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Black holes can have curly hair

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We study equilibrium conditions between a static, spherically symmetric black hole and classical matter in terms of the radial pressure to density ratio $p_r/\rho = w(u)$, where u is the radial coordinate. It is shown that such an equilibrium is possible in two cases: (i) the well-known case $w \to -1$ as $u \to u_h$ (the horizon), i.e., "vacuum" matter, for which $\rho(u_h)$ can be nonzero; (ii) $w \to -1/(1+2k)$ and $\rho \sim (u - u_h)^k$ as $u \to u_h$, where $k > 0$ is a positive integer ($w = -1/3$ in the generic case $k = 1$). A noninteracting mixture of these two kinds of matter can also exist. The whole reasoning is local, hence the results do not depend on any global or asymptotic conditions. They mean, in particular, that a static black hole cannot live inside a star with nonnegative pressure and density. As an example, an exact solution for an isotropic fluid with $w = -1/3$ (that is, a fluid of disordered cosmic strings), with or without vacuum matter, is presented.

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In real astrophysical conditions, black holes do not exist in empty space but are rather surrounded by some kind of matter which is either in equilibrium with the black hole or is falling on it. In other words, real black holes are "dirty." Meanwhile, the famous no-hair theorems (see, e.g., [1,2] and references therein) are not directly applicable to such situations of evident astrophysical interest. The main route in generalizing the possible black hole "hair" in such theorems consists in considering different (dilaton, gauge, etc.) fields, whereas a much simpler but physically and astrophysically more relevant environment, namely, macroscopic matter with certain pressure and density, drops out from consideration.

The aim of the present paper is to partly fill this gap and to prove some statements of this kind in the framework of general relativity. Strange as it may seem, to the best of our knowledge, they were not found (or at least explicitly formulated) before.

The conditions we will rely on are the horizon regularity, the Einstein equations and the conservation law for matter. For simplicity, we restrict ourselves to static, spherically symmetric configurations. The manner of reasoning is close to that of Ref. [3], where we have obtained some model-independent restrictions on the kinds of matter able to support regular cosmological Killing horizons in Kantowski-Sachs geometries.

We begin with writing the general static, spherically symmetric metric in the form

$$ds^2 = A(u)dt^2 - \frac{du^2}{A(u)} - r^2(u)(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where we have chosen the quasiglobal radial coordinate, corresponding to the "gauge" condition $g_{00}g_{11} = -1$. It has the following important properties [4,5]: it always takes a finite value $u = u_h$ at a Killing horizon where $A(u) = 0$[1]; moreover, near a horizon, the increment $u - u_h$ is a multiple (with a nonzero constant factor) of the corresponding increments of manifestly well-behaved Kruskal-type null coordinates, used for analytic continuation of the metric across the horizon. Therefore, with this coordinate, the geometry can be considered jointly on both sides of a horizon in terms of a formally static metric (hence the name "quasiglobal"). On the other hand, both $A(u)$ and $r(u)$ should be analytic (or smooth at least up to derivatives of a certain order $s \geq 2$) functions of u at $u = u_h$. A regular horizon corresponds to a regular zero of $A(u)$, i.e., $A(u) \sim (u - u_h)^n$, where $n \in \mathbb{N}$ is the order of the horizon. In the case of a black hole, the outermost zero of A corresponds to the event horizon.

Consider the Einstein equations[2] $G^\nu_\mu \equiv R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R = -8\pi T^\nu_\mu$ for the metric (1), so that, due to the symmetry of the problem, the stress-energy tensor (SET) for an arbitrary

[1]In principle, u can take an infinite value at a candidate horizon where $A \to 0$, but then, as one can check, the canonical parameter of the geodesics also tends to infinity, so that the space-time is already geodesically complete and no continuation is required. Such cases, which can be termed "remote horizons", can be found, e.g., in some solutions of the Brans-Dicke theory [5]. We will not discuss them here.

[2]We use the units $c = \hbar = G = 1$.

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kind of matter can be written as

$$T_\mu^\nu = \mathrm{diag}(\rho, -p_r, -p_\perp, -p_\perp), \tag{2}$$

where the density ρ, the radial pressure p_r and the transverse pressure $p_\perp$ are functions of u.

One of the Einstein equations reads

$$G_0^0 - G_1^1 \equiv 2A\frac{r''}{r} = -8\pi(\rho + p_r), \tag{3}$$

where the prime stands for d/du. Equation (3) leads to a regularity condition at $u = u_h$ in terms of ρ and p_r:

$$p_r(u_h) + \rho(u_h) = 0, \tag{4}$$

or, more precisely, $p_r + \rho$ must have a zero of at least the same order as $A(u)$ since, by regularity, $|r''| < \infty$.

Regularity thus requires that, at a horizon, the null energy condition (NEC)

$$T_\mu^\nu \xi^\mu \xi_\nu \geq 0, \qquad \xi^\mu \xi_\mu = 0. \tag{5}$$

should be obeyed on the verge. Indeed, for the SET (2), the NEC leads to

$$p_r + \rho \geq 0, \qquad p_\perp + \rho \geq 0. \tag{6}$$

The condition (4) is often discussed in connection with backreaction of quantum fields (see, e.g., [6]). Then, there is no great sense to speak of an equation of state; we simply have three different quantities in (2), obtained from quantum mean values in a fixed background, as functions of the radial coordinate. In what follows, we discuss relations between them, mostly between p_r and ρ, in quite a general form, and our reasoning is equally applicable to quantum mean values, classical fields and (which is astrophysically more relevant) usual matter with certain equations of state $p_r = p_r(\rho)$ and $p_\perp = p_\perp(\rho)$.

We are interested in general properties of matter surrounding a black hole horizon. A question of great importance for astrophysics is which kind of matter is consistent with the existence of a horizon. Although the condition (4) excludes p_r and ρ being both positive, it can be satisfied if both quantities tend smoothly to zero as one approaches the horizon. This will be the subject of our study. Accordingly, mostly bearing in mind small ρ and p_r, we will use the linear relation

$$p_r = w\rho, \qquad w = \mathrm{const}. \tag{7}$$

We will assume $\rho \geq 0$. If, in addition, the NEC is satisfied (so that the weak energy condition is satisfied as well), we call the matter normal, otherwise it is said to be phantom. One can note that the NEC is often violated due to quantum effects [6]; phantom matter is also used in many studies as possible dark energy responsible for the accelerated expansion of the Universe.

Consequences of the conservation law. The only nontrivial component of the conservation law $\nabla_\nu T_\mu^\nu = 0$ can be written in the form (the prime denotes d/du)

$$p_r' + \frac{2r'}{r}(p_r - p_\perp) + \frac{A'}{2A}(\rho + p_r) = 0. \tag{8}$$

Suppose the validity of Eq. (7), at least near the horizon. As to the transverse pressure, we only assume that (at least, in the limit $\rho \to 0$)

$$|p_\perp|/\rho < \infty. \tag{9}$$

It is a very weak restriction: indeed, for comparison, the dominant energy condition would require $|p_\perp|/\rho \leq 1$.

Then, near the horizon, the term with r' in Eq. (8) can be neglected as compared with the third one. In the leading approximation (i.e., retaining terms of the order $\rho/\Delta u$), we obtain, as $A \to 0$,

$$\rho \sim A^{-(w+1)/(2w)}, \qquad w \neq 0. \tag{10}$$

The value $w = 0$ (dust) is naturally excluded since noninteracting dust cannot be in equilibrium in a static gravitational field. For different w, it follows from Eq. (10):

(i) $w > 0$ (normal matter with $p_r > 0$) or $w < -1$ (phantom). The density diverges as $A \to 0$. Thus such matter cannot exist near a horizon.

(ii) $-1 < w < 0$ (normal matter with $p_r < 0$), in this case, both ρ and p_r tend to zero at the horizon, and the condition (4) holds.

(iii) $w = -1$: this special case requires more attention. First, (4) now may hold with $p_r = -\rho \neq 0$, which corresponds to a "vacuum fluid" considered in the next paragraph. Second, if we still assume $\rho \to 0$ as $A \to 0$, one can check that such a solution to Eq. (8) near $u = u_h$ can exist, but it cannot conform to the regularity requirements connected with the Einstein equations to be discussed below. Indeed, let us, going ahead, take ρ in the form (15) and also assume a Taylor expansion of $p_r(\rho)$ at small ρ,

$$p_r = -\rho + b\rho^2 + \ldots. \tag{11}$$

Substituting all this into Eq. (8) and equating coefficients by equal powers of Δu, we immediately obtain $\rho_k = 0$, which means that $\rho(u)$ cannot be represented by a Taylor series near $u = u_h$. It is a general result which does not depend on k or on the behavior of the function $r(\rho)$. One can find an asymptotic solution to Eq. (8) for small A under the assumption (11), having the form $\rho \approx -2/(b \ln A)$, so that really $\rho \to 0$ as $A \to 0$; however, in accord with the above general result, this solution does not have the form (15) and thus should be rejected.

Inclusion of a vacuum fluid. Eqs. (4) and (8) for matter under consideration do not change if we add a "vacuum matter" with the SET [7]

$$T^\nu_{\mu(\mathrm{vac})} = \mathrm{diag}(\rho_{(\mathrm{vac})}, \rho_{r(\mathrm{vac})}, -p_{\perp(\mathrm{vac})}, p_{\perp(\mathrm{vac})}), \tag{12}$$

and if there is no interaction between T_μ^ν and $T^\nu_{\mu(\mathrm{vac})}$, that is, the conservation law holds for each of them separately: $\nabla_\nu T^\nu_\mu = 0$ and $\nabla_\nu T^\nu_{\mu(\mathrm{vac})} = 0$.

The definitive property of vacuum matter is that the SET (12) preserves its form under arbitrary radial boosts [7]. Examples of such matter are usual Maxwell radial electric and magnetic fields for which $p_{\perp(\mathrm{vac})} = \rho_{(\mathrm{vac})}$, their analogs in nonlinear electrodynamics with Lagrangians of the form $L_e = L_e(F)$, $F \equiv F_{\mu\nu}F^{\mu\nu}$ [8], Yang-Mills fields with a similar structure of the SET, and clouds of radially directed cosmic strings [9]. Independently of a particular realization of such vacuum matter, a number of important properties follow from its algebraic structure, $T^0_{0(\mathrm{vac})} = T^u_{u(\mathrm{vac})}$ [7]; it can be used both for constructing globally regular black hole models [7,8] and for describing dark energy [7,10].

By definition, vacuum matter does not contribute to the third term in (8). Therefore, the above conclusions are valid for matter with the SET (2) independently of whether or not there is a noninteracting admixture of a vacuum fluid with the SET (12).

Consequences of the Einstein equations. There are two independent components of the Einstein equations for the metric (1), with the unknown functions $A(u)$ and $r(u)$. Assuming the total SET $T^\nu_{\mu(\mathrm{tot})} = T^\nu_\mu + T^\nu_{\mu(\mathrm{vac})}$, we can choose such two components as Eq. (3) and the equation containing only first-order derivatives,

$$G^1_1 \equiv \frac{1}{r^2}[-1 + A'rr' + Ar'^2] = -8\pi(\rho_{(\mathrm{vac})} - p_r). \quad (13)$$

Now, we require that the metric should be analytic, or at least sufficiently smooth, in terms of the quasiglobal coordinate u (whose distinguished nature is discussed above, after Eq. (1)) and thus admit continuation through the horizon. Therefore, we can write the Taylor expansions in $\Delta u \equiv u - u_h$:

$$\begin{aligned} A(u) &= A_n \Delta u^n[1 + o(1)], \\ r(u) &= r_h + r'_h\Delta u + \frac{1}{2}r''_u\Delta u^2 + o(\Delta u^2), \end{aligned} \quad (14)$$

with finite constants $A_n > 0$, $r_h > 0$, r'_h, and r''_h. Recall that $n \in \mathbb{N}$ is the order of the horizon.

The left-hand side of Eqs. (3) and (13) are also smooth at the horizon. The same then applies to the right-hand side, in other words, both ρ (hence p_r) and $\rho_{(\mathrm{vac})}$ are smooth and, in particular, since we are interested in configurations with $\rho \to 0$ as $u \to u_h$, we can write in the same limit

$$\rho = \rho_k \Delta u^k[1 + o(1)], \qquad k \in \mathbb{N}, \quad (15)$$

where $\rho_k = \mathrm{const} > 0$ and k is the number of the first nonvanishing term of the Taylor series. Combining this with Eq. (10), we obtain:

$$k = -n\frac{w+1}{2w} \Rightarrow w = -\frac{n}{n+2k}, \quad (16)$$

a discrete set of values of $w = p_r/\rho|_{u=u_h}$. This whole set belongs to the range of interest, $-1 < w < 0$.

Let us now substitute these quantities to the Einstein equations. Equation (3) in the main approximation (with the largest terms kept on each side) gives

$$2A_n\frac{r''_h}{r_h}\Delta u^n = -8\pi(w+1)\rho_k\Delta u^k. \quad (17)$$

Evidently, finiteness of r''_h leads to the requirement

$$k \geq n, \quad (18)$$

where $k > n$ corresponds to $r''_h = 0$.

Equation (13), in turn, gives in its main approximation [where the first terms on both sides are simply rewritten while others are represented by their approximate expressions according to (14) and (15)]

$$-1 + nA_n r_h r'_h \Delta u^{n-1} = -8\pi r_h^2[\rho_{(\mathrm{vac})}(u_h) - w\rho_k\Delta u^k], \quad (19)$$

This relation leads to different results depending on the presence or absence of $\rho_{(\mathrm{vac})}$.

Indeed, if $\rho_{(\mathrm{vac})} = 0$ at the horizon, the right-hand side of (19) is zero at $u = u_h$, the only way to satisfy the equation is to require $n = 1$, and then $A_1 r_h r'_h = 1$. The horizon is simple, and Schwarzschild-like. Evidently, the generic case in the expansion (15) is $k = 1$, and Eq. (16) leads to $w = -1/3$ which, in case $p_r = p_\perp$, corresponds to a fluid of chaotically distributed cosmic strings (see [11] and references therein).

In case $\rho_{(\mathrm{vac})} \neq 0$, any $n \geq 1$ is admissible; if $n > 1$, Eq. (19) gives $\rho_{(\mathrm{vac})}(u_h) = 1/(8\pi r_h^2)$, while for $n = 1$ we have $\rho_{(\mathrm{vac})}(u_h) = (1 - A_1 r_h r'_h)/(8\pi r_h^2)$. Again, the generic case is certainly $n = k = 1$ and $w = -1/3$. One can also note that if matter is everywhere nonphantom, Eq. (3) leads to $r'' < 0$, and if, in addition, there is a spatial asymptotic (not necessarily flat) $r \to \infty$ as $u \to \infty$, then $r' > 0$ in the whole space, and $r'_h > 0$ in particular.

We have actually proved the following theorem:

Theorem 1. A static, spherically symmetric black hole can be in equilibrium with a static matter distribution with the SET (2) only if near the event horizon ($u \to u_h$, where u is the quasiglobal radial coordinate) either (i) $w \to -1$ (matter in this case has the form of a vacuum fluid) or (ii) $w \to -1/(1+2k)$, where $w \equiv p_r/\rho$ and k is a positive integer. In case (i), the horizon can be of any order n, and $\rho(u_h)$ is nonzero. In case (ii), the horizon is simple, and $\rho \sim (u - u_h)^k$.

The generic case of such a nonvacuum hairy black hole is $k = 1$, implying $w = -1/3$. In the isotropic case, $p_r = p_\perp$, it corresponds to a fluid of disordered cosmic strings [11]. Since such strings are, in general, arbitrarily curved and may be closed, one can express the meaning of the theorem by the words "nonvacuum black holes can have curly hair". Recall, however, that in general our w characterizes the radial pressure, while the transverse one is only restricted by the condition (9).

Other values of k ($k = 2$, 3 etc.) represent special cases obtainable by fine tuning of the parameter w.

In the presence of vacuum matter with the SET (12), the following theorem holds:

Theorem 2. A static, spherically symmetric black hole can be in equilibrium with a noninteracting mixture of static nonvacuum matter with the SET (2) and vacuum matter with the SET (12) only if, near the event horizon ($u \to u_h$), $w \equiv p_r/\rho \to -n/(n + 2k)$, where $n \in \mathbb{N}$ is the order of the horizon, $n \leq k \in \mathbb{N}$, and $\rho \sim (u - u_h)^k$.

Thus a horizon of a static black hole can in general be surrounded by vacuum matter and matter with $w = -1/3$, which is true for any order of the horizon if $n = k$. (There also can be configurations with $k > n$ and fine-tuned equations of state where $w = -n/(n + 2k) > -1/3$.) An arbitrarily small amount of other kinds of matter, normal or phantom, added to such a configuration, should break its static character by simply falling onto the horizon or maybe even by destroying the black hole. In other words, black holes may be hairy, or dirty, but the possible kinds of hair are rather special in the near-horizon region: normal (with $p_r \geq 0$) or phantom hair are completely excluded. In an equilibrium configuration, all "dirt" is washed away from the near-horizon region, leaving there only vacuum-like or modestly exotic, probably "curly" hair.

In particular, a static black hole cannot live inside a star of normal matter with nonnegative pressure unless there is an accretion region around the horizon or a layer of "string" and/or vacuum matter.

We did not discuss the behavior of $p_\perp$ and $p_{\perp(\mathrm{vac})}$ (except for the restriction (9)). In fact, these quantities are inessential for our reasoning but should be necessarily specified for finding complete solutions in particular models. Our inferences are quite general and hold for all kinds of hair: for instance, in all known examples of black holes with scalar fields (see, e.g., [12] and references therein), the SETs near the horizon must satisfy the above conditions, which may be directly checked.

Also, our approach is relevant to semiclassical black holes in equilibrium with their Hawking radiation (the Hartle-Hawking state), whose SET essentially differs from that of a perfect fluid. Since the density of quantum fields is, in general, nonzero at the horizon (see Sec. 11 of the textbook [1] for details), the regularity condition (4) tells us that such quantum radiation should behave near the horizon like a vacuum fluid. Our results show that a black hole can be in equilibrium with a mixture of Hawking radiation and some kinds of classical matter with $-1 < w < 0$ (including the important case of a Pascal perfect fluid with $p_r = p_\perp$). Possible effects of this circumstance for semiclassical black holes need a further study. Moreover, large enough black holes, for which the Hawking radiation may be neglected, can be in equilibrium with classical matter alone, also including the case of a perfect fluid.

Our reasoning was entirely local, restricted to the neighborhood of the horizon, and the results, which involve the single parameter $w = p_r/\rho$, are in other respects model-independent. Meanwhile, a full analysis of specific systems would require the knowledge of the equation of state and conditions on the metric in the whole space (e.g., the asymptotic flatness condition). Such an analysis depends on the model in an essential way and is beyond the scope of this paper. One can add that the equations of state well-behaved near the horizon are often incompatible with reasonable conditions at infinity (see the example below); it simply means that such matter does not extend to infinity and can only occupy a finite region around the horizon.

It would be of interest to generalize our results to non-spherical and rotating distributions of matter.

Example. In conclusion, let us present an exact solution for a system of utmost interest described by the above theorems. Consider a region of space with a noninteracting mixture of a vacuum fluid specified by $8\pi\rho_{(\mathrm{vac})} = \Lambda(u)$ [$p_{\perp(\mathrm{vac})}$ is then found from the conservation law for the SET (12)] and an isotropic fluid of cosmic strings, such that $p_r = p_\perp = -\rho/3$. Then Eq. (8) leads to $\rho = \rho_0 A$, Eq. (3) takes the form $r'' + \alpha^2 r = 0$, and, without loss of generality, we have

$$r(u) = r_0 \sin\alpha u, \tag{20}$$

where $\rho_0 > 0$, $r_0 > 0$ are arbitrary constants and $\alpha = (8\pi\rho_0/3)^{1/2}$. The remaining unknown function $A(u)$ can be found from Eq. (13), which turns out to be linear,

$$A' r r' + A(r'^2 + \alpha^2 r^2) = 1 - \Lambda(u) r^2, \tag{21}$$

hence easily integrable by quadratures for an arbitrary dependence $\Lambda(u)$ (or $\Lambda(r)$, as was used, e.g., in [7,10]). (Let us stress that our solution is different from that in Ref. [9], obtained for a cloud of unidirectional strings.) In particular, for $\Lambda = \mathrm{const}$ we find

$$A(u) = \frac{1}{\alpha^2 r_0^2}\left[1 - C\cot\alpha u - \Lambda r_0^2(1 - \alpha u \cot\alpha u)\right], \tag{22}$$

with $C = \mathrm{const}$. A horizon corresponds to $A = 0$, e.g., in case $\Lambda = 0$ we obtain $u_h = (1/\alpha)\arctan C$; the horizon is simple: one can verify that $A'(u_h) \neq 0$. As follows from (20), this solution has no large r asymptotic, but it can be incorporated in an asymptotically flat model by matching it at some $u > u_h$ to some intermediate layer (e.g., described by an analytic solution like the one for an incompressible fluid) admitting zero pressure at some surface, at which it can be further matched to the Schwarzschild solution.

It seems instructive to trace the limiting transition from Eqs. (20) and (22) to the vacuum Schwarzschild–(anti) de Sitter metric. As the matter density vanishes, $\rho \to 0$, so that $\alpha \to 0$, it is convenient to write $r_0 = \beta/\alpha$, $C = 2m\alpha/\beta$, where β and m are new constants. Then, after simple calculations, we obtain in the limit $\alpha \to 0$

$$r = \beta u, \qquad A(u) = \frac{1}{\beta^2}\left(1 - \frac{2m}{u\beta} - \frac{\Lambda}{3}\beta^2 u^2\right).$$

Rescaling $t \mapsto \tilde{t} = t/\beta$ and using the coordinate r, we obtain the metric (1) with $A(r) = 1 - 2m/r - \Lambda r^2/3$, as required.

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[1] V. P. Frolov and I. D. Novikov, *Black Hole Physics: Basic Concepts and New Developments* (Kluwer Academic, Boston, 1998).
[2] J. D. Bekenstein, in *Cosmology and Gravitation*, edited by M. Novello (Atlantisciences, France, 2000), arXiv:gr-qc/9808028 (review).
[3] K. A. Bronnikov and O. B. Zaslavskii, Classical Quantum Gravity **25**, 105015 (2008).
[4] K. A. Bronnikov, Phys. Rev. D **64**, 064013 (2001).
[5] K. A. Bronnikov, G. Clément, C. P. Constantinidis, and J. C. Fabris, Gravitation Cosmol. **4**, 128 (1998); Phys. Lett. A **243**, 121 (1998).
[6] M. Visser, Phys. Rev. D **54**, 5103 (1996).
[7] I. G. Dymnikova, Gen. Relativ. Gravit. **24**, 235 (1992); Classical Quantum Gravity **19**, 725 (2002); Int. J. Mod. Phys. D **12**, 1015 (2003).
[8] K. A. Bronnikov, Phys. Rev. D **63**, 044005 (2001).
[9] P. S. Letelier, Phys. Rev. D **20**, 1294 (1979).
[10] K. A. Bronnikov and I. G. Dymnikova, Classical Quantum Gravity **24**, 5803 (2007).
[11] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge Univ. Press, Cambridge, England, 1995); S. Capozziello, V. F. Cardone, G. Lambiase, and A. Troisi, Int. J. Mod. Phys. D **15**, 69 (2006).
[12] K. A. Bronnikov and J. C. Fabris, Phys. Rev. Lett. **96**, 251101 (2006).