Semivariational approach to QCD at finite temperature and baryon density

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Recently a new bosonization method has been used to derive, at zero fermion density, an effective action for relativistic field theories whose partition function is dominated by fermionic composites, chiral mesons in the case of QCD. This approach shares two important features with variational methods: the restriction to the subspace of the composites, and the determination of their structure functions by a variational calculation. But unlike standard variational methods it treats excited states on the same footing as the ground state. I show that this bosonization method is an approximation of an exact procedure in which composites are introduced in the presence of fermionic states with the quantum numbers of the constituents (quasiparticles). This procedure consists of an independent Bogoliubov transformation at each time slice. The time-dependent parameters of the transformation are then associated with composite fields. In this way states of nonvanishing fermion (baryon) number (neglected in the bosonization approach) are retained. By the exact procedure I derive an effective action for QCD at finite temperature and baryon density. I test the result on a four-fermion interaction model.

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I. INTRODUCTION

Increasing temperature and baryon density hadronic matter is expected to undergo one or more crossovers and/or phase transitions. Increasing temperature at zero baryon number one might/should meet a phase in which quarks cohexist with hadronic (possibly colored) states [1]. By increasing baryon density at fixed temperature, one should meet a similar phase and possibly a color superconducting phase [2] due to a weak attractive channel between quarks of different colors. These new states of matter should be at least partially accessible to experimental investigation in heavy ion collision experiments.

Understanding the behavior of hadronic matter at high temperature and baryon density is relevant for the study of early universe and neutron stars. But its theoretical properties can be studied only nonperturbatively and the lattice approach, the most powerful tool for first principles, nonperturbative studies, is affected in the case of finite density QCD by the well-known sign problem.

Some progress was achieved recently [3,4] by simulations at imaginary chemical potential. Other interesting results were obtained [5] within a modified version of the Glasgow reweighting technique and by an approach which makes use of a Taylor expansion in the chemical potential in the small μ/T region [6].

At last a new approach to simulate QCD at finite temperature and baryon density was developed [7], which resembles in some aspects that of the imaginary chemical potential, but seems to have a wider range of applicability [8].

The magnitude of quark masses has large effects in numerical simulations. Several arguments lead to the expectation that the evaluation of the fermion determinant is more stable the larger the fermion masses are [9]. These arguments are relevant to the present work, as explained at the end of Sec. VII.

To tackle the problem of QCD at finite temperature and baryon density I extend a new method of bosonization of fermionic systems whose partition function is dominated by fermionic composites. This is certainly the case of QCD at low temperature and baryon density, in which the relevant degrees of freedom are mesons and nucleons, and also at high temperature and baryon density according to the expectations reported above. This method was first developed in the framework of many-body nonrelativistic theories [10] and then applied to relativistic field theories [11] at finite temperature and zero fermion density. Its heuristic motivation is that reformulation of a theory in terms of fields related to physical degrees of freedom should make it simpler. The starting point of this approach is the partition function in operator form, namely, the trace of the transfer matrix in the Fock space of the fermions. The physical assumption of composite dominance is then implemented by restricting the trace to fermion composites. This requires an approximation of a projection operator on the subspace of the composites, the approximation being the better, the higher the number of fermion states (called index of nilpotency) in the composites. The approximate projection operator is constructed in terms of coherent states of composites, and evaluation of the trace, which is done exactly, generates a bosonic action in terms of the holomorphic variables appearing in the coherent states.

The structure functions of the composites are determined by a variational procedure. So this approach shares two important features with variational methods: The restriction to a subspace of the Fock space of fermions, the space of chiral mesons in the case of QCD, and the deter-

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mination of their structure functions by a variational calculation. But unlike standard variational methods in the present one, excited states are treated on the same footing as the ground state.

The utility of variational methods and bosonization has been widely appreciated in the theory of many-body systems. But their potentiality has also been considered in the framework of relativistic field theories, in particular, gauge theories, for example, by Feynman [12] who, however, was skeptical about their practical applicability, and recently in connection with QCD at high baryon density [13].

The approach just outlined is compatible with any regularization. But in gauge theories the effective action of the composites will involve vacuum expectation values of invariant functions of gauge fields which cannot be evaluated within the present framework. Therefore a lattice formulation was adopted in order to be able to extract such expectation values from numerical simulations. One is then confronted with the well-known difficulty with chiral invariance, which can only in part be overcome by using Kogut-Susskind fermions. However the method can, at least in principle, be used with any other lattice regularization [14] for which a transfer matrix has been explicitly constructed.

The formalism of the transfer matrix does not treat time and space in a symmetric way, and therefore Euclidean invariance of the bosonic action must be checked *a posteriori*. All other symmetries are instead respected.

Before outlining the extension of this approach I will review what has been already done. The validity of the method was tested on a model with a four-fermion interaction in $3 + 1$ dimensions: Euclidean invariance was recovered in the continuum limit and all the known results in the boson sector were exactly reproduced, namely, condensation of a composite boson with the right mass, which breaks the discrete chiral invariance of the model. In addition, the structure function of the composite was determined, and its radial factor, in a polar representation, turned out to be identical with that of the Cooper pairs of the BCS model of superconductivity.

In the present work I show that the bosonization method developed in Refs. [10,11] is an approximation to an exact procedure in which composites are introduced in the presence of fermionic states with the quantum numbers of the constituents, to be called quasiparticles. This procedure consists of an independent Bogoliubov transformation at each time slice. The time-dependent parameters of the transformation are then associated with composite fields. This is part of an old problem: given a Lagrangian which generates bound states, how to replace it by a physically equivalent Lagrangian in which bound states and constituents are treated on equal footing [15]. I solve my problem defining quasiquark states in such a way that quasiquark-quasiantiquark states are orthogonal to meson states. This constraint corresponds to the condition on the wave function renormalization of composite particles in the Lehman spectral representation of composite operators [15].

The bosonization method is obtained by neglecting quasiparticles. To study QCD at finite temperature and baryon density I must use the exact procedure and retain quasiparticles. As a first step I must introduce fermionic states with quark quantum numbers in the presence of mesons, namely, I must construct *a Fock space containing composites and their constituents avoiding double counting*.

The next step, the explicit introduction of baryons and antibaryons constructed in terms of quasiquarks and quasiantiquarks, is desirable but not necessary in a variational calculation, because a space of mesons, quasiquarks, and quasiantiquarks obviously contains baryons and antibaryons. Therefore in the present paper I will not explicitly include in the partition function baryonic states. For a further simplification I will exclude antiquarks, so that my variational space contains mesons and baryons. This amounts to neglect virtual baryons-antibaryons pairs, and it is justified for not too high values of temperature and baryon density. In the resulting effective action the expectation value of the chiral sigma field provides a mass to the quasiquarks. On the ground of the arguments concerning the effects of quark masses on numerical simulations quoted above [9], numerical simulations with such effective action will be more stable, and it should be possible to investigate the limit of zero quark bare mass. I will also investigate in a separate work the possibility of analytical expansions.

The inclusion of quasiantiquark states will be presented in a separate paper [16].

I again test the method on a four-fermion interaction model, reproducing the known results in the fermion sector, namely, existence of a free fermion whose mass is half that of the composite boson, and chiral symmetry restoration with increasing fermion density. The mechanism of this restoration is that quasifermions occupy the lowest energy states, from zero energy up to a maximum energy increasing with density, progressively depleting the condensate. The test on the model in the presence of quasiantiquarks will be presented in Ref. [16] where a study of the dependence on temperature is also performed.

The paper is organized in the following way. In Sec. II I report the general formalism and in Sec. III its application at zero baryon density. In Sec. IV I define quasiquark states and the approximate projection operator in the subspace of mesons and quasiquarks. In Sec. V I derive a first form of their effective action. In Sec. VI I apply this action to the study of the four-fermion interaction model, getting the results described above. In Sec. VII I derive a second form of the effective action, which has a more transparent interpretation, and allows a cross-check of the accuracy of the approximation for the projection operator by comparison of the results for the four-fermion interaction model, which coincide with those obtained by the first effective action.

Moreover I show how one can use this action to investigate the chiral phase transition in QCD assuming a local structure function for the sigma field. In Sec. VIII I summarize my results with an outlook to possible applications.

II. GENERAL FORMALISM

The starting point of our formalism is the standard expression of the partition function of QCD in terms of the transfer matrix

$$Z = \int [dU] \exp[-S_G(U)] \mathrm{Tr}^F \left\{ \prod_{t=0}^{L_0-1} (\hat{T}_t^\dagger \hat{V}_t \exp(\mu \hat{n}_B) \hat{T}_{t+1}) \right\}. \tag{1}$$

L_0 is the number of links in the temporal direction, S_G the gluon action, and

$$\begin{aligned} \hat{T}_t &= \exp[-\hat{u}^\dagger M_t \hat{u} - \hat{v}^\dagger M_t^T \hat{v}] \exp[\hat{v} N_t \hat{u}], \\ \hat{V}_t &= \exp[\hat{u}^\dagger \ln U_{0,t} \hat{u} + \hat{v}^\dagger U_{0,t}^* \hat{v}]. \end{aligned} \tag{2}$$

The $U_{\mu,t}$ are matrices whose matrix elements are the link variables at Euclidean time t

$$(U_{\mu,t})_{\mathbf{x}_1,\mathbf{x}_2} = \delta_{\mathbf{x}_1,\mathbf{x}_2} U_{\mu,t}(\mathbf{x}_1). \tag{3}$$

Because the formalism asymmetrically treats time and space, I use **boldface** letters, as $\mathbf{x}$, to denote spatial coordinates, and *italic* letters to denote space-time coordinates: $x = (t, \mathbf{x})$. $\hat{u}_i^\dagger$ and $\hat{v}_i^\dagger$ are, respectively, creation operators of quarks and antiquarks in state i, obeying canonical anticommutation relations. Tr^F is the trace over the Fock space of quarks, μ the chemical potential, and $\hat{n}_B$ the baryon number operator. The matrices M_t (M_t^T being the transposed of M_t) and N_t are functions of the spatial link variables at time t and possibly of other bosonic fields. They depend on the regularization adopted for the fermions, but what follows is not affected by their explicit expressions, which are reported in Appendix B for Wilson and Kogut-Susskind fermions in the flavor basis.

I include in the gluon action the term

$$\delta S_G = -\sum_t 4\mathrm{tr}_- M_t \tag{4}$$

which comes from transformations on the fermion fields going from the functional form to this operator form of the transfer matrix. I introduced the notation, which I will use for any matrix Λ

$$\mathrm{tr}_\pm \Lambda = \mathrm{tr}(P_0^{(\pm)} \Lambda). \tag{5}$$

The operators $P_0^{(\pm)}$, which project on the quark-antiquark components of the quark field are defined in Appendix B. $\mathrm{tr}_\pm$ is the trace over quarks or antiquarks intrinsic quantum numbers and spatial coordinates (but not over time).

The expression (1) for the partition function was given by Lüscher [17] in the gauge $U_0 \sim 1$, in which $V_t = 1$ (but one has to impose the Gauss constraint).

Now I perform at each time slice an independent, generalized Bogoliubov transformation [18]

$$\begin{aligned} \hat{\alpha}_i &= [R^{1/2}(\hat{u} - \mathcal{F}^\dagger \hat{v}^\dagger)]_i, \\ \hat{\beta}_i &= [(\hat{v} + \hat{u}^\dagger \mathcal{F}^\dagger)(\mathring{R})^{1/2}]_i, \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{\alpha}_i^\dagger &= [(\hat{u}^\dagger - \hat{v}\mathcal{F}) R^{1/2}]_i, \\ \hat{\beta}_i^\dagger &= [(\mathring{R})^{1/2}(\hat{v}^\dagger + \mathcal{F}\hat{u})]_i, \end{aligned} \tag{7}$$

where

$$\begin{aligned} R &= (1 + \mathcal{F}^\dagger \mathcal{F})^{-1}, \\ \mathring{R} &= (1 + \mathcal{F}\mathcal{F}^\dagger)^{-1}. \end{aligned} \tag{8}$$

The quasiparticle operators $\hat{\alpha}$, $\hat{\beta}$ and their Hermitian conjugates satisfy canonical commutation relations for any choice of the matrix $\mathcal{F}$. The quasiparticle vacuum is

$$|\mathcal{F}\rangle = \exp \hat{\mathcal{F}}^\dagger |0\rangle, \tag{9}$$

where

$$\hat{\mathcal{F}} = \hat{v} \mathcal{F} \hat{u}. \tag{10}$$

The standard way to get the functional form of the partition function is to evaluate the trace over the fermion Fock space using coherent states of fermions

$$|\alpha, \beta\rangle = \exp(\alpha \hat{u}^\dagger + \beta \hat{v}^\dagger)|0\rangle, \tag{11}$$

where α, β are Grassmann variables and

$$\alpha \hat{u} = \sum_i \alpha_i \hat{u}_i. \tag{12}$$

I will instead use the quasiparticle coherent states obtained by a Bogoliubov transformation

$$|\alpha, \beta, \mathcal{F}\rangle = \exp(-\alpha \hat{\alpha}^\dagger - \beta \hat{\beta}^\dagger) \exp \hat{\mathcal{F}}^\dagger |0\rangle. \tag{13}$$

To this end I introduce a realization of the identity in terms of these states

$$\mathcal{I} = \int [d\alpha d\beta] \langle \alpha, \beta, \mathcal{F} | \alpha, \beta, \mathcal{F}\rangle^{-1} |\alpha, \beta, \mathcal{F}\rangle \langle \alpha, \beta, \mathcal{F}| \tag{14}$$

and insert it at each time slice in the partition function

$$Z = \int dU \exp[-S_G(U)] \mathrm{Tr}^F \left\{ \prod_{t=0}^{L_0-1} (\mathcal{I} \hat{T}_t^\dagger \hat{V}_t \exp(\mu \hat{n}_B) \hat{T}_{t+1}) \right\}. \tag{15}$$

The explicit expression of the trace is

$$\mathrm{Tr}^F\left\{\prod_{t=0}^{N_0-1}(I\hat{T}_t^\dagger\hat{V}_t\exp(\mu\hat{n}_B)\hat{T}_{t+1})\right\}$$
$$=\int\prod_{t=0}^{L_0-1}[d\alpha_t d\alpha_t^* d\beta_t d\beta_t^*]\langle\alpha_t,\beta_t,\mathcal{F}_t|\alpha_t,\beta_t,\mathcal{F}_t\rangle^{-1}$$
$$\times\langle\alpha_t,\beta_t,\mathcal{F}_t|\hat{T}_t^\dagger\hat{V}_t\exp(\mu\hat{n}_B)\hat{T}_{t+1}|\alpha_{t+1},\beta_{t+1},\mathcal{F}_{t+1}\rangle. \tag{16}$$

The last step is to notice that since nothing depends on the matrix $\mathcal{F}$ I can integrate over it with an arbitrary measure

$$\mathcal{Z}=\int dU\exp[-S_G(U)]\int\prod_{t=0}^{L_0-1}d\mu(\mathcal{F}^\dagger\mathcal{F})$$
$$\times[d\alpha_t d\alpha_t^* d\beta_t d\beta_t^*]\langle\alpha_t,\beta_t,\mathcal{F}_t|\alpha_t,\beta_t,\mathcal{F}_t\rangle^{-1}$$
$$\times\langle\alpha_t,\beta_t,\mathcal{F}_t|\hat{T}_t^\dagger\hat{V}_t\exp(\mu\hat{n}_B)\hat{T}_{t+1}|\alpha_{t+1},\beta_{t+1},\mathcal{F}_{t+1}\rangle. \tag{17}$$

For a physical interpretation I expand the matrix $\mathcal{F}_t$ in a basis of time independent matrices $\Phi_{\mathbf{x}K}$

$$\mathcal{F}_t=\sum_{\mathbf{x}K}\phi_K(t,\mathbf{x})^*\Phi_{\mathbf{x}K}=\phi_t^*\cdot\Phi. \tag{18}$$

The time-dependent coefficients of the expansion $\phi_K(t,\mathbf{x})$ will become dynamical fields associated with mesonic composites at position $\mathbf{x}$ with quantum numbers K such as radial excitations, spin, flavor, and so on. K includes color for colored mesons, which will be important at high temperature and/or baryon density. The choice of the basis matrices Φ_K selects which mesons one wants to include in the calculation in a variational spirit.

I will replace everywhere $\mathcal{F}$ by ϕ, because the basis matrices Φ are chosen once and for all (but will be determined at the end by a variational calculation). From the mathematical point of view the new expression of the partition function is exactly equivalent to the original one. From the physical point of view there is no double counting because the property of quasiparticles of annihilating the vacuum

$$\hat{\alpha}_i|\phi\rangle=\hat{\beta}_i|\phi\rangle=0 \tag{19}$$

can be interpreted as a compositeness condition: Mesonic states are orthogonal to quasiquark-quasiantiquark states. This constraint has the physical meaning of the condition $Z=0$ for bound states in the Lehmann spectral representation of composite operators [15], namely, the condition required to introduce a bound state on the same footing as the constituents in a Lagrangian.

III. THE MESONIC ACTION

For the convenience of the reader I report the effective action which results from the above construction when the contribution of quasiparticles can be neglected. This happens at low energy, when the partition function is dominated by chiral mesons. In this approximation all the coherent states coincide with the quasiparticle vacuum, and we can check *a posteriori* that we get sensible results setting the measure of integration over the chiral fields equal to 1

$$\mathcal{Z}\approx\int dU\exp[-S_G(U)]\mathcal{Z}_{\text{mesons}}(U), \tag{20}$$

where

$$\mathcal{Z}_{\text{mesons}}(U)=\mathrm{Tr}^F\left\{\prod_{t=0}^{L_0-1}(\mathcal{P}_m\hat{T}_t^\dagger\hat{V}_t\exp(\mu\hat{n}_B)\hat{T}_{t+1})\right\}, \tag{21}$$

with

$$\mathcal{P}_m=\int\left[\frac{d\phi d\phi^*}{2\pi i}\right]\langle\phi|\phi\rangle^{-1}|\phi\rangle\langle\phi|, \tag{22}$$

and

$$\left[\frac{d\phi d\phi^*}{2\pi i}\right]=\prod_{\mathbf{x},K}\left[\frac{d\phi_{\mathbf{x}K}d\phi^*_{\mathbf{x}K}}{2\pi i}\right]. \tag{23}$$

$\mathcal{P}_m$ is an approximate projection operator on the subspace of the composites

$$\hat{\Phi}^\dagger_{\mathbf{x},K}=\hat{u}^\dagger\Phi^\dagger_{\mathbf{x}K}\hat{v}^\dagger=\sum_{ij}\hat{u}_i^\dagger(\Phi^\dagger_{\mathbf{x}K})_{ij}\hat{v}_j^\dagger. \tag{24}$$

Since fermion creation operators are nilpotent, composite creation operators $\hat{\Phi}^\dagger_{\mathbf{x},K}$ can be classified according to their index of nilpotency, which is the highest integer exponent Ω such that

$$(\hat{\Phi}^\dagger)^\Omega\neq 0. \tag{25}$$

Ω counts the number of fermion states in the composite. By analogy with systems of elementary bosons the state

$$|\phi\rangle=\exp\left(\sum_{\mathbf{x},K}\phi_{\mathbf{x}K}\hat{\Phi}^\dagger_{\mathbf{x}K}\right)|0\rangle \tag{26}$$

might be considered a coherent state of composites. But since composite operators do not obey canonical commutation relations, the properties of their coherent states differ from those of canonical bosonic coherent states. For instance the basic property of coherent states cannot be exactly satisfied

$$\hat{\Phi}_{\mathbf{x}K}|\phi\rangle\neq\phi_{\mathbf{x}K}|\phi\rangle. \tag{27}$$

However, if the index of nilpotency of the composites is large enough, the composites system resembles a canonical bosonic system, and the properties of canonical boson coherent states will approximately hold for the composite coherent states, as shown in detail in Refs. [10,11].

Hence, under the assumption that the composite operators which dominate the partition function have a large index of nilpotency, an approximate projection operator in the Fock space of the fermions can be defined. It is important to observe that the space selected by this operator includes two physically equivalent states obtained for $\phi = 0, \infty$. They correspond to a completely empty or filled lattice.

The scalar product of coherent states appearing in the definition of the projection operator is

$$\langle\phi|\phi'\rangle = \det{}_{+}[1 + (\phi \cdot \Phi^{\dagger})(\phi'^{*} \cdot \Phi)] \tag{28}$$

and for any matrix Λ

$$\det{}_{\pm}\Lambda = \det(P_0^{\pm}\Lambda). \tag{29}$$

By a little abuse of notation I often write "1" instead of the identity in the space of the matrices Φ.

Following the general procedure outlined in the previous section, I write the meson partition function in the form

$$\begin{aligned}\mathcal{Z}_{\text{mesons}}(U) &= \mathrm{Tr}^F\{\mathcal{P}_m\hat{T}_0^{\dagger}\hat{V}_0\hat{T}_1\mathcal{P}_m\hat{T}_1^{\dagger}\hat{V}_1\cdots\mathcal{P}_m\hat{T}_{L_0-1}^{\dagger}\hat{V}_{L_0-1}\hat{T}_0\}\\ &= \int\prod_{t=0}^{L_0-1}\left[\frac{d\phi_t d\phi_t^*}{2\pi i}\right]\frac{1}{\langle\phi_t|\phi_t\rangle}\langle\phi_t|\hat{T}_t^{\dagger}\hat{V}_t\hat{T}_{t+1}|\phi_{t+1}\rangle,\end{aligned} \tag{30}$$

where a copy of the Fock space of the mesons has been introduced at each time slice. The chemical potential has disappeared because it is not active in a space of only mesons. Explicitly

$$|\phi_t\rangle = \exp\left(\sum_{\mathbf{x},K}\phi_K(t,\mathbf{x})\hat{\Phi}_{\mathbf{x}K}^{\dagger}[U_{i,t}]\right)|0\rangle. \tag{31}$$

I remind one that the structure functions $\Phi_{\mathbf{x},K}$ do not depend explicitly on time, but as they are functions of gauge fields, time will enter as a label of these fields.

In the evaluation of the trace on the fermionic Fock space the only difference with respect to [11] is the presence of the operator $\hat{V}_t$. But I notice that the product of operators $\hat{V}_t\hat{T}_{t+1}$ has an expression similar to that of $\hat{T}_{t+1}$

$$\begin{aligned}\hat{V}_t\hat{T}_{t+1} = {}&\exp[-\hat{u}^{\dagger}\ln(e^{M_{t+1}}U_{0,t}^{\dagger})\hat{u} - \hat{v}^{\dagger}\ln(e^{M_{t+1}^T}U_{0,t}^T)\hat{v}]\\ &\times\exp[\hat{v}N_t\hat{u}].\end{aligned} \tag{32}$$

Then evaluation of the trace over the Fock space proceeds exactly as in [11] with the result

$$\mathcal{Z}_{\text{mesons}}(U) = \int\prod_t\left[\frac{d\phi_t d\phi_t^*}{2\pi i}\right]\exp[-S_{\text{mesons}}(\phi^*,\phi,U)], \tag{33}$$

where

$$\prod_t\left[\frac{d\phi_t d\phi_t^*}{2\pi i}\right] = \prod_{\mathbf{x},K}\left[\frac{d\phi_K(\mathbf{x},t)d\phi_K^*(\mathbf{x},t)}{2\pi i}\right], \tag{34}$$

and

$$S_{\text{mesons}} = \sum_t \mathrm{tr}_{-}[-\ln\mathrm{R}_t + \ln\mathcal{R}_t + M_t^{\dagger}]. \tag{35}$$

In the last equation

$$\begin{aligned}R_t &= (1 + \mathcal{F}^{\dagger}\mathcal{F})_t^{-1},\\ \mathcal{R}_t &= [(1 + \mathcal{F}^{\dagger}N)_{t+1}e^{M_{t+1}}U_{0,t}^{\dagger}e^{M_t^{\dagger}}(1 + N^{\dagger}\mathcal{F})_t\\ &\quad + \mathcal{F}_{t+1}^{\dagger}e^{-M_{t+1}}U_{0,t}^{\dagger}e^{-M_t^*}\mathcal{F}_t]^{-1}e^{M_{t+1}}U_{0,t}^{\dagger}.\end{aligned} \tag{36}$$

Notice that the matrix $\mathcal{R}_t$ involves gauge fields at time t and $t+1$. The notation is somewhat different from that of [11].

It is remarkable that S_{mesons} has been evaluated exactly, so that the only approximations in the partition function are the physical assumption of boson dominance and the form of the projector over the meson subspace. Since the projector depends on the structure functions $\Phi_{\mathbf{x},K}$, the effective action is a functional of these functions which are determined by a variational calculation on the quantities of interest. In simple cases, such as the four-fermion interaction model, the variational calculation provides the exact form of the structure function. In QCD, unless some analytic progress is made along a way similar to that of the four-fermion interaction model, one has to adopt a trial expression.

In Ref. [11] an alternative, equivalent form of the effective action was derived, which has a more transparent interpretation. I will also derive two forms of the effective action at finite baryon density, but for the second one I will follow a somewhat different procedure.

IV. COHERENT STATES OF MESONS PLUS QUASIPARTICLES (WITHOUT QUASIANTIPARTICLES)

In order to extend the formalism to QCD at finite baryon density, I must introduce in the partition function states with nonvanishing baryon number. In the spirit of composites dominance, I should construct baryonic composites and define a projection operator on the subspace of mesons and baryons. In the present framework it is only possible to include, in addition to mesons, quasiparticles states with quark-antiquark quantum numbers, which I will call quasiquarks and quasiantiquarks. Such a space obviously contains the space of mesons and baryons. As anticipated in the Introduction in the present paper I include only mesons and quasiquarks in my variational space, but not quasiantiquarks. My variational space does not contain antibaryons, which are not expected to be important for not too high temperature and baryon density. Inclusion of quasiantiquarks will be presented separately [16].

''Coherent'' states of quasiquarks and mesons are

$$|\alpha, \phi\rangle = \exp(-\alpha \cdot \hat{\alpha}^\dagger) \exp(\phi \cdot \hat{\Phi}^\dagger)|0\rangle. \quad (37)$$

These states can be recast in the form

$$|\alpha, \phi\rangle = \exp(\hat{u}^\dagger \mathrm{R}^{-(1/2)}\alpha + \phi \cdot \hat{\Phi}^\dagger)|0\rangle \quad (38)$$

which is more convenient for calculations. The operator which approximately projects on such states is

$$\mathcal{P}_{m-q} = \int [d\alpha^* d\alpha]\left[\frac{d\phi^* d\phi}{2\pi i}\right]\langle\alpha, \phi|\alpha, \phi\rangle^{-1}|\alpha, \phi\rangle\langle\alpha, \phi|, \quad (39)$$

where the measure is

$$\langle\alpha, \phi|\alpha, \phi\rangle^{-1} = \langle\alpha|\alpha\rangle^{-1}\langle\phi|\phi\rangle^{-1} = \exp\{\mathrm{tr}_- \ln R - \alpha^* \cdot \alpha\}. \quad (40)$$

If $\mathcal{P}_{m-q}$ is an approximate projector it must satisfy the equations

$$\langle 0|\hat{\Phi}^{m_1}\hat{\alpha}^{n_1}\mathcal{P}_{m-q}(\hat{\alpha}^\dagger)^{n_2}(\hat{\Phi}^\dagger)^{m_2}|0\rangle \simeq \langle 0|\hat{\Phi}^{m_1}\hat{\alpha}^{n_1}(\hat{\alpha}^\dagger)^{n_2}(\hat{\Phi}^\dagger)^{m_2}|0\rangle \propto \delta_{m_1,m_2}\delta_{n_1,n_2}. \quad (41)$$

These equations are generated by the following ones:

$$\langle\phi_1\alpha_1|\mathcal{P}_{m-q}|\phi_2\alpha_2\rangle \simeq \langle\phi_1\alpha_1|\phi_2\alpha_2\rangle, \quad (42)$$

by taking derivatives with respect to the variables α_i, ϕ_i and setting them equal to zero. The left-hand side of (42) is

$$\begin{aligned}\langle\alpha_1, \phi_1|\mathcal{P}_{m-q}|\alpha_2, \phi_2\rangle &= \int[d\alpha^* d\alpha]\left[\frac{d\phi^* d\phi}{2\pi i}\right] \\ &\times \exp\{\mathrm{tr}_-[\ln R + \ln(1 + \mathcal{F}^\dagger\mathcal{F}_1) + \ln(1 + \mathcal{F}_2^\dagger\mathcal{F})] \\ &+ \alpha_1^* R_1^{-(1/2)}(1 + \mathcal{F}^\dagger\mathcal{F}_1)^{-1}(1 + \mathcal{F}_2^\dagger\mathcal{F})^{-1}R_2^{-(1/2)}\alpha_2\}. \end{aligned} \quad (43)$$

This shows by inspection that the first member of (41) vanishes unless $m_1 = m_2$, $n_1 = n_2$. Evaluating the integral in the above equation by the saddle point method, as done in [11] for $\mathcal{P}_m$, we see that $\mathcal{P}_{m-q}$ is approximately a projector if we assume

$$\mathrm{tr}\,(\Phi^\dagger\Phi)^n \simeq \Omega^{-n+1}. \quad (44)$$

I remind one that Ω is the index of nilpotency of $\hat{\Phi}$.

V. FIRST FORM OF THE EFFECTIVE ACTION AT FINITE BARYON DENSITY

I follow the derivation of the effective action outlined in Sec. III for zero baryon density, again setting the measure of integration over the composite fields equal to 1. I skip many intermediate steps because calculations of this kind have been reported in detail in [11], and can be easily repeated here by the help of the formulas collected in Appendix A. I start by evaluating the matrix elements of the transfer matrix between coherent states

$$\begin{aligned}&\langle\alpha_t, \phi_t|\hat{T}_t^\dagger\hat{V}_t\exp(\mu\hat{n}_B)\hat{T}_{t+1}|\alpha_{t+1}, \phi_{t+1}\rangle \\ &= \int[d\gamma^* d\gamma][d\delta^* d\delta]e^{-\gamma^*\gamma-\delta^*\delta}\langle\alpha_t, \phi_t|\hat{T}_t^\dagger|\gamma\delta\rangle \\ &\times\langle\gamma\delta|\hat{V}_t\exp(\mu\hat{n}_B)\hat{T}_{t+1}|\alpha_{t+1}, \phi_{t+1}\rangle. \end{aligned} \quad (45)$$

The last factor is

$$\begin{aligned}&\langle\gamma\delta|\hat{V}_t\exp(\mu\hat{n}_B)\hat{T}_{t+1}|\alpha_{t+1}, \phi_{t+1}\rangle \\ &= \det{}_-(1 + \mathcal{F}^\dagger N)_{t+1}\exp\{\gamma^* U_{0,t}e^{-M_{t+1}}(1 + \mathcal{F}^\dagger N)_{t+1}^{-1} \\ &\times[e^\mu \mathrm{R}_{t+1}^{-(1/2)}\alpha_{t+1} + \mathcal{F}_{t+1}^\dagger e^{-M_{t+1}}U_{0,t}^\dagger\delta^*]\}. \end{aligned} \quad (46)$$

A similar result for the other matrix element and integration over γ^*, γ, δ^*, δ leads to the expression

$$\begin{aligned}&\langle\alpha_t, \phi_t|\hat{T}_t^\dagger\hat{V}_t\exp(\mu\hat{n}_B)\hat{T}_{t+1}|\alpha_{t+1}, \phi_{t+1}\rangle \\ &= \det{}_-(e^{-M_t^\dagger}\mathcal{R}_t^{-1}) \\ &\times\exp(\alpha_t^* e^\mu R_t^{-(1/2)}\mathcal{R}_t U_{0,t}e^{-M_{t+1}}R_{t+1}^{-(1/2)}\alpha_{t+1}). \end{aligned} \quad (47)$$

From the measure appearing in the definition of $\mathcal{P}_{m-q}$, Eq. (40), I get the factor

$$\langle\alpha_t, \phi_t|\alpha_t, \phi_t\rangle^{-1} = \det{}_- R_t\exp(-\alpha_t^* \cdot \alpha_t). \quad (48)$$

Putting these pieces together I get the effective action of mesons interacting with quasiquarks

$$S_{\text{mesons quarks}} = S_{\text{mesons}} - \sum_t \alpha_t^*[-\alpha_t + e^\mu R_t^{-(1/2)} \times\mathcal{R}_t U_{0,t}e^{-M_{t+1}}R_{t+1}^{-(1/2)}\alpha_{t+1}], \quad (49)$$

where S_{mesons} is given by Eq. (35). I remind one that α is a 2-spinor with the quark intrinsic quantum numbers. $S_{\text{mesons quarks}}$ can be put in a more transparent form

$$S_{\text{mesons quarks}} = S_{\text{mesons}} - s\sum_t \alpha_t^*(\nabla_t - \mathcal{H}_t)\alpha_{t+1} \quad (50)$$

by introducing the lattice covariant derivative in the presence of a chemical potential and the lattice Hamiltonian

$$\begin{aligned}\nabla_t\alpha_{t+1} &= \frac{1}{s}(e^\mu U_{0,t}\alpha_{t+1} - \alpha_t), \\ \mathcal{H}_t &= \frac{1}{s}e^\mu[U_{0,t} - R_t^{-(1/2)}\mathcal{R}_t U_{0,t}e^{-M_{t+1}}R_{t+1}^{-(1/2)}]. \end{aligned} \quad (51)$$

The factor s in the above equations takes the value 2 in the Kogut-Susskind regularization because the quarks live on blocks, and 1 in the Wilson regularization. I notice that the time derivative is not symmetric, so that this action does not give rise to fermion doubling. Integrating over the Grassmann variables I get the purely bosonic effective action

$$S_{\text{effective}} = -\text{Tr}_{-} \ln(\mathcal{R}^{-1} R e^{M} - e^{\mu} U_0 T_0^{(+)}), \tag{52}$$

where I adopted the following notations: All matrices (with the exception of $T_\mu^{(\pm)}$) which do not have a time label are diagonal in time with matrix elements

$$\begin{aligned}(U_0)_{\mathbf{x}_1,t_1,\mathbf{x}_2,t_2} &= \delta_{t_1,t_2}\delta_{\mathbf{x}_1,\mathbf{x}_2} U_{0,t_1}(\mathbf{x}_1),\\ \mathcal{R}_{i_1,t_1,i_2,t_2} &= \delta_{t_1,t_2}(\mathcal{R}_{t_1})_{i_1,i_2},\end{aligned} \tag{53}$$

while the matrix elements of space-time translation operators are

$$(T_\mu^{(\pm)})_{x_1,x_2} = \delta_{x_2,x_1\pm s\hat{\mu}}. \tag{54}$$

"Tr" is the trace on all entries including time, while I remind one that "tr" is the trace on intrinsic and spatial quantum numbers only.

VI. APPLICATION TO A FOUR-FERMION INTERACTION MODEL WITH KOGUT-SUSSKIND FERMIONS

To get insight from the above result and also to test it I apply it to the four-fermion interaction model adopted as a test at zero fermion density [11]. It is a model in 3 + 1 dimensions regularized on a lattice with Kogut-Susskind fermions in the flavor basis. (I do not know any formulation of the transfer matrix in the spin-diagonal basis which can be used in the present formalism.) For each of the four Kogut-Susskind tastes there are N_f degenerated flavors. Hence, the continuum limit will describe a theory with $4N_f$ flavors. In the flavor basis the action reads

$$\begin{aligned}\mathcal{S} = &\sum_x{}' \sum_y{}' \bar{\psi}(x)[m\mathbb{1}\otimes\mathbb{1} + Q]_{x,y}\psi(y)\\ &+ \frac{1}{2}\frac{g^2}{4N_f}\sum_x{}' (\bar{\psi}(x)\psi(x))^2,\end{aligned} \tag{55}$$

where m is the mass parameter, g^2 the coupling constant, ψ the fermion fields, and Q the hopping matrix:

$$Q = \sum_\mu \gamma_\mu \otimes \mathbb{1}[P_\mu^{(-)}\nabla_\mu^{(+)} + P_\mu^{(+)}\nabla_\mu^{(-)}]. \tag{56}$$

The matrices to the left (right) of the symbol $\otimes$ act on Dirac (taste) indices. I denote by γ and t the matrices acting on these indices, respectively. The operators

$$P_\mu^{(\pm)} = \tfrac{1}{2}[\mathbb{1}\otimes\mathbb{1} \pm \gamma_\mu\gamma_5 \otimes t_5 t_\mu] \tag{57}$$

are orthogonal projectors. The fermion fields are defined on blocks (see Appendix B for details). The right and left derivatives $\nabla_\mu^{(\pm)}$ are given by

$$\nabla_\mu^{(\pm)} = \pm\tfrac{1}{2}(T_\mu^{(\pm)} - 1). \tag{58}$$

The factor $1/2$ is due to the fact that the operators T_μ translate by one block. The model has a discreet chiral symmetry at $m = 0$:

$$\psi \to -\gamma_5 \otimes t_5 \psi, \qquad \bar{\psi} \to \bar{\psi}\gamma_5 \otimes t_5. \tag{59}$$

To have an action bilinear in the fermion fields a scalar field $\sigma(x)$ is introduced, whose integration generates the four-fermion coupling:

$$\begin{aligned}\mathcal{S}' = &\sum_x{}' \sum_y{}' \bar{\psi}(x)[(m+\sigma)\mathbb{1}\otimes\mathbb{1} + Q]_{xy}\psi(y)\\ &+ \frac{4N_f}{2g^2}\sum_x{}' \sigma^2(x).\end{aligned} \tag{60}$$

The partition function now reads

$$\mathcal{Z} = \int [d\sigma][d\bar{\psi} d\psi]\exp[-\mathcal{S}']. \tag{61}$$

Its restriction to fermion composites plus a fermion gas, without antifermions is

$$\begin{aligned}\mathcal{Z}_{C-F} = &\int [d\sigma][d\phi^* d\phi][d\alpha^* d\alpha]\\ &\times \exp\Big[-\frac{4N_f}{2g^2}\sum_x{}' \sigma^2(x) - S_{C-F}\Big].\end{aligned} \tag{62}$$

S_{C-F} is given by Eq. (49), in which one has to insert the expressions of the matrices M, N appropriate to Kogut-Susskind fermions in the flavor basis [19]: The matrix M is equal to zero and the matrix N is reported in Appendix B. Integration over the fermion fields gives the effective action

$$S_{\text{effective}} = -\text{Tr}_{-} \ln[\mathcal{R}^{-1}R - e^{\mu}T_0^{(+)}]. \tag{63}$$

Now I look for constant values of the fields ϕ^*, ϕ, and σ which make the action stationary. I put a bar over constant fields and their functions. Then

$$\bar{S}_{\text{effective}} = -\text{Tr}_{-} \ln[\bar{\mathcal{R}}^{-1}\bar{R} - e^{\mu}T_0^{(+)}], \tag{64}$$

and I can perform the sum over time getting

$$\begin{aligned}\bar{S}_{\text{effective}} = &-\tfrac{1}{2}L_0 \,\text{tr}_{-}\{\mu\theta[e^{\mu} - \bar{\mathcal{R}}^{-1}\bar{R}]\\ &+ \ln(\bar{R}\bar{\mathcal{R}}^{-1})\theta[\bar{\mathcal{R}}^{-1}\bar{R} - e^{\mu}]\},\end{aligned} \tag{65}$$

where θ is the step function which defines the Fermi surface. For $\mu = 0$ I recover the effective action derived [11] at zero fermion density.

To determine the magnitude of the condensate I must perform a variation with respect to the boson fields $\bar{\phi}^*$, $\bar{\phi}$, and to determine the form factor of the composite a variation with respect to the matrices $\Phi^\dagger$, Φ. But $\bar{S}_{\text{effective}}$ does not depend on these variables separately; it is a function of $\bar{\mathcal{F}}^\dagger$ and $\bar{\mathcal{F}}$. The saddle point equations with respect to these matrices for $e^{\mu} < \bar{\mathcal{R}}^{-1}\bar{R}$, are identical to the ones for zero chemical potential. Using the result of [11] I then get

$$\bar{\mathcal{F}}^\dagger = \frac{N}{2H}(\sqrt{1+H^2} + 1), \qquad e^{\mu} < \bar{\mathcal{R}}^{-1}\bar{R}, \tag{66}$$

where

$$H = \frac{1}{2}\sqrt{N^\dagger N} = \sqrt{(m+\bar\sigma)^2 - \Delta} \tag{67}$$

with Δ given by Eq. (B9). I notice that H differs from the lattice Hamiltonian defined above

$$\bar{\mathcal{H}} = \frac{1}{2}e^{\mu}(1 - \bar R^{-1}\bar{\mathcal{R}}) = e^{\mu}H(\sqrt{1+H^2} - H), \tag{68}$$

but they are equal in the formal limit of vanishing lattice spacing. In this limit I can rewrite S_{C-F} in the form

$$S_{C-F} = S_C - 2\sum_t \alpha_t^*[\nabla_t^{(+)} - (H-\mu)]\theta(2H-\mu)\alpha_t. \tag{69}$$

For $m = \mu = 0$, I recover the well-known result that the fermionic system under consideration in the limit of $N_f \to \infty$ contains free fermions of mass $\bar\sigma$ in addition to free bosons of mass $2\bar\sigma$. I emphasize that the result recovered in this way is only formal. Indeed after adding one fermion I should determine the new minimum of the action, namely, the variation of the structure functions. But it can be justified in a concrete way by evaluating the difference of $\bar S_{\rm effective}$ given by Eq. (71) at fermion numbers differing by one unit.

Now I impose the condition on the fermion number which determines the chemical potential. From Eq. (65) I get

$$-\frac{2}{L_0}\frac{\partial}{\partial\mu}\bar S_{\rm effective} = {\rm tr}_-\theta[\exp\mu - \bar{\mathcal{R}}^{-1}\bar R] = {\rm tr}_-\theta[\exp\mu - 1 - 2\bar{\mathcal{H}}] = n_F. \tag{70}$$

For $\mu < 2\bar\sigma$, $n_F = 0$. For $\mu > 2\bar\sigma$, quasifermions occupy the states from zero energy up to a maximum energy E_{n_F} depending on the fermion number n_F.

The effective action at the minimum takes the form

$$\bar S_{\rm effective} = -L_0\,{\rm tr}_-\{\ln(\sqrt{1+H^2}+H)^2 \times \theta(2\bar{\mathcal{H}} + 1 - \exp\mu)\}. \tag{71}$$

Stationarity with respect to $\bar\sigma$ yields the gap equation which determines the masses and therefore the breaking of chiral invariance

$$\frac{4L_0N_f}{g^2}\bar\sigma = -\frac{\partial}{\partial\bar\sigma}\bar S_{\rm effective} = 2L_0\bar\sigma\,{\rm tr}_-\left\{\frac{1}{H\sqrt{1+H^2}}\theta(2\bar{\mathcal{H}} + 1 - \exp\mu)\right\}. \tag{72}$$

Increasing the fermion density, namely, the chemical potential, quasifermions occupy higher and higher energy states depleting the condensate, until only the solution $\bar\sigma = 0$ remains and chiral invariance is restored.

VII. SECOND FORM OF THE EFFECTIVE ACTION

The expression (66) of the form factors is somewhat surprising, because they are increasing functions of momentum. In [11] a more natural form was deduced by performing a unitary transformation in the fermionic Fock space and deriving the corresponding effective action. This transformation changes the empty lattice into the fully occupied one and particles into holes. In this new Fock space the structure functions are decreasing functions of momentum, and in a polar representation their polar factor is equal to that of the Cooper pairs of the BCS model of superconductivity.

But in addition the second form of the action provided a test of consistency of the approximation for the projection operator $\mathcal{P}_m$. I could follow the same path at nonzero baryon density, but instead I will get a similar result in a different way. First I rearrange the trace in Fock space in the following way:

$${\rm Tr}^F\left\{\prod_{t=0}^{N_0-1}(\mathcal{I}\hat T_t^\dagger\hat V_t\exp(\mu\hat n_B)\hat T_{t+1})\right\} = {\rm Tr}^F\{\hat V_0\exp(\mu n_B)\hat T_1\hat T_1^\dagger\hat V_1\exp(\mu n_B)\cdots\hat V_{L_o-1}\exp(\mu n_B)\hat T_0\hat T_0^\dagger\}. \tag{73}$$

Then I insert the projection operator $\mathcal{P}_{m-q}$ in the trace according to

$$\mathcal{Z}'_{\rm mesons\,quarks}(U) = {\rm Tr}^F\left\{\prod_{t=0}^{L_0-1}(\mathcal{P}_{m-q}\exp(\mu\hat n_B)\hat V_t\hat T_{t+1}\hat T_{t+1}^\dagger)\right\}. \tag{74}$$

I emphasize that $\mathcal{Z}'$, $\mathcal{Z}$ need not coincide with each other because $\mathcal{P}_{m-q}$ is not an exact projection operator, but the results obtained by the two forms should agree within the approximation for $\mathcal{P}_{m-q}$. A comparison between these results provides a check of its accuracy.

In the same way as for the first form of the effective action I evaluate the matrix elements

$$\langle\alpha_t,\phi_t|\exp(\mu\hat n_B)\hat V_t\hat T_{t+1}\hat T_{t+1}^\dagger|\alpha_{t+1},\phi_{t+1}\rangle = \exp\{-{\rm tr}_+\mathcal{R}_t' + \alpha_t^* e^{\mu}{\rm R}_t^{-(1/2)}U_{0,t}e^{-M_{t+1}}\mathcal{R}_{t+1}'{\rm R}_{t+1}^{-(1/2)}\alpha_{t+1}\}, \tag{75}$$

where

$$\mathcal{R}'_t = [1 + (N_t + e^{-M_t} U^\dagger_{0,t-1} \mathcal{F}_{t-1} U_{0,t-1} e^{-M_t})^\dagger \times (N_t + e^{-M_t} \mathcal{F}_t e^{-M_t})]^{-1} e^{-M_t^\dagger}. \quad (76)$$

Including the contribution (40) from the measure I get

$$S'_{\text{mesons quarks}} = S'\text{mesons} - s\sum_t \alpha^*_t (\nabla_t - \mathcal{H}'_t)\alpha_{t+1}, \quad (77)$$

where

$$S'_{\text{mesons}} = \sum_t \mathrm{tr}_-[-\ln R_t + \ln \mathcal{R}'_t + M^\dagger_t],$$
$$\mathcal{H}' = \frac{1}{s} e^\mu [U_{0,t} - R_t^{-(1/2)} U_{0,t} e^{-M_{t+1}} \mathcal{R}'_{t+1} R_{t+1}^{-(1/2)}]. \quad (78)$$

Integrating over α^*, α I get the purely bosonic action

$$S'_{\text{effective}} = -\mathrm{Tr}_- \ln[-R(\mathcal{R}')^{-1} e^M + e^\mu U_0 T_0^{(+)}]. \quad (79)$$

By exploiting the cyclic property of the trace it can be rewritten

$$S'_{\text{effective}} = -\mathrm{Tr}_- \ln[(\mathcal{R}')^{-1} R e^M - e^\mu U_0 T_0^{(+)}]. \quad (80)$$

This expression differs from $S_{\text{effective}}$, Eq. (52), by the replacement of $\mathcal{R}$ by $\mathcal{R}'$.

A. Application to the four-fermion interaction model

I use this second form of the effective action for the four-fermion interaction model with Kogut-Susskind fermions in the saddle point approximation. By means of the results of Ref. [11] I find

$$\bar{\mathcal{F}}^\dagger = \frac{N}{2H}(\sqrt{1+H^2} - H), \qquad 2\bar{\mathcal{H}}' > e^\mu - 1. \quad (81)$$

Now the structure function is a decreasing function of the constituent fermions energy. Using the above expression I find that

$$\bar{\mathcal{H}}' = \bar{\mathcal{H}}, \quad (82)$$

so that the results concerning mass of the uncorrelated fermions and restoration of chiral symmetry derived by the first form of the action are recovered.

B. Application to QCD

I start by some heuristic considerations about chiral symmetry breaking and mass generation in QCD in the present formalism. In the effective action the matrices N and $\mathcal{F}$ appear in the combination

$$N + \mathcal{F} = -2\Big\{ m\gamma_0 \otimes \mathbb{1} + \sum_{j=1}^{3} \gamma_0\gamma_j \otimes \mathbb{1}[P_j^{(-)}\nabla_j^{(+)} + P_j^{(+)}\nabla_j^{(-)}]\Big\} + \mathcal{F}.$$

This shows that a nonvanishing expectation value of $\mathcal{F}$ with the quantum numbers of the chiral σ meson at the same time breaks chiral symmetry and gives a mass to the quarks. It is reasonable to think that the present effective action should make numerical simulations easier, because of the arguments of Ref. [9] concerning the stability of such calculations in the presence of massive fermions, and of the arguments of Ref. [20] concerning the importance of extending lattice QCD by including explicitly chiral fields. In particular, one should be able to explore the exact chiral limit in which the bare quark mass vanishes, $m = 0$.

A numerical simulation requires a trial expression for the structure functions of the mesons. For instance one can assume

$$(\Phi_{\mathbf{x}K})_{\mathbf{x}_1 a_1, \mathbf{x}_2 a_2} = f_{\mathbf{x}K}(|\mathbf{x}_1 - \mathbf{x}_2|)\Big(\prod_{\Gamma(\mathbf{x}_1,\mathbf{x}_2)} U_k\Big)_{a_1 a_2}, \quad (83)$$

where the product of the spatial links is along a line Γ joining the quark-antiquark positions $\mathbf{x}_1$, $\mathbf{x}_2$. Waiting for some analytic input from an approximate solution of the saddle point equations the spatial part of the wave function f can be taken from the literature [21] or parametrized in a convenient way. As an extreme illustrative simplification for the study of the chiral condensate one can assume a pointlike structure for the σ meson and neglect all the others

$$\Phi_\sigma = \gamma_0 \otimes \mathbb{1} \otimes \mathbb{1}_c \otimes \mathbb{1}_s, \quad (84)$$

where $\mathbb{1}_c$, $\mathbb{1}_s$ are the identity in color and spatial coordinates, respectively (the normalization of Φ_σ is included in the σ-meson field). I assume that the σ field (but not the gauge field) is constant at the minimum of the action, which then reads

$$\bar{S}'_{\text{effective}} = -\mathrm{Tr}_-[\ln(1 + \bar{\phi}_\sigma^2) - \ln(1 - e^{\mu'} U_0 T_0^{(+)} + NN^\dagger + \bar{\phi}_\sigma^2)], \quad (85)$$

where

$$\mu' = \mu + \ln(1 + \bar{\phi}_\sigma^2). \quad (86)$$

The first term gives the contribution of the σ-meson condensate, while the second one is the contribution of a system of quarks (without antiquarks) interacting with a gauge field in the presence of this condensate which gives a contribution to their mass equal to $\bar{\phi}_\sigma$ (remember that $N^\dagger N$ is essentially the square of the quark Hamiltonian and therefore depends on the spatial link variables). One should now determine the minimum of $\bar{S}'_{\text{effective}}$ with respect to $\bar{\phi}_\sigma$ under the condition on baryon number which becomes

$$-\frac{2}{L_0}\frac{\partial}{\partial \mu'} \bar{S}'_{\text{effective}} = \frac{1}{1 + \bar{\phi}_\sigma^2} n_F. \quad (87)$$

VIII. SUMMARY AND OUTLOOK

I extended the formalism of composite boson dominance to the case of nonvanishing fermion number. This required a definition of fermion and antifermion states in the presence of bosonic composites which avoids double counting. These states, called quasiparticles, satisfy a compositeness condition. The resulting formalism provides a framework to solve the old problem: given a Lagrangian which admits bound states, how to define an equivalent Lagrangian in which the bound states appear on the same footing as the constituents. The problem is separated in two parts: the construction of a Hilbert space in terms of composites and constituents, and the determination of the structure of the composites. The first part has been explicitly solved in a way which allows one to tackle the second by analytical or numerical techniques.

The extension is obtained by performing independent generalized Bogoliubov transformations at each time slice so that the resulting effective Lagrangian is exactly equivalent to the original one. Our previous work of Ref. [11] can therefore be regarded as an approximation in which quasiparticles are altogether neglected. This can be justified at zero fermion density and very low temperature and energy. In the present paper I include quasifermions neglecting quasiantifermions. This amounts to neglecting virtual baryon-antibaryon pairs and can be justified for not too high temperature and baryon density. Quasiantiquarks have been included in a separate work [16], in which the four-fermi interaction model has been investigated analytically at finite temperature and density. An important issue is left for future investigation: the role of the integration measure over the mesonic fields ϕ. In the cases we considered it has been possible to set this measure equal to 1. But certainly at high temperature and/or baryon density it will be necessary to assume a form which will make the integral over mesonic fields convergent, and then we will meet the problem of showing that physical results do not depend on our choice.

Neglecting quasiantifermions I derived two forms of the effective action and I applied both of them to a four-fermi interaction model at zero temperature but finite fermion density. I recovered in both cases the known results, providing a cross-check of the approximation of the projection operator introduced to restrict the fermionic Fock space in the partition function. The discrete chiral invariance of the model (at zero fermion mass) is broken by composite boson condensation and the spectrum of the broken phase contains, in addition to a composite boson, a free fermion whose mass is half that of the boson. Increasing the fermion density, quasifermions occupy the lowest energy states up to an energy which increases with increasing density depleting the condensate, until chiral symmetry is restored. The compositeness condition is crucial to get these results.

I also showed how to use the formalism with QCD. Its effective action can be studied numerically and analytically. In numerical simulations I expect an advantage in the stability of the calculations and in the possibility of considering the limit of zero bare quark mass. To perform such simulations it is at the moment necessary to adopt trial expressions of the mesons structure functions, which should be functions of gauge fields depending on temperature and baryon density, as suggested by the example of the four-fermion model. Analytical investigations, apart from their intrinsic interest, might be of great help in numerical simulations. Among the various possibilities, I am considering an expansion in inverse powers of the index of nilpotency. In this connection I must make two observations. The first is that the index of nilpotency includes the number of momentum fermion states in the composites, and is therefore in general much higher than the number of quark intrinsic degrees of freedom. For instance in the BCS model, the index of nilpotency of Cooper pairs is infinite in the thermodynamic limit, but the number of intrinsic degrees of freedom of the electron is 2. The other observation is that the organization of an expansion in inverse powers of Ω is not *a priori* obvious for all quantities: for instance in the BCS model neglecting terms of order Ω^{-1} in the evaluation of the ground state energy per particle one gets the exact result in the thermodynamic limit, but exactly these terms give rise to the phase transition to the normal state at high temperature [22]. An example of how the properties of such expansions need a careful examination depending on the quantity to which the expansion is applied is given by [23] for the otherwise familiar $1/N$ expansion.

At last, exploiting the variational character of the method I give a simple expression of the effective action assuming a pointlike structure for the chiral σ meson and neglecting all the others.

Possible applications of the present formalism include numerical studies of the evolution of the state of baryon matter with temperature and density and the associated phase transitions.

Also exotic states of baryon matter can be explored. For instance it is not difficult, as will be shown in a separate paper, to introduce in the present formalism diquark states [16]. Among abnormal states of hadronic matter I would like to mention the layered spin-isospin phase [24]. This is a state characterized by one-dimensional crystallization (which distinguishes it from usual pion condensation) in which layers of spin-up protons and spin-down neutrons alternate with layers of spin-up neutrons and spin-down protons. An investigation of a dynamical realization of such a phase in light deformed nuclei showed that the spin-isospin nucleon-nucleon interaction is sufficiently strong to produce a distinctive signature compatible with observation [25], while the critical density for a static phase in neutron stars has been estimated [26] to be 3–4 times normal nuclear density. A first principles calculation might be worthwhile to make an assessment of some simplifications done in the quoted works.

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APPENDIX A: GRASSMANN INTEGRALS AND COHERENT STATES

If $|\alpha\rangle$ is a fermionic coherent state,

$$|\alpha\rangle = \exp(-\alpha\hat{u}^\dagger)|0\rangle, \tag{A1}$$

then

$$\langle\alpha|\alpha\rangle = \exp(\alpha^*\alpha), \tag{A2}$$

and the identity can be written

$$\int[d\alpha d\alpha^*]\langle\alpha|\alpha\rangle^{-1}|\alpha\rangle\langle\alpha| = 1. \tag{A3}$$

I remind one of the fundamental property of coherent states

$$\hat{u}|\alpha\rangle = \alpha|\alpha\rangle, \tag{A4}$$

which implies the relations

$$\begin{aligned}\langle\alpha\beta|\exp(\hat{v}N\hat{u})|\gamma\delta\rangle &= \exp(\delta N\gamma)\langle\alpha\beta|\gamma\delta\rangle\\ &= \exp(\delta N\gamma + \alpha^*\gamma + \beta^*\delta),\end{aligned} \tag{A5}$$

$$\begin{aligned}\langle\gamma\delta|\exp(\hat{u}^\dagger\mathcal{F}^\dagger\hat{v}^\dagger)|0\rangle &= \langle 0|\exp(\hat{v}\mathcal{F}\hat{u})|\gamma\delta\rangle^*\\ &= \exp(\gamma^*\mathcal{F}^\dagger\delta^*).\end{aligned} \tag{A6}$$

With the help of these formulas one can compute matrix elements of the type

$$\langle\alpha\beta|e^{\hat{v}N\hat{u}}e^{\hat{u}^\dagger\mathcal{F}^\dagger\hat{v}^\dagger}|0\rangle \tag{A7}$$

$$= \int\left[\frac{d\gamma^*d\gamma d\delta^*d\delta}{\langle\gamma\delta|\gamma\delta\rangle}\right]\langle\alpha\beta|e^{\hat{v}N\hat{u}}|\gamma\delta\rangle\langle\gamma\delta|e^{\hat{u}^\dagger\mathcal{F}^\dagger\hat{v}^\dagger}|0\rangle \tag{A8}$$

$$= \int[d\gamma^*d\gamma d\delta^*d\delta]e^{-\gamma^*\gamma-\delta^*\delta+\delta N\gamma+\alpha^*\gamma+\beta^*\delta+\gamma^*\mathcal{F}^\dagger\delta^*} \tag{A9}$$

$$= \int[d\delta^*d\delta]e^{-\delta^*(1+\mathcal{F}^*N^T)\delta+\beta^*\delta-\delta^*\mathcal{F}^*\alpha^*} \tag{A10}$$

$$= \exp\{\mathrm{tr}_-\ln(1+\mathcal{F}^*N^T) - \beta^*(1+\mathcal{F}^*N^T)^{-1}\mathcal{F}^*\alpha^*\}, \tag{A11}$$

by use of the identity

$$\begin{aligned}&\int[d\alpha^*d\alpha]\exp(-\alpha^*A\alpha + J^*\alpha + \alpha^*J)\\ &= \det A\exp(J^*A^{-1}J).\end{aligned} \tag{A12}$$

APPENDIX B: THE MATRICES *M*, *N* OF THE TRANSFER MATRIX

In this Appendix I report the expressions of the matrices M, N appearing in the definition of the transfer matrix for the Kogut-Susskind and the Wilson regularization. Their common feature is that they depend only on the spatial link variables.

1. Kogut-Susskind's regularization

Kogut-Susskind fermions in the flavor basis are defined on hypercubes whose sides are twice the basic lattice spacing. While in the text intrinsic quantum numbers and spatial coordinates were comprehensively represented by one index i, here I distinguish the spinorial index $\alpha = \{1, \ldots, 4\}$, the taste index $a = \{1, \ldots, 4\}$, and the flavor index $i = \{1, \ldots, N_f\}$, while $x = \{t, x_1, \ldots, x_3\}$ is a 4-vector of *even* integer coordinates ranging in the intervals $[0, L_t - 1]$ for the time component and $[0, L_s - 1]$ for each of the spatial components. I distinguish summations over basic lattice and hypercubes according to

$$\sum_x{}' := 2^d\sum_x. \tag{B1}$$

The projection operators over fermion-antifermion states are

$$P_0^{(\pm)} = \tfrac{1}{2}(\mathbb{1}\otimes\mathbb{1} \pm \gamma_0\gamma_5\otimes t_5t_0). \tag{B2}$$

The relation between the variables u, v and the quark q field is

$$P_0^{(+)}q = \tfrac{1}{4}u, \qquad P_0^{(-)}q = \tfrac{1}{4}v^\dagger. \tag{B3}$$

In the presence of the scalar field σ and of gauge fields, neglecting an irrelevant constant, $M = 0$, while N is [19]

$$\begin{aligned}N = -2\Big\{&(m+\sigma)\gamma_0\otimes\mathbb{1} + \sum_{j=1}^{3}\gamma_0\gamma_j\\ &\otimes\mathbb{1}[P_j^{(-)}\nabla_j^{(+)} + P_j^{(+)}\nabla_j^{(-)}]\Big\},\end{aligned}$$

where

$$\nabla_j^{(+)} = \tfrac{1}{2}(U_jT_j^{(+)} - 1), \tag{B4}$$

$$\nabla_j^{(-)} = \tfrac{1}{2}(1 - T_j^{(-)}U_j^\dagger) \tag{B5}$$

are the lattice covariant derivative as the $T_\mu^{(\pm)}$ are the forward and backward translation operators of one block, that is of two lattice spacings in the original lattice, in the μ direction (with unit versor $\hat{\mu}$)

$$[T^{(\pm)}_{\mu}]_{x_1,x_2} = \delta_{x_2,x_1\pm 2\hat{\mu}}. \quad \text{(B6)}$$

I set

$$N^{\dagger}N = 4H^2. \quad \text{(B7)}$$

In the absence of gauge fields

$$H^2 = (m + \sigma)^2 - \Delta \quad \text{(B8)}$$

with

$$\Delta = \frac{1}{4}\sum_{i=1,3}(T_i^{(+)} + T_i^{(-)} - 2). \quad \text{(B9)}$$

The eigenvalues of H^2 for constant $\sigma = \bar{\sigma}$ are therefore the fermion energies

$$E_p^2 = (m + \bar{\sigma})^2 + \tilde{p}^2, \quad \text{(B10)}$$

where momentum component $\tilde{p}_i^2$ is

$$\tilde{p}_i^{\,2} = \tfrac{1}{2}(1 - \cos 2p_i), \quad \text{(B11)}$$

and

$$\tilde{p}^{\,2} = \sum_{i=1}^{3} \tilde{p}_i^2. \quad \text{(B12)}$$

2. Wilson's regularization

The projection operators over fermions-antifermions are

$$P_0^{(\pm)} = \tfrac{1}{2}(1 \pm \gamma_0) \quad \text{(B13)}$$

in a basis in which $\gamma_0 = \mathrm{diag}(1, 1, -1, -1)$.

The relations between the quark field q and its upper and lower components u, v are

$$P_0^{(+)}q = B^{-(1/2)}u, \qquad P_0^{(-)}q = B^{-(1/2)}v^{\dagger}, \quad \text{(B14)}$$

where

$$B = 1 - K\sum_{j=1}^{3}(U_jT_j^{(+)} + T_j^{(-)}U_j^{\dagger}) \quad \text{(B15)}$$

and K is the hopping parameter. The matrices M, N are

$$M = \frac{1}{2}\ln\left(\frac{B}{2K}\right), \qquad N = 2KB^{-(1/2)}cB^{-(1/2)}, \quad \text{(B16)}$$

where

$$c = \frac{1}{2}\sum_{j=1}^{3} i(U_jT_j^{(+)} - T_j^{(-)}U_j^{\dagger})\sigma_j. \quad \text{(B17)}$$

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