Gaussian effective potential for the standard model SU(2) × U(1) electroweak theory

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The Gaussian effective potential is derived for the non-Abelian SU(2) × U(1) gauge theory of electroweak interactions. At variance with naive derivations, the Gaussian effective potential is proven to be a genuine variational tool in any gauge. The role of ghosts is discussed and the unitarity gauge is shown to be the only choice which allows calculability without insertion of further approximations. The full non-Abelian calculation confirms the existence of a light Higgs boson in the nonperturbative strong coupling regime of the Higgs sector.

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I. INTRODUCTION

It is now widely believed that the Higgs sector of electroweak interactions can be described by a scalar field with a self-interaction which could be large enough to raise some doubt on the validity of standard perturbative approaches. Thus, while perturbative results might be questioned, nonperturbative calculations would be required at least for comparison. Variational calculations are usually quite reliable for describing strong coupling phenomena, but their use in quantum field theory must face several difficult problems [1]. The problem of calculability can only be solved by use of a Gaussian wave functional, which has its merits as discussed in several papers on the Gaussian effective potential (GEP) [2–11]. Another important problem is the predominance of high momentum fluctuations in the vacuum expectation values. However, the standard model of electroweak interactions is usually regarded as an effective model with a finite energy cutoff which regulates the theory. Thus the role of high momentum fluctuations is in part reduced, as the predictions of the GEP on effective models have been found to be reliable when compared to experimental results [12–14].

It has been pointed out that gauge invariance could be another important challenge for variational calculations, as there is no way to build a gauge invariant Gaussian functional in non-Abelian gauge theories [15]. It has been argued that, in principle, if the states are not gauge invariant, they could be unphysical and span a larger Hilbert space where the unphysical energies could even be lower than the true physical vacuum [15]. However, in this paper we show that a genuine variational GEP can be found for the non-Abelian SU(2) × U(1) standard model of electroweak interactions, and that for any chosen gauge the GEP can be proven to stay above the true effective potential. The genuine variational nature of the GEP makes the choice of gauge a question of taste and numerical convenience, and the physical unitarity gauge may be used without affecting the variational nature of the calculation.

Some further motivation for the work arises from a successful attempt to explain mass generation in the minimal left-right symmetric model of electroweak interactions [16,17], where two scalar Higgs doublets and no bidoublet are present. At tree level, that model predicts a vanishing expectation value for one of the scalar Higgs doublets, and that is a problem since all the fermionic masses turn out to be vanishing as well [18]. In that framework quantum fluctuations have been studied by the GEP and shown [16] to destabilize the symmetric vacuum towards a physical finite expectation value for both the Higgs doublets. While those findings are compatible with the phenomenology, their accuracy could be questioned for the neglect of all the weak couplings. Actually it was a simplified Abelian toy model, with only Higgs and fermionic fields. Thus an extension of the GEP method to the full non-Abelian SU(2) × U(1) gauge group would allow for quantitative predictions in the standard model and in its minimal left-right symmetric versions.

We must mention that this is not the first attempt to apply the GEP to the non-Abelian gauge theory, as previous naive calculations have been reported. It is very important to stress that the reliability of a variational calculation requires that no uncontrolled approximation should be added. The main result of this paper is the rigorous proof of the genuine variational nature of the GEP in the unitarity gauge. In order to avoid problems regarding the gauge dependence of the Hamiltonian, we derive the GEP in the Lagrangian formalism and start from a fully gauge invariant vacuum to vacuum transition amplitude. As in previous works on the U(1) theory [12,13,19], the GEP is derived by a systematic use of Jensen's inequality for expectation values of convex functions. As a consequence the GEP can never fall below the exact effective potential, and its minimum yields the best approximation to the vacuum energy density.

The derivation is useful for clarifying the role played by any gauge choice. In fact Jensen's inequality does not hold for Grassmann anticommuting fields, and when ghost fields are present the naive use of the GEP turns out to be a tree-level perturbative approximation. Thus the gauge must be properly chosen in order to avoid the presence of ghosts, and the unitarity gauge turns out to be a good choice.

In the standard model of weak interactions the method is shown to be a useful nonperturbative tool for the study of the Higgs sector in the strong coupling regime. The GEP predicts the possible existence of a light Higgs boson even if the self-coupling were very large. In other words, a light Higgs boson would not rule out a very large self-coupling which would question most of the perturbative calculations. The role of gauge interactions on the Higgs sector is also discussed and shown to be very small, as expected.

The paper is organized as follows: in Sec. II the full non-Abelian SU(2) × U(1) gauge group is considered, and the main lines of the GEP derivation are outlined; in Sec. III the GEP is derived for the standard model of electroweak interactions; in Sec. IV the gap equations are discussed in detail and some predictions are discussed for the strong coupling regime of the Higgs sector.

II. NON-ABELIAN SU(2) × U(1) THEORY

In the standard model of electroweak interactions the physical vacuum is believed to be at a broken-symmetry minimum of the effective potential. Since the SU(2) × U(1) gauge symmetry is broken to the electromagnetic U(1) group, the full gauge invariance of the GEP is not a real issue, provided that the method is shown to be a genuine variational calculation. The non-Abelian SU(2) × U(1) gauge theory of electroweak interactions is described by the Euclidean Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(D_\mu \Phi)^\dagger (D^\mu \Phi) + V(\Phi^\dagger \Phi) + \mathcal{L}_{\rm YM} \tag{1}$$

where $\vec{A}_\mu$, B_μ are the gauge fields, $\mathcal{L}_{\rm YM}$ is the Yang-Mill Lagrangian

$$\mathcal{L}_{\rm YM} = \tfrac{1}{4}\vec{F}_{\mu\nu}\cdot \vec{F}^{\mu\nu} + \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 \tag{2}$$

in terms of the fields

$$\vec{F}_{\mu\nu} = (\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu) + g\vec{A}_\mu \times \vec{A}_\nu \tag{3}$$

and Φ is a Higgs doublet of complex fields ϕ_1, ϕ_2

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{4}$$

The covariant derivative reads

$$D_\mu = \left[\partial_\mu - ig\vec{A}_\mu \cdot \vec{\mathbf{T}} + ig' B_\mu \frac{\mathbf{Y}}{2}\right] \tag{5}$$

where g, g' are the weak couplings and the generators are defined by the 2×2 matrices $\mathbf{Y} = 1$ and $\vec{\mathbf{T}} = \frac{1}{2}\vec{\sigma}$. As usual the charge operator is $\mathbf{Q} = e(\mathbf{T}_3 + \mathbf{Y}/2)$.

In general, the Higgs doublet Φ can be parametrized according to

$$\Phi = \rho e^{i\gamma} e^{i\sigma_3 \phi} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{6}$$

where $\rho \geq 0$ is a real field, and the three phases γ, ϕ, θ may be taken in the ranges $0 \leq \gamma \leq 2\pi$, $0 \leq \phi \leq 2\pi$, and $0 \leq \theta \leq \pi/2$. Without fixing any special gauge we would like to discuss some general properties of the generating functional

$$Z[J] = \int D[\phi_1, \phi_2, \vec{A}, B] e^{-\int d^4x(\mathcal{L} - \rho J)}. \tag{7}$$

A change of integration variables yields

$$Z[J] = \int D[\rho^4, \sin^2\theta, \gamma, \phi, \vec{A}, B] e^{-\int d^4x(\mathcal{L} - \rho J)} \tag{8}$$

where $\mathcal{L}$ can be written as

$$\mathcal{L} = \mathcal{L}_\rho + \mathcal{L}_{\mathcal{G}} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\rm YM} \tag{9}$$

according to the following definitions: $\mathcal{L}_\rho$ is the Lagrangian of the self-interacting scalar real field ρ

$$\mathcal{L}_\rho = \tfrac{1}{2}(\partial_\mu \rho)^2 + V(\rho^2); \tag{10}$$

$\mathcal{L}_{\mathcal{G}}$ contains the gauge phase quadratic terms

$$\mathcal{L}_{\mathcal{G}} = \tfrac{1}{2}\rho^2[(\partial_\mu \gamma)^2 + (\partial_\mu \phi)^2 + (\partial_\mu \theta)^2]; \tag{11}$$

$\mathcal{L}_2$ contains quadratic interaction terms for the gauge fields

$$\mathcal{L}_2 = \tfrac{1}{8}\rho^2[g^2 \vec{A}_\mu \cdot \vec{A}^\mu - 2gg' B^\mu \vec{A}_\mu \cdot \vec{R} + g'^2 B_\mu B^\mu]; \tag{12}$$

where $\vec{R}$ is the phase dependent vector

$$\vec{R} = \begin{pmatrix} \sin(2\theta)\cos(2\phi) \\ -\sin(2\theta)\sin(2\phi) \\ \cos(2\theta) \end{pmatrix}; \tag{13}$$

$\mathcal{L}_1$ contains linear interaction terms for the gauge fields

$$\mathcal{L}_1 = \tfrac{1}{2}\rho^2 g\vec{A}^\mu \cdot \vec{\Gamma}_\mu + \tfrac{1}{2}\rho^2 g' B^\mu \Theta_\mu \tag{14}$$

where $\vec{\Gamma}_\mu$ and Θ_μ depend on phases and are defined as follows

$$\vec{\Gamma}_\mu = \vec{R}\partial_\mu \gamma + \begin{pmatrix} \sin(2\phi)\partial_\mu \theta \\ \cos(2\phi)\partial_\mu \theta \\ -\partial_\mu \phi \end{pmatrix} \tag{15}$$

$$\Theta_\mu = \partial_\mu \gamma + \cos(2\theta)\partial_\mu \phi. \tag{16}$$

According to the standard De Witt-Faddeev-Popov method [20] the integration over the gauge group can be dealt with by insertion of a gauge fixing term

$$\mathcal{L}_{\rm fix} = -\frac{1}{\epsilon}(f_\alpha)^2 \tag{17}$$

where the index α runs over the four gauge fields

$$f_\alpha = (\vec{f}, f_B). \tag{18}$$

The gauge invariance of the generating functional $Z[J]$ is preserved provided that a factor is also inserted in the integrand, equal to the determinant of the matrix

$$\mathcal{F}_{\alpha,\beta} = \left(\frac{\delta f_\alpha}{\delta \lambda_\beta}\right)_{\lambda_\beta=0} \tag{19}$$

where λ_β is the generic parameter of a gauge transformation [21]. The gauge invariant generating functional now reads

$$Z[J] = \int D[\rho^4, \sin^2\theta, \gamma, \phi, \vec{A}, B] \det\mathcal{F} e^{-\int d^4x(\mathcal{L}+\mathcal{L}_{\rm fix}-\rho J)}. \tag{20}$$

From a formal point of view the determinant can be seen as

$$\det\mathcal{F} = e^{-\int d^4x \mathcal{L}_{gh}} \tag{21}$$

where the *ghost* Lagrangian $\mathcal{L}_{gh}$

$$\mathcal{L}_{gh} = -\mathrm{Tr}\log\mathcal{F} \tag{22}$$

can be written in terms of anticommuting Grassmann ghost fields. Thus the definition of the generating functional $Z[J]$ in Eq. (7) can be made gauge invariant by the replacement $\mathcal{L} \to \mathcal{L} + \mathcal{L}_{\rm fix} + \mathcal{L}_{gh}$.

We can see that the integration over phases yields

$$\int D[\sin^2\theta, \gamma, \phi] e^{-\int d^4x \mathcal{L}_{\mathcal{G}}} \sim \prod_x \frac{1}{\rho^3}. \tag{23}$$

Let us use the shorthand notation $D_\gamma = D[\sin^2\theta, \gamma, \phi]$ and $D_\rho = D[\rho, \vec{A}, B]$, and define the average over phases

$$\langle(\ldots)\rangle_\gamma = \frac{\int D_\gamma e^{-\int d^4x \mathcal{L}_{\mathcal{G}}}(\ldots)}{\int D_\gamma e^{-\int d^4x \mathcal{L}_{\mathcal{G}}}}. \tag{24}$$

The generating functional then reads

$$Z[J] = \int D_\rho e^{\int d^4x \rho J} \langle e^{-\int d^4x(\mathcal{L}+\mathcal{L}_{\rm fix}+\mathcal{L}_{gh}-\mathcal{L}_{\mathcal{G}})}\rangle_\gamma. \tag{25}$$

Moreover, for any trial Gaussian Lagrangian $\mathcal{L}_{\rm GEP}(\rho, \vec{A}, B)$ which does not depend on the phases θ, ϕ, γ, a further average can be defined

$$\langle(\ldots)\rangle_\rho = \frac{\int D_\rho e^{-\int d^4x \mathcal{L}_{\rm GEP}}(\ldots)}{\int D_\rho e^{-\int d^4x \mathcal{L}_{\rm GEP}}}. \tag{26}$$

and the exact gauge invariant generating functional can be written as a double average

$$Z[J] = \langle e^{\int d^4x \rho J} \langle e^{-\int d^4x(\mathcal{L}+\mathcal{L}_{\rm fix}+\mathcal{L}_{gh}-\mathcal{L}_{\mathcal{G}}-\mathcal{L}_{\rm GEP})}\rangle_\gamma\rangle_\rho Z_0 \tag{27}$$

where

$$Z_0 = \int D_\rho e^{-\int d^4x \mathcal{L}_{\rm GEP}}. \tag{28}$$

A variational approximation for the effective potential follows from the use of Jensen's inequality: the approximate generating functional $Z_{\rm GEP}[J]$ is bound by the exact one as

$$Z[J] \geq Z_{\rm GEP}[J] = Z_0 e^{-\int d^4x \langle\langle \mathcal{L}+\mathcal{L}_{\rm fix}+\mathcal{L}_{gh}-\mathcal{L}_{\mathcal{G}}-\mathcal{L}_{\rm GEP}-J\rho\rangle_\gamma\rangle_\rho}. \tag{29}$$

Up to a total volume factor, the exact effective potential is defined as the Legendre transform

$$\mathcal{V}[\bar\rho] = -\log Z[J] + \int d^4x J\bar\rho \tag{30}$$

where $\bar\rho$ is the expectation value of the field ρ in the presence of the source J. We assume that $\langle\rho\rangle_\rho = \bar\rho$ where $\bar\rho$ is a parameter of the trial Lagrangian $\mathcal{L}_{\rm GEP}$. In other words $\bar\rho$ is the central value of the quadratic Lagrangian $\mathcal{L}_{\rm GEP}$. It follows that

$$\mathcal{V}[\bar\rho] \leq \mathcal{V}_{\rm GEP}(\bar\rho) = -\log Z_{\rm GEP}[J] + \int d^4x J\bar\rho. \tag{31}$$

Thus the approximate Gaussian effective potential $\mathcal{V}_{\rm GEP}$ is a genuine variational approximation of the exact effective potential, and can be evaluated by the double average

$$\mathcal{V}_{\rm GEP}(\bar\rho) = -\log\int D_\rho e^{-\int d^4x \mathcal{L}_{\rm GEP}} + \int d^4x \langle\langle \mathcal{L} + \mathcal{L}_{\rm fix} + \mathcal{L}_{gh} - \mathcal{L}_{\mathcal{G}} - \mathcal{L}_{\rm GEP}\rangle_\gamma\rangle_\rho. \tag{32}$$

The present derivation holds for any gauge choice, that means the method can be improved by a gauge change. In fact, as for the Abelian U(1) theory [12,13], the limit $\epsilon \to 0$ should be imposed on $\mathcal{L}_{\rm fix}$ in order to improve the reliability of Jensen's inequality in Eq. (29). Under that limit the integration over the gauge group does not introduce new approximations as the constraint in $\mathcal{L}_{\rm fix}$ yields a δ function and the integration over the gauge group becomes exact (it is not affected by the inequality). On the other hand, a gauge choice should not be a problem as the gauge symmetry is broken anyway in the physical vacuum.

The physics of the non-Abelian SU(2) × U(1) model is more evident in unitarity gauge which seems to be the natural choice for discussing the symmetry breaking mechanism. However, there is a more formal motivation for that choice which has to do with calculability. Provided that we take a simple quadratic shape for the trial Lagrangian $\mathcal{L}_{\rm GEP}$, the Gaussian integral and the averages in Eq. (32) can be all easily evaluated with the important exception of the ghost term $\langle\mathcal{L}_{gh}\rangle$. The existence of this term makes the method useless since we do not know how to calculate its average. In a naive approach we could write $\mathcal{L}_{gh}$ in terms of anticommuting Grassmann ghost fields, but Jensen's inequality cannot be proven for Grassmann variables and the result would not be a genuine variational approximation. There would be no control on the approximation. Another naive approach would consist of the mere neglect of this term, and that can be shown to be the tree-level approximation of a perturbative expansion.

However, in the unitarity gauge the constraint functions f_α do not depend on the gauge fields: the mass of the ghost fields scales like $\epsilon^{-1/2}$ and becomes infinite in the $\epsilon \to 0$ limit, decoupling the ghosts from the physical fields. In other words the factor $\mathrm{Det}\mathcal{F}$ in Eq. (20) becomes a constant and can be carried out of the integral. Thus, in the unitarity gauge the average of $\mathcal{L}_{gh}$ is a constant and can be neglected. We conclude that calculability makes the choice of unitarity gauge the only viable choice.

It is instructive to study the behavior of $\mathcal{L}_{gh}$ in the renormalizable ξ-gauge of Fujikawa, Lee and Sanda [22] which is equivalent to the unitarity gauge in the $\epsilon = 1/\xi \to 0$ limit. The matrix $\mathcal{F}$ can be written as [21]

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_{\mathrm{int}} \tag{33}$$

where $\mathcal{F}_{\mathrm{int}}$ contains a linear coupling with the gauge fields, $\mathcal{F}_0$ is the matrix

$$(\mathcal{F}_0)_{\alpha x, \beta y} = \left[\delta_{\alpha\beta}\partial_\mu\partial^\mu + \frac{1}{\epsilon}M_{\alpha\beta}\right]\delta^4(x-y) \tag{34}$$

and $M_{\alpha\beta}$ is a constant mass matrix. Inserting the definition Eq. (22) in Eq. (32), the double average of $\mathcal{L}_{gh}$ can be written as

$$\langle\langle\mathcal{L}_{gh}\rangle\rangle = -\langle\langle\mathrm{Tr}\log\mathcal{F}_0\rangle\rangle - \langle\langle\mathrm{Tr}\log(1 + \mathcal{F}_0^{-1}\mathcal{F}_{\mathrm{int}})\rangle\rangle. \tag{35}$$

The second term can be expanded yielding the perturbative series

$$\langle\langle\mathrm{Tr}\log(1 + \mathcal{F}_0^{-1}\mathcal{F}_{\mathrm{int}})\rangle\rangle \approx \mathrm{Tr}\langle\langle\mathcal{F}_0^{-1}\mathcal{F}_{\mathrm{int}}\rangle\rangle - \tfrac{1}{2}\mathrm{Tr}\langle\langle\mathcal{F}_0^{-1}\mathcal{F}_{\mathrm{int}}\mathcal{F}_0^{-1}\mathcal{F}_{\mathrm{int}}\rangle\rangle + \ldots \tag{36}$$

According to Eq. (34) the average $\langle\langle\mathcal{F}_0^{-1}\rangle\rangle$ can be regarded as the propagator for a massive particle (a ghost) whose mass scales like $1/\sqrt{\epsilon}$. The interaction vertex $\mathcal{F}_{\mathrm{int}}$ is linear in the gauge fields, and the average of any pair $\langle\langle\mathcal{F}_{\mathrm{int}}\mathcal{F}_{\mathrm{int}}\rangle\rangle$ yields a gauge field propagator. Thus a diagrammatic expansion is recovered by Wick's theorem: Eq. (36) can be regarded as the sum of loop diagrams each consisting of a closed ghost ring crossed by any number of gauge lines. At tree level, neglecting all the interaction lines, the double average of $\mathcal{L}_{gh}$ becomes a constant and can be neglected in the effective potential Eq. (32). Thus the naive neglect of $\mathcal{L}_{gh}$ is equivalent to the tree-level approximation of the perturbative expansion. However, in the $\epsilon \to 0$ limit, the ghost mass becomes infinite and all the terms in the expansion vanish. In the $\epsilon \to 0$ limit the renormalizable ξ-gauge becomes the unitarity gauge, and we recover the result that $\mathcal{L}_{gh}$ can only be neglected in the unitarity gauge.

With that gauge choice understood, the double average in Eq. (32) becomes trivial and the GEP can be easily evaluated provided that a simple quadratic shape is chosen for $\mathcal{L}_{\mathrm{GEP}}$. Moreover, if $\mathcal{L}_{\mathrm{GEP}}$ is an even functional the double average of $\mathcal{L}_1$ also vanishes.

However, in order to get the best approximation from Jensen's inequality, the linear term should be shifted by the best choice of the unbroken electromagnetic U(1) gauge. In the Abelian U(1) model [12] the best choice can be shown to be the transverse gauge which is fixed by the constraint $\partial_\mu A^\mu = 0$. In the unitarity gauge we still have a free overall electromagnetic U(1) phase, and the best approximation arises from the transverse electromagnetic gauge. In order to show that, a linear change of variables is required first from the gauge fields $\vec{A}_\mu$, B_μ to the physical fields $W^\pm_\mu$, Z_μ, A_μ; then the best shift for the electromagnetic phase can be discussed, and eventually the double average will be taken.

III. GEP FOR THE STANDARD MODEL

In the unitarity gauge ($\theta = \pi/2$) the physical massive gauge fields $W^\pm$, Z and the electromagnetic gauge field A are defined according to the linear transformation

$$A^1_\mu = \frac{W^+_\mu + W^-_\mu}{\sqrt{2}} \tag{37}$$

$$A^2_\mu = \frac{W^+_\mu - W^-_\mu}{i\sqrt{2}} \tag{38}$$

$$A^3_\mu = \frac{e}{g}A_\mu - \frac{e}{g'}Z_\mu \tag{39}$$

$$B_\mu = -\frac{e}{g}Z_\mu - \frac{e}{g'}A_\mu \tag{40}$$

where the electromagnetic charge e follows from the constraint

$$\frac{e^2}{g^2} + \frac{e^2}{g'^2} = 1. \tag{41}$$

Insertion of these definitions in the quadratic Lagrangian term Eq. (12) yields

$$\mathcal{L}_2 = \frac{\rho^2}{v^2}M_W^2 W^+_\mu W^{-\mu} + \frac{1}{2}\frac{\rho^2}{v^2}M_Z^2 Z_\mu Z^\mu \tag{42}$$

where the masses M_W and M_Z are given by the standard definitions

$$M_W = \frac{vg}{2} \tag{43}$$

$$M_Z = \frac{1}{2}v\sqrt{g^2 + g'^2} \tag{44}$$

in terms of the free parameter v. The gauge field A_μ remains massless, as it must be, since the electromagnetic U(1) symmetry is unbroken. For the U(1) gauge group we get the best variational approximation [12,13,19] in the transverse gauge $\partial_\mu A^\mu = 0$. That constraint is imposed by still taking the gauge-fixing term to be

$$\mathcal{L}_{\rm fix} = \frac{1}{\epsilon}(\partial_\mu A^\mu)^2 \tag{45}$$

where the limit $\epsilon \to 0$ is understood. This gauge choice is equivalent to a shift of the integration variables before the average, in order to cancel the longitudinal part of the gauge field A_μ. Then the average can be taken in Eq. (32) and, provided that $\mathcal{L}_{\rm GEP}$ is even, the odd Lagrangian terms give a vanishing contribution. Thus we can drop $\mathcal{L}_1$ and the odd terms of $\mathcal{L}_{\rm YM}$ in the average, and the ghost term $\mathcal{L}_{gh}$ which does not contribute in the unitarity gauge. Insertion of Eq. (9) in the effective potential Eq. (32) yields

$$\mathcal{V}_{\rm GEP}(\bar\rho) = -\log\int D_\rho e^{-\int d^4x \mathcal{L}_{\rm GEP}} + \int d^4x \langle \mathcal{L}_{\rm int}\rangle_\rho, \tag{46}$$

where the interaction Lagrangian now reads

$$\mathcal{L}_{\rm int} = \mathcal{L}_\rho + \mathcal{L}_{\rm fix} + \mathcal{L}_2 + \mathcal{L}_{\rm YM}^{\rm even} - \mathcal{L}_{\rm GEP}. \tag{47}$$

Next we take a shift of the scalar field ρ, and as usual [23] we define the scalar Higgs field h according to

$$h = \rho - \bar\rho. \tag{48}$$

A natural choice for the Gaussian trial Lagrangian is the sum of quadratic Gaussian Lagrangians for the gauge fields and the scalar Higgs field

$$\mathcal{L}_{\rm GEP} = \mathcal{L}_{\rm GEP}(h) + \mathcal{L}_{\rm GEP}(W) + \mathcal{L}_{\rm GEP}(Z) + \mathcal{L}_{\rm GEP}(A) \tag{49}$$

with the Lagrangian terms defined according to

$$\mathcal{L}_{\rm GEP}(h) = \tfrac{1}{2}(\partial_\mu h)^2 + \tfrac{1}{2}\Omega_h^2 h^2 \tag{50}$$

$$\mathcal{L}_{\rm GEP}(W) = \tfrac{1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(\partial_\mu W_\nu^- - \partial_\nu W_\mu^-) + \Omega_W^2 W_\mu^+ W^{-\mu} \tag{51}$$

$$\mathcal{L}_{\rm GEP}(Z) = \tfrac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 + \tfrac{1}{2}\Omega_Z^2 Z_\mu Z^\mu \tag{52}$$

$$\mathcal{L}_{\rm GEP}(A) = \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{\epsilon}(\partial_\mu A^\mu)^2. \tag{53}$$

Here the masses Ω_h, Ω_W, and Ω_Z must be regarded as variational parameters. With this choice we get $\langle h\rangle = 0$ and by the definition of h, Eq. (48), then $\langle\rho\rangle = \bar\rho$ as it was required in the derivation of the Gaussian effective potential Eq. (32). In order to evaluate $\mathcal{V}_{\rm GEP}(\bar\rho)$, according to Eq. (46) we also need $\mathcal{L}_{\rm int}$ which now reads

$$\mathcal{L}_{\rm int} = V((\bar\rho + h)^2) - \frac{1}{2}\Omega_h^2 h^2 + \left[\left(\frac{\bar\rho + h}{v}\right)^2 M_W^2 - \Omega_W^2\right] W_\mu^+ W^{-\mu} + \frac{1}{2}\left[\left(\frac{\bar\rho + h}{v}\right)^2 M_Z^2 - \Omega_Z^2\right] Z_\mu Z^\mu + \mathcal{L}_4 \tag{54}$$

where $\mathcal{L}_4$ contains the quartic terms that come out from the product $(\vec A_\mu \times \vec A_\nu)^2$ in $\mathcal{L}_{\rm YM}^{\rm even}$

$$\begin{aligned}\mathcal{L}_4 = {}& e^2[(A_\mu A^\mu)(W_\nu^+ W^{-\nu}) - (W_\mu^+ A^\mu)(W_\nu^- A^\nu)] + e^2\frac{g^2}{g'^2}[(Z_\mu Z^\mu)(W_\nu^+ W^{-\nu}) - (W_\mu^+ Z^\mu)(W_\nu^- Z^\nu)] \\ &+ e^2\frac{g}{g'}[(W_\mu^+ A^\mu)(W_\nu^- Z^\nu) - 2(A_\mu Z^\mu)(W_\nu^+ W^{-\nu}) + (W_\mu^+ Z^\mu)(W_\nu^- A^\nu)] + \frac{1}{2}g^2[(W_\mu^+ W^{-\mu})^2 - (W_\mu^+ W^{+\mu})(W_\nu^- W^{-\nu})].\end{aligned} \tag{55}$$

The couplings can be written in terms of the mass parameters by the standard relations

$$g^2 = \frac{4M_W^2}{v^2} \tag{56}$$

$$e^2 = \frac{4M_W^2}{v^2}\left(1 - \frac{M_W^2}{M_Z^2}\right) \tag{57}$$

$$\frac{g}{g'} = \frac{M_W}{\sqrt{M_Z^2 - M_W^2}}. \tag{58}$$

However, at this stage M_W and M_Z are just an alternative set of parameters, and they are not physical masses.

The explicit evaluation of the Gaussian effective potential then follows by Wick's theorem through Eq. (46). As usual, the classical potential of the standard Higgs sector is written as

$$V(\rho^2) = \frac{1}{2}m^2\rho^2 + \frac{1}{4!}\lambda\rho^4 \tag{59}$$

and denoting by φ the average of the field ρ, $\varphi = \bar\rho$, a straightforward calculation yields the effective potential (GEP)

$$\mathcal{V}_{\text{GEP}}(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{2}m^2 I_0(\Omega_h) + \frac{\lambda}{4!}\varphi^4 + \frac{\lambda}{4}\varphi^2 I_0(\Omega_h) + \frac{\lambda}{8}[I_0(\Omega_h)]^2 - \frac{1}{2}\Omega_h^2 I_0(\Omega_h) + I_1(\Omega_h) + 3I_1(\Omega_z) + 6I_1(\Omega_W) + I(\log\Omega_z + 2\log\Omega_W) + \left[\frac{\varphi^2 + I_0(\Omega_h)}{4}g^2 - \Omega_W^2\right]J(\Omega_W) + \frac{1}{2}\left[\frac{\varphi^2 + I_0(\Omega_h)}{4}(g^2 + g'^2) - \Omega_Z^2\right]J(\Omega_Z) + \left[\frac{9}{4}e^2 I_0(0) + \frac{3}{8}g^2 J(\Omega_W) + \frac{3}{4}\frac{e^2 g^2}{g'^2}J(\Omega_z)\right]J(\Omega_W) \tag{60}$$

where the function $J(X)$ is

$$J(X) = 3I_0(X) + \frac{I}{X^2} \tag{61}$$

and the Euclidean integrals I, I_0, I_1 are defined according to

$$I = \int_\Lambda \frac{d_E^4 k}{(2\pi)^4} \tag{62}$$

$$I_0(X) = \int_\Lambda \frac{d_E^4 k}{(2\pi)^4}\frac{1}{k^2 + X^2} \tag{63}$$

$$I_1(X) = \frac{1}{2}\int_\Lambda \frac{d_E^4 k}{(2\pi)^4}\log(k^2 + X^2). \tag{64}$$

Here the symbol $\int_\Lambda$ means that the integrals are regularized by insertion of a cutoff Λ so that $k < \Lambda$: the Higgs sector is regarded as an effective model with a high energy scale Λ which plays the role of a further free parameter.

IV. GAP EQUATIONS AND PHENOMENOLOGY

The variational parameters Ω_h, Ω_W, and Ω_Z must be determined by requiring that for any value of φ the GEP is at a minimum, thus the three parameters are implicit functions of the average of the field ρ. Once the parameters have been determined, the minimum point of $\mathcal{V}_{\text{GEP}}$ as a function of φ gives the vacuum expectation value of the field ρ. For any φ, the minimum of $\mathcal{V}_{\text{GEP}}$ is obtained by the constraints

$$\frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\Omega_h^2} = \frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\Omega_W^2} = \frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\Omega_Z^2} = 0. \tag{65}$$

We find three coupled equations (gap equations) which define the implicit functions $\Omega_h(\varphi)$, $\Omega_W(\varphi)$, and $\Omega_Z(\varphi)$. Once the parameters have been set at their best value by solving the gap equations, the broken-symmetry vacuum expectation value of the field ρ takes the value φ_0 which is obtained by the vanishing of the total derivative

$$\frac{d\mathcal{V}_{\text{GEP}}}{d\varphi} = \frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\varphi} + \sum_b\left(\frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\Omega_b^2}\right)\left(\frac{d\Omega_b^2}{d\varphi}\right) \tag{66}$$

where the label b runs over the bosons W, Z, and h. If the gap equations are satisfied then

$$\frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\Omega_b^2} = 0 \tag{67}$$

and the total derivative is equal to the partial derivative. Then φ_0 follows from the vanishing of the simple partial derivative

$$\left(\frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\varphi}\right)_{\varphi=\varphi_0} = 0. \tag{68}$$

Equations (65) and (68) are a set of four coupled equations that give the phenomenological predictions of the model. Differentiating Eq. (60) the gap equations Eq. (65) can be written as

$$\Omega_h^2 = m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{2}I_0(\Omega_h) + \frac{g^2}{2}J(\Omega_W) + \frac{g^2 + g'^2}{4}J(\Omega_Z) \tag{69}$$

$$\Omega_Z^2 = (g^2 + g'^2)\frac{\varphi^2 + I_0(\Omega_h)}{4} + \frac{3e^2 g^2}{2g'^2}J(\Omega_W) \tag{70}$$

$$\Omega_W^2 = g^2\frac{\varphi^2 + I_0(\Omega_h)}{4} + \frac{3e^2 g^2}{4g'^2}J(\Omega_Z) + \frac{9}{4}e^2 I_0(0) + \frac{3}{4}g^2 J(\Omega_W). \tag{71}$$

The vacuum expectation value of the field ρ then follows from Eq. (68): the partial derivative reads

$$\frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\varphi} = \varphi\left[m^2 + \frac{\lambda}{6}\varphi^2 + \frac{\lambda}{2}I_0(\Omega_h) + \frac{g^2}{2}J(\Omega_W) + \frac{g^2 + g'^2}{4}J(\Omega_Z)\right] \tag{72}$$

and insertion of Eq. (69) yields

$$\frac{d\mathcal{V}_{\text{GEP}}}{d\varphi} = \frac{\partial\mathcal{V}_{\text{GEP}}}{\partial\varphi} = \varphi\left[\Omega_h^2 - \frac{\lambda\varphi^2}{3}\right]. \tag{73}$$

Then Eq. (68) has two solutions: the unbroken symmetry stationary point $\varphi_0 = 0$ and the physical broken-symmetry vacuum expectation value

$$\varphi_0^2 = \frac{3}{\lambda}\Omega_h^2. \tag{74}$$

According, when φ_0 is set at its phenomenological value v, the self-coupling constant λ turns out to be proportional to the square of the mass parameter Ω_h, and a large Ω_h would not be compatible with perturbation theory. Conversely, the present variational calculation still holds for any large coupling, allowing for a full discussion of the Higgs sector. We notice that Ω_h is not the phenomenological mass M_h of the Higgs Boson which can be smaller than the variational parameter Ω_h. Here Ω_h may be regarded as the bare mass which appears in the zero-order Lagrangian $\mathcal{L}_{\rm GEP}(h)$ in Eq. (50), and in principle it can be very large. The phenomenological mass of the Higgs boson arises from the curvature of the GEP at the broken-symmetry minimum. Strictly speaking we should also check that the curvature is positive, otherwise the solution of the coupled gap equations would not refer to a minimum of the GEP. At tree level, the perturbative result $M_h^2 = \lambda\varphi_0^2/3$ would be equivalent to Eq. (74) only if $\Omega_h = M_h$. In fact we will see that the bare mass Ω_h can be very large compared to the Higgs boson mass M_h, and even a light Higgs boson could be described by a strongly interacting Higgs sector with a very large self-coupling λ [24,25].

The curvature of the GEP follows from the second derivative: from Eq. (73) we see that

$$\frac{d^2\mathcal{V}_{\rm GEP}}{d\varphi^2} = \left[\Omega_h^2 - \frac{\lambda\varphi^2}{3}\right] + 2\varphi^2\left[\frac{d\Omega_h^2}{d\varphi^2} - \frac{\lambda}{3}\right]. \tag{75}$$

At the unbroken symmetry stationary point $\varphi_0 = 0$ the second term vanishes

$$M_0^2 = \left(\frac{d^2\mathcal{V}_{\rm GEP}}{d\varphi^2}\right)_{\varphi=0} = \Omega_h^2 \tag{76}$$

and the physical mass is $M_0 = \Omega_h$. Conversely in the phenomenological broken-symmetry vacuum the first term vanishes and the physical mass M_h is given by

$$M_h^2 = \left(\frac{d^2\mathcal{V}_{\rm GEP}}{d\varphi^2}\right)_{\varphi=\varphi_0} = \frac{6\Omega_h^2}{\lambda}\left[\left(\frac{d\Omega_h^2}{d\varphi^2}\right)_{\varphi=\varphi_0} - \frac{\lambda}{3}\right]. \tag{77}$$

The derivatives of the variational parameters Ω_b can be obtained by differentiating the coupled gap equations Eq. (69)–(71): we get the following set of coupled linear equations

$$\left[1 - \frac{\lambda}{2}\frac{\partial I_0(\Omega_h)}{\partial\Omega_h^2}\right]\left(\frac{d\Omega_h^2}{d\varphi^2}\right) - \frac{1}{4}(g^2 + g'^2)\frac{\partial J(\Omega_Z)}{\partial\Omega_Z^2}\left(\frac{d\Omega_Z^2}{d\varphi^2}\right) - \frac{g^2}{2}\frac{\partial J(\Omega_W)}{\partial\Omega_W^2}\left(\frac{d\Omega_W^2}{d\varphi^2}\right) = \frac{\lambda}{2} \tag{78}$$

$$\frac{g^2}{4}\frac{\partial I_0(\Omega_h)}{\partial\Omega_h^2}\left(\frac{d\Omega_h^2}{d\varphi^2}\right) + \left(\frac{3e^2g^2}{4g'^2}\right)\frac{\partial J(\Omega_Z)}{\partial\Omega_Z^2}\left(\frac{d\Omega_Z^2}{d\varphi^2}\right) - \left[1 + \frac{7g^2}{4}\frac{\partial J(\Omega_W)}{\partial\Omega_W^2}\right]\left(\frac{d\Omega_W^2}{d\varphi^2}\right) = -\frac{g^2}{4} \tag{79}$$

$$\frac{1}{4}(g^2 + g'^2)\frac{\partial I_0(\Omega_h)}{\partial\Omega_h^2}\left(\frac{d\Omega_h^2}{d\varphi^2}\right) - \left(\frac{d\Omega_Z^2}{d\varphi^2}\right) + \left(\frac{3e^2g^2}{2g'^2}\right)\frac{\partial J(\Omega_W)}{\partial\Omega_W^2}\left(\frac{d\Omega_W^2}{d\varphi^2}\right) = -\frac{1}{4}(g^2 + g'^2). \tag{80}$$

The solution is trivial, and insertion of $d\Omega_h^2/d\varphi^2$ in Eq. (77) yields the physical mass of the Higgs boson.

In the weak sector all divergences are known to cancel at one-loop even in the unitarity gauge [26,27] and the perturbative corrections have been reported to be very small, as it should be for any perturbative correction arising from weak couplings. Conversely, in the Higgs sector a large self-coupling λ gives rise to strong nonperturbative effects which can be addressed by the present variational method.

In the Higgs sector, the variational mass parameter Ω_h depends on the self-coupling λ and on the vacuum expectation value $\varphi_0 = v$ through the minimum condition Eq. (74), which gives to Ω_h a clear physical phenomenological meaning: Ω_h sets the scale of the self-coupling λ which reads

$$\lambda = \frac{3\Omega_h^2}{v^2}. \tag{81}$$

Here we do not have any problem at insuring that Ω_h, the solution of the gap equation Eq. (69), takes a finite phenomenological value: in fact the existence of the free mass parameter m^2 makes sure that the solution of the gap equation Eq. (69) can be any number we like. Thus we fix m^2 in order to satisfy the minimum condition Eq. (81) and take Ω_h as a free parameter which gives the strength of the self-coupling λ. We do not need to deal with infinities, but the residual interaction shifts the physical Higgs mass that cannot be taken to be equal to the variational mass parameter Ω_h. In fact we have seen that the shape of the GEP contains nonperturbative effects which can be shown to be the sum of bubble diagrams to all orders [25,28]. The sum is equivalent to Eq. (77) that gives the physical mass M_h of the Higgs boson. As usual we assume that Λ is some very large energy scale and examine the behavior of the physical mass M_h as a function of the self-coupling λ. That is in agreement with the well-known triviality of the scalar theory which requires the existence of a large but finite cutoff.

As shown in Fig. 1 the physical mass M_h is not a monotonous increasing function of the coupling, but it reaches a maximum and then decreases. M_h^2 eventually becomes negative at some large coupling, indicating that the broken-symmetry solution becomes unstable. We get an upper bound for the coupling and, before reaching it, a

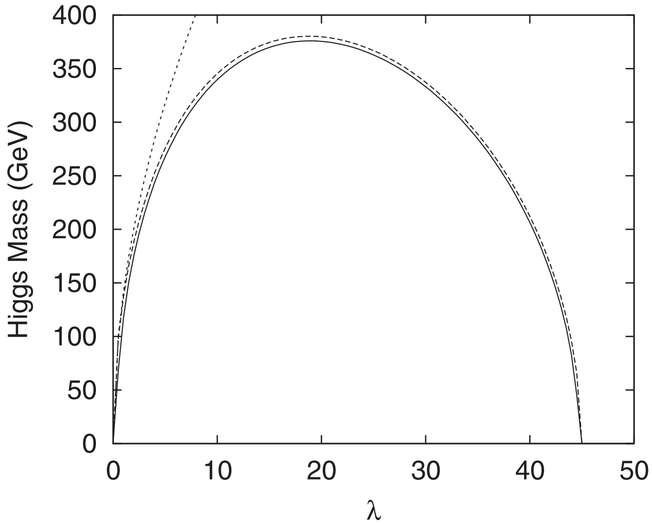

FIG. 1. The Higgs mass M_h according to Eq. (77) as a function of the self-coupling parameter λ for $\Lambda = 9$ TeV (solid line). The dashed line is the simple scalar theory result [24,28]. For comparison the tree-level result $M_h = \Omega_h$ is reported as a dotted line.

low mass nonperturbative strong coupling range. In this scenario a light Higgs can be found for a small coupling (perturbative light Higgs) but also for a large coupling (nonperturbative light Higgs). A very strong self-coupling reduces the mass: this effect cannot be predicted by any perturbative calculation. In the same figure the tree-level approximation $M_h = \Omega_h$ is also reported for comparison: we can see that in the perturbative regime of small λ the variational result is in agreement with the tree-level perturbative result, and the mass increases as the square root of the self-coupling λ. Conversely, in the strong coupling regime the mass of the Higgs boson becomes very small compared to the perturbative prediction which cannot be trusted anymore. In both limits we get a small mass, but we expect a different behavior for the scattering amplitudes in the strong-coupling range [25,28].

The prediction of a light Higgs boson in the strong coupling regime had been discussed in simplified models which neglected the gauge interactions [24,25] and in the Abelian gauge interacting U(1) theory [19]. Here we confirm the same trend in the framework of the full SU(2) $\times$ U(1) gauge theory.

In Fig. 1 the prediction of the GEP for a simple scalar theory [24,28] is reported for comparison. As expected, the effect of gauge interactions is very small and can be neglected for a qualitative discussion of the Higgs sector.

The possible existence of a light Higgs boson with a very strong self-interaction seems to be a nonperturbative feature of the standard model [19,24,25]. Thus the eventual experimental finding of a light Higgs mass $M_h \approx 200$ TeV would not rule out a strongly interacting Higgs sector. However, we expect that a strongly interacting light Higgs boson should show a different behavior when compared with the perturbative predictions: scattering amplitudes should be different and should tell us about the real strength of the self-coupling [29].

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