

Perihelion precession for modified Newtonian gravity

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We calculate the perihelion precession δ for nearly circular orbits in a central potential $V(r)$. Differently from other approaches to this problem, we do not assume that the potential is close to the Newtonian one. The main idea in the deduction is to apply the underlying symmetries of the system to show that δ must be a function of $r \cdot V''(r)/V'(r)$ and to use the transformation behavior of δ in a rotating system of reference. This is equivalent to say that the effective potential can be written in a one-parameter set of possibilities as the sum of centrifugal potential and potential of the central force. We get the following universal formula valid for $V'(r) > 0$ reading $\delta(r) = 2\pi \cdot \left[\frac{1}{\sqrt{3+r \cdot V''(r)/V'(r)}} - 1 \right]$. It has to be read as follows: a circular orbit at this value r exists and is stable if and only if this δ is well-defined as real; and if this is the case, then the angular difference from one perihelion to the next one for nearly circular orbits at this r is exactly $2\pi + \delta(r)$. Then we apply this result to examples of recent interest like modified Newtonian gravity and linearized fourth-order gravity. In the second part of the paper, we generalize this universal formula to static spherically symmetric space-times $ds^2 = -e^{2\lambda(r)} dt^2 + e^{2\mu(r)} dr^2 + r^2 d\Omega^2$; for orbits near r it reads $\delta = 2\pi \cdot \left[\frac{e^{\mu(r)}}{\sqrt{3-2r \cdot \lambda'(r) + r \cdot \lambda''(r)/\lambda'(r)}} - 1 \right]$ and can be applied to a large class of theories. For the Schwarzschild black hole with mass parameter $m > 0$ it leads to $\delta = 2\pi \cdot \left[\frac{1}{\sqrt{1-\frac{6m}{r}}} - 1 \right]$, a surprisingly unknown formula. It represents a strict result and is applicable for all values $r > 6m$ and is in good agreement with the fact that stable circular orbits exist for $r > 6m$ only. For $r \gg m$, one can develop in powers of m and get the well-known approximation $\delta \approx \frac{6\pi m}{r}$.

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I. INTRODUCTION

Adkins and McDonnell [1] “calculate the precession of Keplerian orbits under the influence of arbitrary central-force perturbations.” For some examples including the Yukawa potential they present the result as a hypergeometric function. For nearly circular orbits, they arrive at the formula for the perihelion precession $\Delta\theta_p$, [1], Eq. (11)

$$\Delta\theta_p = -\frac{\pi}{GMmL} \frac{d^2V}{du^2} \Big|_{u=1/L}, \quad (1.1)$$

where G is the gravitational constant, M the mass of the central body, m the mass of the moving test body, L the radius of the orbit, and $u = 1/r$ the inverted radial coordinate. The potential V is the perturbation of the Newtonian potential, so the total potential is then given by $V(r) - GMm/r$. They mention that this formula Eq. (1.1) is “equivalent to a formula for the nearly circular precession that has been used by Dvali, Gruzinov, and Zaldarriaga [2].”

In the fourth section of [1], the Yukawa potential is applied in the form [1], Eq. (31)

$$V(r) = \frac{\alpha}{r} \exp(-r/\lambda), \quad \lambda > 0. \quad (1.2)$$

Using the parameter $\kappa = L/\lambda$ they arrive at [1], Eq. (33)

$$\Delta\theta_p(\kappa, 0) = -\frac{\pi\alpha}{GMm} \kappa^2 \exp(-\kappa). \quad (1.3)$$

In the fifth section of [1], the authors apply the fact, that within this approach, the famous general relativistic perihelion advance can be reproduced by using the first post-Newtonian correction

$$V(r) = -\frac{G^2 M^2 m L}{c^2 r^3}, \quad (1.4)$$

where c is the light velocity. They arrive at [1], Eq. (42)

$$\Delta\theta_p(\text{GR}) = \frac{6\pi GM}{c^2 L}, \quad (1.5)$$

and they also present limits for the value of the cosmological constant by comparing theoretical and measured values of the Mercury perihelion advance.

The authors of [2] investigate those kinds of theories which possess a linearized form of the field equation of the type

$$(\square + f(\square))g_{ij} = T_{ij}$$

and calculate e.g. the anomalous perihelion precession for this kind of theories by perturbations around the Newtonian potential.

In [3], the perturbations in the cosmological context are calculated for several scalar-tensor theories of gravitation, and for the different conformal transformations the distinction between the Einstein and the Jordan frames have been made. They applied the results also to calculate an

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effective gravitational constant for measurements within the Solar System. In [4,5], the authors carefully calculate the possible measurable effects of tensor-multiscalar theories of gravitation, including the secular rate of perihelion advance.

Davies [6] deduces the perihelion precession due to a perturbing central force on an elliptic Keplerian orbit via a perturbation with the Runge-Lenz vector. He mentions that one can mimic the influence of the outer planets to the perihelion shifts of the inner ones by replacing each outer planet by a ring of the same total mass, so that the effective potential can remain rotationally symmetric.

II. PERIHELION PRECESSION

A test mass shall move along a periodic orbit in a central potential $V(r)$. We look for the perihelion precession of this orbit.

Without loss of generality the test mass has unit mass, and the motion takes completely place in the equatorial plane. We parametrize this plane by (r, φ) , denote the time by t , and use the dot to abbreviate for d/dt . Then the Lagrangian reads

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r). \quad (2.1)$$

We assume that $V(r)$ is a C^2 function at all values r which belong to the orbit. For the orbit $(r(t), \varphi(t))$ we define perihelion and aphelion via

$$r_1 = \min_{t \in R} r(t) \quad \text{and} \quad r_2 = \max_{t \in R} r(t), \quad (2.2)$$

respectively, where $0 < r_1 \leq r_2 < \infty$ is assumed.

Let φ_0 be the change of $\varphi(t)$ during the change from $r(t)$ from one perihelion to the next aphelion. Because of time-reversal invariance, the same φ_0 is also the change of $\varphi(t)$ from this aphelion to the next perihelion. For $2\varphi_0 = 2\pi$, the orbit is exactly closed after one revolution. So it is adequate to define the perihelion precession δ by

$$\delta = 2(\varphi_0 - \pi). \quad (2.3)$$

For purely radial oscillations, our definition implies $\varphi_0 = 0$, i.e. $\delta = -2\pi$. If purely radial motion is excluded from the consideration, then all values of δ with $\delta > -2\pi$ may appear as perihelion precession. For $\delta > 0$ we call it perihelion advance.

What happens with δ when the orbit is continuously deformed? For example, let

$$r(\varphi) = 4 + \varepsilon \cos(\varphi) + \cos(2\varphi) \quad (2.4)$$

with some parameter $\varepsilon < 1$. For all values $\varepsilon > 0$ we get $\delta = 0$, but at $\varepsilon = 0$ we get $\delta = -\pi$. This example shows that δ does not always continuously depend on the orbits.

However, in the typical cases, δ is a continuous function and for a given fixed $V(r)$, we have $\delta(r_1, r_2)$, i.e., the prescription of perihelion and aphelion uniquely determine the perihelion precession. We now define

$$\delta(r_0) = \lim_{r_1, r_2 \rightarrow r_0} \delta(r_1, r_2). \quad (2.5)$$

The expression $\delta(r_0, r_0)$ is formally the perihelion precession of an exact circular orbit which does not make any sense. So, what is the interpretation of the limit in Eq. (2.5)? It is just the perihelion precession of nearly circular orbits which should be well defined for those cases where the related exact circular orbit is a stable one.

It is the purpose of the present paper to deduce several formulas for the calculation of δ and to apply them to modified Newtonian gravity.

III. NEARLY CIRCULAR ORBITS

How can we calculate the perihelion precession $\delta(r_0)$ for the nearly circular orbit at $r = r_0$? As we have a second-order equation of motion, it should be a function of r_0 , $V(r_0)$, $V'(r_0)$, and $V''(r_0)$ only, where the dash denotes d/dr . An exact circular orbit at this r value is possible if and only if the repelling centrifugal force is compensated by an attractive central force, i.e., if $V'(r_0) > 0$.¹ Now we start to simplify the problem: Adding a constant to the potential does not alter the orbits, so no dependence on $V(r_0)$ should appear. Similarly we argue as follows: if we multiply the function V by a positive constant, then we can compensate this by multiplying t also by a suitable positive constant without changing the orbits, therefore the dependence on the potential can only be in the form of an expression like

$$\frac{V''(r_0)}{V'(r_0)} = [\ln V'(r_0)],$$

which is invariant with respect to a multiplication of V by a positive constant. Finally, we know that δ is dimensionless, and here we need the last possible argument, r_0 , to produce a dimensionless quantity from it: we define

$$\hat{q} = \frac{r_0 \cdot V''(r_0)}{V'(r_0)}. \quad (3.1)$$

For example, we assume $V(r) = -1/r$, then $\hat{q} = -2$ according to Eq. (3.1); this potential is the exact Newtonian gravitational field, where we know that all the bounded orbits are exact ellipses with the center $r = 0$ being located at one of their focal points, so we get $\delta = 0$ for this case. This motivates the definition $q = \hat{q} + 2$, i.e.,

$$q = \frac{r_0 \cdot V''(r_0)}{V'(r_0)} + 2. \quad (3.2)$$

Then it holds that δ must be a function of q . As no other dependencies exist, it must be a universal function $\delta[q]$

¹This sentence is included to fix the sign convention and to make clear that $V'(r_0)$ may be written in the denominator.

being valid for all potentials, and $\delta[0] = 0$ because for the Newtonian theory, $q = 0$.

The next step is to find out the exact form of this universal function. A first idea to assume exact linearity in q is not justified, because then the restriction $\delta > -2\pi$ deduced in the previous section would not be realized.

To find the exact form of this universal function it suffices to insert a nontrivial one-parameter set of examples for which the solution is known.

To this end we discuss how the perihelion advance changes if φ is replaced by $\tilde{\varphi} = k \cdot \varphi$ with an arbitrary positive parameter k , but r remains unchanged. We get $\tilde{\varphi}_0 = k \cdot \varphi_0$ and with Eq. (2.3) then

$$\tilde{\delta} = -2\pi + k \cdot (\delta + 2\pi) = k \cdot \delta + 2\pi \cdot (k - 1). \quad (3.3)$$

The set of possible δ values is restricted by $\delta > -2\pi$, and by Eq. (3.3), also $\tilde{\delta} > -2\pi$. For $k = 1$ we get, of course, $\tilde{\delta} = \delta$. To find the related q values we need the equation of motion which is deduced in the next section.

IV. THE EQUATION OF MOTION

The Lagrangian Eq. (2.1) reads

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r).$$

The angular momentum M is a conserved quantity:

$$M = \frac{\partial L}{\partial \dot{\varphi}} = r^2\dot{\varphi}, \quad \text{hence } \dot{\varphi} = \frac{M}{r^2}. \quad (4.1)$$

Radial motion is already excluded, so $M \neq 0$. Without loss of generality we assume $M > 0$, otherwise we would change the orientation of the r - φ -plane.

The energy E is also a conserved quantity:

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r). \quad (4.2)$$

Inserting Eq. (4.1) into Eq. (4.2) we get

$$E = \frac{\dot{r}^2}{2} + V(r) + \frac{M^2}{2r^2}. \quad (4.3)$$

We derive Eq. (4.3) with respect to t and divide by \dot{r} afterwards. Then we get the Newtonian force equation

$$0 = \ddot{r} + V'(r) - \frac{M^2}{r^3}, \quad (4.4)$$

the term with M^2 represents the centrifugal force. A circular orbit implies $\ddot{r} = 0$, so by Eq. (4.4), this is possible at $r = r_0$ for $V'(r_0) > 0$ only.

To evaluate stability, we define the effective potential as usual:

$$V_{\text{eff}}(r) = V(r) + \frac{M^2}{2r^2} \quad (4.5)$$

leading to

$$V'_{\text{eff}}(r) = V'(r) - \frac{M^2}{r^3} \quad (4.6)$$

and

$$V''_{\text{eff}}(r) = V''(r) + \frac{3M^2}{r^4}. \quad (4.7)$$

A circular orbit at $r = r_0$ requires $V_{\text{eff}}(r_0) = E$ due to Eqs. (4.3) and (4.5) and $V'_{\text{eff}}(r_0) = 0$ due to Eqs. (4.4) and (4.6). This implies

$$M(r_0) = \sqrt{r_0^3 \cdot V'(r_0)}$$

and

$$E(r_0) = V(r_0) + \frac{r_0}{2} \cdot V'(r_0).$$

A simple calculation shows that the following four inequalities are all equivalent to each other:

$$\frac{dM(r_0)}{dr_0} > 0, \quad \frac{dE(r_0)}{dr_0} > 0, \quad V''_{\text{eff}}(r_0) > 0,$$

and

$$r_0 \cdot V''(r_0) + 3 \cdot V'(r_0) > 0. \quad (4.8)$$

A perturbation of the circular orbit can be parametrized by slightly changed initial conditions, or equivalently by slightly changed values of M and E . In a first step we restrict to perturbations which have the same angular momentum M and a slightly changed energy \tilde{E} instead of E . So we have to solve

$$\tilde{E} = \frac{\dot{r}^2}{2} + V_{\text{eff}}(r). \quad (4.9)$$

To get solutions one needs $\tilde{E} > E$. Thus the problem is now equivalent to a one-dimensional motion in the potential V_{eff} . From Eq. (4.9) we get

$$\dot{r} = \pm \sqrt{2} \cdot \sqrt{\tilde{E} - V_{\text{eff}}(r)}.$$

Together with $\dot{\varphi} = M/r^2$ we find

$$\frac{d\varphi}{dr} = \frac{\dot{\varphi}}{\dot{r}} = \pm \frac{M}{r^2 \cdot \sqrt{2} \cdot \sqrt{\tilde{E} - V_{\text{eff}}(r)}}.$$

If the equation $\tilde{E} = V_{\text{eff}}(r)$ has two solutions r_1, r_2 near r_0 with $r_1 < r_0 < r_2$ we get

$$\varphi_0 = \frac{M}{\sqrt{2}} \cdot \int_{r_1}^{r_2} \frac{dr}{r^2 \cdot \sqrt{\tilde{E} - V_{\text{eff}}(r)}}.$$

In the limit $\tilde{E} \rightarrow E$ we have $r_1, r_2 \rightarrow r_0$. We need a positive finite value for φ_0 in this limit, and this is possible for $V''_{\text{eff}}(r_0) > 0$ only, i.e., if inequality (4.8) is valid. If this is fulfilled, then V_{eff} has a regular quadratic minimum at $r = r_0$ and the limit value of φ_0 depends on $V''_{\text{eff}}(r_0)$ only, not on any higher derivatives of $V(r_0)$. This strictly confirms the

assumption made above that the universal formula for δ does not depend on derivatives of V higher than the second one.²

In a second step we should also look for perturbations where M is slightly changed to \tilde{M} . However, such perturbations can be, due to $dM(r_0)/dr_0 > 0$, rearranged to be perturbations at a slightly changed circular orbit with adequately chosen \tilde{r}_0 instead of r_0 , so this does not lead to new conditions.

Now let $(r(t), \varphi(t))$ be a periodic solution and $k > 0$ a parameter. We define $\tilde{r}(t) = r(t)$ and $\tilde{\varphi}(t) = k \cdot \varphi(t)$. We look for a potential $\tilde{V}(r)$ such that $(\tilde{r}(t), \tilde{\varphi}(t))$ becomes a solution. With Eq. (4.1) we get

$$\tilde{M} = k \cdot M \tag{4.10}$$

and with Eq. (4.3)

$$\tilde{E} = \frac{\dot{r}^2}{2} + \tilde{V}(r) + \frac{\tilde{M}^2}{2r^2}. \tag{4.11}$$

An additive constant to the energy can be compensated by adding a constant to the potential, so we may assume $\tilde{E} = E$. A comparison of Eq. (4.3) with Eq. (4.10) and (4.11) leads to

$$\tilde{V}(r) + \frac{k^2 \cdot M^2}{2r^2} = V(r) + \frac{M^2}{2r^2}$$

i.e. to

$$\tilde{V}(r) = V(r) + \frac{(1 - k^2) \cdot M^2}{2r^2}. \tag{4.12}$$

This means that here we used the transformation behavior of δ in a rotating system of reference, which is equivalent to say that the effective potential can be written in a one-parameter set of possibilities as a sum of centrifugal potential and potential of the central force, and so the knowledge about δ for one element of this set suffices to calculate it for all other elements of this one-parameter set.³

For example, let $V(r) = -1/r$, i.e., again the Newtonian potential; we consider nearly circular orbits at $r_0 = 1$.

²As it represents a key point in the deduction, we give also the idea for a third independent proof of this statement; it is meant as pedagogic remark: if one considers the analogous problem of motion in a 4-dimensional pseudo-Riemannian space-time, then the circular orbits are represented by such geodesics, and the nearly circular orbits are represented by the geodesic deviation equation, which itself has the components of the curvature tensor as coefficients, i.e., no more than second derivatives of the potentials appear; and our classical problem of motion can be given as an adequate limit of space-times.

³This is the key point of the deduction: we give as input only the knowledge of δ for the Newtonian potential at $r_0 = 1$, then by this one-parameter set of transformations we produce the knowledge about δ for a one-parameter class of potentials, and this knowledge suffices to identify the universal function $\delta[q]$ uniquely.

Then $V'(r) = 1/r^2$ and with Eq. (4.4) we get using $\dot{r} = 0$ just $M = 1$. Inserting this into Eq. (4.12) we get

$$\tilde{V}(r) = -\frac{1}{r} + \frac{1 - k^2}{2r^2}. \tag{4.13}$$

Now we apply Eq. (3.3). For $k = 1$ we have, of course, $\delta = 0$, so we get

$$\tilde{\delta} = 2\pi \cdot (k - 1). \tag{4.14}$$

With the help of Eq. (3.2) and (4.13) we calculate at $r_0 = 1$

$$\tilde{q} = \frac{\tilde{V}''(1)}{\tilde{V}'(1)} + 2 = \frac{1}{k^2} - 1.$$

All values \tilde{q} with $\tilde{q} > -1$ can appear this way. We invert this equation and get with the assumption $k > 0$ the result

$$k = \frac{1}{\sqrt{1 + \tilde{q}}}. \tag{4.15}$$

Inserting Eq. (4.15) into Eq. (4.14) and removing the tilde we get for arbitrary values $q > -1$

$$\delta[q] = 2\pi \left(\frac{1}{\sqrt{1 + q}} - 1 \right). \tag{4.16}$$

Indeed, this δ covers all values $\delta > -2\pi$ and it is defined for all values $q > -1$. Applying Eq. (3.2) and the condition $V'(r_0) > 0$ already mentioned in Sec. III this is equivalent to the conditions

$$r_0 \cdot V''(r_0) + 3 \cdot V'(r_0) > 0 \quad \text{and} \quad V'(r_0) > 0. \tag{4.17}$$

Equation (4.16) together with Eq. (3.2) defines the universal function we had looked for in Sec. II. It should be emphasized that we did not solve any integrals or differential equations to deduce it, we only applied the obvious symmetries of the system and the knowledge about the absence of perihelion precession in Newtonian gravity.

It is still unclear what conditions for the potential $V(r)$ have to be met that a periodic orbit with prescribed values of perihelion r_1 and aphelion r_2 exists. To this end let us fix a C^2 function $V(r)$ and values r_1, r_2 with $0 < r_1 < r_2$. Both at r_1 and r_2 we have $\dot{r} = 0$, so we get from Eq. (4.3)

$$V(r_1) + \frac{M^2}{2r_1^2} = V(r_2) + \frac{M^2}{2r_2^2}. \tag{4.18}$$

So one needs the finite version of the condition $V'(r_1) > 0$:

$$\Delta V = V(r_2) - V(r_1) > 0. \tag{4.19}$$

By the way, the purely radial oscillations which are excluded here appear as $M = 0$ in Eq. (4.18) and require $\Delta V = 0$.

Inserting Eq. (4.19) into Eq. (4.18) and solving for M we get

$$M = r_1 r_2 \sqrt{\frac{2\Delta V}{r_2^2 - r_1^2}}. \tag{4.20}$$

Inserting Eq. (4.20) into Eq. (4.3) at $r = r_1$ we get

$$E = \frac{r_2^2 V(r_2) - r_1^2 V(r_1)}{r_2^2 - r_1^2}. \quad (4.21)$$

That the motion between r_1 and r_2 is always possible requires $\dot{r} \neq 0$ in this whole interval,⁴ so we have to fulfil, see Eq. (4.3),

$$E > V(r) + \frac{M^2}{2r^2} \quad \text{for } r_1 < r < r_2. \quad (4.22)$$

Inserting Eqs. (4.20) and (4.21) into this inequality we get

$$V(r) < \frac{r_2^2(r^2 - r_1^2)V(r_2) + r_1^2(r_2^2 - r^2)V(r_1)}{r^2 \cdot (r_2^2 - r_1^2)} \quad (4.23)$$

representing the finite version of the first of the conditions (4.17); one can prove this statement by inserting $r_2 = r_1 + \varepsilon$ into Eq. (4.23) and applying the limit $\varepsilon \rightarrow 0$ afterwards.

Finally it should be mentioned that in some limiting cases, also equality instead of a $<$ relation could lead to some solutions; however, in those cases either $V(r)$ fails to be a C^2 function or the test mass would need infinite time to reach the limit; subsequently, this fails to represent a periodic motion, and both are excluded from our considerations.

V. NEARLY CIRCULAR ORBITS—SECOND ROUND

Now we are ready to formulate the result: Let $V(r)$ be a C^2 function and let $r_0 > 0$ be a fixed value of the radial coordinate. Then an exact circular orbit at this r value is possible if and only if the repelling centrifugal force is compensated by an attractive central force, i.e., if $V'(r_0) > 0$, where the dash at V denotes d/dr . This orbit represents a stable one in the sense that small perturbations of the initial conditions always lead to periodic oscillations around $r = r_0$, if and only if $r_0 \cdot V''(r_0) + 3 \cdot V'(r_0) > 0$. If both inequalities are fulfilled, then the perihelion precession $\delta(r_0)$ of the nearly circular orbits at $r = r_0$ is well defined and can be calculated by use of Eqs. (3.2) and (4.16) to

$$\delta(r_0) = 2\pi \cdot \left(\frac{1}{\sqrt{3 + r_0 \cdot V''(r_0)/V'(r_0)}} - 1 \right). \quad (5.1)$$

This equation represents an exact result and is not restricted to potentials close to the Newtonian one.

⁴*A priori*, an inflexion point might be possible, too, i.e., for instance $\dot{r} > 0$ on an interval but a single point where $\dot{r} = 0$. However, such a behavior is already excluded by our assumption that we consider only periodic orbits; a little bit more sophisticated one can argue: such a behavior would *a priori* allow simultaneously two solutions of the field equations, showing that the Cauchy problem would be ill-defined at this point.

If we rewrite Eq. (5.1) in the form

$$\delta(r_0) = 2\pi \cdot \left(\frac{1}{\sqrt{3 + r_0 \cdot [\ln V'(r_0)]'}} - 1 \right), \quad (5.2)$$

then imposing the validity of the inequalities (4.17) is equivalent to require that Eq. (5.2) represents a well-defined real function. This fact can be explained as follows. In the deduction of Eq. (5.2) we only used symmetry arguments and continuous deformations of the orbits, so in the connected component of the Newtonian potential $V(r) = -1/r$ with $V'(r) > 0$ all regularity conditions will be met; as $V'(r_0) = 0$ represents a singular point for Eq. (5.1), it is also clear from that version of the equation why only $V'(r) > 0$ is allowed.

To ease comparison with the literature it proves useful to work in the inverted radial coordinate $u = 1/r$. We define $W(u) = V(r)$ and a dash at W shall denote d/du . Then it holds

$$\begin{aligned} W'(u) &= -r^2 \cdot V'(r), \\ W''(u) &= 2r^3 \cdot V'(r) + r^4 \cdot V''(r). \end{aligned} \quad (5.3)$$

Because this inversion is a dual transformation we can exchange u with r and simultaneously V with W in Eq. (5.3) and get

$$\begin{aligned} V'(r) &= -u^2 \cdot W'(u), \\ V''(r) &= 2u^3 \cdot W'(u) + u^4 \cdot W''(u). \end{aligned} \quad (5.4)$$

Combining Eq. (3.2) with Eq. (5.4) we get at $u_0 = 1/r_0$

$$q = -\frac{u_0 \cdot W''(u_0)}{W'(u_0)}. \quad (5.5)$$

Then we get with Eq. (5.1)

$$\delta(1/u_0) = 2\pi \cdot \left(\frac{1}{\sqrt{1 - u_0 \cdot W''(u_0)/W'(u_0)}} - 1 \right). \quad (5.6)$$

In this coordinate the inequalities (4.17) read, see Eq. (5.3),

$$u_0 \cdot W''(u_0) - W'(u_0) > 0 \quad \text{and} \quad W'(u_0) < 0. \quad (5.7)$$

Analogously to Eq. (5.2) we can now combine Eq. (5.6) with inequalities (5.7) to get

$$\delta(1/u_0) = 2\pi \cdot \left(\frac{1}{\sqrt{1 - u_0 \cdot [\ln(-W'(u_0))]'}} - 1 \right). \quad (5.8)$$

If we have the additional condition that $|u_0 \cdot W''(u_0)| \ll |W'(u_0)|$, then we get from Eq. (5.6) the following approximation for δ :

$$\delta(1/u_0) = -\pi \cdot u_0 \cdot W''(u_0)/|W'(u_0)|. \quad (5.9)$$

The Newtonian potential for a central mass $m > 0$ reads $V(r) = -mG/r$, where G is the gravitational constant. This leads to $W(u) = -mGu$, hence $W'(u) = -mG$ and $W''(u) = 0$. Therefore, Eq. (5.9) is especially useful if the

central potential under consideration is a small perturbation of the Newtonian potential.

VI. COMPARING WITH ADKINS AND MCDONNELL

In [1], see also the references cited there, the orbital precession due to central-force perturbations has been calculated in details, and applications are given; especially, their Eq. (11) [i.e. our Eq. (1.1) above]

$$\Delta\theta_p = -\frac{\pi}{GMmL} \frac{d^2V}{du^2} \Big|_{u=1/L}$$

is, after adequate transformation of the notation, almost identical to our Eq. (5.9).

Let us check this statement in more detail. To this end we now transform our formulas to the notation used in [1]. This first means, from now on we denote the central mass by M and the mass of the test body by m , and we reintroduce the gravitational constant G and light velocity c into the formulas, even in those cases where units have been chosen with $G = c = 1$. Our $W(u)$ in Eq. (5.9) has then to be replaced by $V(u) - GMmu$, where now this $V(u)$ is the small perturbation of the Newtonian potential according to [1]. This leads to $W'(u) = V'(u) - GMm$ and $W''(u) = V''(u)$. So, for the second derivative, there is no difference. For the first derivative, we have within the used approximation, that $V'(u)$ can be neglected in comparison with GMm . So far, the right-hand side of Eq. (5.9) reads

$$-\frac{\pi u_0 V''(u_0)}{GMm}$$

and evaluating this at $u_0 = 1/L$ exactly leads to the right-hand side of Eq. (1.1).

As an example, let V be according to Eq. (1.2), i.e.

$$V(r) = \frac{\alpha}{r} \exp(-r/\lambda), \quad \lambda > 0.$$

Then we have

$$W(u) = -GMmu \cdot [1 - \beta \exp(-1/(\lambda u))] \quad (6.1)$$

with $\alpha = GMm\beta$. The perihelion shift according to [1] is with $\kappa = L/\lambda$, see Eq. (1.3):

$$\Delta\theta_p = -\pi\beta\kappa^2 \exp(-\kappa), \quad (6.2)$$

an expression which is, as a consequence of the approximation used, completely linear in the parameter β . In fact, the result is valid only in regions, where the perturbation is sufficiently small.

Now we apply our formula Eq. (5.6) to the same problem Eq. (6.1). The factor GMm will cancel out anyhow in Eq. (5.6), so we may put $GMm = 1$, i.e. $\alpha = \beta$, already now. We get

$$\begin{aligned} W(u) &= -u + \beta u \cdot \exp(-1/(\lambda u)), \\ W'(u) &= -1 + \left(1 + \frac{1}{\lambda u}\right) \cdot \exp(-1/(\lambda u)). \end{aligned} \quad (6.3)$$

The more conventional form of this potential W appears when one writes it in dependence on $r = 1/u$ to get

$$W = -\frac{1}{r} + \frac{\beta}{r} \cdot \exp(-r/\lambda). \quad (6.4)$$

In the present case, the second inequality (5.7) evaluated at $u_0 = 1/L$ and using $\kappa = L/\lambda > 0$ reads

$$\beta \cdot (1 + \kappa) < \exp(\kappa) \quad (6.5)$$

and gives a restriction for $\beta > 1$ only. In more detail, let us fix any $\kappa_0 > 0$, then we define

$$\beta = \exp(\kappa_0)/(1 + \kappa_0),$$

and then (6.5) is fulfilled for $\kappa > \kappa_0$ only.

In other words, for every $\beta \leq 1$ all positive radius values L appear for a circular orbit. For each $\beta > 1$ there is a positive r_0 such that a circular orbit exists for $L > r_0$ only.⁵

For the second derivative we get from Eq. (6.3):

$$W''(u) = \frac{\beta}{\lambda^2 u^3} \exp(-1/(\lambda u)). \quad (6.6)$$

Inserting Eqs. (6.3) and (6.6) into Eq. (5.6) we get

$$\delta(L) = 2\pi \cdot \left(\frac{1}{\sqrt{1 - \frac{\kappa^2}{1 + \kappa - \exp(\kappa)/\beta}}} - 1 \right). \quad (6.7)$$

Let us examine Eq. (6.7): For very small values $|\beta| \ll 1$ it is continuous and Eq. (6.2) is a good approximation to it in correspondence with the fact that here the perturbation to the Newtonian potential is small and therefore, Eq. (6.2) is applicable.

Things are quite different for other cases: From Eq. (6.4) one can see that only for $\beta = 1$, the potential remains bounded as $r \rightarrow 0$. For this case and small values of $\kappa \ll 1$ Eq. (6.2) gives $\delta = -\pi\kappa^2$, whereas the exact formula Eq. (6.7) has in the same case $\delta = -2\pi(1 - 1/\sqrt{3})$, a totally different behavior.

In Sec. 3 of [1] it is mentioned that the development of perihelion precession in powers of the eccentricity e contains only even powers of e ; this means that for sufficiently small values of e , where linearization in e is justified, our formulas for nearly circular orbits are applicable, too.

⁵Here, this example was chosen mainly to present how to apply our formula. However, if one wants to interpret the physics behind it, one should note that positive values of β would correspond to ghost degrees of freedom if they have an even helicity. But if the extra Yukawa force is mediated by a vector field (like the W and Z bosons), then even positive values of β are allowed.

VII. APPLICATION TO FOURTH-ORDER GRAVITY

Further examples are as follows. In [7] we wrote without explicit proof, see also [8], page 235–236: “Next, let us study the perihelion advance for distorted circlelike orbits. Besides the general relativistic perihelion advance, which vanishes in the Newtonian limit, we have an additional one of the following behavior: for $r \rightarrow 0$ and $r \rightarrow \infty$ it vanishes and for $r \approx 1/m_0$ and $r \approx 1/m_2$ it has local maxima, i.e., resonances.”

This refers to linearized fourth-order gravity, see Stelle [9] for details, where the gravitational potential for a point mass m reads

$$V(r) = -mr^{-1}(1 + \exp(-m_0r)/3 - 4 \exp(-m_2r)/3). \quad (7.1)$$

The perihelion precession of this and similar theories can be calculated by inserting this potential $V(r)$ into the Eq. (5.1). Of course, in the region of large values r , the known approximations like (5.9) would serve also, but our Eq. (5.1) will give the correct result also for those r values, where $V(r)$ is far from being close to the Newtonian potential.

Here in Eq. (7.1), m_0 is the mass of the massive spin 0 graviton stemming from the R^2 term in the Lagrangian, and m_2 is the mass of the massive spin 2 graviton⁶ stemming from the term $C_{ijkl}C^{ijkl}$ in the Lagrangian. Both m_0 and m_2 are assumed to be positive to exclude the appearance of tachyons, but $m_0 \rightarrow \infty$ and $m_2 \rightarrow \infty$ represent sensible limits.

In the case $m_0 = m_2 > 0$, Eq. (7.1) exactly leads to the case $\beta = 1$ discussed in the previous section, the other cases are similar.

VIII. DISCUSSION—FIRST PART

At the time paper [7] was first published, in 1986, this was a purely theoretical question. However, recently there is a development to take such quadratic gravity theories quite seriously in the sense that their predictions can be confronted with observations, see e.g. [10–14] and the references cited there. Also the cosmological solutions of this kind of theories have been analyzed in more new details recently, see e.g. [15–27].

Further, it should be mentioned, that for very distorted orbits, [28,29] give exact results for the perihelion precession for a perturbed Newtonian potential.

Now, let us look for which potentials $V(r)$ the parameter $\delta(r_0)$ takes values according to Eq. (5.1) not explicitly depending on r_0 . After some calculation we get up to the

⁶This massive spin 2 excitation is a ghost, i.e., it carries negative kinetic energy and thereby spoils the stability of the model. Therefore, this contribution has not a direct phenomenological interpretation.

inessential transformations mentioned above; there is a parameter $c > 0$ such that

$$\delta = 2\pi(c^{-1/2} - 1) \quad (8.1)$$

and

$$V(r) = \ln r \quad \text{for } c = 2, \quad V(r) = \frac{1}{c-2} \cdot r^{c-2} \quad \text{else.} \quad (8.2)$$

As expected, one just gets the self-similar functions as solutions to this problem. The case $c = 1$, i.e., $\delta = 0$ just recovers the Newtonian case. Here the bounded orbits are all exact ellipses, the center of symmetry of the potential, $r = 0$, being at one of their focal points.

For $c = 4$, Eq. (8.2) leads to the harmonic oscillator $V = r^2/2$; here the bounded orbits are also ellipses, but now, the center of symmetry of the potential coincides with the center of the ellipses; therefore, the next perihelion is already after one half rotation, i.e. $\varphi_0 = \pi$ and $\delta = -\pi$ in accordance with Eq. (8.1) for this case. This result once more confirms that our result is a strict one also far from the Newtonian potential.

To prepare for the next part, we now apply units such that light velocity $c = 1$; then it holds that the velocity of the test particle in the exact circular orbit at $r = r_0$ is less than light velocity if

$$r_0 \cdot V'(r_0) < 1. \quad (8.3)$$

IX. CIRCULAR AND NEARLY CIRCULAR GEODESICS

In this second part of the paper we generalize the results of the first part to static spherically symmetric space-times; see also [30–33] for other papers on similar topics.

Let us now generalize the resulting Eq. (5.1) to the analogous situation in a 4-dimensional static spherically symmetric space-time.⁷ We additionally assume that Schwarzschild coordinates are possible, so we consider the metric

$$ds^2 = -e^{2\lambda} dt^2 + e^{2\mu} dr^2 + r^2 d\Omega^2, \quad (9.1)$$

where $d\Omega^2$ is the metric of the standard 2-sphere and λ and μ depend on r only. We look for timelike geodesics in this space-time (9.1). After suitable rotation of the coordinate system this geodesic remains completely in the equatorial plane. Because of the chosen symmetry it holds that geo-

⁷A deduction fully analogous to that one from the first part seems not to be easily done. The second variant, namely, to apply the geodesic deviation equation, also leads to unnecessary complicated expressions. As a third idea one could try to apply general exact solutions of the geodesic equation as found in the textbook literature, e.g. [34], but the elliptic integrals appearing there are not easy to handle, therefore we now choose a fourth method, namely, the direct calculation with nearly circular geodesics.

desics in the equatorial plane of the 4-dimensional space-time (9.1) are exactly the geodesics in the 3-dimensional space-time

$$ds^2 = -e^{2\lambda} dt^2 + e^{2\mu} dr^2 + r^2 d\varphi^2. \quad (9.2)$$

The coordinates in (9.2) are x^i where $i = 0, 1, 2$, and the geodesic shall be parametrized by its natural parameter τ :

$$x^i(\tau) = (t(\tau), r(\tau), \varphi(\tau)). \quad (9.3)$$

With a dot denoting $d/d\tau$ we get from (9.2)

$$-1 = -e^{2\lambda} \dot{t}^2 + e^{2\mu} \dot{r}^2 + r^2 \dot{\varphi}^2. \quad (9.4)$$

We may assume $\dot{t} > 0$. The components of the geodesic equation read

$$0 = \ddot{t} + 2\lambda' \dot{t} \dot{r}, \quad (9.5)$$

$$0 = e^{2\mu} (\ddot{r} + \mu' \dot{r}^2) + \lambda' e^{2\lambda} \dot{t}^2 - r \dot{\varphi}^2, \quad (9.6)$$

and

$$0 = r \ddot{\varphi} + 2\dot{r} \dot{\varphi}. \quad (9.7)$$

We define angular momentum M as usual:

$$M = r^2 \dot{\varphi}. \quad (9.8)$$

Purely radial motion shall not be considered, so we have $M \neq 0$. Without loss of generality we may assume $M > 0$, for otherwise we could reverse the orientation of the equatorial plane.

Because of Eq. (9.7), M is a conserved quantity, and we apply this fact to simplify Eqs. (9.4) and (9.6) to

$$-1 = -e^{2\lambda} \dot{t}^2 + e^{2\mu} \dot{r}^2 + M^2/r^2 \quad (9.9)$$

and

$$0 = e^{2\mu} (\ddot{r} + \mu' \dot{r}^2) + \lambda' e^{2\lambda} \dot{t}^2 - M^2/r^3, \quad (9.10)$$

respectively. Inserting Eq. (9.9) into Eq. (9.10) we can cancel \dot{t} to get

$$0 = e^{2\mu} (\ddot{r} + (\lambda' + \mu') \dot{r}^2) + \lambda' \left(1 + \frac{M^2}{r^2}\right) - \frac{M^2}{r^3}. \quad (9.11)$$

Next we look for a circular orbit at a fixed value r_0 of the radial coordinate $r(\tau) \equiv r_0 > 0$. With Eq. (9.11) we get

$$r_0 \cdot \lambda'(r_0) = \frac{z}{z+1}, \quad \text{where } z = \frac{M^2}{r_0^2} > 0. \quad (9.12)$$

This means that a circular timelike geodesic orbit at $r = r_0$ exists if and only if the inequalities

$$\lambda'(r_0) > 0 \quad \text{and} \quad r_0 \cdot \lambda'(r_0) < 1 \quad (9.13)$$

are fulfilled. They are fully analogous to the second of the inequalities (4.17) and inequality (8.3), respectively.

From Eq. (9.12) we get

$$M^2 = \frac{r_0^3 \cdot \lambda'(r_0)}{1 - r_0 \cdot \lambda'(r_0)}. \quad (9.14)$$

For M Eq. (9.14) the condition $dM/dr_0 > 0$ is equivalent to

$$r_0 \cdot \lambda''(r_0) + 3 \cdot \lambda'(r_0) > 2r_0(\lambda'(r_0))^2. \quad (9.15)$$

Next, let us define energy E by

$$E = e^{2\lambda(r)} \dot{t} > 0. \quad (9.16)$$

Because of Eq. (9.5), E is a conserved quantity. We can apply this equation to remove t from Eq. (9.9), leading to⁸

$$-1 = -E^2 e^{-2\lambda} + e^{2\mu} \dot{r}^2 + M^2/r^2. \quad (9.17)$$

For circular orbits at $r = r_0$ this leads to

$$E^2 = e^{2\lambda(r_0)} \left(1 + \frac{M^2}{r_0^2}\right). \quad (9.18)$$

Inserting Eq. (9.14) into Eq. (9.18) we get for the energy of the circular orbit at $r = r_0$

$$E = \frac{e^{\lambda(r_0)}}{\sqrt{1 - r_0 \cdot \lambda'(r_0)}}. \quad (9.19)$$

The condition $dE/dr_0 > 0$ is equivalent to the condition (9.15).

X. PERIHELION PRECESSION IN SPACE-TIME

We now prescribe a value $r_0 > 0$ such that (9.13) and (9.15) are fulfilled. The circular orbit at $r = r_0$ has angular momentum according to (9.14) and energy E according to (9.19). This circular orbit shall now be perturbed, and the perturbed orbit shall have the same angular momentum M but a different energy $\bar{E} \neq E$. For the radial coordinate r in dependence on τ we make the following ansatz:

$$r = r_0 + \varepsilon \cdot \sin(\alpha\tau), \quad (10.1)$$

where α is a positive parameter and ε shall be small such that higher powers of ε may be neglected. We insert this ansatz (10.1) into Eq. (9.11) and get the following identity:

$$\alpha = e^{-\mu(r_0)} \cdot \frac{\sqrt{r_0 \cdot \lambda''(r_0) + 3 \cdot \lambda'(r_0) - 2r_0(\lambda'(r_0))^2}}{\sqrt{1 - r_0 \cdot \lambda'(r_0)} \cdot \sqrt{r_0}}. \quad (10.2)$$

It is remarkable that just the inequalities deduced before ensure that α becomes a well-defined positive real. From Eqs. (10.1) and (10.2) we get that the time from one perihelion to the next is then τ_0 defined by

⁸The material in this section is essentially textbook standard, as can be found e.g. in [34]; but we presented it here in details to maintain a self-consistent notation.

$$\tau_0 = 2\pi/\alpha. \quad (10.3)$$

The perihelion shift δ is defined as

$$\delta = \varphi(\tau_0) - \varphi(0) - 2\pi; \quad (10.4)$$

it measures how much the change in the angular coordinate φ differs from 2π when the orbit changes from one perihelion to the next one. Clearly, for $\delta = 0$, the orbits are exactly closed after one revolution. From Eqs. (9.8), (10.3), and (10.4) we get

$$\delta = \frac{2\pi}{\alpha} \cdot \frac{M}{r_0^2} - 2\pi \quad (10.5)$$

thus leading to the final result

$$\delta = 2\pi \cdot \left[\frac{e^{\mu(r_0)}}{\sqrt{3 - 2r_0 \cdot \lambda'(r_0) + r_0 \cdot \lambda''(r_0)/\lambda'(r_0)}} - 1 \right]. \quad (10.6)$$

XI. DISCUSSION—SECOND PART

The final formula Eq. (10.6) has a structure quite similar to the corresponding formula Eq. (5.1) from the first part. But a direct change over from one of them to the other one is not easily done, so it was really necessary to deduce both of them. In Eq. (10.6) one can observe, that the spatial metric component encoded by the function μ essentially enters the formula but none of their derivatives do enter here. This is in contrast to the temporal metric component encoded by the function λ from which only the first and second derivative do enter.

As a first test, let us insert the Schwarzschild–de Sitter solution into Eq. (10.6). In units where $c = G = 1$, we have to insert into Eq. (9.1)

$$e^{2\lambda} = e^{-2\mu} = 1 - \frac{2m}{r} - \frac{\Lambda}{3} \cdot r^2, \quad (11.1)$$

where $m > 0$ is the mass of the source and $\Lambda \geq 0$ the

cosmological constant. For $\Lambda = 0$ this leads to

$$\delta = 2\pi \cdot \left[\frac{1}{\sqrt{1 - 6m/r_0}} - 1 \right]. \quad (11.2)$$

This is a strict result and is applicable for all values $r_0 > 6m$. Equation (11.2) is surprisingly unknown up to now. It is in good agreement with the fact that stable circular orbits exist for $r_0 > 6m$ only. For $r_0 \gg m$, one can develop in powers of m and get the well-known approximation

$$\delta \approx \frac{6\pi m}{r_0}. \quad (11.3)$$

For example, for $r_0 = 24m$, the exact Eq. (11.2) leads to $\delta = \pi \cdot (4/\sqrt{3} - 2) \equiv 55,7^\circ$, whereas the approximation (11.3) leads to $\delta = \pi/4 \equiv 45^\circ$.

For $\Lambda > 0$, the condition that a timelike circular orbit exists is the same as for $\Lambda = 0$, namely, $r_0 > 3m$, but the formula for perihelion shift becomes a little bit more complicated,

$$\delta = 2\pi \cdot \left[\frac{1}{\sqrt{1 - 6m/r_0}} \cdot \left(1 + \frac{\Lambda r_0^3}{6m} \cdot \left(3 + \frac{9m}{r_0 - 6m} \right) \right) - 1 \right]. \quad (11.4)$$

However, in the case $\Lambda \ll 1/m^2$ and r_0 is not too large, one gets the useful approximation

$$\delta \approx \frac{6\pi m}{r_0} + \frac{\pi \Lambda r_0^3}{m}.$$

In a future paper, Eq. (10.6) shall be applied also to other spherically symmetric metrics.

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