Gravitoelectromagnetic analogy based on tidal tensors

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We propose a new approach to a physical analogy between general relativity and electromagnetism, based on *tidal tensors* of both theories. Using this approach we write a covariant form for the gravitational analogues of the Maxwell equations, which makes transparent both the similarities and key differences between the two interactions. The following realizations of the analogy are given. The first one matches linearized gravitational tidal tensors to exact electromagnetic tidal tensors in Minkowski spacetime. The second one matches exact magnetic gravitational tidal tensors for ultrastationary metrics to exact magnetic tidal tensors of electromagnetism in curved spaces. In the third we show that our approach leads to a two-step exact derivation of Papapetrou's equation describing the force exerted on a spinning test particle. Analogous scalar invariants built from tidal tensors of both theories are also discussed.

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I. INTRODUCTION

General relativity and electromagnetism are intrinsically very different theories. In spite of that, analogies between them have been unveiled throughout the years; the most popular are the linear approach to gravitoelectromagnetism (GEM) (see e.g. [1–3]) and the analogy based on the decompositions of the Maxwell and Weyl tensors in electric and magnetic parts (see e.g. [4]). The latter is covariant and exact, but purely formal (cf. [5], Sec. 4.3). The former is a physical analogy, which applies intuition and well-known results from electromagnetic phenomena to the description of less familiar gravitational ones. An important outcome is that the dragging of inertial frames caused by currents of mass/energy, which is currently being experimentally tested [6], can be described, to linear order, as a magnetic type effect. This approach is, however, noncovariant, and there is no consensus about its limit of validity. While some authors limit the analogy to stationary configurations [1–3,7–9], others argue it can be extended to time-dependent setups [10–17]. On this last version of the analogy, a set of Maxwell-like equations are derived, which predict the existence of gravitational induction effects similar to the electromagnetic ones; experiments to detect those induced fields have been proposed [16] and, recently, such an experiment has actually been performed [18].

The purpose of this paper is to propose a new, covariant and clarifying approach to gravitoelectromagnetism. We start by outlining the guiding principle: that a physically transparent comparison between the two interactions must be based on quantities common to both theories. The electromagnetic interaction is based on forces; but in gravity, the only covariant forces are tidal forces (since the gravitational force on a *point* test particle can be gauged away by moving to a freely falling frame, due to the equivalence principle); therefore, tidal forces should provide the basis for our approach.

For both theories we define tidal tensors which describe tidal forces in an invariant way. We claim that a *physical analogy* stems from these objects. This analogy stands on universal, covariant equations: the geodesic deviation equation and its analogous electromagnetic worldline deviation equation; the Papapetrou force applied on a gyroscope and the electromagnetic force exerted on a magnetic dipole. We show that Maxwell's equations can be expressed as equations for tidal tensors and sources, which have a straightforward gravitational analogue.

This approach embodies all the correct predictions from the usual linear GEM, while revealing, in an unambiguous way, the regime of validity of the latter. In particular, that it is valid only for stationary setups. On the one hand this sheds light on the ongoing debate; on the other hand it means that the gravitational analogue of Faraday's law of induction was predicted in literature [12,15–17] using an approach that was taken beyond its limit of validity. Indeed, as we shall see, such analogy is ruled out by the symmetries of the gravitational tidal tensors.

II. GRAVITATIONAL AND ELECTROMAGNETIC TIDAL TENSORS

Gravitational tidal forces manifest themselves in an invariant way through the physical effect of geodesic deviation, described by the equation

$$\frac{D^2\delta x^\alpha}{D\tau^2} = -\mathbb{E}^\alpha{}_\beta \delta x^\beta, \qquad \mathbb{E}^\alpha{}_\beta \equiv R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu; \quad (1)$$

where $D/D\tau$ denotes covariant differentiation along a curve parameterized by τ, and δx^α is the connection vector

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between two neighboring geodesics with the *same* tangent vector U^α (see [19,20]).

The analogous electromagnetic problem would be to consider, in an electromagnetic field, two charged particles with the *same* 4-velocity U^α and with the additional condition that both have the same q/m ratio, since there is no electromagnetic counterpart to the equivalence principle. Under these conditions one obtains the deviation equation (see [19])

$$\frac{D^2\delta x^\alpha}{D\tau^2} = \frac{q}{m}E^\alpha{}_\beta \delta x^\beta, \qquad E^\alpha{}_\beta = F^\alpha{}_{\mu;\beta}U^\mu, \tag{2}$$

which suggests the physical analogy $\mathbb{E}_{\alpha\beta} \leftrightarrow E_{\alpha\beta}$. The tensor $E_{\alpha\beta}$ is simply the covariant derivative of the electric field $E^\alpha = F^{\alpha\mu}U_\mu$ measured by the observer with (fixed) 4-velocity U^α; for this reason we will refer to it as the *electric tidal tensor*, and its gravitational counterpart $\mathbb{E}_{\alpha\beta}$, which is known in literature (see e.g. [21]) as *electric part of the Riemann tensor*, as the *electric gravitational tidal tensor*. The different signs in (1) and (2) reflect the different character (attractive or repulsive) of the interaction between masses or charges of the same sign. Given our definition of the electric tidal tensor, it is straightforward to define the *magnetic tidal tensor:*

$$B_{\alpha\beta} \equiv \star F_{\alpha\mu;\beta}U^\mu = \tfrac{1}{2}\epsilon^{\gamma\lambda}{}_{\alpha\mu}F_{\gamma\lambda;\beta}U^\mu, \tag{3}$$

where $\star$ denotes the Hodge dual and $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor. $B_{\alpha\beta}$ measures the tidal effects produced by the magnetic field $B^\alpha = \star F^{\alpha\mu}U_\mu$ seen by the observer of 4-velocity U^α. An analogous procedure applied to the Riemann tensor yields the so-called *magnetic part of the Riemann tensor* (see e.g. [21])

$$\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\sigma}U^\mu U^\sigma = \tfrac{1}{2}\epsilon^{\gamma\lambda}{}_{\alpha\mu}R_{\gamma\lambda\beta\sigma}U^\mu U^\sigma, \tag{4}$$

which we claim, and give evidence throughout this paper, to be the *physical* gravitational analogue of $B_{\alpha\beta}$:

$$\mathbb{H}_{\alpha\beta} \leftrightarrow B_{\alpha\beta}.$$

For this reason $\mathbb{H}_{\alpha\beta}$ will be herein referred to as the *magnetic gravitational tidal tensor.* In (4) the Hodge dual was taken with respect to the first pair of indices of the Riemann tensor; a different choice amounts to changing the order of the indices in $\mathbb{H}_{\alpha\beta}$.

A. Maxwell equations as tidal tensor equations

Maxwell equations are tidal equations. Indeed, using the above defined electromagnetic tidal tensors, they can be cast in the explicitly covariant form:

$$E^\alpha{}_\alpha = 4\pi\rho_c, \tag{5i}$$
$$E_{[\alpha\beta]} = \tfrac{1}{2}F_{\alpha\beta;\gamma}U^\gamma, \tag{5ii}$$
$$B^\alpha{}_\alpha = 0, \tag{5iii}$$
$$B_{[\alpha\beta]} = \tfrac{1}{2}\star F_{\alpha\beta;\gamma}U^\gamma - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^\sigma U^\gamma, \tag{5iv}$$

where j^α and $\rho_c = -j^\alpha U_\alpha$ denote, respectively, the (charge) current 4-vector and the charge density measured by the observer with 4-velocity U^α. Decomposing

$$F_{\alpha\beta;\gamma} = 2U_{[\alpha}E_{\beta]\gamma} + \epsilon_{\alpha\beta\mu\sigma}B^\mu{}_\gamma U^\sigma, \tag{6}$$

$F_{\alpha\beta;\gamma}$ and $\star F_{\alpha\beta;\gamma}$ are readily written in terms of tidal tensors, so that Eqs. (5) can be completely expressed in terms of tidal tensors and sources. Equations (5i) and (5iii) are the covariant forms of $\nabla.\vec{E} = 4\pi\rho_c$ and $\nabla.\vec{B} = 0$; Eqs. (5ii) and (5iv) are covariant forms for $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$ and $\nabla\times\vec{B} = \partial\vec{E}/\partial t + 4\pi\vec{j}$, respectively.

B. The gravitational analogue of Maxwell's equations

By performing, on the gravitational tidal tensors, the same operations that led to Eqs. (5), i.e, taking the traces and antisymmetric parts of the tidal tensors, we obtain the analogous set of equations

$$\mathbb{E}^\alpha{}_\alpha = 4\pi(2\rho_m + T^\alpha{}_\alpha), \tag{7i}$$
$$\mathbb{E}_{[\alpha\beta]} = 0, \tag{7ii}$$
$$\mathbb{H}^\alpha{}_\alpha = 0, \tag{7iii}$$
$$\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma}J^\sigma U^\gamma, \tag{7iv}$$

where $T_{\alpha\beta}$ denotes the energy-momentum tensor, and $J^\alpha = -T^\alpha{}_\beta U^\beta$ and $\rho_m = T_{\alpha\beta}U^\alpha U^\beta$ are, respectively, the mass/energy density current and the mass/energy density measured by the observer of 4-velocity U^α.

Equations (7i) and (7iv) are *exactly* the time-time and time-space projections of Einstein equations:

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^\alpha{}_\alpha).$$

Equations (7ii) and (7iii) are related to the algebraic Bianchi identities [analogously, Eqs. (5ii) and (5iii) follow from the electromagnetic Bianchi identity].

C. Gravity versus electromagnetism

Equations (5) are strikingly similar to Eqs. (7) when the setups are stationary in the observer's rest frame. Otherwise, they tell us that gravitational and electromagnetic interactions must differ significantly, since the tidal tensors do not have the same symmetries.

Charges.—Comparing (5i) and (7i), we see that the gravitational analogue of the electric charge density ρ_c is $2\rho_m + T^\alpha{}_\alpha$ (which becomes $\rho_m + 3p$ in the case of a perfect fluid), pointing out that in gravity, by contrast with electromagnetism, pressure and all material stresses contribute as sources. A perfect analogy exists in the case of Eqs. (5iii) and (7iii): the trace of $B_{\alpha\beta}$ is zero by virtue of the electromagnetic Bianchi identity; likewise, the trace of $\mathbb{H}_{\alpha\beta}$ vanishes by virtue of the first Bianchi identities.

Ampére law.—In stationary (in the observer rest frame) configurations, Eqs. (5iv) and (7iv) match up to a factor of 2; therefore, currents of mass/energy source gravitomagnetism just like currents of charge source magnetism. The extra factor of 2 in (7iv) reflects the different spin of the gravitational and electromagnetic interactions.

Absence of electromagnetic-like induction effects in gravity.—Equations (5ii) and (7ii) reveal a fundamental difference between $\mathbb{E}_{\alpha\beta}$ and $E_{\alpha\beta}$: while the former is always symmetric, the latter is symmetric only if the Maxwell tensor is covariantly constant along the observer's worldline. The physical content of these equations depends crucially on these symmetries: since (5ii) is a covariant form of $\nabla\times\vec{E}=-\partial\vec{B}/\partial t$, the statement encoded in the equation $\mathbb{E}_{[\alpha\beta]}=0$ is that *there is no gravitational analogue to Faraday's law of induction*. Likewise, the fact that the induction term $\star F_{\alpha\beta;\gamma}U^\gamma$ present in (5iv) has no counterpart in (7iv) means that there is no gravitational analogue to the magnetic fields induced, for instance, by the time-varying electric field between the plates of a charging/discharging capacitor.

The different symmetries of the gravitational and electromagnetic tidal tensors are related to a fundamental difference in their tensorial structure: while the former are spatial, the latter are not. As can be seen from decomposition (6), the extra terms in (5ii) and (5iv) as compared to (7ii) and (7iv), indeed consist of temporal projections of the electromagnetic tidal tensors.

Finally, we note that the (noncovariant) set of gravitational Maxwell-like equations derived in the popular linearized theory approach to GEM [1–3,10–13,17] are a special case of the exact Eqs. (7) in the regime of stationary, weak fields and stressless sources.

III. LINEARIZED GRAVITATIONAL PERTURBATIONS

Consider the elementary example of analogous physical problems: the metric outside a rotating spherical mass (which is asymptotically described by the Kerr solution), and the electromagnetic field produced by a rotating charged sphere (in Minkowski spacetime). Let m and J denote the mass and angular momentum of the former, and q and μ denote the charge and magnetic dipole moment of the latter. For an observer at rest relative to the center of the spheres, the gravitational tidal tensors *asymptotically* match their electromagnetic counterparts, identifying $\{m,J\}\leftrightarrow\{q,\mu\}$:

$$\mathbb{E}_{\alpha\beta}dx^\alpha dx^\beta \simeq -\frac{2m}{r^3}dr^2+\frac{m}{r}d\Omega_2 \overset{m\leftrightarrow q}{=} E_{\alpha\beta}dx^\alpha dx^\beta, \quad (8)$$

$$\mathbb{H}_{\alpha\beta}dx^\alpha dx^\beta \simeq \frac{3J\cos\theta}{r^2}\left(d\Omega_2-\frac{2}{r^2}dr^2-\frac{2\tan\theta}{r}drd\theta\right) \overset{J\leftrightarrow\mu}{=} B_{\alpha\beta}dx^\alpha dx^\beta. \quad (9)$$

For an observer moving relative to the central body, however, the electromagnetic tidal tensors will be very different from the gravitational ones (for explicit expressions, see [5], Sec. 2.2.1); they will not even exhibit the same symmetries, as Eqs. (5ii), (5iv), (7ii), and (7iv) make clear.

These results may be seen as a special case of a more general principle. Consider a general electromagnetic field in Minkowski spacetime $[x^\mu\equiv(t,x^k)]$:

$$A=-\phi(x^\mu)dt+A_j(x^\mu)dx^j,\qquad ds^2=-dt^2+\hat{g}_{ij}(x^k)dx^idx^j, \quad (10)$$

where $\hat{g}_{ij}$ is the Euclidean metric in an arbitrary coordinate system. The electric tidal tensor $E_{\alpha\beta}$ is, for an observer with four velocity $U^\alpha=(u^0,u^i)$, given by

$$\begin{aligned} &E_{00}=(\dot{\phi}_{;i}+\ddot{A}_i)u^i, \qquad E_{0i}=(\phi_{;ki}+\dot{A}_{k;i})u^k,\\ &E_{i0}=-(\dot{\phi}_{;i}+\ddot{A}_i)u^0+2\dot{A}_{[j;i]}u^j,\\ &E_{ij}=-(\phi_{;ij}+\dot{A}_{i;j})u^0+2A_{[k;i]j}u^k. \end{aligned} \quad (11)$$

where dots represent time derivatives, the semicolon represents covariant derivatives with respect to $\hat{g}_{ij}$, and $\hat{\epsilon}_{ijk}$ are the components of the Levi-Civita tensor on $\mathbb{R}^3$ in coordinates $\{x^i\}$. We follow, for the Levi-Civita symbol, the orientation defined by $\tilde{\epsilon}_{0123}=-1$. Similarly, the magnetic tidal tensor (3) is given by

$$\begin{aligned} &B_{00}=-\hat{\epsilon}_{ijk}\dot{A}^{j;i}u^k, \qquad B_{0i}=-\hat{\epsilon}_{ljk}A^{j;l}{}_{;i}u^k,\\ &B_{i0}=\hat{\epsilon}_{kji}\dot{A}^{j;k}u^0+\hat{\epsilon}_{jik}(\dot{\phi}^{;j}+\ddot{A}^j)u^k,\\ &B_{ij}=\hat{\epsilon}_i{}^{lm}A_{m;lj}u^0+\hat{\epsilon}_{lik}(\phi^{;l}{}_{;j}+\dot{A}^l{}_{;j})u^k. \end{aligned} \quad (12)$$

Now consider general gravitational perturbations around Minkowski spacetime in the form

$$ds^2=-(1-2\Phi(x^\mu))dt^2-4\mathcal{A}_j(x^\mu)dtdx^j+[\hat{g}_{ij}(x^k)+2\Theta_{ij}(x^\mu)]dx^idx^j, \quad (13)$$

where, as before, $\hat{g}_{ij}$ is the Euclidean metric in an arbitrary coordinate system. For an observer with four velocity $U^\alpha=(u^0,u^i)$, the electric gravitational tidal tensor $\mathbb{E}_{\alpha\beta}$ is given, to linear order in the perturbations, by

$$\mathbb{E}_{00}\simeq-(\Phi_{;ji}+2\dot{\mathcal{A}}_{j;i}+\ddot{\Theta}_{ij})u^iu^j, \quad (14)$$

$$\mathbb{E}_{0i}=\mathbb{E}_{i0}\simeq(\Phi_{;ij}+2\dot{\mathcal{A}}_{(i;j)}+\ddot{\Theta}_{ij})u^0u^j+2(\dot{\Theta}_{j[i;k]}-\mathcal{A}_{[k;i]j})u^ku^j, \quad (15)$$

$$\begin{aligned}\mathbb{E}_{ij}=\mathbb{E}_{ji}\simeq{}&2(\dot{\Theta}_{k(i;j)}-\dot{\Theta}_{ij;k}+\mathcal{A}_{k;ij}-\mathcal{A}_{(i;j)k})u^0u^k\\ &+(2\Theta_{l(i;j)k}u^ku^l-\Theta_{ij;lk}-\Theta_{lk;ij})u^ku^l\\ &-(\Phi_{;ij}+2\dot{\mathcal{A}}_{(i;j)}+\ddot{\Theta}_{ij})(u^0)^2 \end{aligned} \quad (16)$$

(again, the semicolon represents covariant derivatives with respect to $\hat{g}_{ij}$). The magnetic gravitational tidal tensor (4) is, to linear order,

$$\mathbb{H}_{00}\simeq\hat{\epsilon}_{imn}(\mathcal{A}^{n;m}{}_{;j}+\dot{\Theta}_j{}^{n;m})u^iu^j, \quad (17)$$

$$\mathbb{H}_{i0} \simeq \hat{\epsilon}_i{}^{lk}(\dot{\Theta}_{jk;l} - \mathcal{A}_{k;lj})u^0u^j - \hat{\epsilon}_{ik}{}^l(\Phi_{;jl} + 2\dot{\mathcal{A}}_{(l;j)} + \ddot{\Theta}_{lj})u^ku^j, \quad (18)$$

$$\mathbb{H}_{0i} \simeq \hat{\epsilon}_j{}^{lk}(\dot{\Theta}_{ik;l} - \mathcal{A}_{k;li})u^0u^j - \hat{\epsilon}_j{}^{lm}(\Theta_{l[k;i]m} + \Theta_{m[i;k]l})u^ju^k, \quad (19)$$

$$\mathbb{H}_{ij} \simeq \hat{\epsilon}_i{}^{lk}(\mathcal{A}_{k;lj} + \dot{\Theta}_{jk;l})(u^0)^2. \quad (20)$$

As can be now easily verified, the electromagnetic tidal tensors are, generically, very different from their gravitational counterparts. But if one takes time-independent electromagnetic potentials/gravitational perturbations, and a "static observer",[1] $U^\mu = \delta^\mu_0$, then the *linearized* gravitational tidal tensors match their electromagnetic counterparts:

$$\mathbb{E}_{ij} \simeq -\Phi_{;ij} \overset{\Phi\leftrightarrow\phi}{=} E_{ij}, \qquad \mathbb{H}_{ij} \simeq \hat{\epsilon}_i{}^{lk}\mathcal{A}_{k;lj} \overset{\mathcal{A}\leftrightarrow A}{=} B_{ij}. \quad (21)$$

One may regard this matching between the tidal tensors of the two theories as an analogy between the electromagnetic potential A_μ and some components of the metric tensor: $(\phi, A^i) \leftrightarrow (\Phi, \mathcal{A}^i)$, hence defining the fields

$$E^i_G = -\Phi^{;i}, \qquad B^i_G = \hat{\epsilon}^{lki}\mathcal{A}_{k;l} \quad (22)$$

which are equivalent to the "gravito-electromagnetic fields" defined in the usual linear approach to GEM [1–3,10–13,17] (when the latter are assumed time-independent). These help us visualize geodesic motion and frame dragging in analogy with the more familiar picture of a charged particle subject to an electromagnetic Lorentz force. Indeed, the geodesic equation $DU^\alpha/D\tau = 0$ yields, to linear order in the perturbations and in the velocity of the test particle, the space components:

$$\frac{d^2\vec{x}}{dt^2} = -\vec{E}_G - 2\vec{v} \times \vec{B}_G. \quad (23)$$

The "gravito-magnetic" field in (22) also leads directly to the Lense and Thirring precession for test gyroscopes [2,3,22]: as for a magnetic dipole placed in a magnetic field, a "torque" $\tau = -\vec{S} \times \vec{B}_G$ acts on a gyroscope of angular momentum $\vec{S}$ [since in gravity $\vec{S}$ plays the role of the magnetic moment, as relation (9) points out], causing it to precess with angular frequency $\vec{\omega} = \vec{B}_G$, which is accurate to linear order.

The advantage of the tidal tensor formalism is to reveal the regime of validity of such construction in a clear and unambiguous fashion, in particular, that it is limited to a mapping between static electromagnetic fields and stationary gravitational setups: as readily seen by comparing Eqs. (11) and (12) with (14)–(20), in the general case of fields varying with the observer's proper time, the gravitational tidal tensors will be very different from their electromagnetic counterparts, so that the *physical* analogy $(\phi, A^i) \leftrightarrow (\Phi, \mathcal{A}^i)$ no longer holds. This sheds light on the ongoing debate[2] about the limit of validity of the linear GEM, supporting earlier claims by Harris [1], Ohanian and Ruffini (cf. [3], p. 163), and Clark and Tucker [9]. And it implies that the gravitational analogue of Faraday's law of induction was predicted in literature [12,15–17] by taking that analogy beyond its limit of validity.

IV. ULTRASTATIONARY SPACETIMES

Ultrastationary spacetimes are a special class of stationary spacetimes whose metric has a constant g_{00} component in the chart where it is explicitly time independent. The line element is, generically,

$$ds^2 = -(dt + A_i(x^k)dx^i)^2 + \hat{g}_{ij}(x^k)dx^idx^j. \quad (24)$$

In these spacetimes [23,24], the Klein-Gordon equation, $\Box\Psi = m^2\Psi$, with the ansatz $\Psi = e^{-iEt}\psi(x^j)$, reduces to a time-independent Schrödinger equation, $H\psi = \epsilon\psi$, where

$$H = \frac{(\vec{P} + E\vec{A})^2}{2m}, \qquad \epsilon = \frac{E^2 - m^2}{2m},$$

corresponding to the nonrelativistic problem of a particle with "charge" $-E$ and mass m, living in a curved 3-space with metric $\hat{g}_{ij}$, under the action of a magnetic field $\vec{B} = \nabla \times \vec{A}$. The similarity between the physics of these apparently so different setups can be explained in the framework of our approach, through the similarity of tidal forces: the tidal tensor of $\vec{B}$ turns out to be the same, up to a factor of 2, as the *exact* magnetic gravitational tidal tensor of (24) as measured by the static observer $U^\alpha = \delta^\alpha_0$ (precisely the observer for which this construction holds):

$$B_{ij} = \hat{D}_jB_i = \hat{\epsilon}_{lki}\hat{D}_j\hat{D}^lA^k = 2\mathbb{H}_{ij}; \quad (25)$$

$\hat{D}$ denotes covariant derivative with respect to $\hat{g}_{ij}$. This shows that our interpretation of the magnetic part of the Riemann tensor as a magnetic tidal tensor is indeed correct even outside the scope of linearized theory. And it is a highly nontrivial realization of the analogy, since there is an *exact* matching between tidal tensors from a linear theory (electromagnetism) with the ones from a nonlinear theory (gravity), which provides valuable insight for the understanding of some properties of these spacetimes.

An example is the Gödel universe, which in the literature is often discussed as a homogeneous rotating universe—a conceptually hard definition since it means that it rotates around *every* point [2]. The vanishing of the magnetic part of the Weyl tensor in this spacetime has also led to some conceptual difficulties (see [5], Sec. 4.3). Within the anal-

[1]We define the "static observer" as being an observer for which the setup is stationary. An observer at rest relative to the center of mass of the spinning spheres considered above is an example of such an observer.

[2]To follow this debate, in chronological order, see [16,1,3], p. 163, [9,17].

ogy proposed herein both these facts have a straightforward interpretation. The Gödel metric can be written in the form (24) with

$$A_i dx^i = e^{\sqrt{2}\omega x} dy,$$
$$\hat{g}_{ij} dx^i dx^j = dx^2 + \tfrac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + dz^2.$$

The Klein-Gordon equation maps this metric into the magnetic field $\vec{B} = 2\omega \vec{e}_z$ living in the three-space of metric $\hat{g}_{ij}$. This field is *uniform*, since its tidal tensor B_{ij} vanishes. Thus, the physical interpretation for the vanishing of the magnetic part $\mathbb{H}_{\alpha\beta}$ of the Riemann (and hence Weyl) tensor is that the Gödel universe has a *uniform* gravitomagnetic field. The concept of homogeneous rotation is then easily assimilated by an analogy with the more familiar picture of a gas of charged particles subject to a uniform magnetic field: there are Larmor orbits around any point.

V. FORCE ACTING ON A GYROSCOPE

The force acting on a dipole is a purely tidal effect; it is therefore the most obvious physical application for a tidal tensor based analogy. There is no gravitational analogue to the electric dipole, since there are no "negative masses"; for the same reason, there can be no gravitational analogue to a pure magnetic dipole (i.e., without an electric monopole moment). But there is a clear gravitational analogue to a particle with electric monopole plus magnetic dipole moments, which is the ideal gyroscope (i.e., a pole-dipole spinning test particle, as defined in [25]). In gravity no force arises from the monopole term, since a spinless particle moves along a geodesic; hence, the force exerted on a gyroscope should indeed, in the spirit of our approach, be the gravitational counterpart of the electromagnetic force exerted on a magnetic dipole.

In electromagnetism, the force acting on a magnetic dipole when placed in a magnetic field $\vec{B}$ is usually given in literature (see, for instance, [2], p. 318) by

$$\vec{F}_{\rm EM} = \nabla(\vec{\mu}.\vec{B}), \tag{26}$$

where $\vec{\mu}$ denotes the magnetic dipole moment, related to the classical angular momentum $\vec{S}$ by $\vec{\mu} = (q/2m)\vec{S}$.

This expression holds only in the dipole's rest frame; but making use of the tensor $B_{\alpha\beta}$, a simple analysis leads to the corresponding covariant expression (avoiding an otherwise more demanding computation, e.g. [26]):

$$\frac{DP^\beta}{D\tau} = F^\beta_{\rm EM} = \frac{q}{2m} B^{\alpha\beta} S_\alpha, \tag{27}$$

where $B^{\alpha\beta}$ is the magnetic tidal tensor seen by the dipole, and S^α is the "intrinsic angular momentum" (e.g. [27], p. 158), defined as being the 4-vector with components $(0, \vec{S})$ in the dipole's rest frame. In the light of our approach, the analogous gravitational force should then be obtained by replacing $B^{\alpha\beta}$ by $\mathbb{H}^{\alpha\beta}$; dropping the factor $q/2m$ since the gravitational analogue of $\vec{\mu}$ is $\vec{S}$, as can be seen from Eqs. (7iv) or (9); and taking into account the relative minus sign which manifests that "charges" of the same sign attract/repel one another in gravity/electromagnetism, as do charge/mass currents with parallel velocity. Hence, we get

$$\frac{DP^\beta}{D\tau} = F^\beta_G = -\mathbb{H}^{\alpha\beta} S_\alpha, \tag{28}$$

which *exactly* coincides with the equation derived by Papapetrou [25]:

$$\frac{DP^\alpha}{D\tau} = -\frac{1}{2} R^\alpha{}_{\beta\mu\nu} U^\beta S^{\mu\nu}, \tag{29}$$

with Pirani [28] supplementary condition $S^{\mu\nu}U_\nu = 0$, where $S^{\mu\nu}$ denotes the spin tensor (e.g. [27], p. 158). This result may therefore be seen as a definite confirmation that our interpretation of $\mathbb{H}_{\alpha\beta}$ as a magnetic tidal tensor is indeed correct. And we have just proved that the non-geodesic motion of a spinning particle can be understood and *exactly described by a simple application of the analogy based on tidal tensors*.

There have been previous attempts to describe the force exerted on a gyroscope by an analogy with electromagnetism[3]; a first order estimate has been derived in the framework of the linearized theory (e.g. [1,2]), in direct analogy with (26):

$$\vec{F}_G = -\nabla(\vec{S}\cdot\vec{B}_G), \tag{30}$$

where $\vec{B}_G$ is the gravitomagnetic field defined in Sec. III. However, as is asserted in [8], that expression is valid only when the gyroscope is at *rest* in a *stationary*, weak gravitational field, and therefore not suited to describe motion. The reason for that is readily understood in the framework of our approach. Equations (27) and (28) have an important physical interpretation: it is the magnetic tidal tensor as seen by the dipole/gyroscope that determines the force exerted upon it. Hence, as follows Eqs. (5iv) and (7iv), the two forces can be similar only when the fields are stationary (besides weak) in the test particle's frame.

Moreover, Eq. (30) accounts only for the coupling between the intrinsic spin of the source and the spin of the gyroscope, hiding the fact that the gyroscope will indeed deviate from geodesic motion even in the absence of rotating sources; for example, in the Schwarzschild spacetime. This is an effect which is readily visualized with the help of the explicit analogy between (27) and (28): a gyroscope (in nonradial motion) deviates from geodesic

[3]The force on a gyroscope was also recently discussed in [29] in the context of the "quasi-Maxwell" formalism. We note that Eq. (5) therein, valid for a gyroscope at rest in a stationary spacetime, matches its electromagnetic counterpart in two special cases: in the weak field limit, and in ultrastationary metrics. This is in agreement with the results from the approach proposed herein, since it is precisely under those conditions (cf. Secs. III and IV) that a matching between electromagnetic and gravitational tidal tensors was found to exist.

TABLE I. Invariants built on tidal tensors.

Electromagnetic	Gravitational (vacuum)
$L \equiv E_{\alpha\gamma}E^{\alpha\gamma} - B_{\alpha\gamma}B^{\alpha\gamma} = -\frac{1}{2}F^{\sigma\tau;\eta}F_{\sigma\tau;\eta}$	$\mathbb{L} \equiv \mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{8}R^{\sigma\tau\gamma\eta}R_{\sigma\tau\gamma\eta}$
$M \equiv E^{\alpha\gamma}B_{\alpha\gamma} = -\frac{1}{4}\star F^{\sigma\tau;\eta}F_{\sigma\tau;\eta}$	$\mathbb{M} \equiv \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{16}\star R^{\sigma\tau\gamma\eta}R_{\sigma\tau\gamma\eta}$

motion in Schwarzschild spacetime by the very same reason that a magnetic dipole will suffer a force even in the Coulomb field of a point charge: in its "rest" frame, there is a nonvanishing magnetic tidal tensor.

Another important physical content which is lost in the three-dimensional expression (30), but is unveiled in the framework of our approach, concerns the temporal projections of the forces (27) and (28). Let us consider first the electromagnetic force (27). The magnetic dipole may be seen as a small current loop; denoting the area of the loop by a, and its current by I, the magnetic dipole moment is then given by $\vec{\mu} = Ia\vec{n}$, where $\vec{n}$ is the unit vector normal to the plane of the loop. Therefore, in the dipole's rest frame we have

$$F^{\alpha}_{\mathrm{EM}}U_{\alpha} = -B^{i0}\mu_i = \frac{\partial\vec{B}}{\partial t}.\vec{n}aI = \frac{\partial\Phi}{\partial t}I = -I\oint_{\mathrm{loop}}\vec{E}.\vec{d}s,$$

where Φ is the magnetic flux through the loop and $\vec{E}$ is the induced electric field; thus $F^{\alpha}_{\mathrm{EM}}U_{\alpha}$ is *minus* the power transferred to the dipole by Faraday's induction, due to a time-varying magnetic field.

We turn now to the gravitational force (28). Since $\mathbb{H}_{\alpha\beta}$ is a spatial tensor, we *always have* $F^{\alpha}_{G}U_{\alpha} = 0$. This is in accordance with the discussion in Sec. II C: the spatial character of the gravitational tidal tensors *precludes electromagnetic-like induction effects in gravity.*

VI. INVARIANTS

We have seen so far that a matching between gravitational and electromagnetic tidal tensors can only occur when the setups are stationary in the observer's rest frame. But there can be also a matching between observer independent quantities. It is shown in [5] that by using the electromagnetic tidal tensors it is possible to construct invariant (in the sense of being U^{α}-independent) scalars; and that in vacuum the same applies to gravity. These results are summarized in Table I. Recalling the example of analogous physical systems given in Sec. III, one may check that, identifying $\{m, J\} \leftrightarrow \{q, \mu\}$, $\mathbb{L}$ and $\mathbb{M}$ from the Kerr black hole asymptotically match L and M from the spinning spherical charge:

$$\mathbb{L} \simeq \frac{6m}{r^6} \overset{m\leftrightarrow q}{\simeq} L, \qquad \mathbb{M} \simeq \frac{18Jm}{r^7} \overset{\{m,J\}\leftrightarrow\{q,\mu\}}{=} M.$$

And, in the special case: {Schwarzschild black hole} $\leftrightarrow$ {point charge}, that matching is exact.

In the presence of sources, however, this analogy does not hold, because in gravity it is no longer possible to define invariants using $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ only. In vacuum the Riemann tensor has 10 independent components, which are completely encoded in the $5 + 5$ components of $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ (both symmetric and traceless in vacuum). The sources endow $\mathbb{E}_{\alpha\beta}$ with a trace and $\mathbb{H}_{\alpha\beta}$ with an antisymmetric part; thus these tensors combined possess $6 + 8$ independent components which is insufficient to encode all the information in the 20 independent components that the Riemann tensor generically possesses. A third spacial, symmetric tensor, defined by [30] (see also [27] pp. 360–361):

$$\mathbb{F}_{\alpha\gamma} \equiv \star R \star_{\alpha\beta\gamma\delta} U^{\beta}U^{\delta}, \tag{31}$$

where Hodge duality is taken both with respect to the first and second pair of indices, is needed to account for the remaining 6 components. The relevant invariants formed by the Riemann tensor are then completely expressed in terms of these three spatial tensors (for explicit expressions, see [31]).

VII. DISCUSSION

In this paper we have proposed a tensorial description of the *physical* gravitoelectromagnetism—a new analogy between general relativity and electromagnetism, based on tidal tensors. Using this formalism we have written a covariant form for each of Maxwell's equations, and obtained their gravitational analogues. These equations, which are exact and fully general, reveal in a very clear fashion both the similarities and fundamental differences between the two interactions; among the latter the absence of electromagnetic-like induction effects in gravity.[4] In electromagnetism, induction effects manifest themselves

[4]In a recent paper [32] some authors have insisted, in response to [5] and the work presented herein, that time-dependent gravitational setups originate effects which are "on the whole closely analogous" to electromagnetic induction effects. We must note the following: (1) In this paper we present a covariant description of electromagnetic induction in terms of tidal tensors which not only gives a very precise and unambiguous definition of the phenomenon, but it also allows for an immediate comparison with the gravitational case, where it becomes crystal clear that no analogous effects take place. This is in contrast to the vague meaning of the words "induction" and "analogous" used in [32]. (2) It is straightforward to show that, using the linear formalism of [32], time-dependent phenomena are not described by analogous equations in gravity and electromagnetism (see [1], Sec. 3.2). The analogy breaks down [33] even in the case of the very special "toy model" introduced in Ref. [4] of [32] in an attempt to prove their point, as one may easily check.

in the tidal forces: Faraday's law of induction, in its differential form, is covariantly described by Eq. (5ii), which states that a time-varying magnetic field endows the electric tidal tensor with an antisymmetric part. But in gravity, no induction effects are manifest in the tidal forces, since the electric tidal tensor is *always* symmetric; it is also clear that the antisymmetric contribution due to time-varying electric fields in (5iv) has no counterpart in the gravitational Eq. (7iv).

The (noncovariant) Maxwell-like equations derived in the popular linear GEM turn out to be a special case of our *exact* Eqs. (7) in the regime of stationary, weak fields, and stressless sources. That is, therefore, the regime of validity of that approach, which is herein revealed in an unambiguous way, shedding light into an ongoing debate (see introduction and Sec. III).

While naturally embodying the correct results from the linear GEM, the tidal tensor approach takes the gravitoelectromagnetic analogy beyond the scope of the former. It unveils suggestive new similarities: for stationary configurations (in the observer's rest frame), the gravitational and electromagnetic tidal tensors obey strikingly similar equations (note that this is an analogy between exact equations, *not linearized*); in vacuum, the gravitational tidal tensors form scalar invariants in the same way the electromagnetic ones do. A unification within gravitoelectromagnetism was also achieved: the (exact) connection between ultrastationary spacetimes and magnetic fields in some curved manifolds was seen to originate from the same fundamental principle as the analogy from linearized theory: a matching between tidal tensors.

Finally, the nongeodesic motion of a gyroscope is another example of an effect beyond the scope of previous approaches, which we showed in Sec. V that not only can be easily understood, but also exactly described, in analogy with the electromagnetic force exerted on a magnetic dipole. This derivation of Papapetrou's equation has two major strengths. The first, its obvious simplicity by contrast with the lengthy original computation [25]. The second is that, when written in the form (28) explicitly analogous to its electromagnetic counterpart (27), it makes possible, by comparison with the more familiar electromagnetic ones, to visualize effects which are not transparent at all in the form (29) presented in literature. At the same time, it also reveals in a clear fashion significant differences between the electromagnetic and gravitational forces, which arise from the different symmetries of the tidal tensors. In particular, due to these symmetries, as follows from Eqs. (5iv) and (7iv), a similarity between the two forces can only occur when the fields are stationary (besides weak) in the test particle's frame. Another important physical content unveiled by the analogy based on tidal tensors concerns the temporal projections of these forces. The time projection of (27) in the dipole's rest frame is the power transferred to the dipole by Faraday's induction, and the fact that it is zero in the gravitational case (28) may be regarded as another evidence for the absence of electromagnetic-like induction effects in gravity.

We close with a remark on one of the most important aspects of the analogy that stems from the tidal tensors: it *does not* rely on a similarity between tidal tensors. Despite being in general very different (namely in their symmetries and in the fact that gravitational tidal tensors are spatial and nonlinear whereas the electromagnetic ones are not), the electromagnetic and gravitational tidal tensors always play analogous roles in dynamics—that is the statement encoded in Eqs. (1), (2), (27), and (28), and that is also what makes this formalism ideally suited both to compare the two interactions and to apply intuition from electromagnetism to the understanding of gravity. This feature is also implicit in Eqs. (5) and (7); in order to see that, first express Eqs. (5) fully in terms of tidal tensors and sources, using decomposition (6); then, note that by simply replacing the electromagnetic tidal tensors by the gravitational ones, taking into account the usual factor of 2 in (5iv) and the fact that the gravitational analogue of ρ_c is $2\rho_m + T^{\alpha}{}_{\alpha}$, one obtains Eqs. (7) (since the time projections of the gravitational tidal tensors are zero).

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