$$

The subtlety here is that we must include both sides of the potential (allow both positive and negative x) as the solution crosses $x = 0$ at $t = 0$. The corresponding Penrose diagram can be seen in Fig. 1(c). It consists of two pieces, one for $t < 0$ and one for $t > 0$, joined by an identification at the null boundaries. We will have more to say about this in Sec. .

The remarkable fact that the classical effective action is the same for all of the above solutions suggests that perhaps the equivalence holds at the quantum level as well [16]. This turns out to be true, and is one of the main points of this paper.

As a first step, we shall introduce a bit of useful notation. Consider two coordinate systems, which we will refer to as A and B, linked by the following transformation (given here together with its inverse for later convenience):

$$\begin{cases} x_B = \frac{1}{\sqrt{1+e^{2t_A}}}x_A \\ p_B = \sqrt{1+e^{2t_A}}\,p_A - \frac{e^{2t_A}}{\sqrt{1+e^{2t_A}}}x_A \end{cases} \quad \begin{cases} x_A = \frac{1}{\sqrt{1-e^{2t_B}}}x_B \\ p_A = \sqrt{1-e^{2t_B}}\,p_B + \frac{e^{2t_B}}{\sqrt{1-e^{2t_B}}}x_B \end{cases} \quad (e^{2t_A}+1)(1-e^{2t_B}) = 1. \tag{12}$$

The coordinate transformation (12) was chosen so that, if

$$\frac{d}{dt_A}x_A = p_A \quad \text{and} \quad \frac{d}{dt_A}p_A = x_A, \tag{13}$$

then

$$\frac{d}{dt_B}x_B = p_B \quad \text{and} \quad \frac{d}{dt_B}p_B = x_B, \tag{14}$$

and therefore the Hamiltonian in both coordinates is just (1):

$$H = \tfrac{1}{2}p_A^2 - \tfrac{1}{2}x_A^2 = \tfrac{1}{2}p_B^2 - \tfrac{1}{2}x_B^2. \tag{15}$$

As this transformation leaves the Dirac (or the Poisson) bracket invariant

$$[x_A, p_A] = [x_B, p_B] = i, \tag{16}$$

it can be treated as a change in either classical or quantum phase space variables for the fermions.

It is easy to check that the transformation from B to A turns the static solution in Eq. (3) into the closing hyperbola solution in Eq. (7), and that the inverse transformation from A to B turns the static hyperbola into the opening hyperbola. Actually, the latter is only true for $t_B < 0$; to obtain the remainder of the evolution of the opening hyperbola we need to analytically continue (12) in time.

This extended mapping from A to B, valid for $t_B > 0$ and $t_A < 0$, is

$$\begin{cases} x_B = \frac{1}{\sqrt{e^{-2t_A}-1}}x_A \\ p_B = -\sqrt{e^{-2t_A}-1}\,p_A + \frac{e^{-2t_A}}{\sqrt{e^{-2t_A}-1}}x_A \end{cases} \quad \begin{cases} x_A = \frac{1}{\sqrt{e^{2t_B}-1}}x_B \\ p_A = -\sqrt{e^{2t_B}-1}\,p_B + \frac{e^{2t_B}}{\sqrt{e^{2t_B}-1}}x_B \end{cases} \quad (e^{-2t_A}-1)(e^{2t_B}-1) = 1. \tag{17}$$

The above mapping takes a static hyperbola solution in A and turns it into the second half of the evolution of the opening hyperbola in B, when combined with a replacement of $\mu \to -\mu$.

Demanding that our mapping (12) correctly connects the static, opening and closing hyperbola solutions is not enough to uniquely fix it. For example, in [15], a different classical mapping was considered, based on the W_∞ algebra acting on phase space. What distinguishes (12) from all other possible maps is that the collective field $\eta(x)$ transforms trivially under it. Therefore, (12) preserves the form of the effective action (4). We should mention, however, that the mapping used in [15] leads to the same quantum state as ours.

III. TIME-DEPENDENT SOLUTIONS IN FERMIONIC VARIABLES

On the face of it, we have a map between two systems (either classical or quantum) with the *same* Hamiltonian which however evolve on a different time interval, since t_A runs from $-\infty$ to ∞ and t_B runs from $-\infty$ to 0. Using formulas in the Appendix, information contained in the mapping (12) can be summarized by a time-dependent unitary operator which transforms wave functions in the B system of coordinates into those in the A system. Using Eq. (A6), we see that

$$U(t) \equiv \exp\left(i \ln\sqrt{1+e^{2t}}\left(x^2 - \frac{xp+px}{2}\right)\right) \tag{18}$$

$$= \exp\left(-i \ln\sqrt{1-e^{2\tau}}\left(x^2 - \frac{xp+px}{2}\right)\right) \tag{19}$$

does the job. In order to avoid a large number of awkward indices in the following discussion, we have set $t = t_A$ and $\tau = t_B$. $\tau(t) < 0$ is given by $(e^{2t}+1)(1-e^{2\tau}) = 1$. To remind ourselves that U evolves in time, we will write is as either $U(t)$ or $U(\tau)$, whichever seems more natural. As t and τ are linked a one-to-one function, the choice of variable is cosmetic.

The fact that the two systems have the same Hamiltonian is exhibited by the following nontrivial property of U:

$$U(\tau)e^{-i(\tau-\tau_0)H} = e^{-i(t-t_0)H}U(t_0), \tag{20}$$

where $\tau_0 \equiv \tau(t_0)$ (For example, if we take a convenient choice of $t_0 = 0$ then $\tau_0 = -\ln\sqrt{2}$.) The above property of U can be proven using a special case of the Baker-Campbell-Hausdorff formula (see the Appendix for details). To put Eq. (20) in words, a wave function can either be first evolved from τ_0 to τ and then acted upon with U at time τ, or be acted upon with U at time t_0 and then evolved from t_0 to t; the result will be the same.

To go from A to B, we use $U^{-1} = U^\dagger$

$$U^\dagger(t) \equiv \exp\left(-i \ln\sqrt{1+e^{2t}}\left(x^2 - \frac{xp+px}{2}\right)\right)$$
$$= \exp\left(i \ln\sqrt{1-e^{2\tau}}\left(x^2 - \frac{xp+px}{2}\right)\right), \tag{21}$$

which has the property

$$U^\dagger(t)e^{-i(t-t_0)H} = e^{-i(\tau-\tau_0)H}U^\dagger(\tau_0). \tag{22}$$

The unitary operator U allows us to write time-dependent wave functions corresponding to the closing and opening hyperbola solutions, starting from the well-known static eigenfunctions. Denote with $\psi_E(x)$ the eigenfunctions of the Hamiltonian (1) with energy E:

$$-\tfrac{1}{2}(\partial_x^2 + x^2)\psi_E(x) = E\psi_E(x). \tag{23}$$

There are two such eigenfunctions at each eigenenergy, even or odd under parity. Since the change of variables we consider in (12) commutes with taking $x \to -x$, $p \to -p$, it is not necessary to worry about this degeneracy—everything we say will be true for the even and odd eigenfunctions separately.

Using again the formulas in the Appendix, we rewrite U and $U^\dagger$ in the following form:

$$U(\tau) \equiv (1-e^{2\tau})^{1/4}\exp\left(\frac{i}{2}e^{2\tau}x^2\right)\exp\left(\ln\sqrt{1-e^{2\tau}}\,x\frac{\partial}{\partial x}\right), \tag{24}$$

$$U^\dagger(t) \equiv (1+e^{2t})^{1/4}\exp\left(-\frac{i}{2}e^{2t}x^2\right)$$
$$\times \exp\left(\ln\sqrt{1+e^{2t}}\,x\frac{\partial}{\partial x}\right). \tag{25}$$

This form makes the action of $U(t)$ on an arbitrary wave function easy to read off. When acting on the stationary wave function $\psi_E(x)e^{-iE\tau}$ with $U(t)$, we obtain

$$\Psi(x,t) \equiv U(t)\psi_E(x)e^{-iE\tau}$$
$$= (1+e^{2t})^{-1/4}\exp\left(\frac{i}{2}\frac{e^{2t}}{1+e^{2t}}x^2\right)$$
$$\times \psi_E\left(\frac{x}{\sqrt{1+e^{2t}}}\right)e^{-iE\tau(t)}. \tag{26}$$

It is easy to check that (26) is a solution to the time-dependent Schrödinger equation with Hamiltonian (1), as long as we view t as the appropriate time variable. This wave function corresponds to the closing hyperbola, and is valid for all t.

Similarly,

$$\tilde{\Psi}(x,\tau) \equiv U^{\dagger}(\tau)\psi_E(x)e^{-iEt} = (1-e^{2\tau})^{-1/4}\exp\left(-\frac{i}{2}\frac{e^{2\tau}}{1-e^{2\tau}}x^2\right) \times \psi_E\left(\frac{x}{\sqrt{1-e^{2\tau}}}\right)e^{-iEt(\tau)}, \tag{27}$$

corresponds to the first half of the evolution of the opening hyperbola (for $\tau < 0$). To obtain the second half of that evolution, we must analytically continue in τ, which will be done in Sec. .

Let us investigate the form of the time-dependent wave functions in some detail.

The static wave functions ψ_Es are known exactly [17], but let us start with the large x asymptotics. From the WKB approximation, at large x the wave functions approach

$$\psi_E(x) \sim \frac{1}{\sqrt{x}}e^{\pm ix^2/2} \tag{28}$$

for all finite E. Therefore

$$\Psi(x) \sim \frac{1}{\sqrt{x}}\exp\left(\frac{i}{2}\frac{e^{2t}\pm 1}{e^{2t}+1}x^2\right). \tag{29}$$

For the upper sign, the asymptotic behavior is the same as in Eq. (28), but for the lower sign, the behavior is markedly different. This raises doubt about whether (26) can be written as a linear combination of $\psi_E(x)$s. If not, the time-dependent wave functions Ψ would be living in a different Hilbert space from the ψ_Es, and our quantum equivalence would be in trouble.

Fortunately, this is not the right argument. The question whether these wave functions live in the same Hilbert space should be answered by comparing the space of L^2 wave packets that can be built out of the "energy eigenbasis" in either case. A moment of thought reveals that the Hilbert space is indeed the same. Let us make it explicit. Consider a wave packet built out of the static eigenfunctions $\psi_E(x)$, and denote it with $\varphi(x)$. Now let us act on this wave packet with the unitary operator $U(0)$ (taking $t = 0$ to be definite). This gives us a new wave packet

$$\tilde{\varphi}(x) = (\text{phase})(2)^{-1/4}\exp(ix^2/4)\varphi\left(\frac{x}{\sqrt{2}}\right). \tag{30}$$

There is no reason why this new wave packet, also in L^2, but formally in the Hilbert space of the closing hyperbola states, cannot be written as a linear combination of the static eigenfunctions $\psi_E(x)$. We can calculate the Fourier coefficients as we always do, and the integrals must converge, by the virtue of $\tilde{\varphi}$ being in L^2. The two Hilbert spaces are therefore the same. Simply comparing asymptotic behavior was not enough because, for any fixed x, there are energies E sufficiently negative that the asymptotics do not apply. The change of basis formula, which we will derive in Sec. IV, should also be interpreted in the sense of wave packets.

Analytic continuation through $\tau = 0$

Equation (26) describes a complete history of one fermion. A collection of such fermions, one for each E from minus infinity up to some μ, is the quantum state corresponding to the classical solution (7). We have obtained this wave function by a linear transformation from the static wave function, but in doing so, we only used the evolution of the static state up to $\tau = 0$. What about τ positive? Formally, we can analytically continue the change of variables in (12) to positive τ by replacing $t \to i\pi(2n+1)/2 - t$, where n is an arbitrary integer. This will make the argument in ψ_E imaginary, so we need to understand $\psi_E(ix)$.

Fortunately, $\psi_E(ix)$ is easy to deal with. In the differential Eq. (23), the variable x can be thought of as a complex variable. As long as we focus on either odd or even wave functions, the solutions to (23) are unique. Substituting $x \to ix$ in (23) takes us back to the same equation, but with $E \to -E$. Therefore, using uniqueness, we must have $\psi_E(ix) \sim \psi_{-E}(x)$. The magnitude of the proportionality factor can be determined from the known behavior of the properly normalized wave functions at $x \ll \sqrt{|E|}$ [17]

$$\psi_E(x) \sim \frac{e^{-\pi E/2}}{E^{1/4}}\cosh(\sqrt{2E}x) \qquad \text{for even wave functions}, \tag{31}$$

$$\psi_E(x) \sim \frac{e^{-\pi E/2}}{E^{1/4}}\sinh(\sqrt{2E}x) \qquad \text{for odd wave functions}. \tag{32}$$

Therefore,

$$\psi_E(ix) = (\text{phase})e^{\pi E}\psi_{-E}(x), \tag{33}$$

for any ψ_E which is either even or odd under $x \to -x$.

With this result in hand, we can write, up to a constant, the wave function resulting from continuing τ through zero to the positive side in Eq. (26):

$$\Psi(x,t) = (e^{-2t}-1)^{-1/4}\exp\left(\frac{i}{2}\frac{e^{-2t}}{e^{-2t}-1}x^2\right) \times \psi_{-E}\left(\frac{x}{\sqrt{e^{-2t}-1}}\right)e^{-iE\tau(t)}, \tag{34}$$

where now $(e^{2\tau}-1)(e^{-2t}-1) = 1$ and $t < 0$. As can easily be verified, this also is a solution to the time-dependent Schrödinger equation.

The meaning of the analytic continuation through $\tau = 0$ becomes more clear if we analyze the behavior of the opening hyperbola solution. We will ignore again the overall normalization in the discussion, as it has no bearing on the physics.

Equation (27) gives the wave function corresponding to the opening hyperbola, for the first half of its evolution, $\tau < 0$. We can analytically continue this formula to positive τ, where we obtain (up to an overall, irrelevant normalization)

$$\Psi(x,\tau) = (e^{2\tau}-1)^{-1/4}\exp\left(\frac{i}{2}\frac{e^{2\tau}}{e^{2\tau}-1}x^2\right) \times \psi_{-E}\left(\frac{x}{\sqrt{e^{2\tau}-1}}\right)e^{iEt(\tau)}, \tag{35}$$

which also satisfies the Schrödinger equation. To check whether this analytic continuation indeed gives the second half of the evolution of the opening hyperbola, let us compare it with the wave function obtained by transforming the stationary wave function with the second mapping in Sec. II, Eq. (18). Under that transformation, a stationary wave function in system A, $\psi_E(x)e^{-Et}$, becomes

$$\Psi(x,\tau) = (e^{2\tau}-1)^{-1/4}\exp\left(\frac{i}{2}\frac{e^{2\tau}}{e^{2\tau}-1}x^2\right) \times \psi_{E}\left(\frac{x}{\sqrt{e^{2\tau}-1}}\right)e^{-iEt(\tau)}. \tag{36}$$

This is clearly the same wave function as (35), as long as we replace $E\to -E$, in agreement with the $\mu\to-\mu$ replacement which is part of (18).

The meaning of the analytic continuation is now clear: if we are interested in the evolution of the system B over the entire range of t_B, from $-\infty$ to $+\infty$, we must continue past $t_A = +\infty$, or alternatively use a second mapping for the second half of the evolution (which is what was done in [13]). These two approaches will lead to the same answer. Analytic continuation of $\tau \to i\pi(2n+1)/2 - \tau$ is then the meaning we should assign to the identification of boundaries in the Penrose diagrams in Fig. 1(c) (represented by an arrow there).

IV. CHANGE OF BASIS FORMULA

In this section, we will study exact expressions for time-dependent wave functions introduced in Sec. III, culminating in an explicit formula giving the closing hyperbola wave function as a linear combination of the static wave functions. We will perform this analysis in the chiral formalism, (first introduced in [18]), in which the form of the wave functions is simplest.

Define $a_\pm$ to be

$$a^\pm \equiv (x\pm p)/\sqrt{2}, \tag{37}$$

so that $[a^-, a^+] = i$ and $a_\mp = \pm i\partial/\partial_{a_\pm}$. Our mapping (12) in these coordinates is

$$a_B^- = \sqrt{1+e^{2t_A}}a_A^- \tag{38}$$

$$a_B^+ = \frac{1}{\sqrt{1+e^{2t_A}}}a_A^+ - \frac{e^{2t}}{\sqrt{1+e^{2t_A}}}a_A^-. \tag{39}$$

The advantage of the chiral coordinates is that the Hamiltonian is particularly simple

$$H = \mp i\left(a^\pm\frac{\partial}{\partial a^\pm}+\frac{1}{2}\right), \tag{40}$$

and so are its eigenfunctions,

$$\psi_E(a^\pm) = a_\pm^{\pm iE-(1/2)}. \tag{41}$$

Including time evolution is also very simple. Any wave function of the form $e^{\mp t/2}\varphi(e^{\mp t}a^\pm)$, for arbitrary $\varphi(\cdot)$, is a solution to the time-dependent Schrödinger equation. In particular, dressing up energy eigenfunctions in (41) with the proper time dependence gives

$$\psi_E(a^\pm,t) = (a^\pm)^{\pm iE-(1/2)}e^{-iEt} = e^{\mp t/2}(e^{\mp t}a^\pm)^{\pm iE-(1/2)} \tag{42}$$

The unitary operator in Eq. (18), when written in terms of $a^\pm$, is

$$U(t) = \exp\left(i\ln\sqrt{1+e^{2t}}\left(a_-^2+\frac{a_+a_-+a_-a_+}{2}\right)\right). \tag{43}$$

Using the formulas in the Appendix, we can rewrite this as

$$U(t) = (1+e^{2t})^{1/4}\exp\left(\frac{i}{2}e^{2t}a_-^2\right) \times \exp\left(\ln\sqrt{1+e^{2t}}a_-\frac{\partial}{\partial a_-}\right). \tag{44}$$

Acting with $U(t)$ on $\psi_E(a^-,\tau)$, we obtain the wave function of the closing hyperbola in the a_- basis:

$$\Psi(a^-,t) = e^{t/2}\exp\left(\frac{i}{2}e^{2t}(a^-)^2\right)(e^ta^-)^{-iE-(1/2)}, \tag{45}$$

where we have rearranged the wave function to exhibit an appropriate form of time dependence.

In this basis, it turns out to be possible to figure out how to express the wave function (45) as a linear combination of wave functions of the form (42).

To accomplish this, we make use of the following identity:

$$e^{iz} = \int_C \frac{ds}{2\pi}z^{-is}e^{-\pi s/2}\Gamma(is) \tag{46}$$

$$= \lim_{A,B\to+\infty} P.V.\int_{-A}^{B}\frac{ds}{2\pi}z^{-is}e^{-\pi s/2}\Gamma(is)+\frac{1}{2}, \tag{47}$$

Contour C runs along the real-s axis from $-\infty$ to $+\infty$ and below the pole at $s=0$. This formula can be obtained from the integral representation of the Γ-function [[19], Eq. 8.312.2], together with orthogonality conditions for the chiral wave functions (41). It holds for $z>0$, and has been verified numerically.

The convergence for $A, B \to \infty$ is not uniform in z. The limit $B \to \infty$ can be taken in a uniform fashion, since the integrand goes to zero rapidly for large positive s. For s large and negative, the integrand oscillates and only goes to zero as $1/\sqrt{|s|}$. It is then necessary to restrict $A \gg z$. Without uniform convergence, we have to be careful when applying this formula.

Using (47), we have that

$$\exp\left(\frac{i}{2}a^2\right)a^{-(1/2)-iE} = \int_C \frac{d\omega}{4\pi} 2^{(i/2)(E-\omega)} e^{(\pi/4)(E-\omega)} \times \Gamma\left(-\frac{i}{2}(E-\omega)\right) a^{-(1/2)-i\omega} \quad (48)$$

and therefore

$$\Psi(a^-, t) = U(t)\psi_E(a^-, \tau) = \int d\omega K(E-\omega)\psi_\omega(a^-, t), \quad (49)$$

where

$$K(\nu) \equiv \frac{1}{4\pi} 2^{i\nu/2} e^{\pi\nu/4} \Gamma\left(-\frac{i\nu}{2}\right) + \frac{1}{2}\delta(\nu). \quad (50)$$

At the end of Sec. III, we argued that Hilbert spaces of the closing hyperbola states and the static states are the same, since the same L^2 wave packets can be built in both cases. The formula above should be read in that spirit: it links the expansions of any given wave packet in the two basis. Focusing on wave packets removes any difficulty caused by the lack of uniform convergence.

For the sake of completeness, let us rewrite this result in the x-basis:

$$\Psi(x, t) = (1 + e^{2t})^{-1/4} \exp\left(\frac{i}{2} \frac{e^{2t}}{1 + e^{2t}} x^2\right) \times \psi_E\left(\frac{x}{\sqrt{1 + e^{2t}}}\right) e^{-iE\tau(t)} = \int d\omega K(E-\omega)\psi_\omega(x, t). \quad (51)$$

The kernel K is simply a representation of the unitary operator $U(t)$ in the appropriately time-evolving energy eigenbasis.

Notice that $K(\nu)$ decays exponentially for $\nu > 0$. Therefore, energy eigenstates with energy greater than E do not enter into the closing hyperbola solution labeled by E. This fact should is illustrated in Fig. 2(b): there, we can see that the contours for the closing hyperbola state lie within (i.e., at lower x) the static hyperbola contour at the same E.

V. THE FERMI SEA AND CORRELATORS

We can now discuss the quantum state of the doubly scaled matrix model. The fermionic field is defined as

$$\Psi(x, t) = \sum_E \psi_E(x) e^{-iEt} c_E, \quad (52)$$

where c_E is an annihilation operator for a fermion with energy E, $\{c_E, c_{E'}^\dagger\} = \delta_{E,E'}$. The static ground state filled up to the energy μ is defined as

$$|\mu\rangle = \left(\prod_{E<\mu} c_E^\dagger\right)|\mathbf{0}\rangle, \quad (53)$$

where $|\mathbf{0}\rangle$ is the state with no fermions, $c_E|\mathbf{0}\rangle = 0$ for all E. The operator which creates a single fermion with a wave function $\varphi(x)$ at time t is

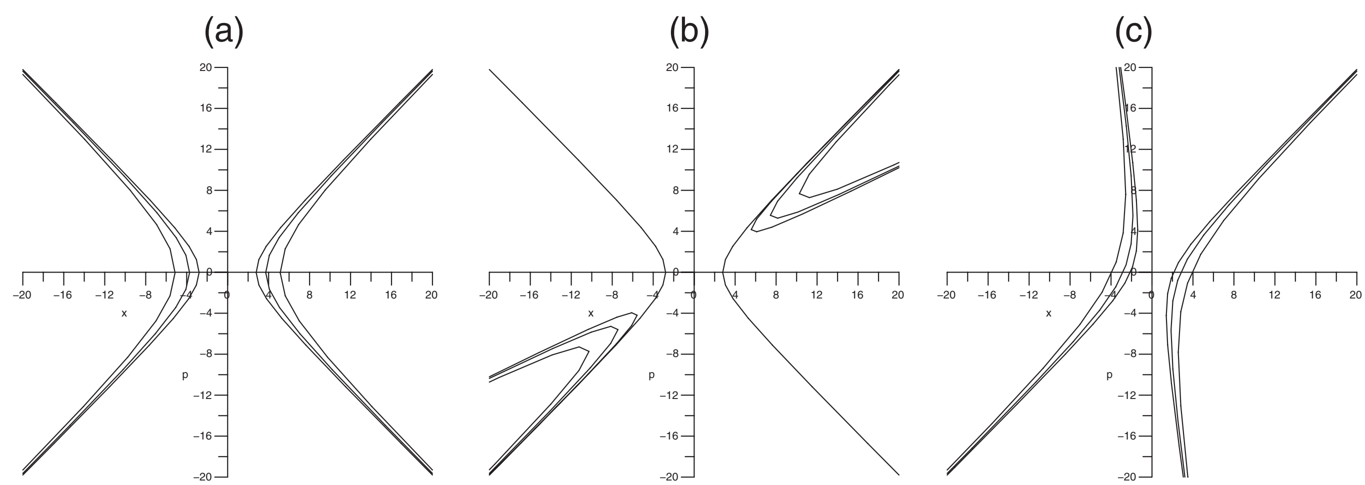

FIG. 2. Rough contour plots of the absolute value of the Wigner wave functions with $E = -10$. In each case, the center hyperbola is where the Wigner wave function peaks and the two other contours are at half height. (a) Static state $\psi_E(x)$; (b) closing hyperbola state (26) at $\exp(2t) = 3$, the additional hyperbola shown is the half-height contour from (a); (c) opening hyperbola state (27) at $\exp(2\tau) = 2/3$. The data for the plots was computed from the definition of the Wigner wave function in the chiral basis.

$$c_{\varphi}^{\dagger} = \int dx \Psi^{\dagger}(x,t)\varphi(x). \tag{54}$$

Therefore, the operator which creates a fermion in one of the closing hyperbola states is given by

$$c_{E,\mathrm{closing}}^{\dagger} \equiv \int dx \Psi^{\dagger}(x,t) \int d\omega K(E-\omega)\psi_{\omega}(x,t) = \int d\omega K(E-\omega) c_{\omega}^{\dagger}, \tag{55}$$

and the state corresponding to the closing hyperbola is

$$|\mu, \mathrm{closing}\rangle = \left(\prod_{E<\mu} \int d\omega K(E-\omega)c_{\omega}^{\dagger}\right)|\mathbf{0}\rangle. \tag{56}$$

This formula shows that the decaying Fermi sea is a state in the Hilbert space of the matrix model, an important fact, but not useful for computation of fermion correlators. More useful formulas can be obtained if we use the linear transformation $U(t)$ instead of the kernel K.

To do that, let us define

$$\tilde{c}_E = c_{E,\mathrm{closing}} \equiv \int dx \Psi(x,t)(U^{\dagger}(t)\bar{\psi}_E(x,\tau)), \tag{57}$$

$$\tilde{c}_E^{\dagger} = c_{E,\mathrm{closing}}^{\dagger} \equiv \int dx \Psi^{\dagger}(x,t)(U(t)\psi_E(x,\tau)), \tag{58}$$

so that

$$|\mu, \mathrm{closing}\rangle = \left(\prod_{E<\mu} \tilde{c}_{\omega}^{\dagger}\right)|\mathbf{0}\rangle \tag{59}$$

and

$$\Psi(x,t) = \sum_E (U(t)\psi_E(x)e^{-iE\tau})\tilde{c}_E. \tag{60}$$

Since $\{\tilde{c}_E, \tilde{c}_{E'}\} = \delta_{E,E'}$, any correlator of the form

$$\begin{aligned} &A^{\mathrm{closing}}(x_1, t_1, \ldots, x_n, t_n; x_1', t_1', \ldots, x_n', t_n') \\ &= \langle \mu, \mathrm{closing}|\Psi^{\dagger}(x_1', t_1')\ldots\Psi^{\dagger}(x_n', t_n') \\ &\times \Psi(x_1, t_1)\ldots\Psi(x_n, t_n)|\mu, \mathrm{closing}\rangle \end{aligned} \tag{61}$$

can be computed as a corresponding correlator in the static state,

$$\begin{aligned} &A^{\mathrm{closing}}(x_1, t_1, \ldots, x_n, t_n; x_1', t_1', \ldots, x_n', t_n') \\ &= \langle \mu|(U^{\dagger}(t_1')\Psi^{\dagger}(x_1', \tau_1'))\ldots(U^{\dagger}(t_n')\Psi^{\dagger}(x_n', \tau_n')) \\ &\times (U(t_1)\Psi(x_1, \tau_1))\ldots(U(t_n)\Psi(x_n, \tau_n))|\mu\rangle. \end{aligned} \tag{62}$$

We have seen that

$$U(t)\Psi(x,\tau) = (1+e^{2t})^{-1/4}\exp\left(\frac{i}{2}\frac{e^{2t}}{1+e^{2t}}x^2\right) \times \Psi\left(\frac{x}{\sqrt{1+e^{2t}}}, \tau(t)\right) \tag{63}$$

and therefore

$$\begin{aligned} &A^{\mathrm{closing}}(x_1, t_1, \ldots, x_n, t_n; x_1', t_1', \ldots, x_n', t_n') \\ &= \prod_{k=1}^{n}(1+e^{2t_k})^{-1/4}\prod_{k=1}^{n}(1+e^{2t_k'})^{-1/4}\prod_{k=1}^{n} \\ &\times \exp\left(\frac{i}{2}\frac{e^{2t_k}}{1+e^{2t_k}}x_n^2\right)\prod_{k=1}^{n}\exp\left(-\frac{i}{2}\frac{e^{2t_k'}}{1+e^{2t_k'}}(x_n')^2\right) \\ &\times A^{\mathrm{static}}\left(\frac{x_1}{\sqrt{1+e^{2t_1}}}, \tau_1, \ldots, \frac{x_n}{\sqrt{1+e^{2t_n}}}, \tau_n; \right. \\ &\left. \frac{x_1'}{\sqrt{1+e^{2t_1'}}}, \tau_1', \ldots, \frac{x_n'}{\sqrt{1+e^{2t_n'}}}, \tau_n'\right). \end{aligned} \tag{64}$$

Correlators in the static background are well known, see for example [17].

This formula is one of the main results of this paper, and is a generalization of the formulas in [15].

As a test, and a demonstration of this result, let us now compute the equal time correlator

$$\begin{aligned} A^{\mathrm{closing}}(x,t;y,t) &= \langle \mu, \mathrm{closing}|\Psi^{\dagger}(y,t)\Psi(x,t)|\mu, \mathrm{closing}\rangle \\ &= (1+e^{2t})^{-1/2}\exp\left(\frac{i}{2}\frac{e^{2t}}{1+e^{2t}}(x^2-y^2)\right) \\ &\times A^{\mathrm{static}}\left(\frac{x}{\sqrt{1+e^{2t}}}, \tau; \frac{y}{\sqrt{1+e^{2t}}}, \tau\right). \end{aligned} \tag{65}$$

This correlator is related to density of fermion eigenvalues in the x-p plane via a well-known formula for the expectation value of the Wigner operator in the context of the $c = 1$ models (see [20] and references therein)

$$\rho(x,p,t) = \int dy \frac{e^{-iyp}}{2\pi}\langle \Psi^{\dagger}(x+y/2,t)\Psi(x-y/2,t)\rangle. \tag{66}$$

After a short calculation, we conclude that

$$\rho^{\mathrm{closing}}(x,p,t) = \rho^{\mathrm{static}}\left(\frac{x}{\sqrt{1+e^{2t}}}, \sqrt{1+e^{2t}}p - \frac{e^{2t}}{\sqrt{1+e^{2t}}}x, \tau\right). \tag{67}$$

Taking the classical approximation where the density for a static hyperbola is simply

$$\rho_{\mu}^{\mathrm{static}}(x,p,t) = \begin{cases} 1 & \text{for } x^2 - p^2 > \mu \\ 0 & \text{otherwise,} \end{cases} \tag{68}$$

the above equation gives

$$\rho^{\mathrm{closing}}(x,p,t) = \begin{cases} 1 & \text{for } (x-p)(x+p-e^{2t}(x-p)) > \mu \\ 0 & \text{otherwise,} \end{cases} \tag{69}$$

which is the same answer we would have obtained if we simply used the classical transformation (12) on ρ^{static}.

A formula analogous to (67) can be derived for products of the Wigner operator, relating their correlators in the closing hyperbola state to the correlators in the static state via the classical mapping (12).

VI. DISCUSSION AND EXTENSIONS

The same quantum evolution has been presented in this paper in several different ways, which leads to the following ambiguity. Let us say someone presents us with a stationary wave function in the upside down harmonic oscillator potential. Without any further information, it is not clear whether this wave function is meant to describe simply the stationary state, or the closing hyperbola state (in which case we should interpret time as ending at zero), or the opening hyperbola state [in which case we should analytically continue the evolution of the system past the time = infinity mark, as can be seen in Fig. 1(c) and was discussed in Sec.].

There is hope that gravity resolves this ambiguity. After all, before it can describe string theory, the matrix model must be augmented by a leg-pole transform, which encodes gravitational and other interactions [2,4]. Our analysis does not capture everything about time-dependent solutions to gravitational effective action. Only once the time-dependent Fermi sea profile is translated into a valid background for dilaton gravity (and string theory) can additional information, such as the conformal factor for the metric, and the behavior of the dilaton, resolve this ambiguity. Unfortunately, such an analysis is beyond the scope of this paper. We have taken a first necessary step towards it, by expressing the closing hyperbola solution as a state in the Hilbert space of the matrix model.

With our explicit formula (56), it should be possible, at least in principle, to bosonize the closing hyperbola quantum state, and to obtain a quantum state in the bosonic collective theory which is closely related to the string theory tachyon. It might even be possible to find the string theory background which corresponds to this solution. The main obstacle is the currently incomplete understanding of the leg-pole transform linking the collective field to the tachyon.

To keep the algebra simple, we have considered here only those time-dependent solutions of the matrix model which approach the static solution in the infinite past. As a result, of the two quantum mechanical descriptions under consideration, one had a time variable running over the entire real line, and the other had semi-infinite time. It is possible to generalize the discussion in the paper to a more involved situation in which the time-dependent solution has time reversal symmetry, and thus diverges from the static one in both its past and its future. Then, one of the quantum mechanical descriptions has compact time: the time interval over which the evolution happens is finite.

In [15], an entire family of opening and closing hyperbola solutions was discussed. For each member of this family there exists a unitary operator U which translates between time evolution of the original matrix model and a new quantum system where the time-dependent solution appears static. The quantum equivalence between the two systems discussed in the present paper can be generalized this way to an entire family of equivalences.

One might wonder whether the results could be extended even further than that. What if we tried to treat more general Fermi surfaces, obtained by acting with higher order operators in the W_∞ algebra? The effective action in that case could not be brought into a static form [14], and therefore any equivalence would have to be between systems with different Hamiltonians. It would nonetheless be interesting to investigate such a possibility. Another interesting extension would be to consider the droplet solution [14].

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APPENDIX

Here we gather, for reference, a number of useful formulas which are used throughout the paper.

1. Baker-Campbell-Hausdorff formula

This formula states that for two elements X and Y of any algebra, we can write $\exp(X)\exp(Y)$ as $\exp(X + Y + \cdots)$, where the $\cdots$ is built entirely out of nested commutators of X and Y. The most widely used version of this formula is

$$[X, Y] = \theta \Rightarrow \exp(X)\exp(Y) = \exp\left(X + Y + \frac{\theta}{2}\right), \quad \text{(A1)}$$

applicable if θ is a c-number (in the center of the algebra).

We are going to need a little more. The following formulas can be derived explicitly, for example, in the sl(2) algebra,

$$[X, Y] = sY \Rightarrow \exp(X)\exp(Y) = \exp\left(X + \frac{s}{1 - e^{-s}} Y\right), \quad \text{and} \quad \exp(Y)\exp(X) = \exp\left(X + \frac{s}{e^{s} - 1} Y\right). \quad \text{(A2)}$$

Using the above two formulas we can also show that

$$[X, Y] = s(X + Y) \Rightarrow \exp(\alpha X)\exp(\beta Y) = \exp\left[(\alpha - \beta)\left(\frac{e^{s\alpha} - 1}{e^{s(\alpha-\beta)} - 1} X + \frac{e^{s\beta} - 1}{1 - e^{s(\beta-\alpha)}} Y\right)\right], \quad \text{(A3)}$$

which is used to prove (20).

2. Change of canonical variables

Let u and v be two canonical variables with $[\hat{u}, \hat{v}] = i$. (We will use hats on operators here to make things more clear.) We want to see how a wave function $\psi(u) \equiv \langle u|\psi\rangle$ corresponding to a state $|\psi\rangle$ is related to $\Psi(U) \equiv \langle U|\psi\rangle$. We will assume that $\hat{U}$, $\hat{V}$ are related to $\hat{u}$, $\hat{v}$ by

$$\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}. \tag{A4}$$

We will take $ad - cb = 1$ so that $[\hat{U}, \hat{V}] = i$, and assume that a is positive.

The relationship between $\psi(u)$ and $\Psi(U)$ depends on whether b is zero or not. Let us start with the simpler case of $b = 0$. We then have $d = 1/a$ and c is arbitrary. It then follows that

$$\Psi(U) = \frac{1}{\sqrt{a}} e^{(i/2)cdU^2} \psi(U/a). \tag{A5}$$

One way to obtain this formula is to define a unitary operator

$$\exp(D) \equiv \exp\left(-i\frac{\ln(a)}{2}(\hat{u}\,\hat{v} + \hat{v}\,\hat{u}) - i\frac{ac\ln(a)}{1-a^2}\hat{u}^2\right) \tag{A6}$$

which has the property that

$$\exp(-D)\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}\exp(D) = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}. \tag{A7}$$

We now make use of one of the special cases of the Baker-Campbell-Hausdorff formula, (A2) to obtain

$$\begin{aligned} \exp(D) &= \exp\left(\frac{i}{2}cdu^2\right)\exp\left(-\frac{\ln(a)}{2}\left(u\frac{\partial}{\partial u} + \frac{\partial}{\partial u}u\right)\right) \\ &= \frac{1}{\sqrt{a}}\exp\left(\frac{i}{2}cdu^2\right)\exp\left(-\ln(a)u\frac{\partial}{\partial u}\right), \end{aligned} \tag{A8}$$

from which Eq. (A5) can be read off easily.

For completeness, let us remark that for $b \neq 0$, we have

$$\langle U|u\rangle = \frac{1}{\sqrt{2\pi b}} e^{(i/2b)(au^2 - 2uU + dU^2)}, \tag{A9}$$

and the relationship between ψ and Ψ is given by

$$\Psi(U) = \int du \langle U|u\rangle \psi(u). \tag{A10}$$

If needed, this formula can be used to relate the wave functions in the chiral basis to those in the position basis.

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