

Skyrmion on a three-cylinder

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The class of static, spherically symmetric, and finite energy hedgehog solutions in the SU(2) Skyrme model is examined on a metric three-cylinder. The exact analytic shape function of the 1-Skyrmion is found. It can be expressed via elliptic integrals. Its energy is calculated, and its stability with respect to radial and spherically symmetric deformations is analyzed. No other topologically nontrivial solutions belonging to this class are possible on the three-cylinder.

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I. INTRODUCTION AND MOTIVATION

We discuss static and spherically symmetric “hedgehog” fields in the SU(2) Skyrme model on a metric three-cylinder ($\mathbb{R} \times \mathbb{S}^2$).¹ A static and spherically symmetric spacetime line element with $\mathbb{R} \times \mathbb{S}^2$ as hypersurfaces of constant time reads

$$ds^2 = -dt^2 + L^2(d\psi^2 + d\Omega^2(\theta, \phi)),$$

$$d\Omega^2(\theta, \phi) = d\theta^2 + \sin^2\theta d\phi^2.$$

Coordinates θ and ϕ are standard spherical angles and $\psi \in (-\infty, +\infty)$.² The positive scale factor L can be interpreted as the three-cylinder’s radius, but its role here is to provide the Skyrme model with additional length scale that can be compared with the characteristic soliton size. This simple field-theoretical setup leads to a single equation for the Skyrme hedgehog that can be solved exactly in three spatial dimensions. The analogous solution on the metric three-sphere (\mathbb{S}^3) can be found only by numerical integration.

By considering fields with appropriate asymptotics, $\mathbb{R} \times \mathbb{S}^2$ (with attached two “points at infinity”) may be topologically identified with \mathbb{S}^3 . In contrast to the \mathbb{S}^3 case, the metric geometry of $\mathbb{R} \times \mathbb{S}^2$ is not isotropic—the sectional Gauss curvature of $\mathbb{R} \times \mathbb{S}^2$ is direction dependent. Nevertheless, these two geometries are conformally identical:

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¹Throughout this paper we distinguish the notion of “metric three-cylinder” from that of “topological three-cylinder.” We use $\mathbb{R} \times \mathbb{S}^2$ as shorthand for metric three-cylinder with line element $d\psi^2 + d\theta^2 + \sin^2\theta d\phi^2$. The same remark applies to a three-sphere— \mathbb{S}^3 is shorthand used in this paper for metric three-sphere with standard line element $d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)$.

²More generally, we could also set $-f^2(\psi)dt^2$ with any f in the line element, but the rate of clocks is irrelevant for static field configurations since the kinetic term of such fields vanishes.

$$d\psi^2 + d\Omega^2(\theta, \phi) = \frac{d\chi^2 + \sin^2\chi d\Omega^2(\theta, \phi)}{\sin^2\chi},$$

$$\chi = 2 \arctan(e^\psi) \in (0, \pi)$$

(length scales on $\mathbb{R} \times \mathbb{S}^2$ and on \mathbb{S}^3 are assumed equal). Coordinate χ is the standard third angle on \mathbb{S}^3 . This conformal identification of metric geometries of $\mathbb{R} \times \mathbb{S}^2$ and \mathbb{S}^3 allows for identification of the translation symmetry Killing vector ∂_ψ on $\mathbb{R} \times \mathbb{S}^2$ with the conformal symmetry Killing vector $\sin\chi\partial_\chi$ on \mathbb{S}^3 .

The conformal symmetry Killing vector is related to bifurcations of static spherically symmetric hedgehogs on \mathbb{S}^3 at characteristic critical length scales. In particular, at the critical radius $L_c = \sqrt{2}$, the 1-Skyrmion (\mathbf{S}_1) on \mathbb{S}^3 separates from the identity map $\mathbf{H}_1: \mathbb{S}^3 \rightarrow \text{SU}(2) \sim \mathbb{S}^3$ by a conformal deformation of this map [1], namely, in the limit $L \searrow L_c$, the shape function of \mathbf{S}_1 reads $\chi \rightarrow e^{\epsilon \sin\chi} \chi = \chi + \epsilon \sin\chi + o(\epsilon)$, $\epsilon \propto \sqrt{L - L_c}$.³ Note that $\sin\chi$ is also the eigenfunction corresponding to the lowest eigenvalue of the Hessian evaluated at \mathbf{H}_1 [3]. The eigenvalue vanishes at $L = L_c$ and is negative for $L > L_c$ and positive for $L < L_c$. The conformal deformation of \mathbf{H}_1 generated by $\sin\chi\partial_\chi$ is thus energetically preferable for $L > L_c$. For such $L > L_c$ it is the \mathbf{S}_1 which minimizes the energy functional.

On $\mathbb{R} \times \mathbb{S}^2$ the situation is qualitatively different. A deformation of a solution generated by ∂_ψ is simply a translation which is also a symmetry of the space. This deformation leaves energy of the 1-Skyrmion on $\mathbb{R} \times \mathbb{S}^2$ and the boundary conditions at infinity unchanged. For that reason one should not expect bifurcations similar to that on \mathbb{S}^3 to occur on $\mathbb{R} \times \mathbb{S}^2$. Moreover, by contrast with the \mathbb{S}^3 case, one may also expect that the 1-Skyrmion on $\mathbb{R} \times \mathbb{S}^2$

³The energy and shape function of \mathbf{S}_1 can be found with arbitrary accuracy by perturbations (however, the resulting series seems to have finite radius of convergence in $L - L_c$, presumably bounded by $\sqrt{2}$; therefore one cannot reconstruct the energy of flat space 1-Skyrmion by taking the limit $L \rightarrow \infty$). To the leading order this perturbative calculation confirms the discussed above conformal deformation of the identity solution [2].

should uniformly tend to a harmonic map of the related sigma model on $\mathbb{R} \times \mathbb{S}^2$.

Translation symmetry of Skyrme equations on $\mathbb{R} \times \mathbb{S}^2$ enables one to use the conserved current associated with this symmetry as the first integral of these equations. In particular, it can be shown that static, spherically symmetric, finite energy and topologically nontrivial hedgehog solutions with topological charge other than ± 1 cannot exist on $\mathbb{R} \times \mathbb{S}^2$. One can also easily examine properties of the 1-Skyrmion, calculate its energy, and analyze its stability. Remarkably, all this can be done without knowing the exact form of the 1-Skyrmion's shape function. Next, we construct an approximate formula for the 1-Skyrmion's shape function, and finally we find its exact form.

II. BASIC EQUATIONS

The standardized form of Lagrangian density of SU(2)-valued Skyrme field U in spacetime with metric signature $(-, +, +, +)$ is [4]

$$\begin{aligned} \mathcal{L}[U] &= \sqrt{-g} \left(\frac{1}{2} \text{Tr}(K_\mu K^\mu) \right. \\ &\quad \left. + \frac{1}{16} \text{Tr}([K_\mu, K_\nu][K^\mu, K^\nu]) \right), \\ K_\mu &\equiv U^{-1} \partial_\mu U. \end{aligned}$$

This Lagrangian simply generalizes to other matrix-valued fields. In the construction of this density, the principle of minimal coupling of matter with gravitation is assumed—metric tensor couples with matter fields in the same way it does in Minkowski spacetime with arbitrary coordinate system. The Skyrme field behaves as a scalar with respect to spacetime transformations. The first summand in the Lagrangian is known as the sigma term. The second term has the opposite scaling with the length scale. It was introduced by Skyrme [5] to ensure the existence of solitons among static solutions.

We use (metrical and topological) isomorphism of the SU(2) group and the unit three-sphere: $\mathbb{S}^3 \ni (\Psi, \Theta, \Phi) \rightarrow U = \exp(i\Psi\vec{\sigma} \circ \vec{n}(\Theta, \Phi)) \in \text{SU}(2)$, where $\vec{n}(\Theta, \Phi)$ is a unit direction determined by spherical angles (Θ, Φ) : $\vec{n} = [\sin\Theta \cos\Phi, \sin\Theta \sin\Phi, \cos\Theta]$; $\vec{\sigma}$ is a vector of Pauli matrices; and Ψ is the third angle on \mathbb{S}^3 .

As general relativity theory teaches us, the stress tensor of matter fields is proportional to the variational derivative of the action functional of these fields with respect to the spacetime metric tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}}{\delta g^{\mu\nu}}, \quad \mathcal{S}[U] = \int \mathcal{L}[U].$$

For the Skyrme Lagrangian

$$\begin{aligned} T_{\mu\nu} &= \frac{\mathcal{L}[U]}{\sqrt{-g}} g_{\mu\nu} - \text{Tr}(K_\mu K_\nu) \\ &\quad - \frac{1}{4} \text{Tr}([K_\mu, K_\alpha][K_\nu, K^\alpha]). \end{aligned}$$

The energy functional of static and spherically symmetric Skyrme hedgehogs on $\mathbb{R} \times \mathbb{S}^2$, reduces to⁴ $[F' \equiv \frac{dF(\psi)}{d\psi}]$

$$\begin{aligned} E[F] &= 4\pi \int_{-\infty}^{+\infty} d\psi \left[L(2\sin^2 F + F'^2) \right. \\ &\quad \left. + \frac{1}{L} \sin^2 F (\sin^2 F + 2F'^2) \right]. \end{aligned} \quad (2.1)$$

We have used the hedgehog ansatz $\Psi = F(\psi)$, $\Theta \equiv \theta$, and $\Phi \equiv \phi$.

To ensure finiteness of energy $E[F]$ we impose the following asymptotic (finite energy) condition: $\sin(F) = o(|\psi|^{-1/2})$ as $|\psi| \rightarrow \infty$. Under this condition $U \rightarrow \pm \mathbf{1}$ at infinity. Solutions with such asymptotics are characterized by the topological charge $Q = (F(+\infty) - F(-\infty))/\pi$, and their energies are bounded from below by a positive number $12\pi^2|Q|$ [this is known as the Faddeev–Bogomol'nyi bound, which is universal for the SU(2) Skyrme model in three spatial dimensions; cf. [6] for a proof]. We employ this distinguished $12\pi^2$ value as the unit of energy.

Critical points of energy functional (2.1) are solutions of the following equation:

$$\left(1 + \frac{2}{L^2} \sin^2 F\right) F'' + \frac{F'^2}{L^2} \sin 2F - \left(1 + \frac{1}{L^2} \sin^2 F\right) \sin 2F = 0. \quad (2.2)$$

A Noether current associated with the Killing vector ∂_ψ of the translation symmetry of $\mathbb{R} \times \mathbb{S}^2$ is $j^\mu = T^\mu{}_\nu (\partial_\psi)^\nu$. Since the Lagrangian is also ψ -translation invariant, this current is conserved for solutions. The conservation equation $0 = \nabla_\mu j^\mu \equiv \partial_\mu (\sqrt{-g} j^\mu) / \sqrt{-g}$ yields on integration with respect to ψ

$$(L^2 + 2\sin^2 F) F'^2 - \sin^2 F (2L^2 + \sin^2 F) = C, \quad (2.3)$$

which is the first integral of Eq. (2.2). The finite energy condition implies the integration constant C in (2.3) must be zero. Hence the first integral of Eq. (2.2) for a finite energy solution reads

$$F'^2 = \frac{2L^2 + \sin^2 F}{L^2 + 2\sin^2 F} \sin^2 F. \quad (2.4)$$

⁴If ξ^μ is a Killing vector and if $T^\mu{}_\nu$ is divergence-free, then the threeform $\omega = -\frac{1}{3!} T^\mu{}_\nu \xi^\nu \sqrt{-g} \epsilon_{\mu\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$ is closed; that is, $d\omega = 0$. In particular, integrated over $\mathbb{R} \times \mathbb{S}^2$ with $\xi^\mu \equiv (\partial_t)^\mu$, this form defines energy functional (2.1). Provided ω vanishes at spatial infinity sufficiently fast, this energy is conserved since $d\omega = 0$.

Inversely, for nonconstant F , on differentiating (2.3) with respect to ψ , we obtain (2.2).

Any nonconstant solution of (2.2) not satisfying (2.4) must be either divergent or oscillate around $\pi/2 \bmod \pi$, depending on the sign of C . Indeed, if $C > 0$ at some point, then $|F'|$ must be bounded from zero for all ψ . Consequently, F must be divergent as $|\psi|$ tends to ∞ . If $C < 0$ then (2.3) implies that $F'^2 \leq 2\sin^2 F - |C|L^{-2}$, and (2.2) implies that $F'' = c(F)\sin 2F$ for all ψ , where $c(F)$ is a strictly positive function. As so, F must stay strictly within the strip $(0, \pi) \bmod \pi$ oscillating around $\pi/2 \bmod \pi$. For such solutions energy defined in (2.1) cannot be finite.

III. PROPERTIES OF STATIC, SPHERICALLY SYMMETRIC, AND FINITE ENERGY HEDGEHOG SOLUTIONS

As we have seen above, the only solutions of Eq. (2.2) that have finite energy are solutions of Eq. (2.4). Without loss of generality we may assume that $0 < F < \pi$ at some point. Suppose that F leaves the strip $[0, \pi]$ at another point. Equation (2.4) then implies that F' vanishes at this point. According to uniqueness theorems applied to Eq. (2.2) the only solution of this initial problem is the vacuum solution $\sin F \equiv 0$. Therefore, $0 < F < \pi$ for all finite ψ . It also follows from (2.4) that F' is bounded from zero within this strip; thus F is monotonic and attains 0 or π only asymptotically. Now it is clear that this solution has unit topological charge. It also follows that no other static, spherically symmetric, and finite energy hedgehog solution with larger than unit topological charge can exist on $\mathbb{R} \times \mathbb{S}^2$.

Within the class of static, spherically symmetric, and finite energy hedgehog solutions, we may restrict our attention to considering only the representative solution for which $F(0) = \pi/2$ and $F'(0) > 0$ [then $F(-\infty) = 0$ and $F(+\infty) = \pi$]. We shall refer to the class of solutions as the 1-Skyrmion on metric three-cylinder.

It follows from the first integral (2.4) that

$$\frac{1}{2}\sin^2 F < \frac{2L^2 + 1}{L^2 + 2}\sin^2 F \leq F'^2 \leq 2\sin^2 F.$$

This implies that the graph of the 1-Skyrmion's shape function, passing through the point $F(0) = \pi/2$, is contained within the region bounded by graphs of the limiting profiles

$$2 \arctan(e^{\psi/\sqrt{2}}) < F < 2 \arctan(e^{\sqrt{2}\psi}). \quad (3.1)$$

The first is attained uniformly at $L \rightarrow 0$, whereas the second one at $L \rightarrow \infty$. This observation suggests that we can assume the following test function

$$\tilde{F} = 2 \arctan(e^{G(L)\psi}) \quad (3.2)$$

to approximate the exact profile of the 1-Skyrmion. The

optimum value of $G(L)$ can be determined at a given L by finding the minimum energy of the test function. Hence $G(L)$ must satisfy the condition $d_G E[\tilde{F}] = 0$, which gives

$$G(L) = \sqrt{\frac{2 + 6L^2}{4 + 3L^2}},$$

$$E[\tilde{F}] = \frac{16\pi\sqrt{2}}{3L} \sqrt{(4 + 3L^2)(1 + 3L^2)}.$$

The energy of \tilde{F} with this $G(L)$ attains its global minimum $\frac{4\sqrt{6}}{3\pi} \times 12\pi^2$ at $L = \sqrt{2/3}$.

IV. STABILITY ANALYSIS OF THE 1-SKYRMION

Let \mathcal{H} denote the Hilbert space of test functions that are both \mathcal{C}^1 and normalizable to $(4\pi L^3)^{-1}$ on \mathbb{R} .⁵ Such functions must vanish at infinity faster than $|\psi|^{-1/2}$. Let $F(\psi)$ be the exact shape function of the 1-Skyrmion and $h \in \mathcal{H}$. The energy functional (2.1) evaluated for the test function $F_\epsilon = F + \epsilon h$, when expanded as a power series in ϵ , reads

$$E[F + \epsilon h] = E[F] + \epsilon \delta_F E[F](h) + \epsilon^2 \delta_F^2 E[F](h, h') + o(\epsilon^2).$$

The first variation $\delta_F E[F](h)$ must vanish for any $h \in \mathcal{H}$ on account of the Euler-Lagrange equations. The Hessian $\delta_F^2 E[F](h, h')$ is a quadratic form in h and h' and provides an energy measure of a perturbation h . To find the lowest bound λ_g of the Hessian attained over \mathcal{H} one has to find a conditional minimum of the Hessian with the condition that $\int h^2 d\mathcal{V} = 1$ and with λ_g being the corresponding Lagrange multiplier. For h to be the minimum energy perturbation in \mathcal{H} , it is necessary that h be the solution of the following eigenvalue problem [7]

$$\frac{\delta(\delta_F^2 E[F](h, h'))}{\delta h(\psi)} = \lambda_g \frac{\delta(\int h^2 d\mathcal{V})}{\delta h(\psi)} = 2\lambda_g h(\psi), \quad h \in \mathcal{H}$$

corresponding to the lowest eigenvalue λ_g . This ordinary differential equation is linear in h , h' , and h'' . By substituting $h \equiv F'$, it can be verified that the variational derivative on the very left in the above equation vanishes identically, provided F satisfies (2.4).⁶ Thus F' is the eigenfunction of the Hessian corresponding to the eigenvalue $\lambda_g = 0$.

We have seen that the 1-Skyrmion's shape function F is monotonic and that $F' = 0$ only asymptotically. Thus F'

⁵Then \mathcal{H} extends to spherically symmetric functions normalizable to unity on $\mathbb{R} \times \mathbb{S}^2$ (with volume element $d\mathcal{V} = L^3 \sin\theta d\psi d\theta d\phi$).

⁶Actually, $\delta_F^2 E[F](h, h')$ vanishes for $h = F'$ by translation invariance of functional (2.1). For if $F_\epsilon(\psi) = F(\psi + \epsilon)$, then $E[F_\epsilon] = E[F + \epsilon F' + O(\epsilon^2)] = E[F] + \epsilon \delta_F E[F](F' + O(\epsilon)) + \epsilon^2 \delta_F^2 E[F](F', F'' + O(\epsilon)) + o(\epsilon^2)$. But for solutions $\delta_F E[F] \equiv 0$, hence, on expanding once again, $E[F_\epsilon] = E[F] + \epsilon^2 \delta_F^2 E[F](F', F'') + o(\epsilon^2)$. By translation invariance $E[F] = E[F_\epsilon]$ for any ϵ , thus $\delta_F^2 E[F](F', F'') = 0$.

has no nodes for finite ψ . It is known that the eigenfunction corresponding to the lowest eigenvalue has no internal nodes [7]. As so, we arrive at the conclusion that the 1-Skyrmion on $\mathbb{R} \times \mathbb{S}^2$ is (marginally) stable against radial and spherically symmetric deformations.

V. THE EXACT ANALYTIC FORMULA FOR THE 1-SKYRMION ON $\mathbb{R} \times \mathbb{S}^2$

With the initial condition $F(0) = \pi/2$ Eq. (2.4) can be rewritten as

$$\psi(F) = \int_{\pi/2}^F \frac{df}{\sin f} \sqrt{\frac{L^2 + 2\sin^2 f}{2L^2 + \sin^2 f}}$$

$$(0, \pi) \ni F \rightarrow \psi \in (-\infty, +\infty).$$

By substituting $z = \sin(\Phi(f))$, with Φ being related to f by

$$k\sqrt{2} \sin(\Phi(f)) = \sqrt{\frac{L^2 + 2\sin^2 f}{2L^2 + \sin^2 f}} \quad k = \sqrt{\frac{2 + L^2}{2 + 4L^2}},$$

this integral can be brought into the form containing canonical elliptic integrals of the first and of the third kind (see Appendix):

$$|\psi(\Phi(F))| = \frac{3\sqrt{2}}{4\sqrt{1 + 2L^2}} \int_{\sin(\Phi(F))}^1 \left(1 + \frac{1}{1 - 4k^2 z^2}\right) \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$

The definitions of k and Φ are correct since $1/2 \leq k < 1$ and $1/2 < 1/(2k) \leq \sin\Phi \leq 1$. We choose in this integral $\psi < 0$ for $0 < 2F < \pi$, and $\psi > 0$ for $\pi < 2F < 2\pi$. The resulting (although implicit) exact formula for the 1-Skyrmion on $\mathbb{R} \times \mathbb{S}^2$ reads

$$|\psi(F)| = \frac{3\sqrt{2}}{4\sqrt{1 + 2L^2}} \left(\mathbf{F}\left(\frac{\pi}{2}, k\right) - \mathbf{F}(\Phi(F), k) + \mathbf{\Pi}(\Phi(F), -4k^2, k) - \mathbf{\Pi}\left(\frac{\pi}{2}, -4k^2, k\right) \right). \quad (5.1)$$

VI. ENERGY OF THE 1-SKYRMION ON $\mathbb{R} \times \mathbb{S}^2$

The exact form of the 1-Skyrmion's shape function is not required to calculate its total energy. By expressing $d\psi$ by $(F')^{-1}dF$ and using the first integral (2.4) in (2.1), we obtain the following formula for the energy of the 1-Skyrmion:

$$E[F] = \frac{16\pi}{L} \int_0^{\pi/2} dF \sin F \sqrt{(2L^2 + \sin^2 F)(L^2 + 2\sin^2 F)}.$$

This integral can be equivalently rewritten as

$$E[F] = E(L) \equiv \frac{16\pi\sqrt{2}}{L} Q\sqrt{P} \int_0^{1/\sqrt{Q}} dz \sqrt{(1 - z^2)(1 - k^2 z^2)},$$

where we have substituted $\sqrt{Q}z(F) = \cos F$, $Q = 1 + L^2/2$, $P = 1 + 2L^2$, and $k = \sqrt{Q/P} < 1$ (k is the same as in Sec. V). This integral, in turn, can be expressed by means of standard elliptic integrals of the first and second kind:

$$E(L) = \frac{16\pi\sqrt{2}}{3L} (\sqrt{P}(P + Q)\mathbf{E}(k, \tilde{\Phi}) - \sqrt{P}(P - Q)\mathbf{F}(k, \tilde{\Phi}) + \sqrt{(P - 1)(Q - 1)}), \quad (6.1)$$

where $\tilde{\Phi} = \arcsin(1/\sqrt{Q})$.

Let us calculate the minimum energy of the 1-Skyrmion. The minimum is attained at a radius L_m defined by the equation $E'(L_m) = 0$. This equation cannot be solved by radicals. However, this equation is analytic and can be expanded about some point. We have already seen that the minimum radius should be close to $L = \sqrt{2/3}$. Therefore, we substitute $L_m = \sqrt{2/3} + \zeta$ and find the Taylor series expansion with respect to the unknown and small ζ . In the first order approximation we obtain

$$L_m = \frac{9\sqrt{42}(11\mathbf{E}(\pi/3, 2/\sqrt{7}) - 3\mathbf{F}(\pi/3, 2/\sqrt{7})) + 30\sqrt{2}}{\sqrt{7}(409\mathbf{E}(\pi/3, 2/\sqrt{7}) - 141\mathbf{F}(\pi/3, 2/\sqrt{7})) - 26\sqrt{3}} + o(\zeta) = 0.81509\dots$$

To find a better approximation we now discard only the $\mathcal{O}(\zeta^4)$ term in the expansion of $E'(\sqrt{2/3} + \zeta)$, obtaining a cubic polynomial in ζ . Cardano's formulas for the roots of the cubic polynomial lead to a monstrous expression for L_m . Therefore, here we show only the decimal expansion of L_m and the corresponding minimum of the 1-Skyrmion's energy:⁷

$$L_m = 0.8150941506\dots,$$

$$E(L_m) = 1.03576803116479882348\dots \times 12\pi^2. \quad (6.2)$$

All of these digits shown are exact and follow from the presented third order calculation. This statement may be verified by carrying out a similar fourth order calculation

⁷Note that $E(L_m)$ is very close to the energy per baryon for the cubic Skyrme crystal found numerically by relaxations to be $\approx 1.036 \times 12\pi^2$ [8] (no error of this value was given, however).

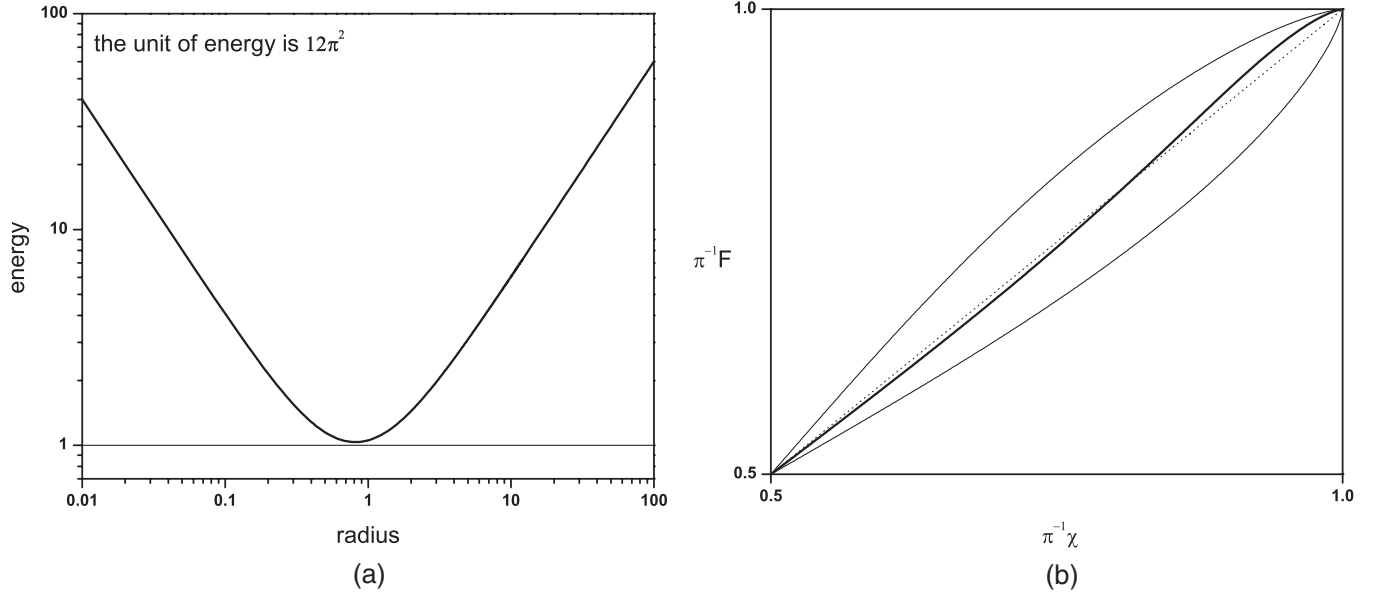


FIG. 1. (a) Energy of the 1-Skyrmion on a metric three-cylinder shown as a function of the scale factor L . The unit of energy is $12\pi^2$. (b) Thick line—the exact shape function (5.1) of the 1-Skyrmion with the minimum possible energy at $L = L_m$ [cf. Eq. (6.2)]; dotted line—the approximate shape function of the 1-Skyrmion defined in Eq. (3.2) [with $L = \sqrt{2/3}$ and $G(L) = 1$]; thin lines—the 1-Skyrmion in the limit $L \rightarrow 0$ (lower line) and in the limit $L \rightarrow \infty$ (upper line). The χ variable is related to the ψ “radial” coordinate by $\chi = 2 \arctan e^\psi$, and only the region $\psi > 0$ is shown.

(the roots of a fourth order polynomial can be still found by radicals).

VII. DISCUSSION

The energy diagram of the 1-Skyrmion on $\mathbb{R} \times \mathbb{S}^2$ is shown in Fig. 1(a). It is divergent both for $L \rightarrow 0$ and for $L \rightarrow \infty$ and has a single minimum. This diagram resembles qualitatively the behavior of the energy of the identity solution $2 \arctan e^\psi$ (\mathbf{H}_1) on \mathbb{S}^3 rather than that of the 1-Skyrmion (\mathbf{S}_1) on \mathbb{S}^3 , of which energy is finite in the limit of infinite radius.⁸ To understand qualitatively why it is so, it must be remembered that \mathbf{H}_1 on \mathbb{S}^3 becomes unstable as the radius passes through its critical value $L = \sqrt{2}$. The instability mode is associated with conformal deformation of \mathbf{H}_1 . Because of the instability \mathbf{H}_1 bifurcates at this radius, and \mathbf{S}_1 separates from it, remaining stable for all radii. The energy of \mathbf{S}_1 tends to a finite limit, whereas the energy of \mathbf{H}_1 diverges in the limit of infinite radius. On $\mathbb{R} \times \mathbb{S}^2$ there is no similar bifurcation—the 1-Skyrmion on $\mathbb{R} \times \mathbb{S}^2$ is always (marginally) stable. As we have seen, the lowest (zero) energy eigenmode is associated with ψ translations, which preserve energy on $\mathbb{R} \times \mathbb{S}^2$. Therefore, no other solution with lower energy and the same symmetry can appear by a bifurcation from the 1-Skyrmion on $\mathbb{R} \times$

⁸We use the same conformal identification $\chi \equiv 2 \arctan(e^\psi) \in (0, \pi)$ of metric geometries of \mathbb{S}^3 and of $\mathbb{R} \times \mathbb{S}^2$ as that defined in Sec. I (the spherical sections of constant χ on \mathbb{S}^3 and of constant ψ on $\mathbb{R} \times \mathbb{S}^2$ were identified).

\mathbb{S}^2 . In the limit of infinite radius the 1-Skyrmion on $\mathbb{R} \times \mathbb{S}^2$ tends to the harmonic map $2 \arctan \exp(\sqrt{2}\psi)$ on $\mathbb{R} \times \mathbb{S}^2$, which is (marginally) stable against radial and spherically symmetric perturbations.⁹ The energy of a harmonic map must diverge as $L \rightarrow \infty$ because the sigma term in the Skyrme Lagrangian scales as L and dominates the Skyrme term, which scales as L^{-1} .

The minimum energy (6.2) of the 1-Skyrmion on $\mathbb{R} \times \mathbb{S}^2$ is over 6 times closer to the Bogomol’nyi bound than the energy of the 1-Skyrmion on flat space, which is $\approx 1.23145 \times 12\pi^2$. We must remember that the Bogomol’nyi bound $12\pi^2$ in the SU(2) Skyrme model is saturated on the unit three-sphere.

The minimal energy of the approximated 1-Skyrmion profile (3.2) is only 0.37% more than the true minimum (6.2). The asymptotics of energies of the approximated and the exact 1-Skyrmion are the same:

⁹The harmonic map on metric $\mathbb{R} \times \mathbb{S}^2$ is $Y = 2 \arctan e^{\sqrt{2}\psi}$ and has energy $E = 16\pi\sqrt{2}$. It is the critical point of the sigma model energy functional on $\mathbb{R} \times \mathbb{S}^2$

$$4\pi \int_{-\infty}^{+\infty} d\psi (Y'^2 + 2\sin^2 Y)$$

(spherical symmetry and the hedgehog ansatz have been assumed). This map is the counterpart of the identity solution on \mathbb{S}^3 . The same reasoning as in the text leads to the conclusion that the map is the only finite energy solution with nonzero topological charge and that it is (marginally) stable against radial and spherically symmetric perturbations.

$$\lim_{L \rightarrow 0} \frac{E(L)}{\tilde{E}(L)} = 1 = \lim_{L \rightarrow \infty} \frac{E(L)}{\tilde{E}(L)}.$$

Also the limiting solutions (3.1) are correctly reproduced by this approximation at $L = 0$ and $L = \infty$. In this sense the profile (3.2) very well approximates the 1-Skyrmion solution on $\mathbb{R} \times \mathbb{S}^2$ (5.1). In Fig. 1(b) the exact shape function is compared with the approximated one at $L = L_m$ [cf. Equation (6.2)].

The metric geometry of $\mathbb{R} \times \mathbb{S}^2$ is conformally identical to that of \mathbb{S}^3 . The manifolds may be also topologically identified if fields with appropriate asymptotics are considered on these manifolds. We have shown that (up to symmetries) only one topologically nontrivial static, spherically symmetric, and finite energy hedgehog solution exists on $\mathbb{R} \times \mathbb{S}^2$ (it has unit topological charge). In the same topological sector on metric \mathbb{S}^3 , arbitrarily many solutions may exist, the number of which increases with the three-sphere's radius. The number of possible solutions of this kind is related to the number of SU(2) harmonic maps on these manifolds. On $\mathbb{R} \times \mathbb{S}^2$ only one harmonic map exists, whereas on \mathbb{S}^3 two countable families of harmonic maps exist [9]. Correspondingly, the structure of solutions on \mathbb{S}^3 is very rich and the number of possible

solutions grows with L [10], whereas on $\mathbb{R} \times \mathbb{S}^2$ the structure is very simple and L independent.

The structure of solutions of the Skyrme model depends on what kind of base space is considered. It is evident that it is not the topology of the space but its metrical properties that are important for this structure. It is also evident that this structure is affected by the number of harmonic maps possible on this space. It would be therefore interesting to analyze Skyrmions on a class of other spaces with the general line element $d\psi^2 + a^2(\psi)(d\theta^2 + \sin^2\theta d\phi^2)$ [the already studied cases include $a(\psi) \propto \psi$, $\sin\psi$, $\sinh\psi$, and at last $a(\psi) \propto 1$ in this paper] and to find out how this structure is related to the function $a(\psi)$.

APPENDIX: THREE FUNDAMENTAL ELLIPTIC FUNCTIONS

We used the following definitions of the standardized elliptic integrals ($k^2 < 1$) [11]:

- (1) $\mathbf{F}(\phi, k) = \int_0^{\sin\phi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$,
- (2) $\mathbf{E}(\phi, k) = \int_0^{\sin\phi} \sqrt{\frac{1-k^2x^2}{1-x^2}} dx$,
- (3) $\mathbf{\Pi}(\phi, \nu, k) = \int_0^{\sin\phi} \frac{dx}{(1+\nu x^2)\sqrt{(1-x^2)(1-k^2x^2)}}$.

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