Remarks on bell-shaped lumps: Stability and fermionic modes

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We consider nontopological, "bell-shaped" localized and regular solutions available in some 1 + 1-dimensional scalar field theories. Several properties of such solutions are studied, namely, their stability and the occurrence of fermion bound states in the background of a kink and a kink-antikink solution of the sine-Gordon model.

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I. INTRODUCTION

Nonlinear phenomena play an important role in many sectors of physics. The most widely known classical solutions obeying nonlinear equations are solitons and topological defects, see e.g. [1,2] for recent reviews. The stability of these objects is usually related to the nontrivial topology of the space of configurations where the fields of the underlying physical models take their values. Nontopological lumps are worth being studied as well. Here topological arguments cannot be invoked to show stability but they can turn out to be stable through the occurrence of conserved bosonic currents related to global symmetries of the underlying model. On the other hand, some models can admit a regular, localized, finite-energy, classical but unstable solution which can be relevant in some specific contexts. Sphalerons, which appear as unstable solutions in the electroweak model [3], are believed to play a role in the baryon asymmetry of the Universe [4] and the computation of their normal modes [5] is an important ingredient for the evaluation of the rate of baryon/lepton-number violating processes. Various extensions of the sphaleron solution have been emphasized. In a very recent paper, families of new axially symmetric solutions representing sphaleron-antisphaleron bound states have been constructed [6].

Among all known nonlinear lumps, the "kink solution" available in the 1 + 1-dimensional scalar field theory with a $\lambda\phi^4$ potential is probably the simplest and the most popular. Recently several authors have constructed different types of localized lump solutions in scalar field theories with specific families of the self-interaction potentials [7]. A special emphasis was set on "bell-shaped" lumps characterized by $\lim_{x\to\pm\infty}\phi(x) = 0$, contrasting with the traditional kink solutions. In particular, the authors of [7] exhibit a family of potential admitting a lump with the same profile as the celebrated Korteweg-de Vries solution. A long list of possible physical applications is given in [7]. Among them, we point out the possibility for these models to be applied in a brane world involving a single extra dimension and for the description of tachyonic states living on the brane (see e.g. [8] and references in this paper). For such applications, as well as for many others, the number of unstable modes of the classical solution constitutes an essential ingredient. The question of stability, which to our knowledge is not emphasized in [7], is considered in the second section of the present paper. The linear equation determining the normal modes turns out to be quasi-exactly-solvable [9]. The normalizable solutions of the equation can be obtained explicitly and one of the eigenvalues is negative, indicating an instability. The corresponding direction of instability can be expressed in terms of elementary functions.

Recently again, kink and antikink solutions were used in a different physical context [10]. In [11], the spectrum on the Dirac equation in the background of several solutions available in the $\lambda\phi^4$ model has been studied. The investigation of fermionic bound states in the background of classical bosonic fields (solitons or sphalerons) has a long history. The Dirac equation in the background of a linear cosmic string was studied in [12,13]. The existence of normalizable fermion zero mode in the background of a sphaleron was addressed in [14,15]. Further studies of fermionic bound states and level crossing were performed in [16]. The spectrum along the full noncontractible loop passing through the sphaleron was reported in [17].

In the third section of this paper, we reconsider the calculation of [11] by replacing the kink and kink-antikink backgrounds by their analogues available in the sine-Gordon model. The advantage of the sine-Gordon trigonometric potential over the quartic polynomial one is that the sine-Gordon model admits an explicit solution describing fully the kink-antikink interaction; as a consequence, the spectrum of the Dirac equation in this background can be computed over the full range of configurations, including the region of the parameters where kink and the antikink interact with each other. The qualitative properties of the analysis reported in [11] are confirmed with the sine-Gordon model for well separated lumps. However the solutions of the sine-Gordon equation allow for configurations where the two lumps are close to each other. The spectral analysis can be extended to this case as well. Our results demonstrate how fermion bound states can emerge from the continuum when the wall and the antiwall become well separated or, equivalently, how existing fermion bound states are absorbed into the continuum when the two lumps come close to each other. Some conclusions and perspective are mentioned in Sec. IV. For completeness, the definition of the Poschl-Teller equation is presented in the Appendix.

II. BELL-SHAPED LUMPS: STABILITY

In this section we study the stability of several new lump solutions which were presented recently in [7]. The models considered are 1 + 1-dimensional scalar field theory of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{2.1}$$

with suitable potential $V(\phi)$. For static fields, the classical equation has the form $d_x^2\phi = \frac{dV}{d\phi}$. The linear stability of a solution, say $\phi_c(x)$, of this equation can be studied by diagonalizing the quadratic part of the Lagrangian in a perturbation around ϕ_c. Concretely, one has to compute the eigenvalues allowing for normalizable eigenfunctions of the spectral equation

$$\left(-\frac{d^2}{dx^2} + \frac{d^2V}{d\phi^2}\bigg|_{\phi=\phi_c}\right)\eta = \omega^2\eta, \qquad \phi(t,x) = \phi_c(x) + e^{i\omega t}\eta(x). \tag{2.2}$$

Negative eigenvectors $\omega^2 < 0$ reveal the existence of instabilities of the classical solution ϕ_c.

Case 1: ϕ^4 potential—The case of a quartic potential is well known but we mention it for completeness and for the purpose of comparison of the eigenmodes with the cases mentioned next. The potential and the corresponding kink-solution read

$$V(\phi) = \frac{1}{2}(\phi^2 - 1)^2, \qquad \phi_c(x) = \pm\tanh(x) \tag{2.3}$$

and the stability equation corresponds to the Poschl-Teller equation (see Appendix) with $N = 2$ and $\omega^2 = \omega_p^2 + 4$. The eigenvalues are then $\omega^2 = 0, 3, 4$. It is well known that the $\lambda\phi^4$ kink has no negative mode and is stable. The corresponding eigenfunctions are

$$\eta_0 = \frac{1}{\cosh(x)^2} = -\frac{d\phi_c}{dx}, \qquad \eta_3 = \frac{\sinh(x)}{\cosh^2(x)}, \qquad \eta_4 = \tanh^2(x) - \frac{1}{3}. \tag{2.4}$$

The zero mode corresponds to infinitesimal translations of the classical solution ϕ_c in the space variable.

Case 2: Inverted ϕ^4 potential—This case corresponds to a quartic potential with inverted sign. The potential and the corresponding lump are given by

$$V(\phi) = \frac{1}{2}\phi^2(1 - \phi^2), \qquad \phi_c(x) = \pm\frac{1}{\cosh(x)}. \tag{2.5}$$

In this case also, the stability equation corresponds to the Poschl-Teller equation with $N = 2$ but with a different shift of the eigenvalue, for instance $\omega^2 = \omega_p^2 + 1$; leading to $\omega^2 = -3, 0, 1$. As a consequence the lump is characterized by one negative mode and one zero mode. Up to a normalization they are given by

$$\tilde{\eta}_{-3} = \frac{1}{\cosh(x)^2}, \qquad \tilde{\eta}_0 = \frac{\sinh(x)}{\cosh^2(x)} = -\frac{d\phi_c}{dx}, \qquad \tilde{\eta}_1 = \tanh^2(x) - \frac{1}{3}. \tag{2.6}$$

As usual, the zero mode is associated with the translation invariance of the underlying field theory.

Case 3: Cubic ϕ^3 potential—The last case investigated corresponds to a potential of third power in the field ϕ. The potential and the corresponding lump are given by

$$V(\phi) = 2\phi^2(1 - \phi), \qquad \phi_c(x) = \pm\frac{1}{\cosh^2(x)}. \tag{2.7}$$

The corresponding normal mode equation corresponds to the Poschl-Teller equation with $N = 3$ with an appropriate shift of the spectrum $\omega^2 = \omega_p^2 + 4$, leading to $\omega^2 = -5, 0, 3, 4$. The lump here is then characterized by one negative mode, one zero mode, and two positive modes. Up to a normalization they are given by

$$\tilde{\eta}_{-5} = \frac{1}{\cosh^3(x)}, \qquad \eta_0 = \frac{\sinh(x)}{\cosh^3(x)} = -\frac{1}{2}\frac{d\phi_c}{dx}, \qquad \eta_3 = \frac{1}{\cosh^2(x)}\left(\tanh^2(x) - \frac{1}{5}\right), \qquad \eta_4 = \tanh(x)\left(\tanh^2(x) - \frac{3}{5}\right). \tag{2.8}$$

In [7], deformations of the three potentials given above are studied as well (e.g. is cases 1 and 3, deformations breaking the reflection symmetry $\phi \to -\phi$). The corresponding stability equation is not of the Poschl-Teller type but their spectrum could be studied perturbatively from the ones obtained above. It should be interesting to see, in particular, how the unstable mode (or the zero mode in case 1) evolve in terms of the coupling constant parametrizing the deformation.

III. FERMION MODES IN A SINE-GORDON KINK-ANTIKINK SYSTEM

Another aspect of lumplike structures of the type discussed here is the possibility to couple them to a fermion field and to generate a mass to the fermion through a Yukawa interaction of the fermion to the scalar field ϕ. Using the notations of Chu-Vachaspati [11], we have

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + i\bar{\psi}\gamma^\mu\partial_\mu\psi - g(\phi - C)\bar{\psi}\psi, \tag{3.1}$$

where ψ is a two-component spinor and the two-dimentional Dirac matrices can be chosen in terms of the Pauli matrices $\gamma^0 = \sigma_3$, $\gamma^2 = i\sigma_1$, g is the Yukawa coupling constant, and the shift constant C has to be chosen appropriately ($C = 0$ for the ϕ^4 model, $C = \pi$ for the sine-Gordon model). To solve the Dirac equation in the background of a classical solution, say ϕ_c, available in the bosonic sector of the model, it is convenient to parametrize [11] the Dirac spinor according to

$$\psi = (\psi_1, \psi_2)^t, \qquad \psi_1 = e^{-iEt}(\beta_+ - \beta_-), \qquad \psi_2 = e^{-iEt}(\beta_+ + \beta_-), \tag{3.2}$$

where $\beta_\pm$ are functions of x. Then the system of first order Dirac equations is transformed into two decoupled second order equations for β_- and β_+:

$$(-\partial_x^2 + V_\pm(\phi_c))\beta_\pm = E^2\beta_\pm, \qquad V_\pm(\phi_c) = g^2\phi_c^2 \mp g\partial_x\phi_c. \tag{3.3}$$

For the Dirac equation in the background, when the one kink corresponding to the model (2.3) was studied in detail in [11], it was found, in particular, that the Eqs. (3.3) are Poschl-Teller equations, respectively, with $N = g$ and $N = g - 1$ and $\omega_p^2 = E^2 - g^2$. For any value of g such that $n - 1 < g \leq n$, there exist n fermionic modes given by

$$E_j = \sqrt{j(2g - j)}, \qquad j = 0, 1, \ldots, n - 1. \tag{3.4}$$

At each integer value of g, a supplementary bound state emerges from the continuum $E = g$ and exists for $g > n$.

Next, the Dirac equation was studied in the background of a kink-antikink configuration, say $\phi_{K\bar{K}}$, represented by means of superposition of a kink centered at $x = -L$ and an antikink centered at $x = L$ with $L \gg 1$:

$$\phi_{K\bar{K}}(x) = \tanh(x + L) - \tanh(x - L) - 1. \tag{3.5}$$

Such configurations are also bell-shaped. The main result is that the spectrum of the Dirac equation in the background of $\phi_{K\bar{K}}$ deviates only a little from the spectrum available in the background of a single kink (or an antikink) for $L \gg 1$. The analysis of the spectrum in the region $L \sim 1$ is unreliable since the linear superposition (3.5) is an approximate solution of the classical equation only for large values of L. In this paper we have reconsidered the Dirac equation (3.3) for the kink and kink-antikink solutions available in the sine-Gordon model. The potential and the form of the fundamental lump in this model are

$$V(\phi) = \frac{(1 - \cos(\phi))}{2}, \qquad \phi(x) = \pm 4\arctan(e^x). \tag{3.6}$$

In Fig. 1, a few fermionic bound states are represented as functions of the parameter g (for clarity, we omitted the modes $n = 4, \ldots, 9$ on the graphic). The figure presents exactly the same pattern as in the case of the ϕ^4 kink. In particular, new bound states emerge regularly from the continuum at critical values of g, say $g = g_n$. Let us mention that the normal modes about the sphaleron and bisphaleron solutions [5] of the standard model of electroweak interactions also lead to a similar pattern.

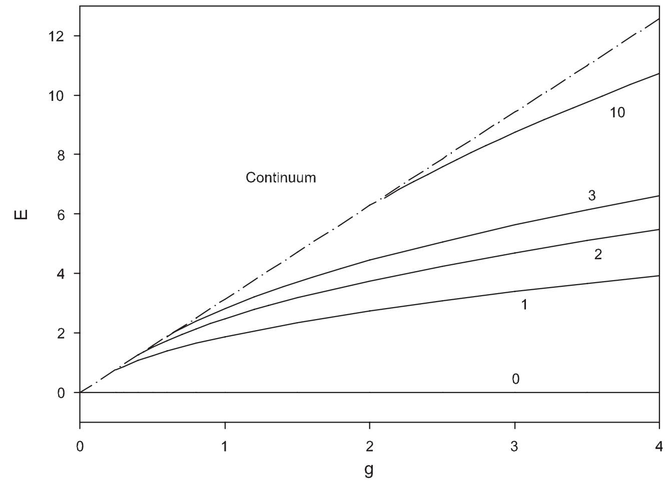

FIG. 1. The fermionic bound state corresponding to the single sine-Gordon kink as a function of the constant g. The level $n = 0, 1, 2, 3$, and 10 are represented.

Because these qualitative properties of the solutions are similar to the one of the ϕ^4 kink we expect the features discovered in [11] to hold in the case of well separated sine-Gordon kink and antikink. In the sine-Gordon model, we can take advantage of the fact that an exact form of the kink-antikink solution is available:

$$\phi_{K\bar{K}} = 4\arctan\frac{\sinh(ut/\beta)}{u\cosh(x/\beta)}, \qquad \beta = \sqrt{1 - u^2}, \tag{3.7}$$

where u is a constant related to the relative velocity of the two lumps. We used this solution to study the Dirac equation in the background of a moving wall-antiwall system (approaching or spreading each other, according to the sign of t) and were able to study the spectrum of the bound states in the domain of the parameter when t becomes small, i.e. when the kink and the antikink interact. Solving Eq. (3.3) in the background (3.7), we implicitly assume that the motion of the kink and of the antikink is treated adiabatically. A time-integration of the full equations will be reported elsewhere [18]. The results are summarized in Fig. 2 where the fermionic eigenvalue E for the ground state $n = 0$ and the first few excited states are represented as functions of t. Our numerical integration of the spectral equations (3.3) for several values of t reveals that only the ground state subsists in the $t \longrightarrow 0$ limit where it enters in the continuum (this could be expected since the classical background solution vanishes in this limit) and that the excited states join the continuum at finite values of t, depending on the level of the excitation. It should be pointed out that the fermionic eigenvalue of the fundamental solution (line $n = 0$) approaches $E = 0$ for $t \longrightarrow \infty$

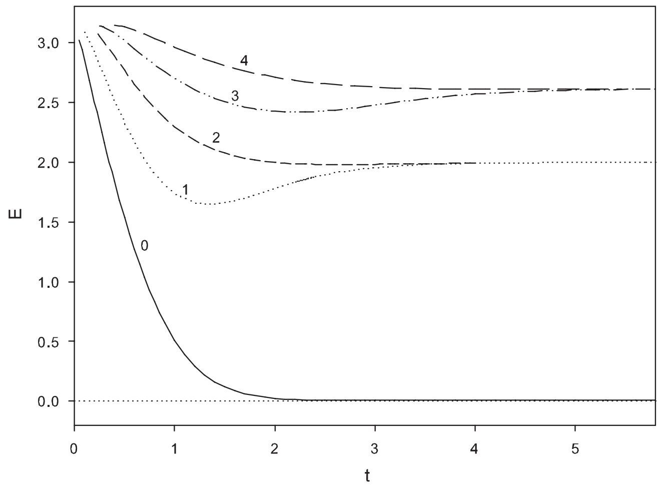

FIG. 2. The fermionic bound state corresponding to the sine-Gordon kink-antikink solution as a function of time for $g = 1$ and $u = 0.5$. The level $n = 0, 1, 2, 3, 4$ are represented.

although staying positive. It decays roughly like $E \sim \exp{-4t}$. The fact that the excited energy levels become pairwise degenerate in the large t limit can be understood from the form of the potential. Indeed, for $t \gg 1$, the potential possesses two well separate valleys centered about the points $x \sim \pm ut$. In the neighborhood of $x = \pm ut$, the form of the potential is given by $V_{\pm} = g^2\phi^2 \mp g\phi'$. Far away from the two regions of the x line situated around $x = \pm ut$ (i.e in the region of the origin and in the asymptotic regions), the potential is exponentially small. In fact the effective potential under investigation results from a superposition of two (suitably shifted) potentials which are supersymmetric partners of each other; accordingly they have the same spectrum apart from the ground state

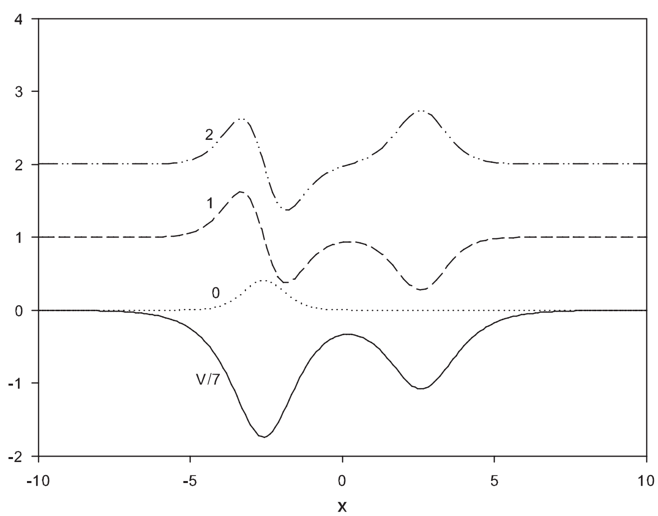

FIG. 3. The profile of the potential and of the first three fermionic bound states corresponding to the single sine-Gordon kink-antikink solution for $t = 4$, $g = 1$, and $u = 0.5$ are represented.

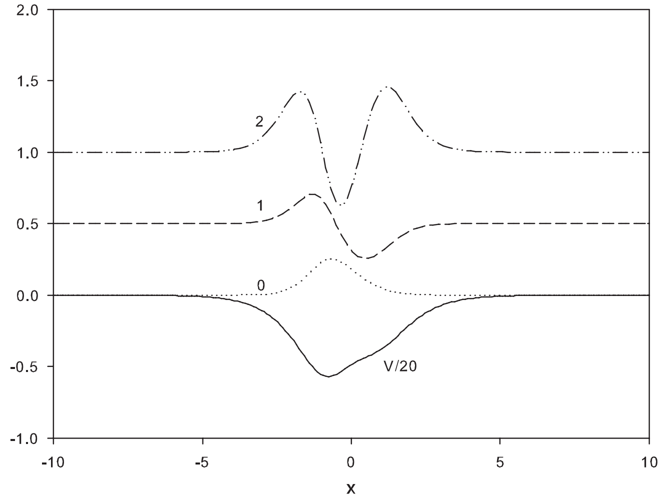

FIG. 4. The profile of the potential and of the first three fermionic bound states corresponding to the single sine-Gordon kink-antikink solution for $t = 1$, $g = 1$, and $u = 0.5$ are represented.

of V_+. It is one of the striking properties of supersymmetric quantum mechanics (see [19] for a review) that if $V_{\pm}(x)$ are supersymmetric partner potentials, the k-th energy level of V_- coincides with the $k + 1$ energy level of V_+. As a consequence, in our case, two eigenvectors exist with roughly the same eigenvalue when the wall and the anti-wall are well separated, like e.g. in Fig. 3. While t decreases, the two valleys of the potential have a tendency to merge in the region of the origin (see Fig. 4) and the degeneracy of the two energy levels corresponding to each other by the supersymmetry is lifted, as shown in Fig. 2. It is tempting to say that the supersymmetry of the spectrum occurring for $|t| \gg 1$ is broken in the $|t| \to 0$ limit. Said in other words, it turns out that when the wall and antiwall approach each other, the system cannot support fermion bound states and their spectra merge into the continuum of the Dirac equation. Seen opposite, fermion bound states (ground state and a number of excited modes, the number of them depending on the coupling constant g) can emerge from the continuum when the wall and antiwall separate from each other. We plan to study this phenomenon in more detail by solving the full Dirac sine-Gordon equation [18].

IV. CONCLUSION AND PERSPECTIVES

The stability equations associated with the bell-shaped lumps recently discussed in [7] turn out to belong to the class of quasi-exactly-solvable equations [9]. In particular their unstable mode can be computed explicitly. Accordingly, the underlying field theory can be used, along with [8], as toy models for studying tachyonic modes in brane world. In Sec. III, the study of the spectrum of the Dirac equation in the background of wall-antiwall configuration has revealed the existence of several fermionic bound states (the exact number depending on the coupling constant) emerging from the continuum while the two walls get more separated in space. We have pointed out the close relation of the pattern of eigenmodes with supersymmetric quantum mechanics. It would be tempting to use this feature in a context of higher dimensional space-time where the extra dimension would be interpreted as the "spatial" coordinate of the walls. Alternative mechanisms of fermion-brane interaction could be looked for in this direction. The main result in this topic was obtained in the celebrated achievement of [20]. Here a kink is used to localize the fermion in a $d = 5$ space-time. Generalizations of this idea to space-times involving more than two codimensions have been emphasized, namely, in [21], where topological solitons available in Yang-Mills and sigma-models are used as localizing mechanisms for fermions.

Finally, let us point out that a lot of activity is devoted to the study of the interactions of solitons, or of more general spatially localized objects, with themselves [22,23]. The interaction of kink solution with an external field of radiations is studied in detail in [22]. It is shown that the radiation exerts a negative pressure on the kink and that, accordingly, the kink is pushed backwards. A similar analysis could be performed in the case of wall-antiwall configurations like the ones emphasized in [11] or in the present paper.

APPENDIX: THE POSCHL-TELLER EQUATION

The Poschl-Teller equation is known as the following one-dimensional eigenvalue Schrodinger equation for the corresponding Poschl-Teller potential.

$$-\frac{d^2}{dx^2}\eta - \frac{N(N+1)}{\cosh^2 x}\eta = \omega_p^2\eta. \qquad (A1)$$

It is considered on an appropriate domain of the Hilbert space of square integrable functions on the real line. It is a standard result that, for integer values of N, the above equation admits $N + 1$ eigenvalues and eigenvectors which can be computed algebraically. For the first few values of N the eigenvalues are given by

$$N = 1\colon\ \omega_p^2 = -1, 0, \qquad N = 2\colon\ \omega_p^2 = -4, -1, 0, \qquad N = 3\colon\ \omega_p^2 = -9, -4, -1, 0. \qquad (A2)$$

[1] N. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2004).
[2] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
[3] F. Klinkhamer and N. Manton, Phys. Rev. D **30**, 2212 (1984).
[4] V. A. Kuzmin, V. A. Rubakov, and M. E. Shaposhnikov, Phys. Lett. **155B**, 36 (1985).
[5] Y. Brihaye and J. Kunz, Phys. Lett. B **249**, 90 (1990).
[6] B. Kleihaus, J. Kunz, and M. Leissner, arXiv:0802.3275.
[7] A. T. Avelar, D. Bazeia, L. Losano, and R. Menezes, arXiv:0711.4721 [Eur. Phys. J. C (to be published)].
[8] F. A. Brito, J. High Energy Phys. 08 (2005) 036.
[9] A. Turbiner, Commun. Math. Phys. **118**, 467 (1988).
[10] M. Postma and B. Hartmann, arXiv:0706.0416.
[11] Yi-Zen Chu and Tanmay Vachaspati, Phys. Rev. D **77**, 025006 (2008).
[12] R. Jackiw and P. Rossi, Nucl. Phys. **B190**, 681 (1981).
[13] E. J. Weinberg, Phys. Rev. D **24**, 2669 (1981).
[14] R. C. Nohl, Phys. Rev. D **12**, 1840 (1975).
[15] J. Boguta and J. Kunz, Phys. Lett. **154B**, 407 (1985).
[16] A. Ringwald, Phys. Lett. B **213**, 61 (1988).
[17] J. Kunz and Y. Brihaye, Phys. Lett. B **304**, 141 (1993).
[18] Y. Brihaye and T. Delsate (unpublished).
[19] F. Cooper, A. Khare, and U. Sukthatme, Phys. Rep. **251**, 267 (1995).
[20] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 136 (1983).
[21] W. Nahm and D. H. Tchrakian, J. High Energy Phys. 04 (2004) 075.
[22] P. Forgacs, A. Lukacs, and T. Romanczukiewicz, Phys. Rev. D **77**, 125012 (2008).
[23] S. Doutta, D. A. Steer, and T. Vachaspati, arxiv:0803.0670.