Light-front dynamics and AdS/QCD correspondence: Gravitational form factors of composite hadrons

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Light-front holography is a remarkable feature of the AdS/CFT correspondence between gravity in AdS space and conformal field theories in physical space-time; it allows string modes $\Phi(z)$ in the anti-de Sitter (AdS) fifth dimension to be precisely mapped to the light-front wave functions of hadrons in physical space-time in terms of a specific light-front impact variable ζ which measures the separation of the quark and gluonic constituents within the hadron. This mapping was originally obtained by matching the exact expression for electromagnetic current matrix elements in AdS space with the corresponding exact expression for the current matrix element using light-front theory in physical space-time. In this paper we show that one obtains the identical holographic mapping using matrix elements of the energy-momentum tensor. To prove this, we show that there exists a correspondence between the matrix elements of the energy-momentum tensor of the fundamental hadronic constituents in QCD with the transition amplitudes describing the interaction of string modes in AdS space with an external graviton field which propagates in the AdS interior. The agreement of the results for electromagnetic and gravitational hadronic transition amplitudes provides an important consistency test and verification of holographic mapping from AdS to physical observables defined on the light front.

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I. INTRODUCTION

One of the most challenging problems of strong interaction dynamics is to determine the composition of hadrons in terms of their fundamental QCD quark and gluon degrees of freedom. Because of the strongly coupling nature of QCD in the infrared domain, it has been difficult to find analytic solutions for the wave functions of hadrons or to make precise predictions for hadronic properties outside of the perturbative regime. Thus an important theoretical goal is to find an initial approximation to bound state problems in QCD which is analytically tractable and which can be systematically improved. Recently the AdS/CFT correspondence [1] between string states in anti-de Sitter (AdS) space and conformal field theories (CFT) in physical space-time, modified for color confinement, has led to a semiclassical model for strongly-coupled QCD which provides analytical insights into its inherently nonperturbative nature including hadronic spectra, decay constants, and wave functions.

As we have shown recently, there is a remarkable mapping between the AdS description of hadrons and the Hamiltonian formulation of QCD in physical space-time quantized on the light front. The light-front wave functions of bound states in QCD are relativistic and frame-independent generalizations of the familiar Schrödinger wave functions of atomic physics, but they are determined at fixed light-cone time $\tau = t + z/c$—the "front form" advocated by Dirac [2]—rather than at fixed ordinary time t. The light-front wave functions of a hadron are independent of the momentum of the hadron, and they are thus boost invariant; Wigner transformations and Melosh rotations are not required. The light-front formalism for gauge theories in light-cone gauge is particularly useful in that there are no ghosts, and one has a direct physical interpretation of orbital angular momentum.

Light-front holography is an important feature of AdS/CFT; it allows string modes $\Phi(z)$ in the AdS fifth dimension to be precisely mapped to the light-front wave functions of hadrons in physical space-time in terms of a specific light-front impact variable ζ which measures the separation of the quark and gluonic constituents within the hadron. This mapping was originally obtained by matching the exact expression for electromagnetic current matrix elements in AdS space with the corresponding exact expression for the current matrix element using light-front theory in physical space-time [3,4]. In this paper we shall show that one obtains the identical holographic mapping using the matrix elements of the energy-momentum tensor. To prove this new result, we will show that there exists a correspondence between the matrix elements of the energy-momentum tensor of the fundamental hadronic constituents in QCD with the transition amplitudes describing the interaction of string modes in anti-de Sitter space with an external graviton field which propagates in the AdS interior.

The AdS/CFT correspondence implies that a strongly-coupled gauge theory is equivalent to the propagation of weakly-coupled strings in a higher dimensional space, where physical quantities are computed in terms of an effective gravitational theory. Thus, the AdS/CFT duality provides a gravity description in a $(d+1)$-dimensional AdS space-time in terms of a d-dimensional conformally-invariant quantum field theory at the AdS asymptotic boundary [5,6].

Holographic duality requires one to consider a higher dimensional warped space with negative curvature and a four-dimensional boundary. In particular, the conformal isometries of the five-dimensional anti-de Sitter space, a maximally symmetric space-time geometry with negative curvature, provides the basis for establishing a duality between a gravity or string theory on AdS_5 space and a conformal gauge theory defined at its four-dimensional space-time boundary. In its original formulation [1], a correspondence was established between the supergravity approximation to type IIB superstring theory on a curved background asymptotic to the product space of $\mathrm{AdS}_5 \times S^5$ [7] and the large N_C, $\mathcal{N}=4$, supersymmetric Yang-Mills (SYM) gauge theory in four dimensions with gauge group $SU(N)$ [8]. The group of conformal transformations $SO(4,2)$ which acts at the asymptotic boundary of AdS space, acts also as the group of isometries of AdS_5, and S^5 corresponds to the $SU(4) \sim SO(6)$ global symmetry which rotates the particles present in the SYM supermultiplet. The supergravity duality requires a large AdS radius R corresponding to a large value of the 't Hooft parameter $g_s N_C$, where $R = (4\pi g_s N_C)^{1/4} \alpha_s'^{1/2}$ and $\alpha_s'^{1/2}$ is the string scale. The classical approximation corresponds to the stiff limit where the string tension $T = R^2/2\pi\alpha' \to \infty$, effectively suppressing string fluctuations.

QCD is fundamentally different from SYM theories where all the matter fields transform in adjoint multiplets of $SU(N_C)$. QCD is also a confining theory in the infrared with a mass gap Λ_{QCD} and a well-defined spectrum of color singlet states. Conformal symmetry is broken in physical QCD by quantum effects and quark masses. There are indications however, both from theory and phenomenology, that the QCD coupling is slowly varying at small momentum transfer [9]. In particular, a new extraction of the effective strong coupling constant $\alpha_s^{g_1}(Q^2)$ from the CEBAF Large Acceptance Spectrometer Collaboration (CLAS) spin structure function data using the Bjorken sum $\Gamma_1^{p-n}(Q^2)$ in an extended Q^2 region [10], indicates the lack of Q^2 dependence of α_s in the low Q^2 limit. One can understand this physically [9]: in a confining theory where gluons have an effective mass or maximal wavelength, all vacuum polarization corrections to the gluon self-energy decouple at long wavelength; thus an infrared fixed point appears to be a natural consequence of confinement [11]. Furthermore, if one considers a semiclassical approximation to QCD with massless quarks and without particle creation or absorption, then the resulting β function is zero, the coupling is constant, and the approximate theory is scale and conformal invariant [12]. One can use conformal symmetry as a *template*, systematically correcting for its nonzero β function as well as higher-twist effects [13].

Different values of the holographic variable z determine the scale of the invariant separation between the partonic constituents. Hard scattering processes occur in the small-z ultraviolet (UV) region of AdS space. In particular, the $Q \to \infty$ zero separation limit corresponds to the $z \to 0$ asymptotic boundary, where the QCD Lagrangian is defined. In the large-z infrared (IR) region a cutoff is introduced to truncate the regime where the AdS modes can propagate. The infrared cutoff breaks conformal invariance, allows the introduction of a scale and a spectrum of particle states. In the hard-wall model [14] a cutoff is placed at a finite value $z_0 = 1/\Lambda_{\mathrm{QCD}}$ and the spectrum of states is linear in the radial and angular momentum quantum numbers: $\mathcal{M} \sim 2n + L$. In the soft-wall model a smooth infrared cutoff is chosen to model confinement and reproduce the usual Regge behavior $\mathcal{M}^2 \sim n + L$ [15]. The resulting models, although *ad hoc*, provide a simple semiclassical approximation to QCD which has both constituent counting rule behavior at short distances and confinement at large distances [9].

It is thus natural, as a useful first approximation, to use the isometries of AdS to map the local interpolating operators at the UV boundary of AdS space to the modes propagating inside AdS. The short-distance behavior of a hadronic state is characterized by its twist (dimension minus spin) $\tau = \Delta - \sigma$, where σ is the sum over the constituent's spin $\sigma = \sum_{i=1}^{n} \sigma_i$. Twist is also equal to the number of partons $\tau = n$. Under conformal transformations the interpolating operators transform according to their twist, and consequently the AdS isometries map the twist scaling dimensions into the AdS modes [16].

The eigenvalues of normalizable modes in AdS give the hadronic spectrum. AdS modes represent also the probability amplitude for the distribution of quarks and gluons at a given scale. There are also non-normalizable modes which are related to external currents: they propagate into the AdS interior and couple to boundary QCD interpolating operators [5,6]. Following this simplified bottom-up approach, a limited set of operators is introduced to construct phenomenological viable five-dimensional dual holographic models [17–20].

In the top-down supergravity approach, one introduces higher dimensional branes to the $\mathrm{AdS}_5 \times S^5$ background [21] in order to have a theory of flavor. One can obtain models with massive quarks in the fundamental representation, compute the hadronic spectrum, and describe chiral symmetry breaking in the context of higher dimensional brane constructs [21–25]. However, a theory dual to QCD is unknown, and this top-down approach is difficult to extend beyond theories exceedingly constrained by their symmetries.

An important feature of light-front quantization is the fact that it provides exact formulas for current matrix elements as a sum of bilinear forms which can be mapped into their AdS/CFT counterparts in the semiclassical approximation. The AdS metric written in terms of light-front coordinates $x^{\pm} = x^0 \pm x^3$ is

$$ds^2 = \frac{R^2}{z^2}(dx^+ dx^- - d\mathbf{x}_\perp^2 - dz^2). \tag{1.1}$$

At fixed light-front time $x^+ = 0$, the metric depends only on the transverse $\mathbf{x}_\perp$ and the holographic variable z. Thus we can find an exact correspondence between the fifth-dimensional coordinate of anti-de Sitter space z and a specific impact variable ζ in the light-front formalism. The new variable ζ measures the separation of the constituents within the hadron in ordinary space-time. The amplitude $\Phi(z)$ describing the hadronic state in AdS_5 can then be precisely mapped to the light-front wave functions $\psi_{n/H}$ of hadrons in physical space-time [3,4], thus providing a relativistic description of hadrons in QCD at the amplitude level.

The correspondence of AdS amplitudes to the QCD wave functions in light-front coordinates was carried out in [3,4] by comparing the expressions for the electromagnetic matrix elements in QCD and AdS for any value of the momentum transfer q^2. It is indeed remarkable that such a correspondence exists, since strings describe extended objects coupled to an electromagnetic field distributed in the AdS interior, whereas QCD degrees of freedom are pointlike particles with individual local couplings to the electromagnetic current. However, as we have shown [3,4], a precise mapping of AdS modes to hadronic light-front wave functions can be found in the strongly-coupled semiclassical approximation to QCD.

The matrix elements of local operators of hadronic composite systems, such as currents, angular momentum, and the energy-momentum tensor, have exact Lorentz invariant representations in the light front in terms of the overlap of light-front wave functions. One may ask, if the holographic mapping found in [3,4] for the electromagnetic current is specific to the charge distribution within a hadron or a general feature of light-front AdS/QCD.

The matrix elements of the energy-momentum tensor $\Theta^{\mu\nu}$ of each constituent define the gravitational form factor of a composite hadron. In this paper we shall use gravitational matrix elements to obtain the holographic mapping of the AdS mode wave functions $\Phi(z)$ in AdS space to the light-front wave functions ψ_H in physical 3 + 1 space-time defined at fixed light-cone time $\tau = t + z/c$. We find the identical holographic mapping from $z \to \zeta$ as in the electromagnetic case. The agreement of the results for electromagnetic and gravitational hadronic transition amplitudes provides an important consistency test and verification of holographic mapping from AdS to physical observables defined on the light front.

This paper is organized as follows. After briefly reviewing the QCD light-front Fock representation in Sec. II, we derive in Sec. III the exact form of matrix elements of the energy-momentum tensor for a n-parton composite object in light-front QCD. In Sec. IV we discuss the gravitational form factors in AdS/QCD. In Sec. V we describe the normalization of the AdS hadronic solutions to the energy-momentum tensor and obtain the corresponding hadronic transition matrix elements in AdS space. The actual mapping from AdS to QCD matrix elements is carried out in Sec. VI, where the Hamiltonian in the holographic light-front representation is related to the light-front Schrödinger equation predicted from AdS/QCD. Some final remarks are given in the conclusions in Sec. VII. Other aspects useful for the discussion of the paper are given in the appendices. In particular we describe in Appendix C the specific AdS/QCD mapping for a two-parton hadronic bound state, which is useful for understanding the n-parton results discussed in this article.

II. LIGHT-FRONT FOCK REPRESENTATION

The light-front expansion of any hadronic system is constructed by quantizing QCD at fixed light-cone time [2] $\tau = t + z/c$. In terms of the hadron four-momentum $P = (P^+, P^-, \mathbf{P}_\perp)$, $P^{\pm} = P^0 \pm P^3$, the light-cone Lorentz invariant Hamiltonian for the composite system $H_{\rm LF}^{\rm QCD} = P^- P^+ - \mathbf{P}_\perp^2$ has eigenvalues given in terms of the eigenmass $\mathcal{M}$ squared corresponding to the mass spectrum of the color-singlet states in QCD [26]

$$H_{\rm LF}|\psi_H\rangle = \mathcal{M}_H^2|\psi_H\rangle, \tag{2.1}$$

where $|\psi_H\rangle$ is an expansion in multiparticle Fock eigenstates $\{|n\rangle\}$ of the free light-front (LF) Hamiltonian: $|\psi_H\rangle = \sum_n \psi_{n/H}|\psi_H\rangle$. The light-front wave functions (LFWFs) $\psi_{n/H}$ provide a *frame-independent* representation of a hadron which relates its quark and gluon degrees of freedom to their asymptotic hadronic state.

The hadron wave function is an eigenstate of the total momentum P^+ and $\mathbf{P}_\perp$ and the longitudinal spin projection S_z, and is normalized according to

$$\langle\psi_H(P^+, \mathbf{P}_\perp, S_z)|\psi_H(P'^+, \mathbf{P}'_\perp, S'_z)\rangle = 2P^+(2\pi)^3\delta_{S_z,S'_z}\delta(P^+ - P'^+)\delta^{(2)}(\mathbf{P}_\perp - \mathbf{P}'_\perp). \tag{2.2}$$

Each hadronic eigenstate $|\psi_H\rangle$ is expanded in a Fock-state complete basis of noninteracting n-particle states $|n\rangle$ with an infinite number of components

$$|\psi_H(P^+, \mathbf{P}_\perp, S_z)\rangle = \sum_{n,\lambda_i}\prod_{i=1}^{n}\int\frac{dx_i d^2\mathbf{k}_{\perp i}}{2\sqrt{x_i}(2\pi)^3}(16\pi^3)\delta\Big(1 - \sum_{j=1}^{n} x_j\Big)\delta^{(2)}\Big(\sum_{j=1}^{n}\mathbf{k}_{\perp j}\Big) \times \psi_{n/H}(x_i, \mathbf{k}_{\perp i}, \lambda_i)|n: x_iP^+, x_i\mathbf{P}_\perp + \mathbf{k}_{\perp i}, \lambda_i\rangle, \tag{2.3}$$

where the sum begins with the valence state; e.g., $n \geq 3$ for baryons. The coefficients of the Fock expansion

$$\psi_{n/H}(x_i, \mathbf{k}_{\perp i}, \lambda_i) = \langle n\colon x_i, \mathbf{k}_{\perp i}, \lambda_i | \psi_H \rangle, \tag{2.4}$$

are independent of the total momentum P^+ and $\mathbf{P}_\perp$ of the hadron and depend only on the relative partonic coordinates, the longitudinal momentum fraction $x_i = k_i^+/P^+$, the relative transverse momentum $\mathbf{k}_{\perp i}$, and λ_i, the projection of the constituent's spin along the z direction. Thus, given the Fock projection (2.4), the wave function of a hadron is determined in any frame. The amplitudes $\psi_{n/H}$ represent the probability amplitudes to find on-mass-shell constituents i with longitudinal momentum $x_i P^+$, transverse momentum $x_i \mathbf{P}_\perp + \mathbf{k}_{\perp i}$, and helicity λ_i in the hadron H. Momentum conservation requires $\sum_{i=1}^n x_i = 1$ and $\sum_{i=1}^n \mathbf{k}_{\perp i} = 0$. In addition, each light-front wave function $\psi_{n/H}(x_i, \mathbf{k}_{\perp i}, \lambda_i)$ obeys the angular momentum sum rule [27] $J^z = \sum_{i=1}^n S_i^z + \sum_{i=1}^{n-1} L_i^z$, where $S_i^z = \lambda_i$ and the $n-1$ orbital angular momenta have the operator form

$$L_i^z = -i\left(\frac{\partial}{\partial k_i^x} k_i^y - \frac{\partial}{\partial k_i^y} k_i^x\right). \tag{2.5}$$

It should be emphasized that the assignment of quark and gluon spin and orbital angular momentum of a hadron is a gauge-dependent concept. The LF framework in light-cone gauge $A^+ = 0$ provides a physical definition since there are no gauge field ghosts and the gluon has spin projection $J^z = \pm 1$; moreover, it is frame-independent.

The LFWFs are normalized according to

$$\sum_n \int [dx_i][d^2\mathbf{k}_{\perp i}] |\psi_{n/H}(x_i, \mathbf{k}_{\perp i})|^2 = 1, \tag{2.6}$$

where the measure of the constituents phase-space momentum integration is

$$\int [dx_i] \equiv \prod_{i=1}^n \int dx_i \delta\left(1 - \sum_{j=1}^n x_j\right), \tag{2.7}$$

$$\int [d^2\mathbf{k}_{\perp i}] \equiv \prod_{i=1}^n \int \frac{d^2\mathbf{k}_{\perp i}}{2(2\pi)^3} 16\pi^3 \delta^{(2)}\left(\sum_{j=1}^n \mathbf{k}_{\perp j}\right). \tag{2.8}$$

The spin indices have been suppressed.

The complete basis of Fock states $|n\rangle$ is constructed by applying free-field creation operators to the vacuum state $|0\rangle$ which has no particle content, $P^+|0\rangle = 0$, $\mathbf{P}_\perp|0\rangle = 0$. The fundamental constituents appear in light-front quantization as the excitations of the dynamical fields, the Dirac field ψ_+, $\psi_\pm = \Lambda_\pm \psi$, $\Lambda_\pm = \gamma^0\gamma^\pm$, and the transverse field $\mathbf{A}_\perp$ in the $A^+ = 0$ gauge, each expanded in terms of quark and gluon creation and annihilation operators on the transverse plane with coordinates $x^- = x^0 - x^3$ and $\mathbf{x}_\perp$ at fixed light-front time $x^+ = x^0 + x^3$ [26]. For each kind of quark f the Dirac field operator is expanded as

$$\psi_f(x)_\alpha = \sum_\lambda \int_{q^+>0} \frac{dq^+}{\sqrt{2q^+}} \frac{d^2\mathbf{q}_\perp}{(2\pi)^3} [b_\lambda^f(q) u_\alpha(q, \lambda) e^{-iq\cdot x} + d_\lambda^f(q)^\dagger v_\alpha(q, \lambda) e^{iq\cdot x}], \tag{2.9}$$

with commutation relations

$$\{b(q), b^\dagger(q')\} = \{d(q), d^\dagger(q')\} = (2\pi)^3 \delta(q^+ - q'^+) \delta^{(2)}(\mathbf{q}_\perp - \mathbf{q}'_\perp). \tag{2.10}$$

Similar expansion follows for the transverse gluon field $\mathbf{A}_\perp$. We shall use the Lepage-Brodsky (LB) conventions [28] for the properties of the light-cone spinors. A one-particle state is defined by $|q\rangle = \sqrt{2q^+} b^\dagger(q)|0\rangle$. Each n-particle Fock state $|p_i^+, \mathbf{p}_{\perp i}\rangle$ is an eigenstate of P^+ and $\mathbf{P}_\perp$ and is normalized according to

$$\langle p_i^+, \mathbf{p}_{\perp i}, \lambda_i | p'^+{}_i, \mathbf{p}'{}_{\perp i}, \lambda'_i\rangle = 2p_i^+ (2\pi)^3 \delta(p_i^+ - p'^+{}_i) \times \delta^{(2)}(\mathbf{p}_{\perp i} - \mathbf{p}'{}_{\perp i}) \delta_{\lambda_i, \lambda'_i}. \tag{2.11}$$

The LFWFs $\psi_n(x_j, \mathbf{k}_{\perp j})$ can be expanded in terms of $n-1$ independent transverse coordinates $\mathbf{b}_{\perp j}$, $j = 1, 2, \ldots, n-1$, conjugate to the relative coordinates $\mathbf{k}_{\perp i}$

$$\psi_n(x_j, \mathbf{k}_{\perp j}) = (4\pi)^{(n-1)/2} \prod_{j=1}^{n-1} \int d^2\mathbf{b}_{\perp j} \times \exp\left(i \sum_{k=1}^{n-1} \mathbf{b}_{\perp k} \cdot \mathbf{k}_{\perp k}\right) \tilde{\psi}_n(x_j, \mathbf{b}_{\perp j}), \tag{2.12}$$

where $\sum_i \mathbf{b}_{\perp i} = 0$. The normalization is defined by

$$\sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} |\tilde{\psi}_n(x_j, \mathbf{b}_{\perp j})|^2 = 1. \tag{2.13}$$

III. GRAVITATIONAL FORM FACTORS OF COMPOSITE HADRONS IN QCD

Matrix elements of the energy-momentum tensor $\Theta^{\mu\nu}$ which define the gravitational form factors play an important role in hadron physics. Since one can define $\Theta^{\mu\nu}$ for each parton, one can identify the momentum fraction and contribution to the orbital angular momentum of each quark flavor and gluon of a hadron. For example, the spin-flip form factor $B(q^2)$, which is the analog of the Pauli form factor $F_2(Q^2)$ of a nucleon, provides a measure of the orbital angular momentum carried by each quark and gluon constituent of a hadron at $q^2 = 0$. Similarly, the spin-conserving form factor $A(q^2)$, the analog of the Dirac form factor $F_1(q^2)$, allows one to measure the momentum fractions carried by each constituent. This is the underlying physics of Ji's sum rule [29]: $\langle J^z\rangle = \frac{1}{2}[A(0) + B(0)]$, which has prompted much of the current interest in the generalized parton distributions (GPDs) measured in deeply virtual Compton scattering [30]. Measurements of the GPDs are of particular relevance for determining the distribution of partons in the transverse impact plane, and thus could be confronted with AdS/QCD predictions which follow from the mapping of AdS modes to the transverse impact representation [3].

An important constraint is $B(0) = \sum_i B_i(0) = 0$; i.e. the anomalous gravitomagnetic moment of a hadron vanishes when summed over all the constituents i. This was originally derived from the equivalence principle of gravity [31]. The explicit verification of these relations, Fock state by Fock state, can be obtained in the light-front quantization of QCD in light-cone gauge [27]. Physically $B(0) = 0$ corresponds to the fact that the sum of the n orbital angular momenta L in an n-parton Fock state must vanish since there are only $n - 1$ independent orbital angular momenta (2.5).

Gravitational form factors can also be computed in AdS/QCD from the overlap integral of hadronic string modes propagating in AdS space with a graviton field $h_{\mu\nu}$ which acts as a source and probes the AdS interior. This has been done very recently for the gravitational form factors of mesons by Abidin and Carlson [32], thus providing restrictions on the GPDs.

Recent applications to the electromagnetic form factors of hadrons [4,9,33–35] in the bottom-up and in the top-down string framework [36] of the AdS/CFT correspondence have followed from the original papers [37,38]. Here we shall extend our previous results [3,4] for the holographic mapping of AdS current matrix elements to gravitational form factors. If both quantities for the gravitational form factors represent the same physical observable for any value of the momentum transfer q^2, then an exact correspondence can be established between the AdS modes $\Phi(z)$ and LFWFs of hadrons $\psi_{n/H}$ as in the case of the electromagnetic form factors. To simplify the discussion, we will consider the holographic mapping of matrix elements of the energy-momentum tensor of mesons, where only one gravitational form factor is present, but the results can be extended to other hadrons as shown in [32].

The QCD Lagrangian density is

$$\mathcal{L}_{\rm QCD} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}, \tag{3.1}$$

where $D_\mu = \partial_\mu - i g_s A^a_\mu T^a$ and $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s c^{abc} A^b_\mu A^c_\nu$, with $[T^a, T^b] = i c^{abc} T^c$ and a, b, c are $SU(3)$ color indices.

We can find a symmetric and gauge-invariant expression for the energy-momentum tensor $\Theta_{\mu\nu}$, the Hilbert energy-momentum tensor, by varying the QCD action with respect to the four-dimensional Minkowski space-time metric $g_{\mu\nu}(x)$

$$\Theta_{\mu\nu}(x) = -\frac{2}{\sqrt{g}} \frac{\delta S_{\rm QCD}}{\delta g^{\mu\nu}(x)}, \tag{3.2}$$

where $S_{\rm QCD} = \int d^4x \sqrt{g} \mathcal{L}_{\rm QCD}$ and $g \equiv |\det g_{\mu\nu}|$. The result is

$$\Theta_{\mu\nu} = \frac{1}{2} \bar{\psi} i(\gamma_\mu D_\nu + \gamma_\nu D_\mu)\psi - g_{\mu\nu}\bar{\psi}(i\not{D} - m)\psi - G^a_{\mu\lambda} G^{a\lambda}_\nu + \frac{1}{4} g_{\mu\nu} G^a_{\lambda\sigma} G^{a\lambda\sigma}. \tag{3.3}$$

The first two terms in (3.3) correspond to the fermionic contribution to the energy-momentum tensor and the last two to the gluonic contribution. In terms of (3.3) the total angular momentum operator J of the composite hadron can be expressed in the gauge-invariant form

$$J_i = \frac{1}{2} \epsilon_{ijk} \int d^3x [\Theta^{0k} x^j - \Theta^{0j} x^k]. \tag{3.4}$$

In the semiclassical AdS/CFT correspondence there are no quantum effects, and only the valence Fock state contributes to the hadronic wave function. In this approximation we need to consider only the quark contribution to the energy-momentum tensor. In the light-front gauge $A^+ = 0$ the fermionic component Θ^{++} is

$$\Theta^{++}(x) = \frac{i}{2} \sum_f \bar{\psi}_f(x) \gamma^+ \overleftrightarrow{\partial}^+ \psi_f(x), \tag{3.5}$$

where an integration by parts is carried out to write Θ^{++} in its Hermitian operator form. The sum in (3.5) extends over all the types of quarks f present in the hadron. Notice that the second term of the energy-momentum tensor (3.3) does not appear in the expression for Θ^{++} since the metric component g^{++} is zero in the light front as discussed in Appendix A.

We will use light-front frame coordinates

$$\begin{aligned} P &= (P^+, P^-, \mathbf{P}_\perp) = \left(P^+, \frac{M^2}{P^+}, \vec{0}_\perp\right), \\ q &= (q^+, q^-, \mathbf{q}_\perp) = \left(0, \frac{2q \cdot P}{P^+}, \mathbf{q}_\perp\right), \end{aligned} \tag{3.6}$$

where $q^2 = -Q^2 = -2q \cdot P = -\mathbf{q}_\perp^2$ is the spacelike four-momentum squared transferred to the composite hadron. The gravitational form factor of a meson is defined in terms of matrix elements of the "plus-plus" components of the energy-momentum tensor evaluated at light-cone time $x^+ = 0$. In the $q^+ = 0$ frame

$$\langle P' | \Theta^{++}(0) | P \rangle = 2(P^+)^2 A(Q^2), \tag{3.7}$$

where $P' = P + q$ and the gravitational form factor $A(Q^2)$ satisfies the momentum sum rule $A(0) = 1$.

The expression for the operator $\Theta^{++}(0)$ in the particle number representation follows from the momentum expansion of the Dirac field $\psi(x)$ in terms of creation and annihilation operators given by (2.9). Using the light-cone metric conventions given in Appendix A and the results listed in Appendix A of [4] for the quark spinor transitions, we find

$$\Theta^{++} = \frac{1}{2}\sum_{f,\lambda}\int \frac{dq^+ d^2\mathbf{q}_\perp}{(2\pi)^3}\int \frac{dq'^+ d^2\mathbf{q}'_\perp}{(2\pi)^3}(q^+ + q'^+) \times \{b_\lambda^{f\dagger}(q)b_\lambda^f(q') + d_\lambda^{f\dagger}(q')d_\lambda^f(q)\}. \tag{3.8}$$

The operator Θ^{++} annihilates a quark (antiquark) with momentum q' (q) and spin projection λ along the z direction and creates a quark (antiquark) with the same spin and momentum q (q').

The matrix element of the energy-momentum tensor $\langle\psi_{P'}|\Theta^{++}(0)|\psi_P\rangle$ can be computed by expanding the initial and final hadronic states in terms of its Fock components using (2.3). The transition amplitude can then be expressed as a sum of overlap integrals with diagonal Θ^{++}-matrix elements in the n-particle Fock-state basis. For each Fock state, we label with $i = n$ the struck constituent quark with light-front longitudinal momentum fraction $x_n = x$ and with $j = 1, 2, \ldots, n-1$ each spectator with longitudinal momentum fraction x_j. Using the normalization condition (2.11) for each individual constituent and after integration over the intermediate variables in the $q^+ = 0$ frame, we find the expression for the gravitational form factor of a meson [27]

$$A(q^2) = \sum_n \int [dx_i][d^2\mathbf{k}_{\perp i}]\sum_{f=1}^{n} x_f \psi^*_{n/H}(x_i, \mathbf{k}'_{\perp i}, \lambda_i) \times \psi_{n/H}(x_i, \mathbf{k}_{\perp i}, \lambda_i), \tag{3.9}$$

where the sum is over all the partons in each Fock state n. The variables of the light-cone Fock components in the final-state are given by $\mathbf{k}'_{\perp i} = \mathbf{k}_{\perp i} + (1 - x_i)\mathbf{q}_\perp$ for a struck constituent quark and $\mathbf{k}'_{\perp i} = \mathbf{k}_{\perp i} - x_i\mathbf{q}_\perp$ for each spectator. Notice that each type of parton contributes to the gravitational form factor with struck constituent light-cone momentum fractions x_f, instead of the electromagnetic constituent charge e_f which appears in the electromagnetic form factor. Since the longitudinal momentum fractions of the constituents add to 1, $\sum_f x_f = 1$, the momentum sum rule is satisfied at $q = 0$: $A(0) = 1$; the formulae are exact if the sum is over all Fock states n. Notice that there is a factor of N_C from a closed quark loop where the graviton is attached and a normalization factor of $1/\sqrt{N_C}$ for each meson wave function; thus color factors cancel out from the expression of the gravitational form factor.

In the light-front formalism matrix elements of local operators are represented as overlaps of light-front wave functions. In order to compare with AdS results it is convenient to express the LF expressions in the transverse impact representation since the bilinear forms may be expressed in terms of the product of light-front wave functions with identical variables. We substitute (2.12) in the formula (3.9). Integration over $k_\perp$ phase space gives us $n-1$ delta functions to integrate over the $n-1$ intermediate transverse variables with the result

$$A(q^2) = \sum_n \prod_{j=1}^{n-1}\int dx_j d^2\mathbf{b}_{\perp j}\sum_{f=1}^{n} x_f \exp\left(i\mathbf{q}_\perp \cdot \sum_{k=1}^{n-1} x_k \mathbf{b}_{\perp k}\right) \times |\tilde{\psi}_n(x_j, \mathbf{b}_{\perp j}, \lambda_j)|^2, \tag{3.10}$$

corresponding to a change of transverse momentum $x_j\mathbf{q}_\perp$ for each of the $n-1$ spectators and is valid for any Fock state n. The results can be summed over n to obtain an exact representation.

A. Effective single-particle distribution

We can define $A_{f/n}(q^2)$ which is the contribution to the gravitational form factor from the struck parton f in Fock state n. In terms of the $n-1$ independent coordinates x_k and $\mathbf{b}_{\perp k}$, $k = 1, 2, \ldots, n$, $k \neq f$ we have

$$A_{f/n}(q^2) = \prod_{k\neq f}\int dx_k d^2\mathbf{b}_{\perp k}\left(1 - \sum_{\ell\neq f} x_\ell\right) \times \exp\left(i\mathbf{q}_\perp \cdot \sum_{m\neq f} x_m \mathbf{b}_{\perp m}\right)|\tilde{\psi}_n(x_k, \mathbf{b}_{\perp k})|^2_{k\neq f}. \tag{3.11}$$

Following [3,4] we can write the gravitational form factor in terms of an effective single-particle density [39] in the light-front frame. Summing over Fock states $A_f(q^2) = \sum_n A_{f/n}(q^2)$, we have

$$A_f(q^2) = \int_0^1 x dx \rho_f(x, \mathbf{q}_\perp), \tag{3.12}$$

where $A(0) = \sum_f A_f(0) = \sum_f \langle x_f\rangle = 1$. The effective density $\rho_f(x, \mathbf{q}_\perp)$ is given by

$$\rho_f(x, \mathbf{q}_\perp) = \sum_n \prod_{k\neq f}\int dx_k d^2\mathbf{b}_{\perp k}\delta\left(1 - x - \sum_{\ell\neq f} x_\ell\right) \times \exp\left(i\mathbf{q}_\perp \cdot \sum_{m\neq f} x_m \mathbf{b}_{\perp m}\right)|\tilde{\psi}_n(x_k, \mathbf{b}_{\perp k})|^2_{k\neq f}. \tag{3.13}$$

The integration is over the coordinates of the $n-1$ spectator partons, and x is the coordinate of the active quark with longitudinal momentum x. We can also write the form factor in terms of an effective single-particle transverse distribution $\tilde{\rho}_f(x, \vec{\eta}_\perp)$

$$A_f(q^2) = \int_0^1 x dx \int d^2\vec{\eta}_\perp e^{i\vec{\eta}_\perp\cdot\mathbf{q}_\perp}\tilde{\rho}_f(x, \vec{\eta}_\perp), \tag{3.14}$$

where $\vec{\eta}_\perp = \sum_{k\neq f} x_k \mathbf{b}_{\perp k}$ is the x-weighted transverse position coordinate of the $n-1$ spectators. The corresponding transverse density is [3,4]

$$\tilde{\rho}_f(x, \vec{\eta}_\perp) = \int \frac{d^2\mathbf{q}_\perp}{(2\pi)^2} e^{-i\vec{\eta}_\perp\cdot\mathbf{q}_\perp} \rho_f(x, \mathbf{q}_\perp)$$
$$= \sum_n \prod_{k\neq f} \int dx_k d^2\mathbf{b}_{\perp k} \delta\Big(1 - x - \sum_{\ell\neq k} x_\ell\Big)$$
$$\times \delta^{(2)}\Big(\sum_{m\neq f} x_m \mathbf{b}_{\perp m} - \vec{\eta}_\perp\Big)|\tilde{\psi}_n(x_k, \mathbf{b}_{\perp k})|^2_{k\neq f}. \tag{3.15}$$

It is useful to integrate (3.14) over angle; we obtain

$$A(q^2) = 2\pi \sum_f \int_0^1 dx(1 - x)$$
$$\times \int \zeta d\zeta J_0\Big(\zeta q \sqrt{\frac{1-x}{x}}\Big)\tilde{\rho}_f(x, \zeta), \tag{3.16}$$

where we have introduced the variable

$$\zeta = \sqrt{\frac{x}{1-x}}\Big|\sum_{k\neq f} x_k \mathbf{b}_{\perp k}\Big|, \tag{3.17}$$

representing the x-weighted transverse impact coordinate of the spectator system.

IV. GRAVITATIONAL FORM FACTORS IN ADS/CFT

AdS coordinates are the $d = 4$ Minkowski coordinates x^μ and z, the holographic coordinate, which we label $x^\ell = (x^\mu, z)$. The metric of AdS_{d+1} space-time is

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \tag{4.1}$$

where the AdS radius is R. Fields propagating in five-dimensional AdS space are represented by capital letters such as Φ. Holographic modes in four-dimensional Minkowski space are represented by ϕ.

A. Gauge/Gravity semiclassical correspondence

The formal statement of the duality between a gravity theory on $(d + 1)$-dimensional anti-de Sitter AdS_{d+1} space and the strong coupling limit of a conformal field theory (CFT) on the d-dimensional asymptotic boundary of AdS_{d+1} at $z = 0$ is expressed in terms of the $d + 1$ partition function for a field $\Phi(x, z)$ propagating in the bulk

$$Z_{\rm grav}[\Phi(x, z)] = e^{iS_{\rm eff}[\Phi]} = \int \mathcal{D}[\Phi] e^{iS[\Phi]}, \tag{4.2}$$

where $S_{\rm eff}$ is the effective action of the AdS_{d+1} theory, and the d-dimensional generating functional of the conformal field theory in presence of an external source $\Phi_0(x)$,

$$Z_{\rm CFT}[\Phi_0(x)] = e^{iW_{\rm CFT}[\Phi_0]} = \Big\langle \exp\Big(i \int d^d x \Phi_0(x)\mathcal{O}(x)\Big)\Big\rangle. \tag{4.3}$$

The functional $W_{\rm CFT}$ is the generator of connected Green's functions of the boundary theory and $\mathcal{O}(x)$ is a QCD local interpolating operator. The precise relation of the gravity theory on AdS space to the conformal field theory at its boundary is [5,6]

$$Z_{\rm grav}[\Phi(x, z)|_{z=0} = \Phi_0(x)] = Z_{\rm CFT}[\Phi_0], \tag{4.4}$$

where the partition function (4.2) on AdS_{d+1} is integrated over all possible configurations Φ in the bulk which approach its boundary value Φ_0. If we neglect the contributions from the nonclassical configurations to the gravity partition function, then the functional $W_{\rm CFT}$ of the four-dimensional gauge theory (4.3) is precisely equal to the classical (on-shell) gravity action (4.2)

$$W_{\rm CFT}[\Phi_0] = S_{\rm eff}[\Phi(x, z)|_{z=0} = \Phi_0(x)]_{\text{on-shell}}, \tag{4.5}$$

evaluated in terms of the classical solution to the bulk equation of motion. This defines the semiclassical approximation to the conformal field theory. In the limit $z \to 0$, the independent solutions behave as

$$\Phi(x, z) \to z^\Delta \Phi_+(x) + z^{d-\Delta}\Phi_-(x), \tag{4.6}$$

where Δ is the conformal dimension. The non-normalizable solution Φ_- is the boundary value of the bulk field Φ which couples to a QCD gauge-invariant operator $\mathcal{O}$ in the $z \to 0$ asymptotic boundary, thus $\Phi_- = \Phi_0$. The normalizable solution $\Phi_+(x)$ is the response function and corresponds to the physical states [40]. The interpolating operators $\mathcal{O}$ of the boundary conformal theory are constructed from local gauge-invariant products of quark and gluon fields and their covariant derivatives, taken at the same point in four-dimensional space-time in the $x^2 \to 0$ limit. Their conformal twist-dimensions are matched to the scaling behavior of the AdS fields in the limit $z \to 0$ and are thus encoded into the propagation of the modes inside AdS space.

Integrating by parts and using the equation of motion for a scalar field in AdS (as discussed below), the bulk contribution to the action vanishes, and one is left with a nonvanishing surface term in the ultraviolet boundary

$$S = R^{d-1} \lim_{z\to 0} \int d^d x \frac{1}{z^{d-1}} \Phi \partial_z \Phi, \tag{4.7}$$

which can be identified with the boundary functional $W_{\rm CFT}$. Substituting the leading dependence (4.6) of Φ near $z = 0$ in the ultraviolet surface action (4.7) and using the functional relation

$$\frac{\delta W_{\rm CFT}}{\delta \Phi_0} = \frac{\delta S_{\rm eff}}{\delta \Phi_0}, \tag{4.8}$$

it follows that $\Phi_+(x)$ is related to the expectation value of $\mathcal{O}$ in the presence of the source Φ_0 [40]

$$\langle 0|\mathcal{O}(x)|0\rangle_{\Phi_0} \sim \Phi_+(x). \tag{4.9}$$

The exact relation depends on the normalization of the fields used [41]. The field Φ_+ thus acts as a classical field, and it is the boundary limit of the normalizable string solution which propagates in the bulk.

B. Gravity action

The action for gravity coupled to a scalar field in AdS_{d+1} space is

$$S = \int d^{d+1}x\sqrt{g}\left(\frac{1}{\kappa^2}(\mathcal{R} - 2\Lambda) + g^{\ell m}\partial_\ell\Phi^*\partial_m\Phi - \mu^2\Phi^*\Phi\right), \tag{4.10}$$

where $\mathcal{R}$ is the scalar curvature, κ is the $d+1$-dimensional Newton constant, and μ is a $d+1$-dimensional mass. The action is written as a sum of two terms $S = S_G + S_M$, where S_G

$$S_G = \frac{1}{\kappa^2}\int d^{d+1}x\sqrt{g}(\mathcal{R} - 2\Lambda), \tag{4.11}$$

describes the dynamics of the gravitational fields $g_{\ell m}$ and determines the AdS background. The dynamics of all other fields, the matter fields, is included in S_M. In the present discussion the matter content is represented by Φ and the action

$$S_M = \int d^{d+1}x\sqrt{g}(g^{\ell m}\partial_\ell\Phi^*\partial_m\Phi - \mu^2\Phi^*\Phi), \tag{4.12}$$

describes a pion mode which propagates in AdS space.

The variation of the action with respect to the metric tensor gives Einstein's equations in the presence of a bulk cosmological constant Λ:

$$\mathcal{R}_{\ell m} - \frac{1}{2}g_{\ell m}\mathcal{R} - \Lambda g_{\ell m} = 0. \tag{4.13}$$

AdS space is a maximally symmetric space with Riemann tensor $R_{ik\ell m}$

$$\mathcal{R}_{ik\ell m} = -\frac{1}{R^2}(g_{i\ell}g_{km} - g_{im}g_{k\ell}). \tag{4.14}$$

By contracting $\mathcal{R}_{ik\ell m}$ we obtain the Ricci tensor $R_{ik} = g^{\ell m}R_{\ell imk}$,

$$R_{ik} = -\frac{d}{R^2}g_{ik}. \tag{4.15}$$

Thus AdS space is an Einstein manifold. By further contracting the Ricci tensor $\mathcal{R} = g^{ik}R_{ik} = g^{i\ell}g^{km}\mathcal{R}_{ik\ell m}$, we obtain the scalar curvature of AdS_{d+1} space $\mathcal{R} = -\frac{d(d+1)}{R^2}$, a constant negative curvature. From the equation of motion (4.13) we find the relation between the cosmological constant and the AdS_{d+1} radius

$$\Lambda = -\frac{d(d-1)}{2R^2}, \tag{4.16}$$

thus $\Lambda = -\frac{6}{R^2}$ for $d = 4$.

Taking the variation of (4.12) with respect to Φ we find the AdS wave equation for the pion mode

$$z^3\partial_z\left(\frac{1}{z^3}\partial_z\Phi\right) - \partial_\rho\partial^\rho\Phi - \left(\frac{\mu R}{z}\right)^2\Phi = 0. \tag{4.17}$$

C. Interaction terms in the gravity action

The expression for the AdS matrix elements describing the interaction of the matter fields in AdS space with an external arbitrary source at the AdS asymptotic boundary follows from the gauge-invariant definition of the energy-momentum tensor

$$\Theta_{\ell m}(x^\ell) = -\frac{2}{\sqrt{g}}\frac{\delta S_M}{\delta g^{\ell m}(x^\ell)}, \tag{4.18}$$

where $g \equiv |\det g_{\ell m}|$. In order to determine the precise form of the transition amplitudes, we shall consider a small deformation of the metric about its AdS background $g_{\ell m}$: $\bar{g}_{\ell m} = g_{\ell m} + h_{\ell m}$; we then expand S_M to first order in $h_{\ell m}$. From (4.12) and (4.18)

$$S_M[h_{\ell m}] = S_M[0] + \frac{1}{2}\int d^{d+1}x\sqrt{g}h_{\ell m}\Theta^{\ell m} + \mathcal{O}(h^2), \tag{4.19}$$

where we have used the relation $\Theta^{\ell m}\delta g_{\ell m} = -\Theta_{\ell m}\delta g^{\ell m}$ which follows from $g^{\ell m}\delta g_{\ell m} = -g_{\ell m}\delta g^{\ell m}$. Thus, in the weak gravitational approximation the coupling of an external graviton field $h_{\ell m}$ to matter is given by the interaction term ($d = 4$)

$$S_I = \frac{1}{2}\int d^4x dz\sqrt{g}h_{\ell m}\Theta^{\ell m}. \tag{4.20}$$

From (4.12) and (4.18) we find the energy-momentum tensor of the matter field Φ

$$\Theta_{\ell m} = \partial_\ell\Phi^*\partial_m\Phi + \partial_m\Phi^*\partial_\ell\Phi - g_{\ell m}(\partial^n\Phi^*\partial_n\Phi - \mu^2\Phi^*\Phi). \tag{4.21}$$

Likewise, we can determine the AdS equation of motion of the graviton field $h_{\ell m}$ by substituting the modified metric $\bar{g}_{\ell m} = g_{\ell m} + h_{\ell m}$ into the gravitational action S_G. We find

$$S_G[h_{\ell m}] = S_G[0] + \frac{1}{4\kappa^2}\int d^{d+1}x\sqrt{g} \times\left(\partial_n h^{\ell m}\partial^n h_{\ell m} - \frac{1}{2}\partial_\ell h\partial^\ell h\right) + \mathcal{O}(h^2), \tag{4.22}$$

where the trace h^ℓ_ℓ is denoted by h. In deriving (4.22) we have made use of the gauge invariance of the theory $h'_{\ell m} = h_{\ell m} + \partial_\ell\epsilon_m + \partial_m\epsilon_\ell$ to impose the harmonic gauge condition $\partial_\ell h^\ell_m = \frac{1}{2}\partial_m h$. The action describing the dynamical fields $h_{\ell m}$ in the weak field approximation ($d = 4$) is given in the linearized form

$$S_h = \frac{1}{4\kappa^2}\int d^4x dz \sqrt{g}\left(\partial_n h^{\ell m}\partial^n h_{\ell m} - \frac{1}{2}\partial_\ell h \partial^\ell h\right), \tag{4.23}$$

resembling the treatment of an ordinary gauge field. The total bulk action describing the coupling of gravity and matter with an external graviton in the weak field approximation thus has two additional terms: $S = S_G + S_M + S_h + S_I$.

V. HADRONIC STATES AND TRANSITION MATRIX ELEMENTS IN ADS/CFT

A physical hadron in four-dimensional Minkowski space has four-momentum P_μ and invariant hadronic mass states determined by the light-front eigenvalue equation $H_{\mathrm{LF}}|\psi_P\rangle = \mathcal{M}^2|\psi_P\rangle$. On AdS space the physical states are represented by normalizable modes

$$\Phi_P(x, z) = e^{-iP\cdot x}\Phi(z), \tag{5.1}$$

with plane waves along the Poincaré coordinates and a profile function $\Phi(z)$ along the holographic coordinate z. The hadronic invariant mass $P_\mu P^\mu = \mathcal{M}^2$ for the string modes (5.1) is found by solving the eigenvalue problem for the AdS wave equation. Each light-front hadronic state $|\psi_P\rangle$ is dual to a normalizable string mode $\Phi_P(x, z)$. To compare a physical observable computed in light-front QCD with the same observable computed in AdS space, we must find a gauge-invariant prescription to relate physical states in both theories. In practice, one compares the results of matrix elements of local operators on the gauge theory side with the corresponding matrix element in the AdS side. A consistent normalization on both sides of the correspondence is determined by the normalization of hadronic states to the energy-momentum tensor.

A. Normalization of hadronic states to the energy-momentum tensor in AdS

We compute the expectation value of the energy-momentum tensor $\Theta_{\ell m}$ along Minkowski coordinates. For $d = 4$

$$\langle\Phi_P|\Theta_\mu^\nu|\Phi_P\rangle = \int d^4x dz \sqrt{g}\,\Theta_\mu^\nu. \tag{5.2}$$

Substituting the plane-wave solution (5.1) in the expression for the energy-momentum tensor (4.21) we find

$$\langle P|\Theta_\mu^\nu|P\rangle = 2P_\mu P^\nu, \tag{5.3}$$

where we have extracted the overall factor $(2\pi)^4\delta^{(4)}(P' - P)$ from the x-integration to compare with the light-front QCD results. We chose the normalization

$$R^3\int_0^{\Lambda_{\mathrm{QCD}}^{-1}}\frac{dz}{z^3}|\Phi(z)|^2 = 1, \tag{5.4}$$

in the cutoff AdS space, consistent with the normalization of AdS solutions to the total charge operator [4] described in Appendix B. In obtaining (5.3) we have dropped the last term in (4.21), a surface term which vanishes by choosing appropriate boundary conditions.

B. Hadronic transition matrix elements in AdS/CFT

The matrix element of the energy-momentum tensor for the hadronic transition $P \to P'$ follows from the interaction term (4.20) describing the coupling of the pion mode with the external graviton field propagating in AdS space

$$\int d^4x dz \sqrt{g} h_{\ell m}(\partial^\ell \Phi_{P'}^* \partial^m \Phi_P + \partial^m \Phi_{P'}^* \partial^\ell \Phi_P), \tag{5.5}$$

where we have dropped the surface term in (4.21).

Since the energy-momentum tensor $\Theta^{\ell m}$ is gauge-invariant, we may impose a more restricted gauge condition in order to simplify the calculations and use the general covariance of the theory to obtain the final result. We choose the harmonic-traceless gauge $\partial_\ell h_m^\ell = \frac{1}{2}\partial_m h = 0$ and we consider the propagation inside AdS space of a graviton probe $h_{\ell m}$ with metric components along Minkowski coordinates $h_{zz} = h_{z\mu} = 0$. The set of linearized Einstein equations from (4.23) reduce to the simple form [32]

$$z^3\partial_z\left(\frac{1}{z^3}\partial_z h_\mu^\nu\right) - \partial_\rho\partial^\rho h_\mu^\nu = 0. \tag{5.6}$$

To solve (5.6) we note that the boundary limit of the graviton probe is a plane wave along the Poincaré coordinates with polarization indices along the physical transverse dimensions $h_\mu^\nu(x, z \to 0) = \epsilon_\mu^\nu e^{-iq\cdot x}$, where $q^2 = -Q^2 < 0$. As discussed in [32] in this particular gauge, the graviton couples to the transverse and traceless part of the energy-momentum tensor. We thus write

$$h_\mu^\nu(x, z) = \epsilon_\mu^\nu e^{-iq\cdot x}H(q^2, z), \tag{5.7}$$

with

$$H(q^2 = 0, z) = H(q^2, z = 0) = 1. \tag{5.8}$$

Substituting h_μ^ν in (5.6) we find the wave equation describing the propagation of the external graviton inside AdS space

$$[z^2\partial_z^2 - 3z\partial_z - z^2Q^2]H(Q^2, z) = 0. \tag{5.9}$$

Its solution subject to the boundary conditions (5.8) is

$$H(Q^2, z) = \frac{1}{2}Q^2z^2K_2(zQ), \tag{5.10}$$

the result obtained by Abidin and Carlson [32].

We can now use the Minkowski space dependence of the normalizable mode $\Phi_P(x, z) = e^{-iP\cdot x}\Phi(z)$ in (5.5). We find the transition amplitude

$$\langle P'|\Theta_\mu^\nu|P\rangle = (P^\nu P'_\mu + P_\mu P'^\nu)A(Q^2), \tag{5.11}$$

where we have extracted the overall factor $(2\pi)^4\delta^{(4)}(P' - P - q)$ from momentum conservation at the vertex from integration over Minkowski variables. We find for $A(Q^2)$

$$A(Q^2) = R^3 \int \frac{dz}{z^3} \Phi(z) H(Q^2, z) \Phi(z), \tag{5.12}$$

with $A(Q^2 = 0) = 1$. The gravitational form factor in AdS is thus represented as the z-overlap of the normalizable modes dual to the incoming and outgoing hadrons Φ_P and $\Phi_{P'}$ with the non-normalizable mode $H(Q^2, z)$ dual to the external graviton source [32]; this provides the form of the gravitational transition matrix element analogous to the electromagnetic form factor in AdS [37]. At small z the string modes scale as $\Phi \sim z^{\Delta}$. At large enough Q, the important contribution to (5.12) is from the region near $z \sim 1/Q$, $A(Q^2) \to (1/Q^2)^{\Delta-1}$, and the ultraviolet pointlike behavior responsible for the power law scaling [42,43] is recovered.

VI. LIGHT-FRONT MAPPING OF STRING AMPLITUDES

The gravitational form factor (5.12) represents the coupling of the graviton to the entire hadron in AdS, independent of the number n of its constituents. Since (5.12) gives $A(0) = 1$, it implicitly includes the sum over the coupling of the graviton to all n massless constituents. The gravitational coupling, like a number operator, sums over all particles. Similarly, the electromagnetic current sums over constituents, but weighted by their fractional charge. To simplify the discussion we will establish the connection of the AdS/CFT results for the gravitational form factor and the light-front results for the lowest Fock state $n = 2$ using the effective single-particle distribution discussed in Sec. III A. This is particularly useful for extending the results to arbitrary n, subject to the requirement that one normalizes the hadronic matrix element of the energy-momentum tensor so that $A(0) = 1$.

For $n = 2$, there are two terms which contribute to the light-front result in the f-sum in (3.16). Exchanging $x \leftrightarrow 1 - x$ in the second integral we find

$$A_{n=2}(q^2) = 4\pi \int_0^1 dx(1-x) \times \int \zeta d\zeta J_0\left(\zeta q \sqrt{\frac{1-x}{x}}\right) |\tilde{\rho}_{n=2}(x,\zeta)|^2, \tag{6.1}$$

where $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$. It is simple to prove that if $\tilde{\rho}$ is a symmetric function of x and $1 - x$ then

$$2\pi \int_0^1 dx(1-x) \int \zeta d\zeta |\tilde{\rho}_{n=2}(x,\zeta)|^2 = \frac{1}{2}, \tag{6.2}$$

and thus $A(q^2)$ satisfies the sum rule $A(0) = 1$.

To compare with the light-front QCD results we express the bulk-to-boundary propagator $H(Q^2, z)$ (5.10) for the graviton probe using the Hankel-Nicholson integral representation (Appendix A of Ref. [4])

$$H(Q^2, z) = 4Q^4 \int_0^\infty \frac{tJ_0(zt)}{(t^2+Q^2)^3} dt. \tag{6.3}$$

Introducing a new variable $x = \frac{Q^2}{t^2+Q^2}$ we find

$$H(Q^2, z) = 2 \int_0^1 x dx J_0\left(zQ\sqrt{\frac{1-x}{x}}\right), \tag{6.4}$$

and thus

$$A(Q^2) = 2R^3 \int_0^1 x dx \int \frac{dz}{z^3} J_0\left(zQ\sqrt{\frac{1-x}{x}}\right) |\Phi(z)|^2. \tag{6.5}$$

We can now compare the above expression with the light-front expression (6.1), and can identify the spectator density function appearing in the light-front formalism with the corresponding AdS density

$$\tilde{\rho}(x,\zeta) = \frac{R^3}{2\pi} \frac{x}{1-x} \frac{|\Phi(\zeta)|^2}{\zeta^4}. \tag{6.6}$$

Extension to arbitrary n follows from the x-weighted definition of the transverse impact variable of the $n - 1$ spectator system given by (3.17): $\zeta = \sqrt{\frac{x}{1-x}} |\sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j}|$.

Equation (6.6) holds for all momentum transfer Q^2 and gives the same relation between string modes $\Phi(\zeta)$ in AdS_5 and the QCD transverse charge density $\tilde{\rho}(x,\zeta)$ obtained previously by mapping the electromagnetic current matrix elements [4]. The variable ζ, $0 \le \zeta \le \Lambda_{\mathrm{QCD}}^{-1}$, represents a measure of the transverse separation between pointlike constituents, and it is also the holographic variable z.

In the case of a two-parton system the correspondence between the string amplitude $\Phi(z)$ in AdS space and the QCD light-front wave function $\tilde{\psi}(x, \mathbf{b}_\perp)$ follows from (6.6). For two partons the transverse density (3.15) has the simple form

$$\tilde{\rho}_{n=2}(x,\zeta) = \frac{|\tilde{\psi}(x,\zeta)|^2}{(1-x)^2}, \tag{6.7}$$

and a closed form solution for the two-constituent bound state light-front wave function is obtained

$$|\tilde{\psi}(x,\zeta)|^2 = \frac{R^3}{2\pi} x(1-x) \frac{|\Phi(\zeta)|^2}{\zeta^4}, \tag{6.8}$$

with $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$. For a two-parton system the light-front mapping can also be carried out directly from (3.10). This is done in Appendix C, where the consistency with the LF mapping results from electromagnetic current matrix elements is also pointed out.

A. Holographic light-front Hamiltonian and Schrödinger equation

The above analysis provides an exact correspondence between the holographic variable z and an impact variable

ζ which measures the transverse separation between point-like constituents within a hadron; we can identify $\zeta = z$. The mapping of z from AdS space to ζ in light-front frame allows the equations of motion in AdS space to be recast in the form of a light-front Hamiltonian equation [26] with eigenvalues given in terms of the hadronic eigenmass $\mathcal{M}^2$

$$H_{\mathrm{LF}}|\phi\rangle = \mathcal{M}^2|\phi\rangle, \tag{6.9}$$

a remarkable result which allows the discussion of the AdS/CFT solutions in terms of light-front equations in physical 3 + 1 space-time.

Factoring out the plane-wave dependence of the hadronic mode $\Phi_P(x, z) = e^{-iP\cdot x}\Phi(z)$ and substituting in (4.17) we find

$$[z^2\partial_z^2 - (d-1)z\partial_z + z^2\mathcal{M}^2 - (\mu R)^2]\Phi(z) = 0, \tag{6.10}$$

the wave equation describing the propagation of a scalar mode in AdS. The allowed values of μ are determined by requiring that asymptotically the dimensions become separated by integers according to the spectral relation $(\mu R)^2 = \Delta(\Delta - d)$ and the stability condition dictated by the Breitenlohner-Freedman bound $(\mu R)^2 \geq -d^2/4$ for a scalar field [44]. We find $(\mu R)^2 = -4 + L^2$ for $\Delta = 2 + L$ and $d = 4$. Thus the stability bound requires $L^2 \geq 0$.

By substituting

$$\phi(\zeta) = \left(\frac{\zeta}{R}\right)^{-3/2}\Phi(\zeta) \tag{6.11}$$

in the AdS scalar wave equation (6.10) we find an effective Schrödinger equation as a function of the weighted impact variable ζ [3,4]

$$\left[-\frac{d^2}{d\zeta^2} + V(\zeta)\right]\phi(\zeta) = \mathcal{M}^2\phi(\zeta), \tag{6.12}$$

with $-\frac{d^2}{d\zeta^2}$ the light-front kinetic energy operator and conformal potential

$$V(\zeta) \to -\frac{1-4L^2}{4\zeta^2}, \tag{6.13}$$

an effective relativistic two-particle light-front wave equation for mesons defined at $x^+ = 0$. Its eigenmodes determine the hadronic mass spectrum.

In the transverse impact holographic representation with holographic light-front wave functions $\phi(\zeta) = \langle\zeta|\phi\rangle$, the LC eigenvalue equation thus reads

$$\langle\zeta|H_{\mathrm{LC}}|\phi\rangle = \mathcal{M}^2\langle\zeta|\phi\rangle, \tag{6.14}$$

with

$$\langle\zeta|H_{\mathrm{LC}}|\phi\rangle = \left[-\frac{d^2}{d\zeta^2} - \frac{1-4\nu^2}{4\zeta^2}\right]\langle\zeta|\phi\rangle, \tag{6.15}$$

in the conformal limit. The light-front modes $\phi(\zeta) = \langle\zeta|\phi\rangle$ are normalized according to

$$\langle\phi|\phi\rangle = \int d\zeta|\langle\zeta|\phi\rangle|^2 = 1, \tag{6.16}$$

and represent the probability amplitude to find n-partons at transverse impact separation $\zeta = z$. Its eigenvalues are determined by the boundary conditions $\phi(z = 1/\Lambda_{\mathrm{QCD}}) = 0$ and are given in terms of the roots of Bessel functions: $\mathcal{M}_{L,k} = \beta_{L,k}\Lambda_{\mathrm{QCD}}$. The normalizable modes are

$$\phi_{L,k}(\zeta) = \frac{\sqrt{2}\Lambda_{\mathrm{QCD}}}{J_{1+L}(\beta_{L,k})}\sqrt{\zeta}J_L(\zeta\beta_{L,k}\Lambda_{\mathrm{QCD}})\theta(\zeta \leq \Lambda_{\mathrm{QCD}}^{-1}). \tag{6.17}$$

The lowest stable state $L = 0$ is determined by the Breitenlohner-Freedman bound. Higher excitations are matched to the small z asymptotic behavior of each string mode to the corresponding conformal dimension of the boundary operators of each hadronic state. The AdS metric ds^2 (1.1) is invariant if $\mathbf{x}_\perp^2 \to \lambda^2\mathbf{x}_\perp^2$ and $z \to \lambda z$ at equal light-front time $x^+ = 0$. The effective wave equation (6.12) has the Casimir representation L^2 corresponding to the $SO(2)$ group of rotations in the transverse light-front

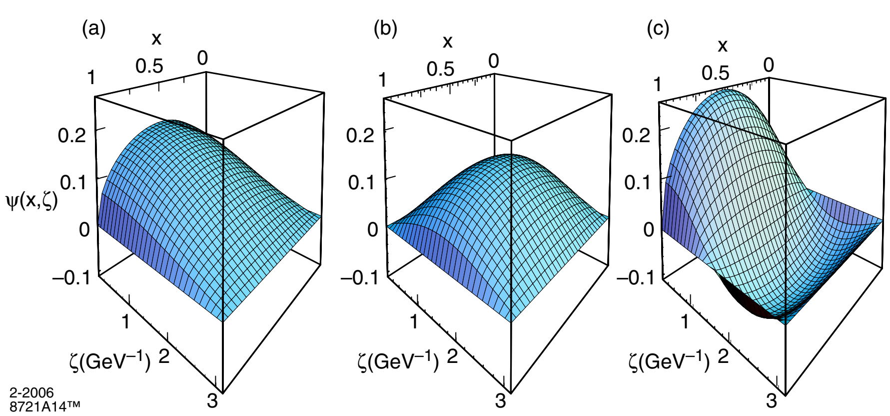

FIG. 1 (color online). AdS/QCD predictions for the light-front wave functions of a meson in the hard-wall model: (a) $n = 0$, $L = 0$; (b) $n = 0$, $L = 1$; (c) $n = 1$, $L = 0$.

plane. Indeed, the Casimir operator for $SO(N) \sim S^{N-1}$ is $L(L+N-2)$. This shows the natural holographic connection to the light front. The fundamental light-front equation of AdS/CFT has the appearance of a Schrödinger equation, but it is relativistic, covariant, and analytically tractable.

A closed form of the light-front wave functions $\tilde{\psi}(x, \mathbf{b}_\perp)$ for a two-parton bound state follows from (6.8)

$$\tilde{\psi}_{L,k}(x, \mathbf{b}_\perp) = \frac{\Lambda_{\rm QCD}}{\sqrt{\pi} J_{1+L}(\beta_{L,k})} \sqrt{x(1-x)} J_L(\sqrt{x(1-x)} |\mathbf{b}_\perp| \beta_{L,k} \Lambda_{\rm QCD}) \theta\left(\mathbf{b}_\perp^2 \le \frac{\Lambda_{\rm QCD}^{-2}}{x(1-x)}\right). \tag{6.18}$$

The resulting wave function depicted in Fig. 1 displays confinement at large interquark separation and conformal symmetry at short distances, reproducing dimensional counting rules for hard exclusive amplitudes and the conformal properties of the LFWFs at high relative momenta [42,43].

For the soft-wall model [15] we can obtain the basis set of light-front wave functions by comparing the QCD expression for the gravitational form factor (3.16) with the corresponding expression for the AdS form factor, where the graviton probe propagates in the distorted metric. In the large Q limit we can identify the light-front transverse density with the corresponding AdS density, with identical results as obtained for the mapping of the electromagnetic form factor in [4].

VII. CONCLUSIONS

Light-front holography is one of the most remarkable features of AdS/CFT. It allows one to project the functional dependence of the wave function $\Phi(z)$ computed in the single AdS fifth dimension to the hadronic frame-independent light-front wave function $\psi(x_i, \mathbf{b}_{\perp i})$ in $3+1$ physical space-time. The variable z maps to $\zeta(x_i, \mathbf{b}_{\perp i})$. As we have discussed, this correspondence is a consequence of the fact that the metric ds^2 for AdS$_5$ at fixed light-front time τ is invariant under the simultaneous scale change $\mathbf{x}_\perp^2 \to \lambda^2 \mathbf{x}_\perp^2$ in transverse space and $z^2 \to \lambda^2 z^2$. The transverse coordinate ζ is closely related to the invariant mass squared of the constituents in the LFWF and its off-shellness in the light-front kinetic energy, and it is thus the natural variable to characterize the hadronic wave function. In fact ζ is the only variable to appear in the light-front Schrödinger equations predicted from AdS/QCD. These equations for both meson and baryons give a good representation of the observed hadronic spectrum, especially in the case of the soft-wall model. The resulting LFWFs also have excellent phenomenological features, including predictions for the electromagnetic form factors and decay constants.

It is interesting to note that the form of the nonperturbative pion distribution amplitude $\phi_\pi(x)$ obtained from integrating the $q\bar{q}$ valence LFWF $\psi(x, \mathbf{k}_\perp)$ over $\mathbf{k}_\perp$ has a quite different x behavior than the asymptotic distribution amplitude predicted from the PQCD evolution [45] of the pion distribution amplitude. The AdS prediction $\phi_\pi(x) = \sqrt{3} f_\pi \sqrt{x(1-x)}$ has a broader distribution than expected from solving the Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution equation in perturbative QCD. This observation appears to be consistent with the results of the Fermilab diffractive dijet experiment [46], the moments obtained from lattice QCD [9], and pion form factor data [47].

Nonzero quark masses are naturally incorporated into the AdS predictions [9] by including them explicitly in the LF kinetic energy $\sum_i \frac{\mathbf{k}_{\perp i}^2 + m_i^2}{x_i}$. Given the nonperturbative LFWFs one can predict many interesting phenomenological quantities such as heavy quark decays, generalized parton distributions, and parton structure functions. The AdS/QCD model is semiclassical and thus only predicts the lowest valence Fock-state structure of the hadron LFWF. In principle, the model can be systematically improved by diagonalizing the full QCD light-front Hamiltonian on the AdS/QCD basis.

Another interesting application is hadronization at the amplitude level. In this case one uses light-front time-ordered perturbation theory for the QCD light-front Hamiltonian to generate the off-shell quark and gluon T-matrix helicity amplitudes such as $e^+e^- \to \gamma^* \to X$. If at any stage a set of color-singlet partons has light-front kinetic energy $\sum_i \mathbf{k}_{\perp i}^2 / x_i < \Lambda_{\rm QCD}^2$, then one coalesces the virtual partons into a hadron state using the AdS/QCD LFWFs. A similar approach was used to predict antihydrogen formation from virtual positron-antiproton states produced in $\bar{p}A$ collisions [48].

The hard-wall AdS/QCD model resembles bag models where a boundary condition is introduced to implement confinement. However, unlike traditional bag models, the AdS/QCD model is frame-independent. An important property of bag models is the dominance of quark interchange as the underlying dynamics of large-angle elastic scattering. This agrees with the survey of two-hadron exclusive reactions [49]. In addition the AdS/QCD model implies a maximal wavelength for confined quarks and gluons and thus a finite IR fixed point for the QCD coupling.

We originally derived the light-front holographic mapping by matching the exact expression for current matrix elements in AdS space with the corresponding exact expression for the electromagnetic current matrix element using light-front theory in physical space-time. In this paper we have shown that one obtains the identical holographic mapping using the hadronic matrix elements of the energy-momentum tensor. This is a highly nontrivial test of the consistency of the light-front holographic mapping.

Our analysis also allows one to predict the individual quark and gluon contributions to the gravitational form factors $A(q^2)$ and $B(q^2)$. Thus we can predict the momentum fractions for quarks q and gluons g, $A_{q,g}(0) = \langle x_{q,g} \rangle$, and orbital angular momenta $B_{q,g}(0) = \langle L_{q,g} \rangle$ carried by each quark flavor and gluon in the hadron with sum rules $\sum_{q,g} A_{q,g}(0) = A(0) = 1$ and $\sum_{q,g} B_{q,g}(0) = B(0) = 0$. The last sum rule corresponds to the vanishing of the anomalous gravitational moment which is true Fock state by Fock state [27] in light-front theory.

The mathematical consistency of light-front holography for both the electromagnetic and gravitational hadronic transition matrix elements demonstrates that the mapping between the single AdS space dimension z and the transverse light-front variable ζ, which is a function of the multidimensional coordinates of the partons in a given light-front Fock state x_i, $\mathbf{b}_{\perp i}$ at fixed light-front time τ, is a general principle. The holographic mapping from $\Phi(z)$ to the light-front wave functions of relativistic composite systems provides a new tool for extending the AdS/CFT correspondence to theories such as QCD which are not conformally invariant.

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APPENDIX A: METRIC CONVENTIONS

1. Light-cone metric and Minkowski space

The Minkowski metric written in terms of light-front coordinates is

$$d\sigma^2 = dx^+ dx^- - d\mathbf{x}_\perp^2 - dz^2, \qquad \text{(A1)}$$

with timelike and spacelike components $x^+ = x^0 + x^3$ and $x^- = x^0 - x^3$ respectively. We write contravariant four-vectors such as x^μ as

$$x^\mu = (x^+, x^-, x^1, x^2) = (x^+, x^-, \mathbf{x}_\perp). \qquad \text{(A2)}$$

Scalar products are

$$\begin{aligned} x \cdot p = x_\mu p^\nu &= g_{\mu\nu} x^\mu p^\nu \\ &= x_+ p^+ + x_- p^- + x_1 p^1 + x_2 p^2 \\ &= \frac{1}{2}(x^+ p^- + x^- p^+) - \mathbf{x}_\perp \cdot \mathbf{p}_\perp, \end{aligned} \qquad \text{(A3)}$$

with front-form metrics

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \text{(A4)}$$

and

$$g^{\mu\nu} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad \text{(A5)}$$

A covariant vector such as ∂_μ is

$$\partial_\mu = (\partial_+, \partial_-, \partial_1, \partial_2) = (\partial_+, \partial_-, \vec{\partial}_\perp). \qquad \text{(A6)}$$

Thus $\partial^+ = 2\partial_-$ and $\partial^- = 2\partial_+$.

2. AdS space

AdS coordinates are the Minkowski coordinates x^ℓ and z labeled $x^\ell = (x^\ell, z)$. The AdS_{d+1} metric is

$$ds^2 = g_{\ell m} dx^\ell dx^m = \frac{R^2}{z^2}(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \qquad \text{(A7)}$$

with conformal metrics

$$g_{\ell m} = \frac{R^2}{z^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \text{(A8)}$$

and

$$g^{\ell m} = \frac{z^2}{R^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad \text{(A9)}$$

The AdS metric is conveniently written $g_{\ell m} = \frac{R^2}{z^2} \eta_{\ell m}$ and $g^{\ell m} = \frac{z^2}{R^2} \eta^{\ell m}$, where $\eta_{\ell m}$ has diagonal components $(1, -1, \cdots, -1)$. The metric determinant $g = |g_{\ell m}|$ is $g = (\frac{R^2}{z^2})^{d+1}$.

APPENDIX B: NORMALIZATION OF HADRONIC STATES TO THE CHARGE OPERATOR IN ADS

We compute the expectation value of the electromagnetic current J_ℓ

$$J_\ell = i(\Phi^* \partial_\ell \Phi - \Phi \partial_\ell \Phi^*), \qquad \text{(B1)}$$

along Minkowski coordinates for a hadronic state $\Phi_P(x, z) = e^{-iP\cdot x}\Phi(z)$ in AdS_5 space

$$\langle \Phi_P | J^\mu | \Phi_P \rangle = \int d^4x dz \sqrt{g} J^\mu. \tag{B2}$$

Substituting the hadronic plane-wave solution we obtain

$$\langle P | J^\mu | P \rangle = 2P^\mu, \tag{B3}$$

where we have extracted the overall factor $(2\pi)^4\delta^{(4)}(P' - P)$ from the x-integration to compare with the light-front QCD results described in [4]. We use the normalization

$$R^3 \int_0^{\Lambda_{\rm QCD}^{-1}} \frac{dz}{z^3} |\Phi(z)|^2 = 1, \tag{B4}$$

in the cut-off AdS space. The total charge operator is a diagonal operator in the AdS hadronic representation.

APPENDIX C: TWO-PARTON EXAMPLE

The mapping of AdS transition amplitudes to light-front QCD transition matrix elements is much simplified for two-parton hadronic states. It further illustrates important technical aspects for extending the results to the n-parton case. We describe in this appendix the actual two-parton mapping for the electromagnetic and gravitational transition amplitudes.

1. Electromagnetic form factor

The Drell-Yan-West expression for the electromagnetic form-factor in impact space [3,4]

$$F(q^2) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} \sum_q e_q \exp\Big(i\mathbf{q}_\perp \cdot \sum_{k=1}^{n-1} x_k \mathbf{b}_{\perp k}\Big) \times |\tilde{\psi}_n(x_j, \mathbf{b}_{\perp j})|^2, \tag{C1}$$

is written as a sum of overlap integrals of light-front wave functions of the $j = 1, 2, \ldots, n-1$ spectator constituents. We have included explicitly in Eq. (C1) the contribution from each active constituent q with charge e_q. The formula is exact if the sum is over all Fock states n.

For definiteness we shall consider a two-quark π^+ valence Fock state $|u\bar{d}\rangle$ with charges $e_u = \frac{2}{3}$ and $e_{\bar{d}} = \frac{1}{3}$. For $n = 2$, there are two terms which contribute to the q-sum in (C1). Exchanging $x \leftrightarrow 1 - x$ in the second integral we find ($e_u + e_{\bar{d}} = 1$)

$$F_{\pi^+}(q^2) = \int_0^1 dx \int d^2\mathbf{b}_\perp e^{i\mathbf{q}_\perp \cdot \mathbf{b}_\perp (1-x)} |\tilde{\psi}_{u\bar{d}/\pi}(x, \mathbf{b}_\perp)|^2, \tag{C2}$$

with normalization $F_\pi^+(q = 0) = 1$. Integrating over angle we find

$$F_{\pi^+}(q^2) = 2\pi \int_0^1 \frac{dx}{x(1-x)} \times \int \zeta d\zeta J_0\Big(\zeta q \sqrt{\frac{1-x}{x}}\Big) |\tilde{\psi}_{u\bar{d}/\pi}(x, \zeta)|^2, \tag{C3}$$

where $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$. Notice that by performing an identical calculation for the π^0 meson the result is $F_{\pi^0}(q^2) = 0$ for any q, as expected from C-charge conjugation invariance.

We now compare this result with the electromagnetic form-factor in AdS space [3,4]:

$$F(Q^2) = R^3 \int \frac{dz}{z^3} J(Q^2, z) |\Phi_{\pi^+}(z)|^2, \tag{C4}$$

where $F(Q^2 = 0) = 1$ and the bulk-to-boundary propagator $J(Q^2, z) = zQK_1(zQ)$ describes the propagation of the external electromagnetic current inside AdS. Using the integral representation of $J(Q^2, z)$

$$J(Q^2, z) = \int_0^1 dx J_0\Big(\zeta Q \sqrt{\frac{1-x}{x}}\Big), \tag{C5}$$

we can write the AdS electromagnetic form factor as

$$F(Q^2) = R^3 \int_0^1 dx \int \frac{dz}{z^3} J_0\Big(zQ \sqrt{\frac{1-x}{x}}\Big) |\Phi_{\pi^+}(z)|^2. \tag{C6}$$

Comparing with the expression for the electromagnetic form factor in light-front QCD (C3) for arbitrary values of Q, we find the relation between the pion LFWF $\tilde{\psi}$ and the hadronic string mode Φ_π [3,4]

$$|\tilde{\psi}_{u\bar{d}/\pi}(x, \zeta)|^2 = \frac{R^3}{2\pi} x(1-x) \frac{|\Phi_\pi(\zeta)|^2}{\zeta^4}, \tag{C7}$$

where we identify the transverse light-front variable ζ, $0 \le \zeta \le \Lambda_{\rm QCD}$, with the holographic variable z.

2. Gravitational form factor

The light-front expression for the helicity-conserving gravitational form factor in impact space is (3.10)

$$A(q^2) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} \sum_f x_f \exp\Big(i\mathbf{q}_\perp \cdot \sum_{k=1}^{n-1} x_k \mathbf{b}_{\perp k}\Big) \times |\tilde{\psi}_n(x_j, \mathbf{b}_{\perp j})|^2, \tag{C8}$$

which includes the contribution of each struck parton with longitudinal momentum fraction x_f and corresponds to a change of transverse momentum $x_j\mathbf{q}$ for each of the $j = 1, 2, \ldots, n-1$ spectators. For $n = 2$, there are two terms which contribute to the f-sum in (C8). Exchanging $x \leftrightarrow 1 - x$ in the second integral we find

$$A_\pi(q^2) = 2 \int_0^1 x dx \int d^2\mathbf{b}_\perp e^{i\mathbf{q}_\perp \cdot \mathbf{b}_\perp (1-x)} |\tilde{\psi}_{q\bar{q}/\pi}(x, \mathbf{b}_\perp)|^2. \tag{C9}$$

Using the light-front wave function normalization

$$\int_0^1 dx \int d^2\mathbf{b}_\perp |\tilde{\psi}(x, \mathbf{b}_\perp)|^2 = 1, \tag{C10}$$

it is simple to prove that if ψ is a symmetric function of x and $1 - x$ the first x-moment

$$\int_0^1 x dx \int d^2\mathbf{b}_\perp |\tilde{\psi}(x, \mathbf{b}_\perp)|^2 = \frac{1}{2}, \tag{C11}$$

and thus $A_\pi(q^2)$ satisfies the sum rule $A_\pi(0) = 1$. Integrating (C9) over angle we find

$$A_\pi(Q^2) = 4\pi \int_0^1 \frac{dx}{(1-x)} \int \zeta d\zeta J_0\left(\zeta q \sqrt{\frac{1-x}{x}}\right) \times |\tilde{\psi}_{q\bar{q}/\pi}(x, \zeta)|^2, \tag{C12}$$

where $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$.

We now consider the expression for the hadronic gravitational form factor in AdS space (5.12)

$$A(Q^2) = R^3 \int \frac{dz}{z^3} H(Q^2, z) |\Phi_\pi(z)|^2, \tag{C13}$$

where $A(Q)$ is normalized to one at $Q = 0$ and $H(Q^2, z) = \frac{1}{2} Q^2 z^2 K_2(zQ)$ describes the propagation of the external graviton inside AdS space. Using the integral representation of $H(Q^2, z)$ (6.4)

$$H(Q^2, z) = 2 \int_0^1 x dx J_0\left(zQ\sqrt{\frac{1-x}{x}}\right), \tag{C14}$$

the AdS gravitational form factor can be expressed as

$$A(Q^2) = 2R^3 \int_0^1 x dx \int \frac{dz}{z^3} J_0\left(zQ\sqrt{\frac{1-x}{x}}\right) |\Phi_\pi(z)|^2. \tag{C15}$$

Comparing with the QCD light-front gravitational form factor (C12) we find ($\zeta = z$)

$$|\tilde{\psi}_{q\bar{q}/\pi}(x, \zeta)|^2 = \frac{R^3}{2\pi} x(1-x) \frac{|\Phi_\pi(\zeta)|^2}{\zeta^4}, \tag{C16}$$

which is identical to the result (C7) obtained from the mapping of the pion electromagnetic transition amplitude.

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