Dyons with potentials: Duality and black hole thermodynamics

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(Received 14 May 2007; published 18 July 2008)

A modified version of the double potential formalism for the electrodynamics of dyons is constructed. Besides the two vector potentials, this manifestly duality invariant formulation involves four additional potentials, scalar potentials which appear as Lagrange multipliers for the electric and magnetic Gauss constraints and potentials for the longitudinal electric and magnetic fields. In this framework, a static dyon appears as a Coulomb-like solution without string singularities. Dirac strings are needed only for the Lorentz force law, not for Maxwell's equations. The magnetic charge no longer appears as a topological conservation law but as a surface integral on a par with electric charge. The theory is generalized to curved space. As in flat space, the string singularities of dyonic black holes are resolved. As a consequence all singularities are protected by the horizon and the thermodynamics is shown to follow from standard arguments in the grand canonical ensemble.

DOI: 10.1103/PhysRevD.78.025025 PACS numbers: 14.80.Hv, 03.50.De, 04.20.Fy, 04.70.Dy

I. INTRODUCTION

Reissner-Nordstrøm black holes with both electric and magnetic charge

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$N = \sqrt{1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}}, \tag{1.1}$$

$$A = -\frac{Q}{r}dt + P(1 - \cos\theta)d\phi \tag{1.2}$$

are generally excluded in a discussion of uniqueness theorems and geometric derivations of the first law because the gauge potential is singular along a string that intersects the horizon and goes to infinity [1,2]. Exceptions can be found in [3] for stationary and axisymmetric perturbations and in [4] where dipole charge contributions to the first law for five dimensional black ring solutions are investigated by dealing directly with divergent potentials on the horizon. Dyonic solutions were also excluded in an investigation of duality of electric and magnetic black holes using Euclidean methods [5].

Nevertheless, for variations of the three parameters the first law

$$\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A} + \phi_H\delta Q + \psi_H\delta P, \tag{1.3}$$

where

$$\Delta = M^2 - (Q^2 + P^2), \qquad r_\pm = M \pm \sqrt{\Delta}, \tag{1.4}$$

$$\kappa = \frac{r_+ - r_-}{2r_+^2}, \qquad \mathcal{A} = 8\pi\left[M^2 - \frac{Q^2 + P^2}{2} + M\sqrt{\Delta}\right], \tag{1.5}$$

$$\phi_H = \frac{Q}{r_+}, \qquad \psi_H = \frac{P}{r_+}, \tag{1.6}$$

can easily be inferred from the purely electric case by using a duality argument. Furthermore, electric-magnetic black hole duality has been extended to the case of dyons in the canonical ensemble by using the manifestly duality invariant double potential formalism [6]. In its original version [7], this formalism involves as dynamical degrees of freedom two vector potentials. An independent rederivation [8] has been written with two additional scalar potentials which are spurious because they appear only as a part of a total derivative of the action. In the black hole context [6], coupling to external static sources can be made either through fixed strings, spherically symmetric nondynamical longitudinal fields, or intermediate combinations. Finally, the coupling to dynamical dyons with the help of dynamical strings has been studied in detail in [9,10], including a proof of equivalence with Dirac's original theory [11,12] and a derivation of the appropriate quantization condition [13,14].

What we will do in this paper is introduce potentials for longitudinal components of electric and magnetic fields. This has the effect of making the two scalar potentials nonspurious as they now appear as the Lagrange multipliers for the divergence constraints on electric and magnetic fields. We thus increase the redundancy of the description in such a way as to have twice as much gauge invariance as in standard Maxwell theory.

Now, taking into account all the results described above, this extension of the double potential formalism is rather straightforward and seems hardly worth the effort. We beg to differ.

First of all, the electric and magnetic potentials produced by a static dyon both appear as Coulomb-like solutions in a single, manifestly duality invariant formulation without any stringlike singularities. In this framework Dirac strings are only needed in order to produce the correct Lorentz force law from an action principle for dynamical point-particle dyons.

In curved space, the new formulation is ideally suited for the description of black hole dyons. As in flat space, their string singularity is resolved and a geometric derivation of the first law can be done along standard lines because all singularities are now protected by the horizon. This is a direct consequence of the intriguing transmutation into a surface integral of the magnetic charge which appears as a topological conservation law in the standard approach. Since there is no quantization condition on magnetic or on electric charge for a single dyon and because of the presence in the formalism of both chemical potentials, thermodynamics and Euclidean computations can be performed in the grand canonical ensemble, thus circumventing arguments of [2,5].

In the next section, we discuss our formulation in Minkowski space in the case of fixed external sources. Section III is devoted to Dirac strings and dynamical point-particle dyons. We finally write down and analyze the appropriate action for curved space and discuss applications in the context of black hole physics.

II. EXTENDED DOUBLE POTENTIAL FORMALISM IN FLAT SPACE

In this section we present an action principle for electromagnetism in the presence of electric and magnetic sources which is manifestly duality invariant. Both electric and magnetic Gauss constraints are dynamical and appear in the action with their corresponding Lagrange multipliers. For a static dyon, the solution of the field equations is Coulomb-like, both in the electric and the magnetic sector. We show that the theory can be gauge fixed so as to coincide with standard electromagnetism and conclude the section by showing that Lorentz invariance, while not manifest, is nevertheless realized through canonical generators very much as in the standard Hamiltonian formulation of electromagnetism. This suggests, as we will explicitly show in the last section, that the theory can be generalized to curved space.

A. Action, duality, and gauge symmetries

The dynamical fields of the theory are A^a_μ, C^a, $a = 1, 2$. Here, $A^a_\mu \equiv (A_\mu, Z_\mu)$ are the standard and new potentials. The additional fields $C^a \equiv (C, Y)$ make up the longitudinal parts of magnetic and electric fields $\vec{B}^a \equiv (\vec{B}, \vec{E})$ according to

$$\vec{B}^a = \vec{\nabla} \times \vec{A}^a + \vec{\nabla} C^a. \tag{2.1}$$

The external magnetic and electric currents $j^{a\mu} \equiv (k^\mu, j^\mu)$ are conserved, $\partial_\mu j^{a\mu} = 0$. In this section, we assume that they correspond to the currents produced by a single point-particle dyon. We consider the action

$$I[A^a_\mu, C^a] = I_M[A^a_\mu, C^a] + I_I[A^a_\mu; j^{a\mu}], \tag{2.2}$$

where

$$I_M[A^a_\mu, C^a] = \frac{1}{2}\int d^4x[\epsilon_{ab}(\vec{B}^a + \vec{\nabla}C^a)\cdot(\partial_0\vec{A}^b - \vec{\nabla}A^b_0) - \vec{B}^a\cdot\vec{B}_a] \tag{2.3}$$

is the substitute for the usual Maxwell action and

$$I_I[A^a_\mu; j^{a\mu}] = \int d^4x \epsilon_{ab} A^a_\mu j^{b\mu} \tag{2.4}$$

is the "interaction" action. Here ϵ_{ab} is skew-symmetric with $\epsilon_{12} = 1$, and indices $a, b, \ldots$ raised and lowered with the Kronecker delta. The action (2.2) is manifestly invariant under simultaneous duality rotations on $(A^a_\mu, C^a; j^{a\mu})$

$$\delta_D A^a_\mu = \epsilon^{ab}A_{b\mu}, \qquad \delta_D C^a = \epsilon^{ab}C_b, \qquad \delta_D j^{a\mu} = \epsilon^{ab} j^\mu_b. \tag{2.5}$$

It is also gauge invariant under

$$\delta_\lambda A^a_\mu = \partial_\mu \lambda^a, \qquad \delta_\lambda C^a = 0. \tag{2.6}$$

B. Equations of motion and point-particle dyon

The Euler-Lagrange equations of motion associated with (2.3) are easily shown to be equivalent to Maxwell's equation with magnetic and electric currents. Indeed, variations with respect to A^a_0 give the constraints

$$\vec{\nabla}\cdot\vec{B}^a \equiv \nabla^2 C^a = j^{0a}. \tag{2.7}$$

Variations with respect to C^a imply the equations

$$\nabla^2 C_a = \epsilon_{ab}(\vec{\nabla}\cdot\partial_0\vec{A}^b - \nabla^2 A^b_0). \tag{2.8}$$

The fields A^0_a, C^a are auxiliary in the sense that, under suitably boundary conditions at spatial infinity, their equations of motion can be solved for A^0_a, C^a in terms of all other fields, without the need for initial conditions.

Variations with respect to $\vec{A}^a$ yield Maxwell's equations in the form

$$-\epsilon_{ab}\partial_0\vec{B}^b + \vec{\nabla}\times\vec{B}_a = \epsilon_{ab}\vec{j}^b. \tag{2.9}$$

As a consequence, if the electromagnetic field tensor F is expressed in the usual way in terms of electric and magnetic fields, $F_{0i} = -B^2_i$, $F_{ij} = \epsilon_{ijk}B^{1k}$, it follows that both

dF and d^*F vanish outside of sources on account of the Euler-Lagrange equations of motion.

In the case of a single point-particle dyon at the origin with charges $Q^a \equiv (P, Q)$, for example,

$$j^{a\mu}(x) = 4\pi Q^a \delta_0^\mu \delta^3(x), \tag{2.10}$$

instead of (1.2), Maxwell's equations in the above form are now solved by

$$A^a = -\frac{\epsilon^{ab}Q_b}{r}dt, \qquad C^a = -\frac{Q^a}{r}. \tag{2.11}$$

This solution resolves the string-singularity of the standard formulation. It is unique in the transverse gauge $\vec{\nabla}\cdot\vec{A}^a = 0$ with vanishing boundary conditions on A^a_μ, C^a.

C. Canonical structure and degrees of freedom

By using integrations by parts and decomposing $\vec{A}^a = \vec{A}^{aT} + \vec{\nabla}M^a$, with $M^a = (M_A, M_Z)$, the free action (2.3) can be written in the form

$$\begin{aligned} I_M[\vec{A}^{aT}, A_0^a, M^a, C^a] = \int d^4x[&-\vec{\nabla}\times\vec{Z}_T\cdot\partial_0\vec{A}_T \\ &+ \nabla^2 Y\partial_0 M_A - \nabla^2 C\partial_0 M_Z \\ &- \frac{1}{2}\vec{E}\cdot\vec{E} - \frac{1}{2}\vec{B}\cdot\vec{B} - A_0\nabla^2 Y \\ &+ Z_0\nabla^2 C], \end{aligned} \tag{2.12}$$

where $\vec{E} = \vec{\nabla}\times\vec{Z} + \vec{\nabla}Y$, $\vec{B} = \vec{\nabla}\times\vec{A} + \vec{\nabla}C$. This shows that the canonically conjugate pairs are $(\vec{A}_T, -\vec{\nabla}\times\vec{Z}_T)$, $(\nabla^2 Y, M_A)$, and $(-\nabla^2 C, M_Z)$ so that there are 4 conjugate pairs per spacetime point.

Variation with respect to the Lagrange multipliers Z_0 imposes the first class constraint $\nabla^2 C = 0$. Partial gauge fixing to the standard covariant description can be achieved by requiring the longitudinal part of the second vector potentials to vanish, $M_Z = 0$, and gives back the usual Hamiltonian description of electromagnetism. Complete gauge fixation is then achieved, as usual, by solving the electric Gauss constraint $\nabla^2 Y = 0$ associated with the Lagrange multiplier A_0 together with the gauge condition $M_A = 0$. The gauge fixed theory contains 2 physical degrees of freedom per spacetime point described by the transverse vector potential $\vec{A}^T$ and its canonically conjugate variable $-\vec{E}^T = -\vec{\nabla}\times\vec{Z}^T$, as it should.

For later use, we note that

$$\begin{aligned} \{A^{ai}(x), B^{bj}(x')\} &= -\epsilon^{ab}\delta^{ij}\delta^3(x,x'), \\ \{C^a(x), B^{bj}(x')\} &= 0, \\ \{M^a(x), C^b(x')\} &= \epsilon^{ab}\nabla^{-2}\delta^3(x,x'), \\ \{B^{ai}(x), B^{bj}(x')\} &= \epsilon^{ab}\epsilon^{ijk}\partial_k\delta^3(x,x'). \end{aligned} \tag{2.13}$$

D. Duality, gauge, and Poincaré generators

The Hamiltonian and constraints associated with the first order action $I_M[A^a_\mu, C^a]$ are

$$H = \int d^3x\frac{1}{2}\vec{B}^a\cdot\vec{B}^a, \qquad g_a = \epsilon_{ab}\vec{\nabla}\cdot\vec{B}^b. \tag{2.14}$$

The duality generator is the $SO(2)$ Chern-Simons term [7] suitably extended to the longitudinal potentials,

$$D = -\frac{1}{2}\int d^3x(\vec{B}^a + \vec{\nabla}C^a)\cdot\vec{A}_a. \tag{2.15}$$

It commutes with the Hamiltonian and the other Poincaré generators introduced below, but is only weakly gauge invariant,

$$\{g_a, D\} = \epsilon_{ab}g^b. \tag{2.16}$$

The duality transformations (2.5) on the canonical variables A^a_i, C^a are generated through $\delta_D A^a_i = \{A^a_i, D\}$, $\delta_D C^a = \{C^a, D\}$. The extension to the Lagrange multipliers is dictated by (2.15) and the requirement that the first order action $I_M[A^a_\mu, C^a]$ is invariant. In the same way, the gauge transformations δ_λ in (2.6) are generated by

$$\Upsilon[\lambda] = \int d^3x g_a\lambda^a. \tag{2.17}$$

In this expression, the generators are smeared with the arbitrary functions λ^a defining the gauge transformation in (2.6).

A general Poincaré generator may be written as

$$T(\omega, a) = \tfrac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - a_\mu P^\mu. \tag{2.18}$$

Here $J^{\mu\nu}$ and P^μ are the individual Poincaré generators, and $\omega_{\mu\nu}$, a_μ the corresponding parameters defining the transformation. The generator of time translations is the Hamiltonian, $P^0 = H$. The Lorentz generators may be decomposed as

$$\tfrac{1}{2}\omega_{\mu\nu}J^{\mu\nu} = \tfrac{1}{2}\omega_{ij}\epsilon^{ijk}J_k - \omega_{i0}K^i. \tag{2.19}$$

The Poincaré generators are related to the symmetric energy-momentum tensor with complete electric and magnetic fields as follows:

$$T^{00} = \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2), \qquad T^{i0} = (\vec{E}\times\vec{B})^i, \tag{2.20}$$

$$P^\mu = \int d^3x T^{\mu 0}, \qquad J'^{\mu\nu} = -\int d^3x(x^\mu T^{\nu 0} - x^\nu T^{\mu 0}), \tag{2.21}$$

where $J'_k = J_k$ and $\vec{K}' = \vec{K} - x^0\vec{P}$,

$$\vec{P} = -\frac{1}{2}\int d^3x \epsilon_{ab}\vec{B}^a \times \vec{B}^b,$$
$$\vec{J} = \int d^3x \epsilon_{ab}\vec{B}^a(\vec{x}\cdot\vec{B}^b), \qquad \vec{K} = \int d^3x \vec{x}\left(\frac{1}{2}\vec{B}^a\cdot\vec{B}^a\right). \tag{2.22}$$

One can then show by direct computation of the Poisson brackets that these generators form a representation of the Poincaré algebra, up to terms involving the constraints. We will prove this explicitly in Sec. IV C and show that

$$\{T(\omega, a), T(\theta, b)\} = T([\omega, \theta], \omega b - \theta a) + \Upsilon[[\xi, \eta]_B], \tag{2.23}$$

where

$$\xi(\omega, a)^\mu = -\omega^\mu{}_i x^i + a^\mu,$$
$$\eta(\theta, b)^\mu = -\theta^\mu{}_i x^i + b^\mu, \tag{2.24}$$
$$[\xi, \eta]_B^a = B^{ai}\epsilon_{ijk}\xi^j\eta^k - \epsilon^{ac}B_{ci}(\xi^0\eta^i - \eta^0\xi^i).$$

If we then define

$$a_i' = a_i + \omega_{0i}x^0 \tag{2.25}$$

and all other parameters unchanged, the conserved Noether charges generating the Poincaré transformations as canonical transformations on the fields are

$$Q(\omega, a) = T(\omega, a'). \tag{2.26}$$

Indeed, deriving (2.23) in terms of $a'(a)$ with respect to b_0 and putting $\theta = 0$ gives

$$\{H, Q(\omega, a)\} = \frac{\partial}{\partial t}Q(\omega, a) - \int d^3x g_a \epsilon^{ab}B_{bi}\xi^i(\omega, a'). \tag{2.27}$$

As a consequence, the Noether charges are conserved on the constraint surface, as they should, and the Poincaré transformations of the canonical variables, $\delta_Q C^a = \{C^a, Q\}$ and $\delta_Q A_i^a = \{A_i^a, Q\}$, can be extended to the Lagrange multipliers so as to leave the action invariant. Explicitly, with the understanding that $\xi = \xi(\omega, a'(a))$,

$$\delta_Q C^a = 0, \tag{2.28}$$

$$\delta_Q A_i^a = \partial_i\lambda_Q^a - \epsilon^{ab}B_{bi}\xi^0 - \epsilon_{ijk}\xi^j B^{ak}, \tag{2.29}$$

$$\delta_Q B_i^a = -\epsilon^{ijk}\partial_j(\epsilon^{ab}B_{bk}\xi^0) - \partial_j(B^{aj}\xi^i) + \partial_j(B^{ai}\xi^j), \tag{2.30}$$

$$\delta_Q A_0^a = \partial_0\lambda_Q^a + \epsilon^{ab}B_{bi}\xi^i, \tag{2.31}$$

where

$$\lambda_Q^a = -\epsilon^{ab}\nabla^{-2}\partial_i(B_b^i\xi^0) + \nabla^{-2}\partial_i(\epsilon^{ijk}B_j^a\xi_k). \tag{2.32}$$

III. DYNAMICAL POINT-PARTICLE DYONS

We show in this section that for sources that correspond to dynamical point-particle dyons, a consistent action principle that makes the dyons evolve according to the Lorentz force law needs Dirac-type strings and requires a veto giving rise to the standard quantization condition. We then show equivalence with Dirac's original, manifestly Lorentz invariant formulation.

A. Dyons and Dirac strings

We begin by reviewing the use of Dirac strings in the theory of magnetic monopoles. Let us first fix conventions. Define $\epsilon_{a_1\ldots a_n} = \epsilon^{a_1\ldots a_n}$ to be totally skew-symmetric with $\epsilon_{1\ldots n} = 1$. The Levi-Civita tensor is $\boldsymbol{\varepsilon}_{a_1\ldots a_n} = \sqrt{|g|}\epsilon_{a_1\ldots a_n}$. Indices on this tensor are raised with the metric, which implies that $\boldsymbol{\varepsilon}^{a_1\ldots a_n} = \frac{(-)^\sigma}{\sqrt{|g|}}\epsilon^{a_1\ldots a_n}$ where σ is the signature of the metric. Our convention for the dual is $({}^*\omega^p)_{a_1\ldots a_{n-1}} = \frac{1}{p!}\omega^{b_1\ldots b_p}\boldsymbol{\varepsilon}_{b_1\ldots b_p a_1\ldots a_{n-p}}$.

Consider a $(d+1)$-dimensional surface Σ_{d+1} in flat 4-dimensional spacetime parameterized by $(\tau, \sigma_1, \ldots, \sigma_d)$,

$$x^\mu = v^\mu(\tau, \sigma_1, \ldots, \sigma_d).$$

Associated with this surface, define the $d+1$ form $H_{\Sigma_{d+1}}$ with contravariant components

$$H_\Sigma^{\mu_1\cdots\mu_{d+1}}(x) = \int_\Sigma \delta^{(4)}(x - v)dv^{\mu_1}\wedge\cdots\wedge dv^{\mu_{d+1}}. \tag{3.1}$$

It is straightforward to show that if $\partial\Sigma$ is the boundary of Σ, then,

$$d^*H_{\Sigma_{d+1}} = {}^*H_{\partial\Sigma_{d+1}}. \tag{3.2}$$

In the Dirac theory, the worldline $\Gamma: x^\mu = z^\mu(\tau)$ of a magnetic pole of charge g defines the magnetic current

$$j^\mu_{mag} = gH^\mu_\Gamma. \tag{3.3}$$

The worldline is the boundary of the worldsheet of a Dirac string Σ: $x^\mu = y^\mu(\tau, \sigma)$. Hence, if $G^{\mu\nu} = gH^{\mu\nu}_\Sigma$,

$$d^*G = {}^*j_{\rm mag}. \tag{3.4}$$

Dirac defines the electromagnetic field by

$$F = da + {}^*G \tag{3.5}$$

and gets the desired modified Bianchi identity $dF = {}^*j_{\rm mag}$. Note that we have used the lowercase a_μ for the electromagnetic potential here. This is to distinguish it from the potentials A_i^a in our formalism [see Eq. (2.1)]. In particular, A_i^1 in our formulation is not equal to a_i, which arises in other two-potential formulations to be discussed below.

In the Dirac formulation, the theory has an extra gauge symmetry associated with the freedom of arbitrarily choosing the position of the strings while keeping its boundary (worldline of monopole) fixed. To see this, consider the displacement of a string defined by

$$x^\mu = w^\mu(\tau, \sigma, \lambda), \tag{3.6}$$

where the initial string worldsheet Σ is at $\lambda = 0$ and the final, Σ', at $\lambda = 1$. The boundary of the 3-dimensional surface Υ defined by (3.6) is $\Delta\Sigma = \Sigma - \Sigma'$. Hence, if

$$\Delta H^{\mu\nu}_{\Sigma} = H^{\mu\nu}_{\Sigma} - H^{\mu\nu}_{\Sigma'}, \tag{3.7}$$

then from (3.2),

$$^*\Delta G = d^*K, \tag{3.8}$$

where $K^{\alpha\beta\gamma} = gH^{\alpha\beta\gamma}_{\Upsilon}$. Therefore, we see that the electromagnetic field F in (3.5) is invariant under the displacement of the string if, while moving the string, we also vary a by

$$\Delta a = -^*K. \tag{3.9}$$

The Dirac action, which depends on the string only through $F^{\mu\nu}$ is invariant under this gauge symmetry, up to the anomaly that gives rise to the quantization condition, which will be explained in more detail below when discussing the double potential formalism.

In a manifestly duality invariant theory, magnetic and electric charges are treated on the same footing. In general one considers n dynamical dyons with magnetic and electric charges $q^a_n \equiv (g_n, e_n)$. The current is then defined as

$$j^{a\mu}(x) = \sum_n q^a_n H^\mu_{\Gamma_n}(x) = \sum_n q^a_n \int_{\Gamma_n} \delta^4(x - z_n) dz^\mu_n, \tag{3.10}$$

where the sum in n is over the worldlines Γ_n of every dyon of charge q^a_n [parameterized by $z^\mu_n(\tau)$ with an arbitrary parameter τ]. For the Dirac strings attached to them, we define

$$G^{a\mu\nu}(x) = \sum_n q^a_n H^{\mu\nu}_{\Sigma_n}(x) = \sum_n q^a_n \int_{\Sigma_n} \delta^4(x - y_n) dy^\mu_n \wedge dy^\nu_n, \tag{3.11}$$

where Σ_n is the worldsheet of the Dirac string whose boundary is Γ_n [parameterized by $y^\mu_n(\tau, \sigma)$ with arbitrary parameters τ and σ]. The analogs of Eqs. (3.4) and (3.8) in this case are

$$d^*G^a = {}^*j^a, \qquad {}^*\Delta G^a = d^*K^a, \tag{3.12}$$

where

$$K^{a\ \alpha\beta\gamma} = \sum_n q^a_n H^{\alpha\beta\gamma}_{\Upsilon_n}, \tag{3.13}$$

and Υ_n is the surface defined by the displacement of the string attached to the dyon q^a_n.

When splitting space and time, as in the different manifestly duality invariant formulations, it is convenient to also split the space and time components of the string currents, defining

$$\alpha^a_i = \tfrac{1}{2}\epsilon_{ijk} G^{a\ jk} = {}^*G^a_{0i}, \qquad \beta^{ai} = G^{a\ 0i} = \tfrac{1}{2}\epsilon^{ijk*}G^a_{jk}. \tag{3.14}$$

Explicitly,

$$\vec{\alpha}^{\ a} = \sum_n q^a_n \int_{\Sigma_n} \delta^4(x - y_n) \frac{1}{2} d\vec{y}_n \times \wedge d\vec{y}_n, \tag{3.15}$$

$$\vec{\beta}^{\ a} = \sum_n q^a_n \int_{\Sigma_n} \delta^4(x - y_n) dy^0_n \wedge d\vec{y}_n, \tag{3.16}$$

where $(d\vec{y}_n \times \wedge d\vec{y}_n)_i = \epsilon_{ijk} dy^i \wedge dy^j$ and the first identity in (3.12) becomes

$$\vec{\nabla} \cdot \vec{\beta}^a = j^{a0}, \qquad \vec{\nabla} \times \vec{\alpha}^a - \partial_0 \vec{\beta}^a = \vec{j}^a. \tag{3.17}$$

It is also convenient to work with the dual of $K^{a\ \alpha\beta\gamma}$, the one-form v^a_α. In terms of it, we may derive the way the vectors $\vec{\alpha}^a$ and $\vec{\beta}^a$ in (3.14) transform under displacement of the strings. Using the second identity in (3.12),

$$\Delta\alpha^a_i = {}^*\Delta G^a_{0i} = (dv^a)_{0i} = \partial_0 v^a_i - \partial_i v^a_0, \tag{3.18}$$

$$\Delta\beta^{ai} = \tfrac{1}{2}\epsilon^{ijk*}\Delta G^a_{jk} = \tfrac{1}{2}\epsilon^{ijk}(dv^a)_{jk} = (\vec{\nabla} \times \vec{v}^a)^i. \tag{3.19}$$

For dynamical dyons, the action must be supplemented with the kinetic term

$$I_k[z^\mu_n] = -\sum_n \int_{\Gamma_n} \sqrt{-dz^\mu_n dz_{n\mu}}. \tag{3.20}$$

The total action I' that includes the dynamics of the dyons and produces the correct Lorentz force law is

$$I'[A^a_\mu, C^a, y^\mu_n] = I_M + I_I + I_k + \frac{1}{2}\int d^4x \epsilon_{ab}[2\vec{\nabla} C^a \vec{\alpha}^b - \vec{\beta}^a \vec{\alpha}^b - \vec{\beta}^a \nabla^{-2} \vec{\nabla} \times \partial_0 \vec{\beta}^b]. \tag{3.21}$$

The constraints (2.7) and the electromagnetic Eqs. (2.9) are clearly unchanged, for the extra piece in the action does not depend on A^a_μ. The equations obtained from the variation of C^a are modified with respect to the result of (2.8) to

$$\nabla^2 C_a = \epsilon_{ab}(\vec{\nabla} \cdot [\partial_0 \vec{A}^b + \vec{\alpha}^b] - \nabla^2 A^b_0). \tag{3.22}$$

For later use we note that, applying $\vec{\nabla}\times$ to (2.9) and using (3.22), together with the boundary condition that $\vec{B}^a$ falls off at least as fast as r^{-1} at infinity,

$$\vec{B}_a \approx \epsilon_{ab}(\partial_0 \vec{A}^b - \vec{\nabla} A^b_0 + \vec{\alpha}^b + \nabla^{-2}\vec{\nabla} \times \partial_0 \vec{\beta}^b). \tag{3.23}$$

As a side remark, we also note that the definitions

$$\mathbf{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + {}^*\mathcal{G}^a_{\mu\nu}, \qquad {}^*\mathcal{G}^a_{ij} = \epsilon_{ijk}\partial^k C^a, \tag{3.24}$$

$$ {}^*\mathcal{G}^a_{0i} = \alpha^a_i + \nabla^{-2}(\vec{\nabla}\times\partial_0\vec{\beta}^a)_i, \tag{3.25}$$

are such that

$$B_{ai} = \tfrac{1}{2}\epsilon_{ijk}\mathbf{F}^{jk}_a = {}^*\mathbf{F}_{a0i}. \tag{3.26}$$

Furthermore, they allow us to write the on-shell Eqs. (3.23) as

$$B_{ai} \approx \epsilon_{ab}\mathbf{F}^b_{0i} = -\tfrac{1}{2}\epsilon_{ijk}\epsilon_{ab}{}^*\mathbf{F}^{jkb}, \tag{3.27}$$

while the equations of motion for A^a_μ take the covariant form

$$\partial_\nu\mathbf{F}^{\mu\nu}_a = \epsilon_{ab}j^{b\mu}, \qquad \partial_\nu{}^*\mathbf{F}^{\mu\nu}_a = -j^\mu_a. \tag{3.28}$$

In the case of a single dyon, one assumes without loss of generality that the string terms in the last line of (3.21) are absent. Indeed, in this case one can perform a duality rotation so that, say, the magnetic charge vanishes. The string terms then reduce to $-\int d^4x\vec{\nabla}C^1\alpha^2$ and can be dropped because they only affect the irrelevant auxiliary equation used to determine A^2_0. This justifies *a posteriori* the coupling to the sources considered in the first section.

B. Lorentz force law and veto

We still need to vary y^μ, z^μ in action (3.21) in order to derive the Lorentz force law. We will see below that in order to obtain it, we need to impose the so called "Dirac veto." This demand was introduced by Dirac in his original treatment of magnetic monopoles [12] to obtain the desired classical equations. It consists of the requirement that no electric charge can touch a Dirac string. At the quantum level, Dirac showed that the veto modifies the topology of phase space, giving rise to his celebrated quantization condition. In our formalism the Dirac veto is required as well, as we show below. The difference resides in that here we will need to ask that no dyon can touch the string of any other dyon. This generalized version of the Dirac veto was also used in [9].

Variations of I_I with respect to z^μ_n give

$$\delta_z I_I = \sum_n \epsilon_{ab}q^b_n\int_{\Gamma_n}((\partial_0\vec{A}^a - \vec{\nabla}A^a_0)\cdot(\delta z^0_n d\vec{z}_n - \delta\vec{z}_n dz^0_n) + (\vec{\nabla}\times\vec{A}^a)\cdot(\delta\vec{z}_n\times d\vec{z}_n)). \tag{3.29}$$

Before varying I_M with respect to y^μ_n, we establish the following identities. For all smooth vector fields $\vec{V}^a$, $\vec{W}^a$ one has

$$\int d^4x\vec{V}_b\delta_y\vec{\alpha}^b = \sum_n q^b_n\Big[\int_{\Gamma_n}\vec{V}_b\cdot(\delta\vec{z}_n\times d\vec{z}_n) + \int_{\Sigma_n}\Big(\vec{\nabla}\cdot\vec{V}_b\delta\vec{y}_n\cdot\frac{1}{2}(d\vec{y}_n\times\wedge d\vec{y}_n) - \partial_0\vec{V}_b\cdot\Big(\delta\vec{y}_n\times(dy^0_n\wedge d\vec{y}_n) - \delta y^0_n\frac{1}{2}(d\vec{y}_n\times\wedge d\vec{y}_n)\Big)\Big)\Big] \tag{3.30}$$

and

$$\int d^4x\delta_y\vec{\beta}^a\vec{W}_a = \sum_n q^a_n\Big[\int_{\Gamma_n}(\delta z^0_n d\vec{z}_n - \delta\vec{z}_n dz^0_n)\cdot\vec{W}_a + \int_{\Sigma_n}\Big(\delta\vec{y}_n\times(dy^0_n\wedge d\vec{y}_n) - \frac{1}{2}\delta y^0_n(d\vec{y}_n\times\wedge d\vec{y}_n)\cdot(\vec{\nabla}\times\vec{W}_a)\Big)\Big]. \tag{3.31}$$

The variation of I_M with respect to y^μ_n may be computed by specializing for the fields

$$\vec{V}_b = \tfrac{1}{2}\epsilon_{ab}(2\vec{\nabla}C^a - \vec{\beta}^a), \tag{3.32}$$

$$\vec{W}_a = -\tfrac{1}{2}\epsilon_{ab}(\vec{\alpha}^b + 2\nabla^{-2}\vec{\nabla}\times\partial_0\vec{\beta}^b). \tag{3.33}$$

Combining all terms,

$$\delta_z I_I + \delta_y I_M = \sum_n q^a_n\Big[\int_{\Gamma_n}((\vec{W}_a - \epsilon_{ab}(\partial_0\vec{A}^b - \vec{\nabla}A^b_0))\cdot(\delta z^0_n d\vec{z}_n - \delta\vec{z}_n dz^0_n) + (\vec{V}_a - \epsilon_{ab}(\vec{\nabla}\times\vec{A}^b))\cdot(\delta\vec{z}_n\times d\vec{z}_n)) + \int_{\Sigma_n}\Big(\vec{\nabla}\cdot\vec{V}_a\delta\vec{y}_n\cdot\frac{1}{2}(d\vec{y}_n\times\wedge d\vec{y}_n) + (\vec{\nabla}\times\vec{W}_a - \partial_0\vec{V}_a)\cdot\Big(\delta\vec{y}_n\times(dy^0_n\wedge d\vec{y}_n) - \delta y^0_n\frac{1}{2}(d\vec{y}_n\times\wedge d\vec{y}_n)\Big)\Big)\Big]. \tag{3.34}$$

Now, taking the divergence of (3.32) and making use of the first identity in (3.17) and the constraints (2.7) one gets,

$$\vec{\nabla}\cdot\vec{V}_a = \tfrac{1}{2}\epsilon_{ab}j^{b0}. \tag{3.35}$$

It follows that the second term in (3.34) vanishes provided the string attached to dyon n does not cross any other dyon (Dirac veto). This is due to the fact that the Dirac veto ensures that $j^{\mu a} = 0$ on the worldsheet of the strings. Similarly, from (3.32) and (3.33) and the identities (3.17) it is straightforward to show that

$$\vec{\nabla}\times\vec{W}_a - \partial_0\vec{V}_a = -\tfrac{1}{2}\epsilon_{ab}\vec{j}^b. \tag{3.36}$$

Hence, the last term in (3.34) also vanishes on account of Dirac's veto. The string piece in the first and second terms of (3.34) again vanish because of the veto. This may be seen from the fact that (3.32) may be written as

$$\vec{V}_a - \epsilon_{ab}(\vec{\nabla}\times\vec{A}^b) = -\epsilon_{ab}\vec{B}^b + \tfrac{1}{2}\epsilon_{ab}\beta^b, \tag{3.37}$$

and therefore, due to the veto, the integral on the worldline of a dyon only sees the first term. In the same way, using (3.23),

$$\vec{W}_a - \epsilon_{ab}(\partial_0\vec{A}^b - \vec{\nabla}A_0^b) = -\vec{B}_a + \tfrac{1}{2}\epsilon_{ab}\vec{\alpha}^b, \tag{3.38}$$

and the second term vanishes on the worldline of a dyon.

Combining the remaining terms with those from the variation of I_k, extremization of the total action now implies the Lorentz force law

$$m_n\frac{d}{d\tau}\left(\frac{\frac{dz_n^0}{d\tau}}{\sqrt{-\frac{dz_{n\mu}}{d\tau}\frac{dz_n^\mu}{d\tau}}}\right) = q_n^a\vec{B}_a(z_n)\cdot\frac{d\vec{z}_n}{d\tau}, \tag{3.39}$$

$$m_n\frac{d}{d\tau}\left(\frac{\frac{d\vec{z}_n}{d\tau}}{\sqrt{-\frac{dz_{n\mu}}{d\tau}\frac{dz_n^\mu}{d\tau}}}\right) = q_n^a\vec{B}_a(z_n)\cdot\frac{dz_n^0}{d\tau} + \frac{d\vec{z}_n}{ds}\times\vec{B}^a(z_n)\epsilon_{ab}q_n^b, \tag{3.40}$$

as it should.

C. Equivalence with Dirac's covariant formulation and quantization condition

We end this section by showing that the theory presented above is equivalent to Dirac's theory. This shows that the theory with dyons is Lorentz invariant. We will actually show that our action (3.21) is equivalent to an action found in [9] which, in turn, has been shown in [10] to be equivalent to a generalization of Dirac's covariant formulation allowing for dyons.

Explicitly, this action reads

$$\bar{I}[\vec{a}^a, y_n^\mu] = \frac{1}{2}\int d^4x[\epsilon_{ab}\vec{b}^a(\partial_0\vec{a}^b + \vec{\alpha}^b) - \vec{b}^a\cdot\vec{b}_a + \epsilon_{ab}\vec{a}^a\cdot\vec{j}^b] + I_k, \tag{3.41}$$

where

$$\vec{b}^a = \vec{\nabla}\times\vec{a}^a + \vec{\beta}^a. \tag{3.42}$$

This formulation makes use of Dirac strings in the same way as our formulation does. That is, each dyon q_n is attached to a string parameterized by $y_n^\mu(\tau,\sigma)$. The quantities $\vec{\alpha}^a$ and $\vec{\beta}^a$ appearing in (3.41) are the same ones as defined in Eqs. (3.15) and (3.16) above and satisfy the identities (3.17). Note that on account of these identities, the longitudinal part of $\vec{a}$ drops out of this action principle. The field $\vec{b}^a$ is the magnetic/electric field appearing in Maxwell's equations. It must, therefore, be the same as our $\vec{B}^a$.

In this formulation, Gauss's law appears as an identity on taking the divergence of $\vec{b}^a$ in (3.42). The field $\vec{\nabla}\times\vec{a}^a$ is transversal but has a stringlike singularity which is removed by $\vec{\beta}^a$. In our formulation $\vec{B}^a$ has two nonsingular pieces, namely, the transverse and longitudinal components of it. To show equivalence, we decompose $\vec{\beta}^a$ accordingly so that

$$\begin{aligned}\vec{b}^a &= \vec{\nabla}\times\vec{a}^a + \vec{\beta}^a\\ &= \vec{\nabla}\times(\vec{a}^a - \nabla^{-2}\vec{\nabla}\times\vec{\beta}^a) + \vec{\nabla}\nabla^{-2}\vec{\nabla}\cdot\vec{\beta}^a.\end{aligned} \tag{3.43}$$

From the constraints (2.7) and the first identity in (3.17), the longitudinal piece is precisely $\vec{\nabla}C^a$. We are then lead to the following identifications:

$$\vec{A}^a = \vec{a}^a - \nabla^{-2}\vec{\nabla}\times\vec{\beta}^a, \tag{3.44}$$

$$C^a = \nabla^{-2}\vec{\nabla}\cdot\vec{\beta}^a. \tag{3.45}$$

(The first equation is true up to an irrelevant longitudinal field, in order for $\vec{B}^a = \vec{b}^a$ to hold.)

To establish equivalence is now straightforward. We start with the action (3.21) of our formulation. Assuming vanishing boundary conditions at spatial infinity, (A_0^a, C^a) are auxiliary fields because their equations of motions (2.7) and (3.22) can be used to algebraically determine them in terms of the other fields. We can thus solve for them in action (3.21). Then we use (3.44) to write $\vec{A}^a$ in terms of $\vec{a}^a$, and after a bit of algebra involving the identities (3.17) we get precisely action (3.41).

In the double potential formulation of [9], i.e., for action (3.41), the symmetry corresponding to shifts in the string is realized by transforming $\vec{\alpha}^a$ and $\vec{\beta}^a$ in (3.18) and (3.19), with $v_0^a = 0$ and

$$\Delta\vec{a}^a = -\vec{v}^a. \tag{3.46}$$

Let us do that for the case in which there are only two dyons, q^a, $\bar{q}^a$. We will only change the position of the string attached to q^a. Varying action (3.41) and using the identity $\vec{\nabla}\cdot\vec{b}^a = j^{a0}$ we get

$$\delta\bar{I} = \frac{1}{2}\epsilon_{ab}\int d^4x j^{a\mu}v_\mu^b. \tag{3.47}$$

This is zero unless the worldline of dyon $\bar{q}^a$ crosses the 3-dimensional manifold Υ swept by the string attached to q^a. In that case the variation is

$$\delta\bar{I} = \tfrac{1}{2}\epsilon_{ab}q^a\bar{q}^b. \tag{3.48}$$

This will not affect the quantum mechanical system if the variation is proportional to $2\pi\hbar n$, for some integer n. This leads us to the Dirac-Schwinger-Zwanziger quantization condition

$$\bar{e}g - e\bar{g} = 2\pi n\hbar, \tag{3.49}$$

up to a factor of $1/2$. This factor is removed by a careful analysis of the topology of the system. We will not discuss this here. More details can be found in [9,10].

Finally, we study what happens in our formulation. First, let us compute how the field $\vec{A}^a$ transforms under the movement of the string. From Eq. (3.19), (3.44), and (3.46) we get (3.44),

$$\Delta\vec{A}^a = -\vec{v}^a - \nabla^{-2}\vec{\nabla}\times\nabla\times\vec{v}^a = -\vec{\nabla}(\nabla^{-2}\vec{\nabla}\cdot\vec{v}^a). \tag{3.50}$$

If we now take

$$\Delta A_0^a = -v_0^a = -\partial_0(\nabla^{-2}\vec{\nabla}\cdot\vec{v}^a) + \partial_0(\nabla^{-2}\vec{\nabla}\cdot\vec{v}^a) - v_0^a, \tag{3.51}$$

the variation defined by (3.50) and the first term of (3.51) is a gauge transformation of the form (2.6) which leaves action I' in (3.21) invariant. We thus only need to compute the variation under the movement of the strings and the second part of (3.51). Using identities (3.17) one obtains precisely the same result as in the previous case, namely, the right-hand side of Eq. (3.47). The argument leading to the quantization condition is therefore the same.

IV. EXTENDED DOUBLE POTENTIAL FORMALISM IN CURVED SPACE

We generalize the first order action to curved spacetimes and discuss the canonical and gauge structure of the theory, including diffeomorphism invariance. In particular, we show that the standard algebra of surface deformations of the purely gravitational case now involves both Gauss-type constraints with structure functions depending on electric and magnetic fields. We proceed to the equations of motion deriving from the generalized action principle and show that they are equivalent to the covariant Einstein-Maxwell equations. We then show how the string singularity of the Reissner-Nordstrøm dyonic black hole solution gets resolved in our formalism. We compute the electric and magnetic surface integrals following the Regge-Teitelboim approach, discuss how they appear in a geometric derivation of the first law and in the Euclidean approach to black hole thermodynamics. Finally, we apply these results to the resolved Reissner-Nordstrøm black hole.

A. Action and canonical structure

The first order action I_M can be generalized to curved spacetimes. We consider a globally hyperbolic spacetime, foliated by a spacelike family of hypersurfaces, each labeled by the value of a timelike coordinate t. The induced metric on each surface is $g_{ij}(t)$. We follow the conventions of MTW [15], chapter 21, where spatial indices are lowered and raised with the 3-metric g_{ij} and g is its determinant. We denote by ϵ_{ijk} the completely antisymmetric symbol, which differs from the $[ijk]$ notation used in MTW.

Adapting the results derived in [6–8,16] and defining $\mathcal{B}^{ai} = \epsilon^{ijk}\partial_j A_k^a + \sqrt{g}\partial^i C^a$, we get the following manifestly duality invariant action in the absence of sources:

$$I_M[A_\mu^a, C^a, g_{ij}, N, N^i] = \frac{1}{8\pi}\int d^4x\Big[(\mathcal{B}^{ai} + \sqrt{g}\partial^i C^a)\times\epsilon_{ab}(\partial_0 A_i^b - \partial_i A_0^b) - \frac{N}{\sqrt{g}}\mathcal{B}_a^i\mathcal{B}_i^a - \epsilon_{ab}\epsilon_{ijk}N^i\mathcal{B}^{aj}\mathcal{B}^{bk}\Big], \tag{4.1}$$

where $N = (-^{(4)}g^{00})^{1/2}$ and $N^i = {}^{(4)}g^{ij(4)}g_{0i}$ are the lapse and shift functions and ${}^{(4)}g_{\mu\nu}$ is the 4-dimensional metric.

We are interested in solutions to the equations of motion derived from $I = I_{\rm ADM} + I_M$, where $I_{\rm ADM}$ is the first order action for pure general relativity. Introducing the collective notation $z^A = (g_{ij}, \pi^{ij}, A_i^a, C^a)$ for the different fields in our system, this action principle takes the form

$$I[z,u] = \int d^4x[a_A(z)\partial_0 z^A - u^\alpha\gamma_\alpha], \tag{4.2}$$

$$a_A(z)\partial_0 z^A = \frac{\pi^{ij}}{16\pi}\partial_0 g_{ij} - \frac{\mathcal{E}^i}{4\pi}\partial_0 A_i + \frac{\sqrt{g}\partial^i C}{4\pi}\partial_0 Z_i. \tag{4.3}$$

The constraints $\gamma_\alpha \equiv (\mathcal{H}_\perp, \mathcal{H}_i, \mathcal{G}_a)$ are associated with the Lagrange multipliers $u^\alpha \equiv (N, N^i, A_0^a)$ and given by[1]

$$\mathcal{H}_\perp = \frac{1}{16\pi}(\mathcal{H}_\perp^{\rm ADM} + \mathcal{H}_\perp^{\rm mat}),$$
$$\mathcal{H}_i = \frac{1}{16\pi}(\mathcal{H}_i^{\rm ADM} + \mathcal{H}_i^{\rm mat}), \qquad \mathcal{G}_a = \frac{1}{4\pi}\epsilon_{ab}\partial_i\mathcal{B}^{bi}, \tag{4.4}$$

where $\mathcal{H}_\perp^{\rm ADM}$, $\mathcal{H}_i^{\rm ADM}$ are given in [15,16] and

$$\mathcal{H}_\perp^{\rm mat} = \frac{2g_{ij}}{\sqrt{g}}\mathcal{B}_a^i\mathcal{B}^{aj}, \qquad \mathcal{H}_i^{\rm mat} = 2\epsilon_{ab}\epsilon_{ijk}\mathcal{B}^{aj}\mathcal{B}^{bk}. \tag{4.5}$$

The first two sets of constraints in (4.4) above are the gravitational Hamiltonian and momentum constraints, while the last set are the two electromagnetic Gauss constraints.

In order to disentangle the canonical structure we begin by writing this action as

$$I_M = \frac{1}{4\pi}\int d^4x\Big[-\mathcal{E}^i\partial_0 A_i + \sqrt{g}\partial^i C\partial_0 Z_i - A_0\partial_i\mathcal{E}^i + Z_0\partial_i\mathcal{B}^i - \frac{N}{2\sqrt{g}}(\mathcal{E}^i\mathcal{E}_i + \mathcal{B}^i\mathcal{B}_i) + \epsilon_{ijk}N^i\mathcal{E}^j\mathcal{B}^k\Big], \tag{4.6}$$

[1]Note the misprint in Eq. (21.116) of [15], where there should be no lapse function on the right-hand side.

where $\mathcal{E}^i = \epsilon^{ijk}\partial_j Z_k + \sqrt{g}\partial^i Y$ and $\mathcal{B}^i = \epsilon^{ijk}\partial_j A_k + \sqrt{g}\partial^i C$. We assume here and below that every 3-vector admits a unique orthogonal, spatially covariant decomposition (see e.g. [17]) $X^i = X^{Ti} + X^{Li}$, where

$$X^{Li} = \partial^i M, \qquad X^{Ti} = \frac{1}{\sqrt{g}}\epsilon^{ijk}\partial_j L_k, \tag{4.7}$$

for some M, L_k. In terms of the inverse of the spatially covariant Laplacian ∇^{-2} and the spatially covariant derivative ∇_i, we have

$$M = \nabla^{-2}\nabla_j X^j, \qquad X^{Ti} = X^i - \partial^i M. \tag{4.8}$$

A vector is transverse if its divergence vanishes and longitudinal if its curl vanishes,

$$\begin{aligned} &\partial_i(\sqrt{g}X^i) = 0 \Rightarrow X^i = X^{Ti}, \\ &\epsilon^{ijk}\partial_j X_k = 0 \Rightarrow X^i = X^{Li}. \end{aligned} \tag{4.9}$$

We then have

$$\int d^3x\sqrt{g}X^i g_{ij}Y^j = \int d^3x\sqrt{g}(X^{Li}g_{ij}Y^{Lj} + X^{Ti}g_{ij}Y^{Tj}).$$

Using such a decomposition for A^i_a, $A^a_i = \partial_i M^a + A^{aT}_i$, $M^a = (M_A, M_z)$, the kinetic term becomes

$$\begin{aligned} \int d^4x a_A(z)\partial_0 z^A = \int d^4x\bigg(\bigg[\frac{\pi^{ij}}{16\pi} + \frac{\sqrt{g}D^{ijkl}}{4\pi}(Z^T_k\sqrt{g}\partial_l C \\ - A^T_k\sqrt{g}\partial_l Y)\bigg]\partial_0 g_{ij} - \frac{\epsilon^{ijk}\partial_j Z^T_k}{4\pi}\partial_0 A^T_i \\ + \frac{\partial_i(\sqrt{g}\partial^i Y)}{4\pi}\partial_0 M_A - \frac{\partial_i(\sqrt{g}\partial iC)}{4\pi}\partial_0 M_Z\bigg), \end{aligned} \tag{4.10}$$

where $D_{ijkl} = \frac{1}{2\sqrt{g}}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl})$ is the DeWitt supermetric. Note that D^{ijkl} is not the inverse DeWitt supermetric, but the result of raising all the indices with the metric. Let us define

$$\pi^a = (\partial_i(\sqrt{g}\partial^i C), \partial_i(\sqrt{g}\partial^i Y)) = (-\pi_Z, \pi_A), \tag{4.11}$$

as new independent variables, so that $C^a = \nabla^{-2}\frac{\pi^a}{\sqrt{g}}$. We also define

$$\tilde{\pi}^{ij} = \pi^{ij} + 4\sqrt{g}D^{ijkl}\epsilon_{ab}A^{aT}_k\sqrt{g}\partial_l C^b, \tag{4.12}$$

The independent phase space variables are thus $(g_{ij}, A^T_i, M_A, M_Z, \tilde{\pi}^{ij}, Z^T_i, \pi_A, \pi_Z)$ in terms of which the canonically conjugate pairs are

$$\begin{aligned} &\bigg(g_{ij}, \frac{\tilde{\pi}^{kl}}{16\pi}\bigg), \qquad \bigg(A^T_i, -\frac{\epsilon^{ijk}\partial_j Z^T_k}{4\pi}\bigg), \\ &\bigg(M_A, \frac{\pi_A}{4\pi}\bigg), \qquad \bigg(M_Z, \frac{\pi_Z}{4\pi}\bigg). \end{aligned} \tag{4.13}$$

In particular,

$$\begin{aligned} \{\mathcal{B}^{ai}(x), \mathcal{B}^{bj}(y)\} = 4\pi\epsilon^{ijk}\epsilon^{ab}\partial^x_k\delta^3(x,y), \\ \{M^a(x), \pi^b(y)\} = 4\pi\epsilon^{ab}\delta^3(x,y). \end{aligned} \tag{4.14}$$

B. Gauge structure

Before turning to the equations of motions and their solutions, let us discuss the gauge structure of the theory. We want to show that the constraints γ_α are first class. Defining $\epsilon^\alpha \equiv (\xi^\perp, \xi^i, \lambda^a)$ with ϵ^α vanishing at the boundary and $\Gamma[\epsilon] = \int d^3x\gamma_\alpha\epsilon^\alpha$, this means that

$$\{\Gamma[\epsilon_1], \Gamma[\epsilon_2]\} = \Gamma[[\epsilon_1, \epsilon_2]], \tag{4.15}$$

for a suitably defined $[\epsilon_1, \epsilon_2]$. In this case, the gauge transformations leaving action (4.2) invariant are given by

$$\delta_\epsilon z^A = \{z^A, \Gamma[\epsilon]\}, \qquad \delta u^\alpha = \partial_0\epsilon^\alpha + [\epsilon, u]^\alpha. \tag{4.16}$$

In order to compute these brackets it is useful to go to the Darboux coordinates identified in the previous subsection in terms of which the Gauss constraints become

$$\mathcal{G}_a = \frac{1}{4\pi}\epsilon_{ab}\pi^b. \tag{4.17}$$

Now one should do the change of coordinates in $\mathcal{H}_\perp$, $\mathcal{H}_i$, i.e., perform the replacement $\pi^{ij} = \tilde{\pi}^{ij} - 4\sqrt{g}D^{ijkl}\epsilon_{ab}A^{aT}_k\sqrt{g}\partial_l C^b$. Since the additional terms that are generated in this way are proportional to C^a and thus vanish on the constraint surface defined by $\mathcal{G}_a$ they can safely be discarded in the source-free situation. In the following, we will drop the tilde on π^{ij}.

In particular, we have

$$\delta_\epsilon\pi^a = 0, \tag{4.18}$$

$$\delta_\epsilon A^a_i = \partial_i\lambda^a_T - \frac{g_{ij}}{\sqrt{g}}\epsilon^{ab}\mathcal{B}^j_b\xi^\perp - \epsilon_{ijk}\xi^j\mathcal{B}^{ak}, \tag{4.19}$$

$$\delta_\epsilon g_{ij} = \nabla_i\xi_j + \nabla_j\xi_i + 2D_{ijkl}\pi^{kl}\xi^\perp, \tag{4.20}$$

where

$$\begin{aligned} \lambda^a_T = \lambda^a - \epsilon^{ab}\frac{1}{\sqrt{g}}\nabla^{-2}\partial_i(\mathcal{B}^i_b\xi^\perp) \\ + \frac{1}{\sqrt{g}}\nabla^{-2}\partial_i\bigg(\frac{\epsilon^{ijk}}{\sqrt{g}}\mathcal{B}^a_j\xi_k\bigg). \end{aligned} \tag{4.21}$$

Equation (4.18) implies that π^a are constants of motion, which is consistent with the longitudinal part of the source-free Maxwell equations. From (4.19) we also find

$$\begin{aligned} \delta_\epsilon\mathcal{B}^{ia} = -\epsilon^{ijk}\partial_j\bigg(\frac{1}{\sqrt{g}}\epsilon^{ab}\mathcal{B}_{bk}\xi^\perp\bigg) - \partial_j(\mathcal{B}^{aj}\xi^i) \\ + \partial_j(\mathcal{B}^{ai}\xi^j). \end{aligned} \tag{4.22}$$

Infinitesimal diffeomorphisms along η^μ are recovered by using $\xi^\perp = N\eta^0$, $\xi_i = g_{i\mu}\eta^\mu$. Indeed, with this choice of parameters, $\mathcal{L}_\eta g_{\mu\nu} \approx \delta_\xi g_{\mu\nu}$. This can be seen for instance on (4.20) by using the (auxiliary) equations of motion for π^{ij} together with the definitions of lapse N and shift N^i in terms of the 4-metric $g_{\mu\nu}$. In other words, diffeomorphism invariance in the Hamiltonian framework is implemented through the gauge transformations generated by $H[\xi] = \int d^3x(\mathcal{H}_\perp \xi^\perp + \mathcal{H}_i \xi^i)$. In particular, using as gauge parameters the Lagrange multipliers, $\epsilon^\alpha = u^\alpha$, amounts to performing an infinitesimal time-translation on account of the Hamiltonian equations of motion.

Let us end this discussion by determining $[\epsilon_1, \epsilon_2]$. The constraints $\mathcal{G}_a$, have vanishing Poisson brackets among themselves and with all other constraints because the $\mathcal{B}^{ai}$ do not depend on M_A, M_Z. It follows that $[\lambda, \epsilon_2]^\alpha = 0$ and also, from (4.16) and (4.19) that the associated gauge transformations δ_λ generated by $G[\lambda] = \int d^3x \mathcal{G}_a \lambda^a$ are the double electromagnetic gauge transformations of (2.6), while all other variables are left invariant.

We still have to compute $\{H[\xi], H[\eta]\}$. We notice first of all that the purely gravitational part satisfies the algebra of surface deformations [18,19],

$$\{H^{\rm ADM}[\xi], H^{\rm ADM}[\eta]\} = H^{\rm ADM}[[\xi, \eta]_{\rm SD}], \tag{4.23}$$

$$[\xi, \eta]^\perp_{\rm SD} = \xi^i \partial_i \eta^\perp - \eta^i \partial_i \xi^\perp, \tag{4.24}$$

$$[\xi, \eta]^i_{\rm SD} = g^{ij}(\xi^\perp \partial_j \eta^\perp - \eta^\perp \partial_j \xi^\perp) + \xi^j \partial_j \eta^i - \eta^j \partial_j \xi^i. \tag{4.25}$$

From (4.20) and (4.22), we find that

$$\{H^{\rm ADM}[\xi], H^{\rm mat}[\eta]\} - (\xi \leftrightarrow \eta) + \{H^{\rm mat}[\xi], H^{\rm mat}[\eta]\} = H^{\rm mat}[[\xi, \eta]_{\rm SD}] + G[[\xi, \eta]_B], \tag{4.26}$$

where

$$[\xi, \eta]^a_B = \mathcal{B}^{ai} \epsilon_{ijk} \xi^j \eta^k - \frac{\epsilon^{ac} \mathcal{B}_{ci}}{\sqrt{g}} (\xi^\perp \eta^i - \eta^\perp \xi^i). \tag{4.27}$$

Combining with (4.23), we finally get

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]_{\rm SD}] + G[[\xi, \eta]_B]. \tag{4.28}$$

According to [20], such a constraint algebra provides the integrability conditions that guarantee that "the evolution of a three geometry can be viewed as the deformation of a three-dimensional cut in a four-dimensional space-time."

C. Derivation of the Poisson algebra of Poincaré generators in flat spacetime

In this subsection, we derive the Poisson algebra of the Poincaré generators in flat spacetime as given in (2.23) by restricting the results of the previous subsection to flat spacetime.

We thus assume in this subsection that $N = 1$, $N^i = 0$, $g_{ij} = \delta_{ij}$. Greek indices take values from 0 to 3 with $\mu = (\perp, i)$. Indices are lowered and raised with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and its inverse. Let $\tilde{\omega}_{\mu\nu} = -\tilde{\omega}_{\nu\mu}$. In this case, the Lie algebra of vector fields $\xi(\tilde{\omega}, \tilde{a}) = (-\tilde{\omega}^\mu{}_i x^i + \tilde{a}^\mu)\frac{\partial}{\partial x^\mu}$ with bracket the surface-deformation bracket (4.24) and (4.25) forms a representation of the Poincaré algebra [21],

$$[\xi(\tilde{\omega}_1, \tilde{a}_1), \xi(\tilde{\omega}_2, \tilde{a}_2)]_{\rm SD} = \xi([\tilde{\omega}_1, \tilde{\omega}_2], \tilde{\omega}_1 \tilde{a}_2 - \tilde{\omega}_2 \tilde{a}_1). \tag{4.29}$$

It then follows from (4.28) that this is also the case for the canonical generators $\mathcal{H}[\xi(\tilde{\omega}, \tilde{a})]$ equipped with the Poisson bracket, when one considers the restriction to the constraint surface defined by $\mathcal{G}_a = 0$.

Comparing with Sec. II D, we find that

$$\tfrac{1}{2}\omega_{\mu\nu} J^{\mu\nu} - a_\mu P^\mu = \mathcal{H}[\xi(\tilde{\omega}, \tilde{a})] \Leftrightarrow \tilde{\omega}_{\mu\nu} = 4\pi \omega_{\mu\nu}, \quad \tilde{a}_0 = 4\pi a_0, \qquad \tilde{a}_i = 4\pi(a_i + \omega_{0i} x^0). \tag{4.30}$$

This concludes the proof that the generators defined in (2.22) form a representation of the Poincaré algebra and the algebra in (2.24) is a direct consequence of (4.28).

D. Equations of motion with sources and comparison to covariant formalism

The standard Einstein-Maxwell equations, now in the presence of external, magnetic, and electric conserved current densities $j^{a\mu}$, $\partial_\mu j^{a\mu} = 0$ given by (3.10) with associated string terms defined in (3.11), derive from extremizing the action $I_{\rm geom} + I'_M$, where

$$I_{\rm geom}[g_{\mu\nu}] = \frac{1}{16\pi} \int d^4x \sqrt{-{}^{(4)}g} R, \tag{4.31}$$

and

$$I'_M[g_{\mu\nu}, \mathcal{F}^{\mu\nu}, a_\mu, y^\mu] = \frac{1}{4\pi} \int d^4x \Big[-\frac{1}{2}(\partial_\mu a_\nu - \partial_\nu a_\mu + {}^*G_{\mu\nu})\mathcal{F}^{\mu\nu} + \frac{1}{4} \frac{1}{\sqrt{-{}^{(4)}g}} \times \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + a_\mu j^\mu \Big]. \tag{4.32}$$

To make connection with the ADM formalism, we have followed [16] and introduced the auxiliary tensor densities $\mathcal{F}^{\mu\nu}$. On the one hand, one can solve the equations of motion for $\mathcal{F}^{\mu\nu}$ algebraically, $\mathcal{F}^{\mu\nu} = \sqrt{-{}^{(4)}g} g^{\mu\alpha} g^{\nu\beta}(\partial_\mu a_\nu - \partial_\nu a_\mu + {}^*G_{\mu\nu})$. When substituted into the action, one recovers the Einstein-Maxwell theory with Dirac strings. On the other hand, one can introduce $\mathcal{F}^{0i} = \mathcal{E}^i_{\rm ADM}$, eliminate the auxiliary $\mathcal{F}^{ij}$ and use the decomposition of the 4-metric into the 3-metric, lapse and shift to find the standard Hamiltonian form

$$I'_M[\mathcal{E}^i_{\rm ADM}, a_\mu, g_{ij}, N, N^i, y^\mu] = \frac{1}{4\pi}\int d^4x\Big[-\mathcal{E}^i_{\rm ADM}(\partial_0 a_i + \alpha_i) - a_0\partial_i\mathcal{E}^i_{\rm ADM} - \frac{N}{2\sqrt{g}}(\mathcal{E}^i_{\rm ADM}\mathcal{E}_i^{\rm ADM} + \mathcal{B}^i_{\rm ADM}\mathcal{B}_i^{\rm ADM}) + \epsilon_{ijk}N^i\mathcal{E}^j_{\rm ADM}\mathcal{B}^k_{\rm ADM} + a_\mu j^\mu\Big], \tag{4.33}$$

where $\mathcal{B}^i_{\rm ADM} = \epsilon^{ijk}\partial_j a_k + \beta^k$. The equations of motion for $\mathcal{E}^i_{\rm ADM}$ read

$$\mathcal{E}^{\,i}_{\rm ADM} = \frac{\sqrt{g}}{N}g^{ij}(\partial_j a_0 - \partial_0 a_j - \alpha_j - \epsilon_{jkl}N^k\mathcal{B}^l_{\rm ADM}). \tag{4.34}$$

They determine $\mathcal{E}^i_{\rm ADM}$ in terms of the other variables and the sources. Similarily, in the gravitational sector, the equations of motion following from varying π^{ij} are auxiliary in the sense that they can be solved algebraically for π^{ij} in terms of the other variables. After this has been done, the constraints and the equations of motion following from variation of g_{ij} are equivalent to the covariant Einstein-Maxwell equations with Dirac strings. Hence, every solution $g_{\mu\nu}$, a_μ to the covariant equations of motions is a solution to the ADM equations of motion with electric and magnetic fields $\mathcal{E}^i_{\rm ADM}$, $\mathcal{B}^i_{\rm ADM}$ and momenta π^{ij} determined in terms of $g_{\mu\nu}$, a_μ. Conversely, every solution of the ADM equations of motion gives a solution to the covariant equations of motion.

Alternatively, one can multiply (4.34) by N. The longitudinal part of this equation is solved uniquely for a_0, while the transverse part gives, after using $\epsilon^{ijk}\partial_j\alpha_k - \partial_0\beta^i = k^i$,

$$\partial_0\mathcal{B}^i_{\rm ADM} = -\epsilon^{ijk}\partial_j\Big(\frac{N}{\sqrt{g}}\mathcal{E}_k^{\rm ADM}\Big) - \partial_j(N^i\mathcal{B}^j_{\rm ADM}) + \partial_j(N^j\mathcal{B}^i_{\rm ADM}) - k^i. \tag{4.35}$$

Finally, Maxwell's equations for a_i are

$$\partial_0\mathcal{E}^i_{\rm ADM} = \epsilon^{ijk}\partial_j\Big(\frac{N}{\sqrt{g}}\mathcal{B}_k^{\rm ADM}\Big) - \partial_j(N^i\mathcal{E}^j_{\rm ADM}) + \partial_j(N^j\mathcal{E}^i_{\rm ADM}) - j^i. \tag{4.36}$$

As a side remark, note that the constraint algebra in the absence of sources and strings in the standard ADM approach to Einstein-Maxwell theory can be directly rederived from our result (4.28) and is given by

$$\{H[\xi], H[\eta]\} = H[[\xi,\eta]_{\rm SD}] + G_{\rm ADM}[[\xi,\eta]_{\rm ADM}], \tag{4.37}$$

where

$$G_{\rm ADM}[\lambda] = \int d^3x\partial_i\mathcal{E}^i_{\rm ADM}\lambda,$$
$$[\xi,\eta]_{\rm ADM} = \mathcal{B}^i_{\rm ADM}\epsilon_{ijk}\xi^j\eta^k - \frac{\mathcal{E}_i^{\rm ADM}}{\sqrt{g}}(\xi^\perp\eta^i - \eta^\perp\xi^i). \tag{4.38}$$

Indeed, the algebra rests only on the constraints and the transformation properties (4.14) and (4.20). Provided that $\mathcal{E}^i_{\rm ADM} = \mathcal{E}^i$ and $\mathcal{B}^i_{\rm ADM} = \mathcal{B}^i$, which we always assume in the following, these are the same in both descriptions, except that the constraint $\mathcal{G}_2 = -\partial_i\mathcal{B}^i_{\rm ADM}$ is absent because $\partial_i\mathcal{B}^i_{\rm ADM}$ vanishes identically in the absence of sources and strings.

In the presence of sources, (3.41) generalizes readily to curved space, where a manifestly duality invariant action principle is defined by $I_{\rm ADM} + \bar{I}_M$, with

$$\bar{I}_M[a^a_i, g_{ij}, N, N^i, y^\mu] = \frac{1}{8\pi}\int d^4x\Big[(b^{ai}\epsilon_{ab}(\partial_0 a^b_i + \alpha^b_i) - \tag{4.39}$$

$$-\frac{N}{\sqrt{g}}b^i_a b^a_i - \epsilon_{ab}\epsilon_{ijk}N^i b^{aj}b^{bk} + \epsilon_{ab}a^a_i j^{bi}\Big], \tag{4.40}$$

and $b^a_i = \epsilon^{ijk}\partial_j a^a_k + \beta^{ai}$. Indeed, equivalence of the associated equations of motion to the ADM/covariant ones is obvious when $b^{1i} = \mathcal{B}^i_{\rm ADM}$, $b^{2i} = \mathcal{E}^i_{\rm ADM}$ since the gravitational equations of motion are unaffected while those for a^a_i read

$$\partial_0 b^{ai} = -\epsilon^{ijk}\partial_j\Big(\frac{N}{\sqrt{g}}\epsilon^{ab}b_{bk}\Big) - \partial_j(N^i b^{aj}) + \partial_j(N^j b^{ai}) - j^{ai}, \tag{4.41}$$

and coincide with the relevant Eqs. (4.35) and (4.36).

With the longitudinal electric and magnetic fields produced by the potentials C^a, the appropriate action principle is $I_{\rm ADM} + I_M + I_J$, where I_M is defined in (4.1) and

$$I_J[A^a_\mu, C^a, y^\mu] = \frac{1}{4\pi}\int d^4x\epsilon_{ab}(A^a_\mu j^{b\mu} + \sqrt{g}\partial^i C^a\alpha^b_i - \frac{1}{2}\beta^{ai}\alpha^b_i + \frac{1}{2}\beta^{aTi}\partial_0\gamma^b_i). \tag{4.42}$$

Here γ^a_i is the potential for the transverse part of β^{ai}, $\beta^{aTi} = \epsilon^{ijk}\partial_j\gamma^a_k$.

In this case the equations of motion for A^a_i are given by

$$\partial_0\mathcal{B}^{ai} = -\epsilon^{ijk}\partial_j\Big(\frac{N}{\sqrt{g}}\epsilon^{ab}\mathcal{B}_{bk}\Big) - \partial_j(N^i\mathcal{B}^{aj}) + \partial_j(N^j\mathcal{B}^{ai}) - j^{ai}. \tag{4.43}$$

They are the correct matter field equations provided that $\mathcal{B}^{ai} = b^{ai}$ are the magnetic and electric fields. This implies on the one hand $\epsilon^{ijk}\partial_j A^a_k = \epsilon^{ijk}\partial_j a^a_k + \beta^{aTi}$, and in turn $A^a_k = a^a_k + \gamma^a_k$, up to an irrelevant longitudinal part, and

$\sqrt{g}\partial^i C^a = \beta^{aLi}$ on the other hand. Again, the equations of motion for A_0^a, C^a,

$$\partial_i(\sqrt{g}\partial^i C^a) = j^{a0}, \tag{4.44}$$

$$\partial_i[N\mathcal{B}^i_a - \epsilon_{ab}\sqrt{g}g^{il}(\partial_0 A_l^b - \partial_l A_0^b + \alpha_l^b + \epsilon_{ljk}N^j\mathcal{B}^{bk})] = 0, \tag{4.45}$$

are auxiliary because they can be used to solve these fields in terms of the others. This can be done in the action principle and gives back (4.39).

In conclusion, if the lapse N is nonvanishing and the covariant decomposition of spatial vectors into longitudinal and transverse components is unique then there is a one-to-one and onto correspondence between solutions of the covariant/ADM equations of motion and solutions to the equations of motion deriving from $I_{\rm ADM} + I_M + I_J$.

In the case of a single dyon, one can again drop all string terms in I_J, which then simplifies to

$$I_J[A^a_\mu; j^{a\mu}] = \frac{1}{4\pi}\int d^4x \epsilon_{ab} A^a_\mu j^{b\mu}. \tag{4.46}$$

All relevant matter equations of motion are correct in this case, but one has to face the fact that the metric dependence in the longitudinal part of $\mathcal{B}^{ai}$ implies an additional term in the equations of motion associated with variations of g_{kl},

$$\frac{\delta(I_{\rm ADM} + I_M)}{\delta g_{kl}} - \frac{\delta(I_{\rm ADM} + \bar{I}_M)}{\delta g_{kl}} = \frac{\sqrt{g}}{4\pi} D^{ijkl}\partial_j C^a X_{ai}, \tag{4.47}$$

$$X^i_a = \frac{N}{\sqrt{g}}\mathcal{B}^i_a - \epsilon_{ab}g^{il}(\partial_0 A^b_l - \partial_l A^b_0 + \epsilon_{lkm}N^k\mathcal{B}^{bm}). \tag{4.48}$$

Again, one can use a duality rotation to make the magnetic charge vanish in which case the equations of motion imply $C^1 = 0$. We thus only need to consider X^i_2. But in the purely electric case $\alpha_i = 0$ and $A_\mu = a_\mu$ so that X^i_2 vanishes on account of the matter equation of motion (4.34).

E. String-singularity free dyonic black holes

Consider now the case of a dyonic Reissner-Nordstrøm solution with charge Q^a. The dyon defined by (1.1) and (1.2) is a solution outside of the location of the dyon (at $r = 0$ in our coordinate system) with a Dirac-string singularity to the equations deriving from $I_{\rm geom} + I'_M$ given in (4.31) and (4.32). It is thus also a solution to the equations of motion derived from $I_{\rm ADM} + I'_M$ for which

$$\mathcal{E}^i_{\rm ADM} = \delta^i_r Q\sin\theta, \qquad \mathcal{B}^i_{\rm ADM} = \delta^i_r P\sin\theta. \tag{4.49}$$

In the simplified duality invariant formulation defined by $I_{\rm ADM} + I_M + I_J$, with I_J given in (4.46), we have to determine the vector and scalar potentials giving rise to $\mathcal{B}^{ai} = \delta^i_r Q^a\sin\theta$, where $Q^1 = P$, $Q^2 = Q$ with the metric given by (1.1). This is easily seen to be the case for $A^{ai} = 0$ and

$$\begin{aligned} C^a &= -Q^a\int_r^\infty \frac{dr'}{r'^2 N(r')} \\ &= \frac{Q^a}{\sqrt{Q^bQ_b}}\ln\frac{r(M-\sqrt{Q^fQ_f})}{Mr - Q^cQ_c - \sqrt{Q^dQ_d(r^2 - 2Mr + Q^eQ_e)}} \\ &= -\frac{Q^a}{r} + O(r^{-2}). \end{aligned} \tag{4.50}$$

In the gauge where the scalar potentials vanish at infinity, it is then straightforward to see that all matter equations of motions are solved by

$$A^a_0 = -\frac{\epsilon^{ab}Q_b}{r}, \tag{4.51}$$

and one can directly check that in this case $X^i_a = 0$. In conclusion, in the new formulation, the Reissner-Nordstrøm dyon is described by the metric (1.1) and the potentials (4.50) and (4.51). The string singularity of the standard approach has thus been resolved in the new formulation.

In the gauge where the scalar potentials vanish at infinity, let us define

$$\phi = -A_0, \qquad \psi = Z_0, \tag{4.52}$$

with ϕ_H, ψ_H denoting these quantities evaluated at the horizon, in agreement with (1.6). For the resolved Reissner-Nordstrøm dyon, this gives

$$\phi = \frac{Q}{r}, \qquad \psi = \frac{P}{r}. \tag{4.53}$$

In the Euclidean methods discussed below, it is useful to choose a gauge where the scalar potentials vanish on the horizon. In this case,

$$A^a_0 = -\epsilon^{ab}Q_b\left(\frac{1}{r} - \frac{1}{r_+}\right), \tag{4.54}$$

$$A_0 = -\frac{Q}{r} + \phi_H, \qquad Z_0 = \frac{P}{r} - \psi_H. \tag{4.55}$$

F. Surface charges

The Regge-Teitelboim analysis [21] allows one to derive the correct variational principle in the presence of nonvanishing surface charges at infinity. Consider an arbitrary gauge transformation generator $\Gamma[\epsilon] = \int d^3x\gamma_\alpha\epsilon^\alpha$ with ϵ^α not necessarily vanishing at the boundary (a condition we demanded in Sec. IV B above). The variation of this generator under a change of phase space variables may be written

$$\delta\Gamma[\epsilon] = \int d^3x \delta_z(\gamma_\alpha \epsilon^\alpha) = \int d^3x\left(\delta z^A \frac{\delta(\gamma_\alpha\epsilon^\alpha)}{\delta z^A} - \partial_i k^i_\epsilon\right), \quad (4.56)$$

where $\delta/\delta z^A$ is the Euler-Lagrange derivative. The second piece is a boundary term and arises from integration by parts. The expression $k^i_\varepsilon[z^A, \delta z^A]$ depends on the phase space variables and linearly on their variations and the gauge parameters.

For the simplest application, consider phase space variables z^A_s that satisfy the constraints and variations δz^A_s obeying the linearized constraints. In this case, the left-hand side of (4.56) vanishes. Suppose then that the associated solution z^a_s, u^α_s to the evolution equations is time independent, $\partial_0 z^A_s = 0$. In particular, this means that the associated vector field $\eta^\mu = \delta^\mu_0$ is the timelike Killing vector field of the metric $g_{\mu\nu}$. In this case, the evolution equations following from (4.2),

$$\frac{\delta a_B}{\delta z^A}\partial_0 z^A - \partial_0 a_A = \frac{\delta(\gamma_\alpha u^\alpha)}{\delta z^A}, \quad (4.57)$$

imply that the first term on the right-hand side of (4.56) vanishes as well. We thus find

$$\partial_i k^i_{u_s}[z^A_s, \delta z^A_s] = 0. \quad (4.58)$$

Using Stokes's theorem, it follows that the integral over a sphere at radius r and fixed time t does not depend on the radius r,

$$\oint_{S_{r_1}} d^3x_i k^i_{u_s}[z^A_s, \delta z^A_s] = \oint_{S_{r_2}} d^3x_i k^i_{u_s}[z^A_s, \delta z^A_s]. \quad (4.59)$$

Here $d^3x_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$. The explicit expression for $k^i_\varepsilon[z^A; \delta z^A]$ can be easily worked out by integrations by parts. It is defined up to the divergence of an arbitrary superpotential, $\partial_j t^{[ij]}$, which does not play any role for our purpose. It splits into a standard purely gravitational part and a matter part,

$$k^i_\epsilon[z^A; \delta z^A] = k^{\mathrm{grav},i}_\epsilon[g_{ij}, \pi^{ij}; \delta g_{ij}, \delta\pi^{ij}] + k^{\mathrm{mat},i}_\epsilon[z^A; \delta z^A]. \quad (4.60)$$

The former has been derived in [21] and reads

$$k^{\mathrm{grav},i}_\epsilon = \frac{1}{16\pi}[G^{ljki}(\xi^\perp\nabla_k\delta g_{lj} - \partial_k\xi^\perp\delta g_{lj}) + 2\xi_k\delta\pi^{ki} + (2\xi^k\pi^{ji} - \xi^i\pi^{jk})\delta g_{jk}],$$
$$G^{ljki} = \frac{1}{2}\sqrt{g}(g^{lk}g^{ji} + g^{il}g^{jk} - 2g^{lj}g^{ki}), \quad (4.61)$$

where G^{ijkl} is the inverse of the DeWitt supermetric, $D_{ijkl}G^{klmn} = \frac{1}{2}(\delta^m_i\delta^n_j + \delta^m_j\delta^n_i)$. The matter part now involves, besides the electric contributions, the sought for magnetic ones:

$$k^{\mathrm{mat},i}_\varepsilon = \frac{1}{4\pi}\Big(\frac{\xi^\perp}{\sqrt{g}}\epsilon^{ijk}\mathcal{B}^a_j\delta A_{ak} - \xi^\perp\mathcal{B}^{ai}\delta C_a + \epsilon_{ab}(\xi^k\mathcal{B}^{ai} - \xi^i\mathcal{B}^{ak})\delta A^b_k - \epsilon_{ab}\sqrt{g}g^{il}\epsilon_{ljk}\xi^j\mathcal{B}^{ak}\delta C^b + \epsilon_{ab}(\sqrt{g}\partial^i\lambda^a\delta C^b - \lambda^a\delta\mathcal{B}^{bLi})\Big). \quad (4.62)$$

In the sequel, we are interested in asymptotically flat gravitational field configurations carrying finite charges associated with energy momentum. We will not need to consider the more general boundary conditions guaranteeing finite charges associated with rotations or boosts. The appropriate fall-off conditions on the gravitational variables and lapse and shift have been discussed in detail in [21],

$$g_{rr} = 1 + O(r^{-1}), \qquad g_{\theta\theta} = r^2 + O(r), \qquad g_{\phi\phi} = r^2\sin^2\theta + O(r), \quad (4.63)$$

$$g_{r\theta} = O(r^0) = g_{r\phi}, \qquad g_{\theta\phi} = O(r), \quad (4.64)$$

$$\pi^{rr} = O(r^0), \qquad \pi^{\theta\theta} = O(r^{-2}) = \pi^{\phi\phi} = \pi^{\theta\phi}, \qquad \pi^{r\theta} = O(r^{-1}) = \pi^{r\phi}, \quad (4.65)$$

$$N = 1 + O(r^{-1}), \qquad N^\phi = O(r^{-2}) = N^\theta, \qquad N^r = O(r^{-1}). \quad (4.66)$$

For the matter variables, we assume

$$A^a_r = O(r^{-1}), \qquad A^a_\theta = O(r^0) = A^a_\phi, \qquad C^a = O(r^{-1}), \qquad A^a_0 = k^a + O(r^{-1}). \quad (4.67)$$

In particular, these fall-off conditions include the background solutions $\bar{z}$, $\bar{u}$ described by

$$\bar{g}_{rr} = 1, \qquad \bar{g}_{\theta\theta} = r^2, \qquad \bar{g}_{\phi\phi} = r^2\sin^2\theta, \quad (4.68)$$

$$\bar{N} = 1, \qquad \bar{N}^\phi = 0 = \bar{N}^\theta = \bar{N}^r, \qquad \bar{A}^a_0 = k^a, \quad (4.69)$$

and all other variables vanishing. For later use we introduce the additional notation

$$k^1 = \phi^c, \qquad k^2 = -\psi^c. \quad (4.70)$$

In order to allow configurations satisfying the fall-off conditions to be extrema of the variational principle, action (4.2) needs to be supplemented by the addition of a suitable surface term at the boundary at infinity, i.e., the surface r, t constant with $r \to \infty$,

$$I^T[z, u] = \int d^4x[a_A(z)\partial_0 z^A - u^\alpha\gamma_\alpha] - Q_u[z]. \quad (4.71)$$

The surface term $Q_u[z]$ is determined by the requirement that, under variations of the fields z^A satisfying the fall-off conditions, its variation $\delta_z Q_u$ should precisely cancel the spatial boundary term arising when deriving the Hamiltonian equations of motion, i.e., the term due to the right-hand side of (4.56),

$$\delta_z Q_u[z] = \oint_{S^\infty} d^3x_i k^i_u[z, \delta z]. \quad (4.72)$$

For the purely gravitational part, this problem was solved in [21], the appropriate boundary term being the ADM mass:

$$\oint_{S^\infty} d^3x_i k^{\mathrm{grav},i}_u[z, \delta z] = \oint_{S^\infty} d^3x_i k^{\mathrm{grav},i}_{\bar{u}}[\bar{z}, \delta z] = \delta_z \oint_{S^\infty} d^3x_i k^{\mathrm{grav},i}_{\bar{u}}[\bar{z}, z - \bar{z}], \quad (4.73)$$

so that

$$Q^{\mathrm{grav}}_u[g, \pi] = \oint_{S^\infty} d^3x_i k^{\mathrm{grav},i}_{\bar{u}}[\bar{z}, z - \bar{z}] = \mathcal{M}, \quad (4.74)$$

$$\mathcal{M} = \oint d^3x_i \sqrt{\bar{g}}(\bar{g}^{lk}\bar{g}^{ji} - \bar{g}^{lj}\bar{g}^{ki})\bar{D}_k(g_{lj} - \bar{g}_{lj}), \quad (4.75)$$

where the covariant derivative is taken with respect to the flat background metric $\bar{g}_{ij}$.

For the matter part, the boundary conditions imply also that

$$\oint_{S^\infty} d^3x_i k^{\mathrm{mat},i}_u[z, \delta z] = \oint_{S^\infty} d^3x_i k^{\mathrm{mat},i}_{\bar{u}}[\bar{z}, \delta z] = \delta_z \oint_{S^\infty} d^3x_i k^{\mathrm{mat},i}_{\bar{u}}[\bar{z}, z - \bar{z}]. \quad (4.76)$$

In particular, the boundary conditions (4.63), (4.64), (4.65), and (4.66) are such that, when $(\xi^\perp, \xi^i, \lambda^a)$ are replaced by (N, N^i, A^a_0), the contributions proportional to N, N^i from the matter part (4.62) vanish. There is thus no correction to the ADM mass for the adopted boundary conditions. This will not remain true for more general boundary conditions where the matter part (4.62) can contribute both to the ADM energy momentum and the Lorentz generators. For the boundary conditions at hand, only the last term survives and combines into magnetic and electric charge $\mathcal{Q}^a = (\mathcal{P}, \mathcal{Q})$, as expected,

$$Q^{\mathrm{mat}}_u[g, C] = \oint_{S^\infty} d^3x_i k^{\mathrm{mat},i}_{\bar{u}}[\bar{z}, z - \bar{z}] = -k^a \epsilon_{ab} \mathcal{Q}^b, \quad (4.77)$$

$$\mathcal{Q}^b = \frac{1}{4\pi} \oint_{S^\infty} d^3x_i \mathcal{B}^{biL}. \quad (4.78)$$

In other words, off shell, the correct Hamiltonian for the boundary condition under considerations is

$$\boldsymbol{H} = \int d^3x(\mathcal{H}_\perp N + \mathcal{H}_i N^i) + \mathcal{M}, \quad (4.79)$$

while electric and magnetic charges are given by

$$\boldsymbol{Q} = -\frac{1}{\phi^c} \int d^3x(\mathcal{G}_1 A_0) + \mathcal{Q}, \qquad \boldsymbol{P} = -\frac{1}{\psi^c} \int d^3x(\mathcal{G}_2 Z_0) + \mathcal{P}. \quad (4.80)$$

These observables commute in the Poisson bracket,

$$\{\boldsymbol{H}, \boldsymbol{Q}\} = 0 = \{\boldsymbol{H}, \boldsymbol{P}\} = \{\boldsymbol{Q}, \boldsymbol{P}\}, \quad (4.81)$$

and the total action (4.71) can be written as

$$I^T[z, u] = \int dt\Big(\int d^3x a_A(z)\partial_0 z^A - (\boldsymbol{H} - \phi^c \boldsymbol{Q} - \psi^c \boldsymbol{P})\Big). \quad (4.82)$$

G. First law

For the resolved Reissner-Nordstrøm dyon z, u given by (1.1), (4.50), and (4.55), the first law of thermodynamics can now be derived as a consequence of using identity (4.59) between infinity, $r_1 \longrightarrow \infty$ and the outer horizon $r_2 = r_+$,

$$\oint_{S^\infty} d^3x_i k^i_u[z, \delta z] = \oint_{S_{r_+}} d^3x_i k^i_u[z, \delta z], \quad (4.83)$$

where δ_z describes a variation around the dyon satisfying the constraints. Indeed, in this case, $k^\phi = 0 = k^\theta = k^r$, while $k^a = \frac{\epsilon^{ab}Q_b}{r_+}$. In other words $k^1 = \phi_H$ is the electric potential on the horizon, while $k^2 = -\psi_H$ is minus the magnetic potential on the horizon. Now, the results of the previous subsection imply that we get at infinity,

$$\oint_{S^\infty} d^3x_i k^i_u[z, \delta z] = \oint_{S^\infty} d^3x_i k^i_{\bar{u}}[\bar{z}, z - \bar{z}] = \delta_z \mathcal{M} - \phi_H \delta_z \mathcal{Q} - \psi_H \delta_z \mathcal{P}. \quad (4.84)$$

For the matter part, we have

$$\oint_{S_r} d^{n-1}x_i k^{\mathrm{mat},i}_u = -\frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} d\phi \epsilon_{ab} A^a_0 \delta \mathcal{B}^{bLi}, \quad (4.85)$$

which vanishes on the horizon $r = r_+$ where A^a_0 vanishes. Note that in the gauge where A^a_0 vanishes at infinity, the matter part gives no contribution at infinity, but $\phi_H \delta_z \mathcal{Q} + \psi_H \delta_z \mathcal{P}$ at the horizon, as it should.

Finally at the horizon, the purely gravitational part gives

$$\oint_{S_{r_+}} d^{n-1}x_i k^{\mathrm{grav},i}_u = \frac{\kappa}{8\pi}\delta_z \mathcal{A}. \quad (4.86)$$

This can be shown for instance by using the fact that $k^{\mathrm{grav},i}_u$ is the time-space component of a conserved superpotential $k^{[\mu\nu]}_{\partial/\partial t}$ that can be proved to coincide, for variations satisfying the linearized field equations and up to an irrelevant term of the form $\partial_\sigma t^{[\sigma\mu\nu]}_{\partial/\partial t}$, with the conserved superpotential considered in [22]. In turn the latter has been shown to contribute $\frac{\kappa}{8\pi}\delta_z\mathcal{A}$ at the horizon.

This concludes the geometric discussion of the first law

$$\delta_z\mathcal{M} = \frac{\kappa}{8\pi}\delta_z\mathcal{A} + \phi_H\delta_z\mathcal{Q} + \psi_H\delta_z\mathcal{P}, \tag{4.87}$$

for variations satisfying the linearized equations of motion around the Reissner-Nordstrøm dyon.

H. Euclidean approach

Our setup also allows us to complement the work of [5,6], by evaluating the partition function in the grand canonical ensemble, along the lines of [23].

For the three commuting observables $\hat{\boldsymbol{H}}$, $\hat{\boldsymbol{Q}}$, $\hat{\boldsymbol{P}}$, we thus would like to compute

$$Z[\beta, \phi^c, \psi^c] = \mathrm{Tr}e^{-\beta(\hat{\boldsymbol{H}}-\phi^c\hat{\boldsymbol{Q}}-\psi^c\hat{\boldsymbol{P}})} = e^{\Psi_G}, \tag{4.88}$$

where $\Psi_G(\beta, -\beta\phi^c, -\beta\psi^c)$ is the Massieu potential for the grand canonical ensemble (see e.g. [24] in the present context),

$$\Psi_G(\beta, -\beta\phi, -\beta\psi) = S(\langle\hat{H}\rangle, \langle\hat{Q}\rangle, \langle\hat{P}\rangle) - \beta\langle\hat{H}\rangle + \beta\phi\langle\hat{Q}\rangle + \beta\psi\langle\hat{P}\rangle, \tag{4.89}$$

$$d\Psi_G = -\langle\hat{H}\rangle d\beta + \langle\hat{Q}\rangle d(\beta\phi) + \langle\hat{P}\rangle d(\beta\psi). \tag{4.90}$$

The path integral representation for this partition function is

$$Z[\beta, \phi, \psi] = \int \mathcal{D}\Phi e^{I_e^T}, \tag{4.91}$$

where Φ represents all the fields (z^A, u^α) together with appropriate ghost fields $\bar{C}^\alpha$, C^α [25] (see e.g. [26] for a review). The appropriate action is

$$I_e^T = \int_0^\beta d\tau\Big(i\int d^3x a_A(z)\partial_0 z^A - (\boldsymbol{H} - \phi^c\boldsymbol{Q} - \psi^c\boldsymbol{P})\Big) + \text{ghost terms}. \tag{4.92}$$

The path integral is taken over all periodic paths in τ with periodicity β and $N \to 1$, $A_0 \to \phi^c$, $Z_0 \to -\psi^c$ for $r \to \infty$.

We now notice that the transformation defined by

$$\pi^{ij} \to -i\pi^{ij}, \qquad A_i \to -iA_i, \qquad Z_i^L \to -iZ_i^L, \qquad N^i \to -iN^i \tag{4.93}$$

with all other variables unchanged maps the action I_e^T to a real action when all (transformed) variables are real. The latter action differs from the Lorentzian action (4.82) by the fact that the terms involving π^{ij} and $\mathcal{B}^{iT}$ in $N\mathcal{H}_\perp$ and $N^i\mathcal{H}_i$ have the opposite signs. For the purely gravitational part, this is as it should be in order that the path integral corresponds to one over Euclidean metrics after integration over the momenta π^{ij}.

The leading contribution to the path integral is given by the value of $e^{I_e^T}$ evaluated at the classical solutions satisfying the specified boundary conditions, that is, (i) the falloff conditions (4.63), (4.64), (4.65), (4.66), and (4.67), (ii) fixed values of the potentials (ϕ^c, ψ^c), and (iii) a fixed inverse temperature β. The Reissner-Nordstrøm dyon (RND) described by the lapse and the spatial metric given in (1.1) and the matter fields (4.50) and (4.55) is such a solution if $\phi^c = \phi_H$ and $\psi^c = \psi_H$ with ϕ_H, ψ_H defined in (1.6) since all variables affected by the above transformation vanish in this case. Furthermore, this solution is time independent and satisfies the (modified) constraints so that I_e^T reduces to surface integrals. For the matter part, we find directly that $I_e^{\mathrm{mat}}(\mathrm{RND}) = \beta\phi_H Q + \beta\psi_H P$. For the gravitational part, it has been shown for instance in [27] in the current Hamiltonian context that $I_e^{\mathrm{grav}}(\mathrm{RND}) = -\beta M + \frac{1}{4}\mathcal{A}$, with $\mathcal{A}$ given in (1.5).

Assuming then that the dyon is the only extremum, it follows that to leading order,

$$\Psi_G = -\beta M + \tfrac{1}{4}\mathcal{A} + \beta\phi_H Q + \beta\psi_H P, \tag{4.94}$$

which is the expected result.

V. CONCLUSION

In this paper we have generalized the manifestly duality invariant double potential formalism [7,8] to include potentials for the longitudinal electric and magnetic fields thus turning the scalar potentials into nonspurious Lagrange multipliers. By introducing additional pure gauge degrees of freedom on the classical level, which corresponds to an additional quartet [28] on the quantum level, we have turned a topological conservation law, the magnetic charge, into a dynamical one.

We have shown on the example of the Reissner-Nordstrøm dyon that the formalism is tailor-made for a treatment of black hole dyons by standard action based methods and allows one to compute in the grand canonical ensemble. How to explicitly resolve the string singularity of the Kerr-Newman dyon and derive its thermodynamics will be discussed elsewhere.

In our approach Dirac strings are only needed for the coupling to dynamical dyons and the derivation of the Lorentz force law. It would be interesting to understand whether there are applications of the formalism in the non-Abelian case or extensions to gravitational magnetic charge.

ACKNOWLEDGMENTS

The authors thank G. Compère for useful discussions. A. G. is grateful to the International Solvay Institutes for hospitality during various stages of this project. This work is supported in part by a "Pôle d'Attraction Interuniversitaire" (Belgium), by IISN-Belgium, convention

4.4505.86, by the Fund for Scientific Research-FNRS (Belgium) (with which G. B. is associated), by Proyectos FONDECYT 1051084, 7070183, and 1051064, by the research Grant No. 26-05/R of Universidad Andrés Bello and by the European Commission programme MRTN-CT-2004-005104, in which G. B. is associated with V. U. Brussel.

Note added in proof.—After the present work was accepted for publication, Refs. [29,30] were called to our attention. In these references, a manifestly covariant double potential formalism is developed. As explained there, the problem is the occurrence of a second ''photon'' that has to be removed in an *ad hoc* manner.

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