Role of twisted statistics in the noncommutative degenerate electron gas

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We consider the problem of a degenerate electron gas in the background of a uniformly distributed positive charge, ensuring overall neutrality of the system, in the presence of noncommutativity. In contrast to previous calculations [F. S. Bemfica and H. O. Girotti, J. Phys. A **38**, L539 (2005)] that did not include twisted formalism, we study the effects of noncommutativity from the points of view of both usual and braided twisted symmetry. We find corrections to the ground-state energy already at first order in perturbation theory when the usual twisted statistics is taken into account. The effects of noncommutativity, however, disappears if braided twisted symmetry is considered. In the former case these corrections arise since the interaction energy is sensitive to two-particle correlations, which are modified for the usual twisted anticommutation relations.

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I. INTRODUCTION

The study of noncommutative (NC) geometry and its implications have gained considerable importance in recent times as these studies are motivated from string theory and certain condensed matter systems like the quantum Hall effect. Despite this, the physical consequences of noncommutativity remains unclear. In free space (no boundaries) and in the absence of interactions, noncommutativity seems to have no observable physical consequences [1,2]. On the other hand, in [3] it was found that noncommutativity does affect the ground-state energy of a degenerate electron gas to second order in perturbation theory. However, in this computation the role of twisted statistics was not taken into account. As is well known by now it is necessary to twist the anticommutation relations in order to restore the Galilean [4] or, more generally, Poincaré invariance [5,6]. Here we revisit this calculation taking also due care of the twisted formalism, both in the usual and braided frame work. One expects that the twisted statistics, arising in the usual framework, will also have an effect on the ground-state energy as it changes the two-particle correlations [4], and it is well known that the two-particle interaction energy is sensitive to the two-particle correlations. Thus there are two possible ways in which noncommutativity may have a physical effect. The first is due to the noncommutative nature of space *per se* and the second is due to the modification of two-particle correlations due to the twisted statistics required to restore Poincaré invariance.

On the other hand, as shown in [2], the implementation of braided twisted symmetry does not give rise to any twisted statistics. One therefore does not expect to find any noncommutative correction in this case. It should be mentioned in this context that the braided twisted symmetry [2] entails the compositions of functions at different points resulting in noncommutativity between space-time coordinates even for two different particles, labeled by α and β: $[\hat{x}^\mu_\alpha, \hat{x}^\nu_\beta] = i\theta^{\mu\nu}$ in contrast to the usual case, where it is given by $[\hat{x}^\mu_\alpha, \hat{x}^\nu_\beta] = i\theta^{\mu\nu}\delta_{\alpha\beta}$. Although it is well known that the noncommutativity in the lowest Landau level of the Landau problem takes the form of the latter, with $1/B$ playing the role of the noncommutative parameter [7], it is not clear what kind of form the fundamental noncommutativity between space-time coordinates will take. Indeed, there is an ongoing controversy regarding whether one should implement the usual or braided twisted symmetry [2,8]. It is therefore quite desirable to find an example, where the difference between these two examples show up explicitly.

Here we demonstrate this in the specific setting of [3], namely, a nonrelativistic degenerate electron gas in the presence of a uniform neutralizing background charge, interacting through a screened Coulomb potential. One of the motivations for studying this particular system is that this is the typical setting encountered in astro physical or quantum Hall systems.

It is generally believed that the noncommutative parameter $\theta^{\mu\nu}$ is of the order of the Planck area. It, therefore, seems very unlikely that any experimental signature of its presence can be detected by any terrestrial observation, unless one finds a way of amplifying it by several orders of magnitude. One of the motivations for this paper is to study whether this is at all feasible by considering a nonrelativistic degenerate electron gas, which presumably can contain few orders higher than Avogadro's number ($\sim 10^{23}$) of particles, so that the effects of noncommutativity may be amplified.

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The plan of the paper is as follows. We first introduce the basic notations and conventions of NC geometry in Sec. II. In Sec. III we provide a brief review on the origin of twisted (anti)commutation relations and the related twisted bosons/fermions. We also discuss how the implementation of braided twisted symmetry can restore the conventional bosons/fermions, associated with their usual symmetry/antisymmetry properties of the wave functions. In Sec. IV we consider the effect of the noncommutativity by computing the energy shift, arising from the screened interparticle Coulomb potential, to first order in perturbation theory. Again here we discuss the effects of noncommutativity from the points of view of conventional and braided twisted symmetry. Here we use the technology developed by Fetter and Walecka [9] for the same commutative problem ($\theta = 0$). Finally, we conclude in Sec. V.

II. NC GEOMETRY

To fix our notation and conventions, we briefly recall the essentials of a NC geometry. The canonical NC structure is given by the following operator-valued space-time coordinates,

$$[x^\mu_{\rm op}, x^\nu_{\rm op}] = i\theta^{\mu\nu}. \tag{1}$$

Instead of working with functions of these operator-valued coordinates, one can alternatively work with functions of c-numbered coordinates provided one composes the functions through the Moyal star ($*$) product defined as [7]

$$\alpha *_\theta \beta(x) = \left[\alpha \exp\left(\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu}\overrightarrow{\partial}_\nu\right)\beta\right](x), \tag{2}$$

$$\theta^{\mu\nu} = -\theta^{\nu\mu} \in R, \qquad x = (x^0, x^1, \ldots, x^d). \tag{3}$$

This definition applies to a restricted class of functions in which at least one of α, β is smooth [10]. A more general integral definition can be given, which applies to a more general class of functions that includes distributions [10,11], and the relation between the two definitions was discussed in [10]. The definition (2) will, however, suffice for our present purposes as the functions α, β will be taken to be periodic (see Sec. IV). With this definition, the Moyal bracket $[x^\mu, x^\nu]_* \equiv (x^\mu * x^\nu - x^\nu * x^\mu) = i\theta^{\mu\nu}$ is isomorphic to the corresponding commutator (1) involving $x^\mu_{\rm op}$. In this paper, we consider only the problem of spatial noncommutativity and therefore set $\theta^{0i} = 0$. The Poincaré group $\mathcal{P}$ or the diffeomorphism group $\mathcal{D}$, which acts on the NC space-time R^{d+1}, defines a natural action on smooth functions $\alpha \in \mathcal{C}^\infty(R^{d+1})$ as

$$(g\alpha)(x) = \alpha(g^{-1}x), \tag{4}$$

for $g \in \mathcal{P}$ or $\in \mathcal{D}$. However, in general,

$$(g\alpha) *_\theta (g\beta) \neq g(\alpha *_\theta \beta), \tag{5}$$

showing that the action of the group $\mathcal{P}$ or $\mathcal{D}$ is not an automorphism of the Algebra $\mathcal{A}_\theta(R^{d+1})$, unless one considers the translational subgroup. The Poincaré symmetry can, however, be restored by the twisted implementation of the Lorentz group, as has been shown recently in the literature [5,6]. This in turn implies that the permutation group, used for defining bosons or fermions, gets twisted as well in order to maintain the statistics operator as a superselected observable. One thus ends up defining twisted fermionic or bosonic fields, the Fourier modes of which are subjected to the following (anti)commutation relations:

$$\hat{\tilde{a}}_{\mathbf{k}_1\lambda_1}\hat{\tilde{a}}_{\mathbf{k}_2\lambda_2} = \eta e^{2i\mathbf{k}_1\wedge\mathbf{k}_2}\hat{\tilde{a}}_{\mathbf{k}_2\lambda_2}\hat{\tilde{a}}_{\mathbf{k}_1\lambda_1}, \tag{6}$$

$$\hat{\tilde{a}}^\dagger_{\mathbf{k}_1\lambda_1}\hat{\tilde{a}}^\dagger_{\mathbf{k}_2\lambda_2} = \eta e^{2i\mathbf{k}_1\wedge\mathbf{k}_2}\hat{\tilde{a}}^\dagger_{\mathbf{k}_2\lambda_2}\hat{\tilde{a}}^\dagger_{\mathbf{k}_1\lambda_1}, \tag{7}$$

$$\hat{\tilde{a}}^\dagger_{\mathbf{k}_1\lambda_1}\hat{\tilde{a}}_{\mathbf{k}_2\lambda_2} = \frac{1}{\eta}(\hat{\tilde{a}}_{\mathbf{k}_2\lambda_2}\hat{\tilde{a}}^\dagger_{\mathbf{k}_1\lambda_1} - (2\pi)^3\delta(\mathbf{k}_2 - \mathbf{k}_1)\delta_{\lambda_1\lambda_2}) \times e^{2i\mathbf{k}_2\wedge\mathbf{k}_1}, \tag{8}$$

where $\hbar\mathbf{k}$ represents the spatial components of momentum and the $\{\lambda\}$'s represent the spin degree of freedom. Incidentally, this structure of twisted (anti)commutation relation can also be obtained in the nonrelativistic domain, if one considers the twisted action of the Galileo group [4]. As has been shown in the paper just mentioned, one has to just restore the symmetry corresponding to spatial rotation [SO(3)] here, as the Galileo boost generator does not get affected by the twist. Also the wedge symbol ($\wedge$) inserted between $\mathbf{k}_1$ and $\mathbf{k}_2$ denotes

$$\mathbf{k}_1 \wedge \mathbf{k}_2 = \tfrac{1}{2}k_{1i}\theta^{ij}k_{2j}. \tag{9}$$

Finally, $\eta = \pm 1$ as dictated by Boson or Fermion statistics. As we are interested in fermionic systems $\eta = -1$. These operators go over to usual Bose/Fermi ladder operators in the limit $\theta \to 0$: $\hat{\tilde{a}} \to \hat{a}$. In fact, it has been shown in [6] that $\hat{\tilde{a}}$ and $\hat{a}$ are related as $\hat{\tilde{a}} = e^{-((i/2)p_\mu\theta^{\mu\nu}P_\nu)}\hat{a}$.

III. A BRIEF REVIEW OF TWISTED (ANTI) COMMUTATION RELATION

It has been shown earlier that in a nonrelativistic system, the presence of spatial noncommutativity given by θ^{ij} spoils the SO(3) rotational symmetry. This can be seen easily from the fact that the vector $\boldsymbol{\theta} = \{\theta_i\}$, dual to the second rank antisymmetric tensor θ^{ij}

$$\theta_i = \tfrac{1}{2}\epsilon_{ijk}\theta^{jk}, \tag{10}$$

fixes a direction in the 3-dimensional space. Thus the only surviving symmetry is the SO(2) rotation around the $\boldsymbol{\theta}$ direction. However, applying the Drinfeld twist to the Hopf algebra, the entire SO(3) symmetry can be restored. This is done by considering a twisted co-product, $\Delta_\theta(M_{ij})$, corresponding to SO(3) generators M_{ij}, with action defined as

$$\Delta_\theta(M_{ij}) = F_\theta^{-1}\Delta_0(M_{ij})F_\theta, \tag{11}$$

where

$$\Delta_0(M_{ij}) = M_{ij} \bigotimes 1 + 1 \bigotimes M_{ij} \tag{12}$$

is the co-product in the commutative case ($\theta = 0$) and $F_\theta = e^{-(i/2)\theta^{ij}P_i \bigotimes P_j}$ is the twist operator. The corresponding co-product for the rotation group SO(3) in the commutative and noncommutative cases are

$$\Delta_0(g) = g \bigotimes g, \qquad \Delta_\theta(g) = F_\theta^{-1}\Delta_0(g)F_\theta. \tag{13}$$

Note that the star product, defined in (2), can now be written alternatively as

$$(\alpha * \beta)(x) = m_\theta(\alpha \bigotimes \beta) = m_0(F_\theta \alpha \bigotimes \beta), \tag{14}$$

where m_0 represents the composition of a pair of functions in the commutative ($\theta = 0$) case, which is nothing but pointwise multiplication of these functions, i.e., $\alpha(x)\beta(x)$.

It can be easily seen at this stage that the usual projection operator for a two-particle system, $P_0 = \frac{1}{2}(1 \pm \tau_0)$, involving the flip map $\tau_0(\alpha \bigotimes \beta) = \beta \bigotimes \alpha$, and which projects onto the symmetric (antisymmetric) subspace describing bosonic (fermionic) statistics, does no longer commute with Δ_θ: $[\Delta_\theta, P_0] \neq 0$. This implies that one must twist the flip map as

$$\tau_\theta = F_\theta^{-1}\tau_0 F_\theta = F_\theta^{-2}\tau_0, \tag{15}$$

so that the corresponding projection operator $P_\theta = \frac{1}{2}(1 \pm \tau_\theta) = F_\theta^{-1}P_0F_\theta$ commutes with Δ_θ

$$[\Delta_\theta, P_\theta] = 0. \tag{16}$$

This ensures that the new statistics, i.e. the flip operator τ_θ, remains superselected, i.e. it commutes with all Galilean generators, and defines twisted bosons or fermions. We now provide a heuristic argument how the algebra involving the corresponding creation and annihilation operators also get modified.

To begin with, let us apply the twisted projection operator P_θ on the tensor product of two momentum eigenstates $|k> \bigotimes |l>$ and make use of (15):

$$P_\theta(|k> \bigotimes |l>) = \tfrac{1}{2}(|k> \bigotimes |l> + \eta F_\theta^{-2}|l> \bigotimes |k>). \tag{17}$$

Up to an overall phase, this can be rewritten in a symmetric or antisymmetric form as in the $\theta = 0$ case

$$P_\theta(|k> \bigotimes |l>) = e^{ik\wedge l}[\tfrac{1}{2}(|k, l \gg + \eta|l, k \gg)], \qquad |k, l \gg \equiv e^{-ik\wedge l}|k> \bigotimes |l>. \tag{18}$$

Now identifying $\hat{\tilde{a}}_l^\dagger \hat{\tilde{a}}_k^\dagger|0> = P_\theta(|k> \bigotimes |l>)$ it easily follows that $\hat{\tilde{a}}_k^\dagger \hat{\tilde{a}}_l^\dagger = \eta e^{2ik\wedge l}\hat{\tilde{a}}_l^\dagger \hat{\tilde{a}}_k^\dagger$. The other phase $e^{-2ik\wedge l}$, occurring in $\hat{\tilde{a}}_k^\dagger \hat{\tilde{a}}_l$ of Eq. (8), can be easily understood from the fact that the annihilation operator $\hat{\tilde{a}}_l$ is associated with a momentum $(-l)$, in contrast to the operator $\hat{\tilde{a}}_l^\dagger$, for which the associated momentum is $(+l)$.

Despite the appearance of these twisted commutation/anticommutation relations (6)–(8), the implementation of braided twisted symmetry, as prescribed in [2], can restore the usual symmetry/antisymmetry of the wave function under permutation, so that they become usual bosons/fermions.

To see this more clearly, note that the conventional twisted symmetric/antisymmetric 2-particle wave function is obtained as

$$\Psi_T(\mathbf{x}, \mathbf{y}) \equiv (<\mathbf{x}| \bigotimes <\mathbf{y}|)P_\theta(|\psi> \bigotimes |\phi>) = \tfrac{1}{2}(\psi(\mathbf{x})\phi(\mathbf{y}) \pm F_\theta^{-2}\phi(\mathbf{x})\psi(\mathbf{y})), \tag{19}$$

which clearly has eigenvalues ± 1 under τ_θ (15) and the usual symmetric/antisymmetric form is restored in the commutative $\theta \to 0$ limit. Now the implementation of braided twisted symmetry of Ref. [2] is tantamount to a $*$-composing pair of functions even at distinct points. This implies that we have to replace the above as

$$\Psi_{\mathrm{BT}}(\mathbf{x}, \mathbf{y}) = \tfrac{1}{2}(\psi(\mathbf{x}) * \phi(\mathbf{y}) \pm F_\theta^{-2}\phi(\mathbf{x}) * \psi(\mathbf{y})) \tag{20}$$

to obtain a(n) (anti)symmetric 2-particle wave function under braided twisted symmetry. Now using the fact that

$$\psi(\mathbf{x}) * \phi(\mathbf{y}) = F_\theta\psi(\mathbf{x})\phi(\mathbf{y}) = e^{(i/2)\theta^{ij}\partial_i^x\partial_j^y}\psi(\mathbf{x})\phi(\mathbf{y}) \tag{21}$$

one obtains for (20)

$$\Psi_{\mathrm{BT}}(\mathbf{x}, \mathbf{y}) = \tfrac{1}{2}(e^{(i/2)\theta^{ij}\partial_i^x\partial_j^y}\psi(\mathbf{x})\phi(\mathbf{y}) \pm e^{(i/2)\theta^{ij}\partial_i^y\partial_j^x}\psi(\mathbf{y})\phi(\mathbf{x})). \tag{22}$$

Clearly, under $\mathbf{x} \leftrightarrow \mathbf{y}$ interchange, this is symmetric/antisymmetric in the ordinary ($\theta = 0$) sense. One therefore does not expect to have any effect of noncommutativity, stemming solely from twisted statistics, if braided twisted symmetry is implemented.

IV. NONCOMMUTATIVE DEGENERATE ELECTRON GAS

To begin with, the Hamiltonian operator for the electronic system in the second-quantized formulation can be written as

$$\hat{H}_{\rm el} = \int d^3x \hat{\psi}^\dagger(\mathbf{x}) * \hat{T}\, \hat{\psi}(\mathbf{x}) + \frac{1}{2}\iint d^3x d^3y \hat{\psi}^\dagger(\mathbf{x}) * \hat{\psi}^\dagger(\mathbf{y}) * V(\mathbf{x},\mathbf{y}) * \hat{\psi}(\mathbf{y}) * \hat{\psi}(\mathbf{x}). \tag{23}$$

Before we explain the terms and notations, first note that in order to get a well-defined thermodynamic limit, we introduce a box of volume $V = L^3$, containing N particles, as a regulator so that when $V \to \infty$ and $N \to \infty$, the density (N/V) is held fixed. Upon introducing the box, we have to specify boundary conditions. We take these to be periodic, which implies compactifying to the three torus T^3. The field operators

$$\hat{\psi}(\mathbf{x}) = \sum_{\mathbf{k},\lambda} \psi_{\mathbf{k}\lambda}(\mathbf{x}) \hat{\tilde{a}}_{\mathbf{k}\lambda}, \tag{24}$$

occurring in (23) then act on the Fock-space of states obtained by superposing annihilation operators in terms of the single-particle wave functions

$$\psi_{\mathbf{k}\lambda}(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{x}} \eta_\lambda \tag{25}$$

with $\mathbf{k}$ taking the discrete set of values $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$ and $\mathbf{n}$ represents a triplet of positive or negative integers, arising from the imposition of periodic boundary conditions in the box, and η_λ stands for two spin functions,

$$\eta_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \eta_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Furthermore, the arguments $\mathbf{x}$ are now the usual c-numbered coordinates, so that they compose through $*$ products. In this form the action is Hermitian, as follows from the fact that $(f * g)^\dagger = g^\dagger * f^\dagger$ [7]. Since we have compactified to T^3 the star product can be taken as the one appropriate to T^3 [7,12]. However, since the original twisted (anti)commutation relations (6)–(8) were obtained in [6] for noncompact spaces like R^3, we prefer to define the star product on R^3. This can be done by periodic continuation of the functions over R^3 and defining the star product on the continued functions.

The kinetic and interaction energy operators are denoted by $\hat{T}(\mathbf{x})$ and $V(\mathbf{x},\mathbf{y})$, respectively:

$$\hat{T} = \frac{1}{2m}\hat{\mathbf{p}}^2 = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2, \tag{26}$$

$$V(\mathbf{x},\mathbf{y}) = \frac{e^2}{2}\frac{e^{-\mu|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}. \tag{27}$$

This has to be augmented by the Hamiltonian of the positive inert background having the particle density $n(\mathbf{x})$

$$H_b = \frac{e^2}{2}\int d^3x d^3x' \frac{n(x)n(x')e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{28}$$

and the Hamiltonian

$$H_{\text{el-}b} = -e^2 \sum_{i=1}^{N} \int d^3x \frac{n(x)e^{-\mu|\mathbf{x}-\mathbf{x}_i|}}{|\mathbf{x}-\mathbf{x}_i|}, \tag{29}$$

representing the energy between the electrons and positive background.[1] With this the total Hamiltonian for the system becomes

$$H = H_{\rm el} + H_b + H_{\text{el-}b}. \tag{30}$$

Here we have introduced a screened Coulomb potential through an exponentially damping factor having a regulator μ of dimension $[L]^{-1}$. This renders the integrals appearing in H_b and $H_{\text{el-}b}$ finite in the thermodynamic limit. It should, however, be kept in mind that the limit $\mu \to 0$ should be taken after the $L \to \infty$ limit, so that one can ensure $\mu^{-1} \ll L$ at each step of the computation. This also facilitates the shifting of origin of integration, as dictated by convenience, as the surface terms are negligibly small in this limit.

Taking the distribution to be uniform $n(\mathbf{x}) = N/V$, we can now compute the pair of nondynamical (inert) terms

$$H_b = \frac{1}{2}e^2\frac{N^2}{V}\frac{4\pi}{\mu^2}, \tag{31}$$

$$H_{\text{el-}b} = -e^2\frac{N^2}{V}\frac{4\pi}{\mu^2}, \tag{32}$$

where we have made use of the translational invariance. As expected, these are c-number terms. The total Hamiltonian (30) thus reduces to

$$H = -\frac{1}{2}e^2\frac{N^2}{V}\frac{4\pi}{\mu^2} + H_{\rm el} \tag{33}$$

with all the interesting physical effects being buried in $H_{\rm el}$.

We therefore turn our attention to $H_{\rm el}$. To begin with, let us consider the kinetic term in (23) first. This can be simplified as

$$\int d^3x \hat{\psi}^\dagger(\mathbf{x}) * \hat{T}\, \hat{\psi}(\mathbf{x}) = \sum_{\mathbf{k}\lambda\mathbf{k}'\lambda'} \frac{\hbar^2\mathbf{k}^2}{2m} \hat{\tilde{a}}^\dagger_{\mathbf{k}'\lambda'} \hat{\tilde{a}}_{\mathbf{k}\lambda} \int d^3x \psi^\dagger_{\mathbf{k}'\lambda'}(\mathbf{x}) * \psi_{\mathbf{k}\lambda}(\mathbf{x}). \tag{34}$$

[1] Strictly speaking a $*$ operator has to be inserted in H_b (28) and $H_{\text{el-}b}$ (29) also, if the braided twisted symmetry [2] is implemented. However, this has no effect in our case, as we shall be concerned here only with a uniform distribution of particle density.

Using the definition of the $*$ product given in (2), we can easily see that this brings in an exponential factor $e^{-i\mathbf{k}'\wedge\mathbf{k}}$, as the $*$ operator is sandwiched between a pair of plane-wave states of the form (25). However, the integration yields $\delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}$, thereby reducing this θ-dependent exponential factor to identity. As far as this kinetic term is concerned, this therefore yields the same result as in the commutative ($\theta = 0$) case:

$$\int d^3x\hat{\psi}^\dagger(\mathbf{x}) * \hat{T}\,\hat{\psi}(\mathbf{x}) = \sum_{\mathbf{k}\lambda}\frac{\hbar^2k^2}{2m}\hat{\tilde{a}}^\dagger_{\mathbf{k}\lambda}\hat{\tilde{a}}_{\mathbf{k}\lambda}, \qquad (35)$$

which can be interpreted as the kinetic energy of each mode multiplied by the corresponding number operator. Almost the same thing also happens for the potential energy term as all θ-dependent exponential factors coming from the Moyal star product reduces to the identity here as well. Seemingly, the noncommutative structure of space, encoded in the Moyal star product, plays no role, at least to first order in perturbation theory. This was also found in [3]. There the effects of noncommutativity only showed up in second order, which is not unexpected as the exchange correlations of a noncommutative theory will generically be different from those of a commutative theory. However, as we proceed to show now, a θ dependence, stemming from the twisted anticommutation relation, survives to give a NC modification to the ground-state energy already to first order in perturbation theory.

To this end consider the potential term in (23). First of all this requires a proper interpretation. Given that $\mathbf{x}$ and $\mathbf{y}$ represent the position coordinates of two distinct particles in a pair, we must have $[x^\mu, y^\nu]_* = 0$ if conventional twisted symmetry is implemented but will be nonzero if braided twisted symmetry is implemented [2]. Let us consider the conventional twisted symmetry first. Consequently, in this case, only functions involving coordinates of a single particle will compose through the $*$ product. This implies that the appropriate way of writing this potential term is

$$\int d^3x\hat{\psi}^\dagger(\mathbf{x}) * \left(\int d^3y\hat{\psi}^\dagger(\mathbf{y}) * V(\mathbf{x},\mathbf{y}) * \hat{\psi}(\mathbf{y})\right) * \hat{\psi}(\mathbf{x}), \qquad (36)$$

where the $\mathbf{y}$ integral is performed first, after composing the three functions of $\mathbf{y}$, with $\mathbf{x}$ being held fixed, to yield a function of $\mathbf{x}$. With this interpretation, we can write

$$\iint d^3xd^3y\hat{\psi}^\dagger(\mathbf{x}) * \hat{\psi}^\dagger(\mathbf{y}) * V(\mathbf{x},\mathbf{y}) * \hat{\psi}(\mathbf{y}) * \hat{\psi}(\mathbf{x}) = \frac{1}{2}\sum_{\{k\}\{\lambda\}}\hat{\tilde{a}}^\dagger_{\mathbf{k}_1\lambda_1}\hat{\tilde{a}}^\dagger_{\mathbf{k}_2\lambda_2}\hat{\tilde{a}}_{\mathbf{k}_4\lambda_4}\hat{\tilde{a}}_{\mathbf{k}_3\lambda_3}\langle\mathbf{k}_1\lambda_1\mathbf{k}_2\lambda_2|V|\mathbf{k}_3\lambda_3\mathbf{k}_4\lambda_4\rangle, \qquad (37)$$

where the matrix element is given by

$$\langle\mathbf{k}_1\lambda_1\mathbf{k}_2\lambda_2|V|\mathbf{k}_3\lambda_3\mathbf{k}_4\lambda_4\rangle = \frac{e^2}{2V^2}\int d^3xe^{-i\mathbf{k}_1.\mathbf{x}}\eta_{\lambda_1}(1)^\dagger * \left(\int d^3ye^{-i\mathbf{k}_2.\mathbf{y}}\eta_{\lambda_2}(2)^\dagger * \frac{e^{-\mu|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} * e^{i\mathbf{k}_4.\mathbf{y}}\eta_{\lambda_4}(2)\right) * e^{i\mathbf{k}_3.\mathbf{x}}\eta_{\lambda_3(1)}. \qquad (38)$$

We now make use of the identity[2]

$$\frac{e^{-\mu|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = \frac{4\pi}{(2\pi)^3}\int\frac{d^3k}{\mu^2+k^2}e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})} \qquad (39)$$

and then convert the integration over " k" to a sum by the standard replacement $\int d^3k \to (\frac{2\pi}{L})^3\sum_{\mathbf{k}}$, so that the parity with other momentum variables can be restored. We can simplify this matrix element to get

$$\langle\mathbf{k}_1\lambda_1\mathbf{k}_2\lambda_2|V|\mathbf{k}_3\lambda_3\mathbf{k}_4\lambda_4\rangle = \frac{2\pi e^2}{V}\sum_{\mathbf{k}}\frac{1}{\mu^2+k^2}e^{-i\mathbf{k}_1\wedge\mathbf{k}_3}e^{-i\mathbf{k}_2\wedge\mathbf{k}_4}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\delta_{\mathbf{k},(\mathbf{k}_1-\mathbf{k}_3)} \times \delta_{\mathbf{k},(\mathbf{k}_4-\mathbf{k}_2)}, \qquad (40)$$

where $\mathbf{k}$ clearly gets restricted to $\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_3 = \mathbf{k}_4 - \mathbf{k}_2$, so that $\mathbf{k}$, occurring in the Fourier transform of the screened Coulomb potential, can be identified with the momentum transfer. On the other hand, the Kronecker deltas involving momenta levels also enforce the momentum conservation. The potential energy operator then becomes

$$\frac{2\pi e^2}{V}\sum_{\{\mathbf{k}\}\{\lambda\}}\frac{1}{\mu^2+k^2}e^{-i\mathbf{k}_1\wedge\mathbf{k}_3}e^{-i\mathbf{k}_2\wedge\mathbf{k}_4}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\delta_{\mathbf{k},(\mathbf{k}_1-\mathbf{k}_3)} \times \delta_{\mathbf{k},(\mathbf{k}_4-\mathbf{k}_2)}\tilde{a}^\dagger_{\mathbf{k}_1\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_2\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_4}\tilde{a}_{\mathbf{k}_3\lambda_3}, \qquad (41)$$

where $\{\mathbf{k}\}$ represents the set $(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ on which the summation has to be performed. The total Hamiltonian can now be written as

$$\hat{H} = -\frac{1}{2}e^2\frac{N^2}{V}\frac{4\pi}{\mu^2} + \sum_{\mathbf{k}\lambda}\frac{\hbar^2k^2}{2m}\tilde{a}^\dagger_{\mathbf{k}\lambda}\tilde{a}_{\mathbf{k}\lambda} + \frac{2\pi e^2}{V}\sum_{\{\mathbf{k}\}\{\lambda\}}\frac{1}{\mu^2+k^2}e^{-i\mathbf{k}_1\wedge\mathbf{k}_3}e^{-i\mathbf{k}_2\wedge\mathbf{k}_4}\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \times \delta_{\mathbf{k},(\mathbf{k}_1-\mathbf{k}_3)}\delta_{\mathbf{k},(\mathbf{k}_4-\mathbf{k}_2)}\tilde{a}^\dagger_{\mathbf{k}_1\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_2\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_4}\tilde{a}_{\mathbf{k}_3\lambda_3}. \qquad (42)$$

[2]This can easily be seen by noting that the screened Coulomb potential is a Green's function of the Laplacian augmented by a mass term: $(-\nabla^2 + \mu^2)(\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}) = 4\pi\delta^3(\mathbf{x})$.

The electrical neutrality of the system makes it possible to eliminate μ from the Hamiltonian. To that end, consider the last term of the Eq. (42), which can now be recast in terms of the momentum variables $\mathbf{k}$, $\mathbf{k}_3$, and $\mathbf{k}_4$ by eliminating $\mathbf{k}_1$ and $\mathbf{k}_2$ through the Kronecker deltas as

$$\frac{2\pi e^2}{V}\sum_{[\mathbf{k}\mathbf{k}_3\mathbf{k}_4\lambda_1\lambda_2]}\frac{1}{\mu^2+k^2}e^{-i\mathbf{k}\wedge\mathbf{k}_3}e^{i\mathbf{k}\wedge\mathbf{k}_4}\tilde{a}^\dagger_{\mathbf{k}+\mathbf{k}_3,\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_4-\mathbf{k},\lambda_2}\times\tilde{a}_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_3\lambda_1}, \tag{43}$$

where the two spin summations have been evaluated with the Kronecker deltas. At this stage, it is convenient to separate the above expression into two terms, referring to $\mathbf{k}\neq 0$ and $\mathbf{k}=0$, respectively,

$$\frac{2\pi e^2}{V}\sum'_{[\mathbf{k}\mathbf{k}_3\mathbf{k}_4\lambda_1\lambda_2]}\frac{1}{\mu^2+k^2}e^{-i\mathbf{k}\wedge\mathbf{k}_3}e^{i\mathbf{k}\wedge\mathbf{k}_4}\tilde{a}^\dagger_{\mathbf{k}+\mathbf{k}_3,\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_4-\mathbf{k},\lambda_2}\times\tilde{a}_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_3\lambda_1}+\frac{2\pi e^2}{V}\sum_{[\mathbf{k}_3\mathbf{k}_4\lambda_1\lambda_2]}\frac{1}{\mu^2}\tilde{a}^\dagger_{\mathbf{k}_3\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_3\lambda_1} \tag{44}$$

where the prime on the first summation means the $\mathbf{k}=0$ term is omitted. The second term may be rewritten with the discrete version of the twisted anticommutation relation given by (6) and (8). In this discrete version, (6) remains unaffected, while one has to just replace $(2\pi)^3\delta^3(\mathbf{k}_1-\mathbf{k}_2)\to\delta_{\mathbf{k}_1\mathbf{k}_2}$ in (8). Using this the second term takes the form

$$\begin{aligned}&\frac{2\pi e^2}{V}\sum_{[\mathbf{k}_3\mathbf{k}_4\lambda_1\lambda_2]}\frac{1}{\mu^2}\frac{1}{\eta}(\tilde{a}^\dagger_{\mathbf{k}_3\lambda_1}\tilde{a}_{\mathbf{k}_3\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_2}\\&\quad-\tilde{a}^\dagger_{\mathbf{k}_3\lambda_1}\delta_{\mathbf{k}_3\mathbf{k}_4}\delta_{\lambda_1\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_2})\eta e^{2i\mathbf{k}_3\wedge\mathbf{k}_4}e^{2i\mathbf{k}_4\wedge\mathbf{k}_3}\\&=\frac{2\pi e^2}{V}\frac{4\pi}{\mu^2}(\hat{N}^2-\hat{N}),\end{aligned} \tag{45}$$

where

$$\begin{aligned}\hat{N}&=\int d^3x\hat{n}(x)=\sum_r\hat{\tilde{a}}^\dagger_r\hat{\tilde{a}}_r=\sum_r\hat{a}^\dagger_r\hat{a}_r=\sum_r\hat{n}_r\\&=\int d^3x\hat{\psi}^\dagger(x)\hat{\psi}(x)\end{aligned} \tag{46}$$

represents the number operator. Here too the effect of noncommutativity disappears.[3] Since we always deal with states of fixed N, the operator $\hat{N}$ may be replaced by its eigenvalue N, thereby yielding a c-number contribution to the Hamiltonian

$$\frac{N^2e^2}{V}\frac{2\pi}{\mu^2}-\frac{Ne^2}{V}\frac{2\pi}{\mu^2}. \tag{47}$$

[3]As was mentioned earlier, it was shown in Ref. [6] that the twisted and untwisted ladder operators are related by a unitary transformation, thereby preserving the number operators.

The first term of the above expression cancels the first term of the Hamiltonian in (33). The second term represents an energy $-2\pi e^2(V\mu^2)^{-1}$ per particle and vanishes in the proper physical limit: first $L\to\infty$ and $\mu\to 0$ as discussed earlier. Thus the explicit μ^{-2} divergence cancels identically in the thermodynamic limit, which reflects the electrical neutrality of the total system; furthermore, it is now permissible to set $\mu=0$ in the first term of (43), since the resulting expression is well defined. We therefore obtain the final Hamiltonian for a bulk electron gas in a uniform positive background

$$\hat{H}=\sum_{\mathbf{k}\lambda}\frac{\hbar^2k^2}{2m}\tilde{a}^\dagger_{\mathbf{k}\lambda}\tilde{a}_{\mathbf{k}\lambda}+\frac{2\pi e^2}{V}\sum'_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4}\sum_{\lambda_1\lambda_2}\frac{1}{k^2}e^{-i\mathbf{k}\wedge\mathbf{k}_3}e^{i\mathbf{k}\wedge\mathbf{k}_4}\times\tilde{a}^\dagger_{\mathbf{k}+\mathbf{k}_3,\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_4-\mathbf{k},\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_3\lambda_1}. \tag{48}$$

We can now introduce the interparticle spacing r_0, defined through $V\equiv\frac{4}{3}\pi r_0^3N$ and measure it in units of the Bohr radius $a_0=\frac{\hbar^2}{me^2}$, thus yielding a dimensionless variable $r_s=\frac{r_0}{a_0}$.

With r_0 as the unit of length, we define the following quantities:

$$\begin{aligned}\bar{V}&=r_0^{-3}V,&\bar{\mathbf{k}}&=r_0\mathbf{k},\\\bar{\mathbf{k}}_3&=r_0\mathbf{k}_3,&\bar{\mathbf{k}}_4&=r_0\mathbf{k}_4,\end{aligned} \tag{49}$$

and thus we obtain the following dimensionless form of the Hamiltonian operator:

$$\hat{H}=\frac{e^2}{a_0r_s^2}\bigg(\sum_{\bar{\mathbf{k}}\lambda}\frac{1}{2}\bar{\mathbf{k}}^2\tilde{a}^\dagger_{\bar{\mathbf{k}}\lambda}\tilde{a}_{\bar{\mathbf{k}}\lambda}+\frac{2\pi r_s}{\bar{V}}\sum'_{[\bar{\mathbf{k}}\bar{\mathbf{k}}_3\bar{\mathbf{k}}_4]}\sum_{\lambda_1\lambda_2}\frac{1}{\bar{k}^2}\tilde{a}^\dagger_{\bar{\mathbf{k}}+\bar{\mathbf{k}}_3,\lambda_1}\times\tilde{a}^\dagger_{\bar{\mathbf{k}}_4-\bar{\mathbf{k}},\lambda_2}\tilde{a}_{\bar{\mathbf{k}}_4\lambda_2}\tilde{a}_{\bar{\mathbf{k}}_3\lambda_1}e^{-ir_0^2\bar{\mathbf{k}}\wedge(\bar{\mathbf{k}}_3-\bar{\mathbf{k}}_4)}\bigg). \tag{50}$$

In the limit $r_s\to 0$, corresponding to the high-density limit ($r_0\to 0$), the potential energy becomes a small perturbation even though it is neither weak nor short ranged, while the leading term comes from the kinetic-energy term. Thus the leading term in the interaction energy of a high-density electron gas can be obtained with first-order perturbation theory. In the high-density limit, we can therefore separate the original dimensional form of the Hamiltonian (48) into two parts:

$$\hat{H}_0=\sum_{\mathbf{k}\lambda}\frac{\hbar^2k^2}{2m}\tilde{a}^\dagger_{\mathbf{k}\lambda}\tilde{a}_{\mathbf{k}\lambda}, \tag{51}$$

$$\hat{H}_1=\frac{2\pi e^2}{V}\sum'_{[\mathbf{k}\mathbf{k}_3\mathbf{k}_4]}\sum_{\lambda_1\lambda_2}\frac{1}{k^2}\tilde{a}^\dagger_{\mathbf{k}+\mathbf{k}_3,\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_4-\mathbf{k},\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_3\lambda_1}\times e^{-i\mathbf{k}\wedge(\mathbf{k}_3-\mathbf{k}_4)}, \tag{52}$$

where $\hat{H}_0$ is the unperturbed Hamiltonian, representing a noninteracting Fermi system, and $\hat{H}_1$ is the (small) perturbation. Correspondingly, the ground-state energy E may be written as $E^0 + E^1 + \ldots$, where E^0 is the ground-state energy of a free Fermi gas, while E^1 is the first-order energy shift. Since the Pauli exclusion principle allows only two fermions in each momentum eigenstate (this also holds in noncommutative space),[4] one with spin up and one with spin down, the normalized ground-state $|F>$ is obtained by filling the momentum states up to a maximum value, the Fermi momentum $p_F = \hbar k_F$. In the thermodynamic limit one can again replace summation by integration, so that k_F can be determined by computing the expectation value of the number operator in the ground-state $|F>$

$$N = \langle F|\hat{N}|F\rangle = \sum_{\mathbf{k}\lambda}\langle F|\hat{n}_{\mathbf{k}\lambda}|F\rangle = \sum_{\mathbf{k}\lambda}\Theta(k_F - k) = (3\pi^2)^{-1}Vk_F^3 = N, \tag{53}$$

where $\Theta(x)$ denotes the step function

$$\Theta(x) = 1 \quad \text{for } x \geq 0, \qquad \Theta(x) = 0 \quad \text{for } x < 0. \tag{54}$$

Equivalently k_F can be expressed as

$$k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} = \left(\frac{9\pi}{4}\right)^{1/3} r_0^{-1}. \tag{55}$$

Now the expectation value of $\hat{H}_0$ may be evaluated as

$$\begin{aligned} E^{(0)} &= \langle F|\hat{H}_0|F\rangle = \frac{\hbar^2}{2m}\sum_{\mathbf{k}\lambda}k^2\langle F|\hat{n}_{\mathbf{k}\lambda}|F\rangle \\ &= \frac{\hbar^2}{2m}\sum_{\mathbf{k}\lambda}k^2\Theta(k_F - k) \\ &= \frac{\hbar^2}{2m}\sum_{\lambda}\frac{V}{(2\pi)^3}\int d^3k k^2\Theta(k_F - k) = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}N \\ &= \frac{e^2}{2a_0}\frac{N}{r_s^2}\frac{3}{5}\left(\frac{9\pi}{4}\right)^{2/3}. \end{aligned} \tag{56}$$

In a free Fermi gas, the ground-state energy per particle $E^{(0)}/N$ is $\frac{3}{5}$ of the Fermi energy $\epsilon_F^0 = \hbar^2 k_F^2/2m$. We now compute the shift in the ground-state energy to first order in perturbation theory:

$$\begin{aligned} E^{(1)} &= \langle F|\hat{H}_1|F\rangle \\ &= \frac{2\pi e^2}{V}\sum_{\mathbf{k}\mathbf{k}_3\mathbf{k}_4}{}' \sum_{\lambda_1\lambda_2}\frac{1}{k^2} \\ &\quad \times \langle F|\tilde{a}^\dagger_{\mathbf{k}+\mathbf{k}_3,\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_4-\mathbf{k},\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_3\lambda_1}e^{-i\mathbf{k}\wedge(\mathbf{k}_3-\mathbf{k}_4)}|F\rangle. \end{aligned} \tag{57}$$

For a nonzero matrix element, all the states $|\mathbf{k}_4\lambda_2>$, $|\mathbf{k}_3\lambda_1>$, $|\mathbf{k}+\mathbf{k}_3, \lambda_1>$, $|\mathbf{k}_4 - \mathbf{k}, \lambda_2>$ should be occupied in $|F>$. Also, since $\mathbf{k} = 0$ is excluded, we must have the pairing

$$\mathbf{k} + \mathbf{k}_3, \qquad \lambda_1 = \mathbf{k}_4, \lambda_2 \tag{58}$$

so that the matrix element becomes

$$\delta_{\mathbf{k}+\mathbf{k}_3,\mathbf{k}_4}\delta_{\lambda_1\lambda_2}\langle F|\tilde{a}^\dagger_{\mathbf{k}+\mathbf{k}_3,\lambda_1}\tilde{a}^\dagger_{\mathbf{k}_4-\mathbf{k},\lambda_2}\tilde{a}_{\mathbf{k}_4\lambda_2}\tilde{a}_{\mathbf{k}_3\lambda_1}|F\rangle.$$

The exponential factor in the previous expression for $E^{(1)}$ (57) again becomes the identity as $\mathbf{k} + \mathbf{k}_3 = \mathbf{k}_4 (i.e. \mathbf{k} = \mathbf{k}_4 - \mathbf{k}_3)$. Finally, using the twisted commutation relation given by (6) and (8) and also considering the fact that the term $\mathbf{k} = 0$ is excluded from the sum, the above expression becomes

$$\begin{aligned} &-\delta_{\mathbf{k}+\mathbf{k}_3,\mathbf{k}_4}\delta_{\lambda_1\lambda_2}\langle F|\hat{n}_{\mathbf{k}+\mathbf{k}_3,\lambda_1}\hat{n}_{\mathbf{k}_3\lambda_1}e^{-2i\mathbf{k}_3\wedge(\mathbf{k}+\mathbf{k}_3)}|F\rangle \\ &\quad = -\delta_{\mathbf{k}+\mathbf{k}_3,\mathbf{k}_4}\delta_{\lambda_1\lambda_2}\Theta(k_F - |\mathbf{k}+\mathbf{k}_3|)\Theta(k_F - k_3)e^{-2i\mathbf{k}_3\wedge\mathbf{k}}. \end{aligned} \tag{59}$$

This clearly demonstrates that the effect of noncommutativity in the conventional twisted framework appears only in this phase factor, which on turn stems solely from the twisted anticommutation relations (6)–(8). At this stage, we can therefore check the status of this observation, if braided twisted symmetry [2] is implemented. As mentioned earlier, this is tantamount to defining a star product also between functions at different points $\mathbf{x}$ and $\mathbf{y}$. Thus we have to reevaluate the matrix element $\langle \mathbf{k}_1\lambda_1, \mathbf{k}_2\lambda_2|V|\mathbf{k}_3\lambda_3, \mathbf{k}_4\lambda_4\rangle$ in (38). To really zoom in to the points of deviation, note that this matrix element involves the following integral:

$$I = \int d^3x d^3y e^{-i\mathbf{k}_1\cdot\mathbf{x}} * e^{-i\mathbf{k}_2\cdot\mathbf{y}} * e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} * e^{i\mathbf{k}_4\cdot\mathbf{y}} * e^{i\mathbf{k}_3\cdot\mathbf{x}}.$$

First note that we can still write $e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} = e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{y}}$. To group together all the exponential factors involving $\mathbf{y}$, we will have to shift the factor $e^{i\mathbf{k}\cdot\mathbf{x}}$ to the right of $e^{i\mathbf{k}_4\cdot\mathbf{y}}$, thereby generating an additional phase $e^{-2i\mathbf{k}\wedge\mathbf{k}_4}$:

$$I = \int d^3x\left(e^{-i\mathbf{k}_1\cdot\mathbf{x}} * \left(\left(\int d^3y e^{-i\mathbf{k}_2\cdot\mathbf{y}} * e^{-i\mathbf{k}\cdot\mathbf{y}} * e^{i\mathbf{k}_4\cdot\mathbf{y}}\right) * e^{i\mathbf{k}\cdot\mathbf{x}} * e^{i\mathbf{k}_3\cdot\mathbf{x}}\right)e^{-2i\mathbf{k}\wedge\mathbf{k}_4}.$$

This phase will then combine with the one in (59), which arises from twisted anti(commutation) relations to yield

[4] As one can see that the Pauli's exclusion principle remains valid in momentum space, as can be seen from (6) that in the case of $\eta \to -1$, $\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} = 0$. Consequently, the concepts like Fermi label etc. remain valid in momentum space even in the presence of noncommutativity. However, this does not remain true in configuration space, as $\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})$ does not vanish necessarily as follows from the relation $\psi(x)\psi(y) = \int d^2x' d^2y' \Gamma_\theta(x, y, x', y')\psi(y')\psi(x'); \theta \neq 0$ [4]. Indeed it has been shown there, that a repulsive statistical potential between a pair of identical (twisted) fermions can saturate to a finite value at coincident points, thereby violating the Pauli principle in real space.

$e^{-2i\mathbf{k}\wedge(\mathbf{k}_4-\mathbf{k}_3)}$ as the final phase factor. This, however, becomes the identity in view of the above mentioned identification $\mathbf{k}=\mathbf{k}_4-\mathbf{k}_3$ (58), leaving no trace of the noncommutativity in the first order of perturbation theory, if the braided twisted symmetry is considered. On the other hand, the nontrivial phase occurring in (59) shows that there exists a nontrivial effect of noncommutativity if the conventional twisted symmetry is considered. We now carry out the corresponding shift in the ground-state energy $E^{(1)}$ (57) using (59) to get

$$\begin{aligned} E^{(1)} &= -\frac{2\pi e^2}{V}\sum_{[\mathbf{k}\mathbf{k}_3\lambda_1]}\frac{1}{k^2}\Theta(k_F-|\mathbf{k}+\mathbf{k}_3|)\Theta(k_F-k_3) \\ &\quad\times e^{-2i\mathbf{k}_3\wedge\mathbf{k}} \\ &= -\frac{2\pi e^2}{V}\sum_{\lambda_1}\left(\frac{V}{(2\pi)^3}\right)^2\int d^3k d^3k_3\frac{1}{k^2} \\ &\quad\times\Theta(k_F-|\mathbf{k}+\mathbf{k}_3|)\Theta(k_F-k_3)e^{-2i\mathbf{k}_3\wedge\mathbf{k}}. \end{aligned}$$

Note that we have again included the $\mathbf{k}=0$ term at the level of integration; as being a set of measure zero, it does not contribute anything to the integral. Now taking the summation over λ_1 the above expression becomes

$$\begin{aligned} E^{(1)} &= -\frac{4\pi e^2 V}{(2\pi)^6}\int d^3k d^3k_3 k^{-2}\Theta(k_F-|\mathbf{k}+\mathbf{k}_3|) \\ &\quad\times\Theta(k_F-k_3)e^{-2i\mathbf{k}_3\wedge\mathbf{k}}. \end{aligned}$$

It is convenient to change variables: $\mathbf{k}_3\rightarrow\mathbf{k}_3+\frac{\mathbf{k}}{2}$, which reduces the above expression into the symmetrical form

$$\begin{aligned} E^{(1)} &= -\frac{4\pi e^2 V}{(2\pi)^6}\int\frac{d^3k}{k^2}d^3k_3\Theta\left(k_F-\left|\mathbf{k}_3+\frac{1}{2}\mathbf{k}\right|\right) \\ &\quad\times\Theta\left(k_F-\left|\mathbf{k}_3-\frac{1}{2}\mathbf{k}\right|\right)e^{-ik_{3i}\theta^{ij}k_j}. \end{aligned} \tag{60}$$

The computation of this integral is quite involved. In the following we present the computation for which only $\theta^{12}\neq 0$. Although a bit lengthy the integral can then be computed in a straightforward manner (see appendix) to yield

$$E^{(1)} = -\frac{16\pi^3 Ve^2}{(2\pi)^6}k_F^4\sum_{r=0}^{\infty}\frac{(-1)^r(k_F^2\theta)^{2r}4^{-1-r}}{(1+r)(1+2r)(\Gamma(\frac{3}{2}+r))^2}. \tag{61}$$

This is an infinite series in θ. Now as θ is of the order of the Plank-length scale, hence very small, we sum the series up to the first contributing term in θ, neglecting all other higher-order terms. This gives

$$\begin{aligned} E^{(1)} &= -\frac{16\pi^3 Ve^2}{(2\pi)^6}k_F^4\left(1-\frac{(k_F^2\theta)^2}{54}\right) \\ &= -\frac{e^2}{2a_0}\left[\left(\frac{9\pi}{4}\right)^{1/3}\frac{3}{2\pi}\frac{1}{r_s}-\left(\frac{9\pi}{256}\right)^{2/3}\left(\frac{\theta}{a_0^2}\right)^2\frac{1}{r_s^5}\right]. \end{aligned} \tag{62}$$

Thus the ground-state energy per particle in the high-density limit is given approximately by

$$\frac{E}{N}\underset{r_s\rightarrow 0}{=}\frac{e^2}{2a_0}\left[\frac{2.21}{r_s^2}-\frac{0.916}{r_s}+\left(\frac{\theta}{a_0^2}\right)^2\frac{0.2302}{r_s^5}+\ldots\right]. \tag{63}$$

As can be seen from the above expression, the ground-state energy has θ corrections and deviates from the commutative result. The ground-state energy per particle is increased by an order of $(\sim(\frac{\theta}{a_0^2})^2)$, which is a dimensionless quantity. Note, however, that this θ dependency has its roots in the twisted anticommutation relations, rather than the noncommutative structure of space *per se*, as the θ dependency arising from the star product, which encodes the noncommutativity of space, always dropped out. This is true to first order in perturbation theory. However, as was shown in [3] both of these effects plays a role to higher order in perturbation theory.

It is also quite clear from the above expression that when r_s becomes very small, i.e. in the high-density limit, the effect of spatial noncommutativity on the ground-state energy of the degenerate electron gas becomes more significant. Taking the noncommutativity parameter (θ) to be of the order of plank length, it can be seen that the effect of noncommutativity in any terrestrial experiments may not be found, but in the case of astrophysical objects, where matter density is very high, the effect of noncommutativity may be found by experiments.

V. CONCLUSIONS

We have shown that, in contrast to [3], the ground-state energy of a noncommutative degenerate electron gas in a neutralizing background acquires noncommutative corrections to first order in perturbation theory if the conventional twisted symmetry is implemented. But no noncommutative correction is obtained if the braided twisted symmetry is implemented. These corrections, in the former case, arise from the modified two-particle correlations resulting from the twisted anticommutation relations. All θ dependency arising directly from the star product dropped out to this order in perturbation theory. On the other hand, in the case of braided twisted symmetry, the nontrivial correction arising out of twisted anticommutation relations is neutralized by the $*$ composition of exponential functions at different points. Our final observation is that any observable effect, as far as the energy shift is concerned, can only be obtained when the system is extremely dense—a situation that can presumably arise only in an astrophysical setting. These experiments, we feel, can determine whether the usual twisted symmetry or braided twisted symmetry is realized in nature.

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APPENDIX

Here we provide some of the basic steps leading to the evaluation of the integral

$$I = \int \frac{d^3k}{k^2} d^3k_3 \Theta\left(k_F - \left|\mathbf{k}_3 + \frac{1}{2}\mathbf{k}\right|\right) \times \Theta\left(k_F - \left|\mathbf{k}_3 - \frac{1}{2}\mathbf{k}\right|\right) e^{-ik_{3i}\theta^{ij}k_j}. \tag{A1}$$

These Θ functions indicate that the exponential function has to be integrated first over the overlapping region "R" of the two spheres of equal radii k_F centered at $\pm(\frac{1}{2}\mathbf{k})$ with respect to the variable $\mathbf{k}_3$ holding $\mathbf{k}$ fixed. The resultant function of $\mathbf{k}$ is then integrated over the solid sphere $|\mathbf{k}| \leq 2k_F$. Considering the region $R_+(k_{3z} > 0)$ initially, we can write the corresponding integral over k_3 with proper limit as

$$L_+ = \int_{R_+} d^3k_3 e^{-ik_{3i}\alpha^i} \quad \text{where } \alpha^i = \theta^{ij}k_j = \int_0^{(k_F-(k/2))} dz e^{-i\alpha^3 z} \int_{-\sqrt{(k_F^2-(z+(k/2))^2)}}^{\sqrt{(k_F^2-(z+(k/2))^2)}} dy e^{-i\alpha^2 y} \int_{-\sqrt{(k_F^2-(z+(k/2))^2-y^2)}}^{\sqrt{(k_F^2-(z+(k/2))^2-y^2)}} dx e^{-i\alpha^1 x}.$$

Here we have substituted $k_{3x} = x$, $k_{3y} = y$, and $k_{3z} = z$ for convenience.

Upon simplification, this takes the following form:

$$L_+ = \frac{2}{\alpha^1} \int_0^{(k_F-(k/2))} dz e^{-i\alpha^3 z} \int_{-p}^{+p} dy e^{-i\alpha^2 y} \sin(\alpha^1\sqrt{p^2 - y^2}) \quad \text{where } p = \sqrt{k_F^2 - \left(z + \frac{k}{2}\right)^2}.$$

Substituting the above expression for L_+ in (62), one gets for the corresponding I_+

$$I_+ \equiv \int \frac{d^3k}{k^2} L_+ = \int \frac{d^3k}{k^2}\left[\frac{4}{\alpha^1}\int_0^{(k_F-(k/2))} dz \int_0^p dy \cos(\alpha^2 y)\sin(\alpha^1\sqrt{p^2-y^2})\right],$$

where we have taken $\theta^{13} = \theta^{23} = 0$, by orienting the 3-axis in the $\mathbf{k}$ frame in the direction of $\boldsymbol{\theta}$, where $\theta_i = \frac{1}{2}\epsilon_{ijk}\theta_{jk}$ [see Eq. [10]] so that the only surviving components of θ^{ij} are $\theta^{12} = -\theta^{21} = \theta$, and consequently $\alpha^3 = 0$ and $\alpha^1 = \theta k_2$ and $\alpha^2 = -\theta k_1$. The above integral can now be expressed in terms of Bessel functions [13]

$$J_\nu(z) = \frac{z^\nu}{2^\nu}\sum_{r=0}^{\infty}(-1)^r \frac{z^{2r}}{2^{2r} r! \Gamma(\nu + r + 1)}$$

as

$$I_+ = 2\pi \int \frac{d^3k}{k^2}\int_0^{(k_F-(k/2))} dz \frac{p}{\sqrt{(\alpha^1)^2 + (\alpha^2)^2}} \times J_1(p\sqrt{(\alpha^1)^2+(\alpha^2)^2}).$$

Now we can integrate term by term by expanding the series of Bessel functions. By making use of the integral

$$\int_0^{(k_F-(k/2))} dz\left(k_F^2 - \left(z+\frac{k}{2}\right)^2\right)^{r+1} = (k_F)^{(2r+3)}\sum_{j=0}^{r+1}\frac{r+1C_j(-1)^j}{2j+1}\left(1 - \left(\frac{k}{2k_F}\right)^{(2j+1)}\right),$$

one gets after a lengthy but straightforward computation

$$I_+ = -k_F^4 \sum_{r=0}^{\infty}\frac{(-1)^r(k_F^2\theta)^{2r}}{r!(r+1)!(2r+1)!}\sum_{j=0}^{r+1}\frac{r+1C_j(-1)^j}{(r+j+1)} \times \sum_{l=0}^{r}\frac{rC_l(-1)^l}{(2l+1)}.$$

The last two factors involving series summation can now be carried out exactly to get

$$\sum_{l=0}^{r}\frac{rC_l(-1)^l}{(2l+1)} = \frac{\sqrt{\pi}r!}{2\Gamma(\frac{3}{2}+r)},$$

$$\sum_{j=0}^{r+1}\frac{r+1C_j(-1)^j}{(r+j+1)} = \frac{4^{(-1-r)}\sqrt{\pi}r!}{\Gamma(\frac{3}{2}+r)}.$$

With this I_+ takes the following form:

$$I_+ = 4\pi^2 k_F^4 \sum_{r=0}^{\infty} \frac{(-1)^r (k_F^2\theta)^{2r} 4^{-1-r}}{2(1+r)(1+2r)(\Gamma(\frac{3}{2}+r))^2}. \tag{A2}$$

A similar result is obtained for I_-, when one considers the region R_- corresponding to $k_{3z} < 0$. Consequently one has $I = 2I_+ = 2I_-$. Substituting this back in (60), one gets the desired expression of $E^{(1)}$.

[1] J. Govaerts and F. G. Scholtz, J. Phys. A **40**, 12415 (2007).
[2] G. Fiore and J. Wess, Phys. Rev. D **75**, 105022 (2007).
[3] F. S. Bemfica and H. O. Girotti, J. Phys. A **38**, L539 (2005).
[4] B. Chakraborty, S. Gangopadhyay, A. G. Hazra, and F. G. Scholtz, J. Phys. A **39**, 9557 (2006).
[5] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, Phys. Lett. B **604**, 98 (2004).
[6] A. P. Balachandran, G. Mangano, A. Pinzul, and S. Vaidya, Int. J. Mod. Phys. A **21**, 3111 (2006).
[7] R. J. Szabo, Phys. Rep. **378**, 207 (2003); Int. J. Mod. Phys. A **19**, 1837 (2004).
[8] Tureanu, Phys. Lett. B **638**, 296 (2006); A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. Qureshi, and S. Vaidya, Phys. Rev. D **75**, 045009 (2007).
[9] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (Dover, New York, 2003).
[10] R. Estrada, J. M. Gracia-Bondia, and J. C. Varilly, J. Math. Phys. (N.Y.) **30**, 2789 (1989).
[11] J. M. Gracia-Bondia and J. C. Varilly, J. Math. Phys. (N.Y.) **29**, 869 (1988).
[12] M. Rieffel, Contemp Math. **105**, 191 (1990).
[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1980).