Very special (de Sitter) relativity

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The effects of a nonvanishing value for the cosmological constant in the scenario of Lorentz symmetry breaking recently proposed by Cohen and Glashow (which they denote as very special relativity) are explored and observable consequences are pointed out.

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I. INTRODUCTION

There is a growing consensus that the most likely interpretation of the observed acceleration of the Universe (cf. [1] for a recent reference) is the presence of a cosmological constant, of mass dimension two, whose energy scale corresponds roughly to the inverse of the Hubble radius, $R_H \sim 10^{10}$ ly. This means $\lambda \sim \frac{1}{R_H^2} \sim 10^{-65}$ eV^2, so that the corresponding mass scale is the *Hubble mass* $M_\lambda \equiv \lambda^{1/2} \sim 10^{-33}$ eV.

When interpreted as a vacuum energy density, the quantity of interest is $M_p^2 M_\lambda^2 \equiv M_{DE}^4$, (where M_P is the Planck mass). This yields the *dark energy* scale $M_{DE} \sim 10^{-3}$ eV.

It should be remarked that these two scales M_λ and M_{DE} are quite different; the former is the one that determines the corresponding de Sitter geometry, whereas the latter is related to possible dynamical effects of the matter.

The propagation of all massive particles is then affected by powers of $\xi \equiv \frac{M_\lambda}{m}$, the dimensionless ratio of the mass scale associated with the particle. Massless particles are also affected in subtle ways, and the only known particle for which a large value of ξ is not excluded is the lightest neutrino.

A new line of thought concerning the neutrino masses has been recently put forward by Cohen and Glashow [2]. They disposed of the hypothesis that the full rotation group ought to be preserved in any realistic scenario of Lorentz symmetry violation. In that way, they pointed out the possibility of a breaking from $SO(1,3)$ to a proper subgroup, a four-parameter subgroup. The scenario in which the fundamental symmetry of nature is just $SIM(2)$ plus translations was dubbed by them Very Special Relativity (VSR), and in this framework it is possible to postulate a new origin for the neutrino masses.

A fascinating point of this approach is that Lorentz symmetry breaking appears inextricably entangled with parity breaking, in the sense that the addition of just parity to VSR is enough to restore the full Lorentz symmetry. Another curious consequence is that there are no new invariant tensors for $SIM(2)$, so that any local term that can be written in the Lagrangian enjoys enhanced full Lorentz symmetry. It seems that intrinsic Lorentz breaking effects in this scheme must be both parity violating and non local in nature.

In this paper, the generalization of VSR ideas to de Sitter spacetime is studied in two different ways. In Sec. II, we explore several possibilities to break its isometry group, depending on whether preeminence is given to boost invariance or else to rotation invariance. On the other hand, the nonlocal equation of motion postulated by Cohen and Glashow [2] can easily be derived from a local Lagrangian, with auxiliary fields. We generalize it to the de Sitter space in Sec. III, and we study its symmetries in different cases with some care.

II. BREAKING DE SITTER INVARIANCE

The difference between the Poincaré and the de Sitter groups are the nonvanishing commutation relations between translations in the latter. Then, one possible approach in order to break the de Sitter invariance is to start with the $SIM(2)$ algebra[1]

$$[J_3, T_A] = i\epsilon_{AB} T_B \qquad [K_3, T_A] = iT_A \tag{1}$$

and try to include as many translations as we can. Let us coin the provisional name ΛVSR for the resulting group.

Given the fact that

$$\begin{aligned}
[K_3, H] = iP_3, &\qquad [K_3, P_3] = iH \\
[T_A, P_3] = -iP_A, &\qquad [T_A, P_B] = iP_+\delta_{AB}, \\
[T_A, H] = iP_A &\qquad [J_3, P_A] = i\epsilon_{AB}P_B \\
[P_i, P_j] = i\epsilon_{ijk}J_k, &\qquad [H, P_i] = iK_i,
\end{aligned} \tag{2}$$

where $P_\pm \equiv H \pm P_3$, it easily follows that we can include only P_+ as a (Hamiltonian) translation

$$[T_A, P_+] = 0, \qquad [K_3, P_+] = iP_+, \qquad [J_3, P_+] = 0. \tag{3}$$

If we include P_A, then we have to include K_A as well, because $[H, P_A] = K_A$. If we include P_3, this forces us to include P_A, owing to $[T_A, P_3] = iP_A$. Even if we include H

[1] Remember that $T_1 = K_1 - J_2$, $T_2 = K_2 + J_1$, and $A, B = 1,$, 2. Along the full paper only nonvanishing commutators will appear.

by itself, we are forced to include P_A again, because $[T_A, H] = iP_A$.

The appropriate subgroup of de Sitter that corresponds to ΛVSR is then generated by $\{T_A, K_3, J_3, P_+\}$

$$[K_3, T_A] = iT_A \qquad [J_3, T_A] = i\epsilon_{AB}T_B \\ [K_3, P_+] = iP_+ \tag{4}$$

This is obviously a subgroup of the Lorentz group as well; the only thing that is specific to de Sitter in this context is the inability to consider all components of the momentum as quantum numbers.

To summarize, the net effect of the cosmological constant is to reduce the number of available generators from eight ($SIM(2)$ plus translations) down to five. That is, if we follow the same philosophy of breaking the rotations and keeping the boost K_3 that lead to VSR in flat space. But there is another possibility. Since $SIM(2)$ is defined by the requirement of leaving invariant a certain null direction, and since the de Sitter group is the five-dimensional equivalent of a Lorentz group, we can exploit this analogy.

Let us split the generators into four *boosts* $M_{0I} \equiv K_I$ and six *rotations* M_{IJ}, where the five-dimensional spatial indices run from $I, J, \ldots = 1, 2, 3, 4$. Next, define for the four-dimensional spatial indices $i, j, \ldots = 1, 2, 3$, $M_{ij} = \epsilon_{ijk}L_k$, and $M_{4i} = N_i$. The commutators read

$$\begin{aligned} &[K_4, K_i] = -iN_i, \qquad &&[K_4, N_i] = -iK_i, \\ &[K_i, K_j] = -i\epsilon_{ijk}L_k, \qquad &&[K_i, L_j] = i\epsilon_{ijk}K_k, \\ &[K_i, N_j] = i\delta_{ij}K_4, \qquad &&[L_i, L_j] = i\epsilon_{ijk}L_k, \\ &[L_i, N_j] = i\epsilon_{ijk}N_k, \qquad &&[N_i, N_j] = i\epsilon_{ijk}L_k, \end{aligned} \tag{5}$$

so that if we define $2J_i^{\pm} \equiv L_i \pm N_i$, there are two commuting $SO(3)$ algebras

$$[J_i^a, J_j^b] = i\epsilon_{ijk}J_k^a\delta^{ab}. \tag{6}$$

It is plain to verify that the little group of a null vector is now the Euclidean three-dimensional group $E(3)$, generated by the six elements

$$[L_i, L_j] = i\epsilon_{ijk}L_k \qquad [L_i, T_j] = i\epsilon_{ijk}T_k, \tag{7}$$

where $T_i \equiv K_i + N_i$, and the group that takes a null vector into a multiple of itself is none other than $SIM(3)$. The Lie algebra is augmented with the new generator K_4

$$[K_4, T_i] = -iT_i. \tag{8}$$

As noted in [3], where the authors suggested already the idea of breaking an isometry group of a global spacetime, this is the maximal subgroup of the de Sitter group. In that work, they pointed out that there are no continuous deformations of the VSR group, such that the translations become noncommutative. Since our search has been limited to de Sitter subgroups, we knew in advance that we are not allowed to keep the full translation invariance.

It should be remarked that $SIM(3)$ is *not* equivalent to ΛVSR, which only included five generators, whereas we now enjoy seven, that is, the cosmological constant just kills one generator with respect to the flat case. In the language appropriated for taking the flat limit $l \to \infty$, the L_i are equivalent to our former J_i, so that we are keeping rotational invariance, $T_i \equiv K_i + P_i$, and K_4 is our former H. The theory breaks all boosts: K_3 is no longer a symmetry of the theory.

III. VSR FERMION PROPAGATION IN DE SITTER

Cohen and Glashow [2] proposed neutrino masses that neither violate lepton number nor require additional sterile states. The nonlocal Dirac equation would be

$$\left(\not{p} - \frac{1}{2}m_\nu^2\frac{\not{n}}{p\cdot n}\right)\nu_L = 0, \tag{9}$$

where n points to the preferred null direction. In this description, all massive particles, including neutrinos, enjoy a standard dispersion relation $p^2 = m^2$.

Lorentz violating effects are argued by the aforementioned authors to be of order $o(\gamma^{-2})$, except in a narrow cone about the privileged direction. Near the endpoint of the electron spectrum of beta decay, novel effects could be experimentally accessible.

The Cohen-Glashow equation can be formally derived from the local action for three chiral spinors

$$\mathcal{L} = i\bar{\nu}\not{\partial}\nu + i\bar{\chi}\partial_n\psi + i\bar{\psi}\partial_n\chi + im\bar{\chi}\not{n}\nu + im\bar{\psi}\nu + \text{c.c.} \tag{10}$$

Although it seems at first sight that the symmetry of this Lagrangian consists only on the Euclidean group $E(2)$, that is, the set of transformations that leave invariant the null vector n, the symmetry is actually enhanced toward $SIM(2)$, provided fermions transform as

$$\chi_L \to \frac{1}{\lambda}D_s(\Lambda)\chi_L, \tag{11}$$

where $D_s(\Lambda)$ is the usual $(0, 1/2)$ spinor representation of the Lorentz group. The parameter λ is the rescaling of n caused by the $SIM(2)$ transformation. This is a new representation of $SIM(2)$, but not the one induced as a subgroup of the Lorentz group.

Although it is needed to specify a new transformation law in order to recover the full $SIM(2)$ symmetry, the VSR scheme can be seen partially (at least, for the $E(2)$ subgroup) as a dynamical consequence of this Lagrangian. Coupling this system to gravity will give us a new possibility to break the de Sitter invariance.

Let us now turn our attention to the description of the fermion propagation in de Sitter. We have reviewed in the appendix for the convenience of the reader some facts on fermions in de Sitter space. Let us start from the Lagrangian (written in a free-falling local frame)

$$\mathcal{L} = \sqrt{-g}(i\bar{\nu}\alpha^\mu\nabla_\mu\nu + i\bar{\chi}\nabla_n\psi + i\bar{\psi}\nabla_n\chi + im\bar{\chi}\not{n}\nu \\ + im\bar{\psi}\nu + \text{c.c.}), \tag{12}$$

where $\alpha^\mu = e_a{}^\mu \gamma^a$ is a set of curved gamma matrices, n is a null vector field over the de Sitter space, and $\not{n}$ stands for $g_{\mu\nu}\alpha^\mu n^\nu$. This seems the simplest generalization of the description we just introduced in flat space.

The Lagrangian defined using the Weyl technique (i.e., in a local inertial frame or tetrad) is diffeomorphism invariant, and enjoys local Lorentz invariance as well. It is natural to assume that the symmetry group of the action is the subgroup of the set of all isometries (de Sitter) that leave the null vector n invariant $\mathcal{L}(k)n = 0$, that is, local $E(2)$ invariance.

In order to recover the flat space action, we would demand n to be covariantly constant $\nabla n = 0$, which in fact implies that it is a Killing vector by itself, but there are not null Killing vectors over the de Sitter space.

Minimal coupling applied to $\partial n = 0$ is then too restrictive. Let us assume instead $\nabla n \to 0$ if $l \to \infty$.

In the Riemannian coordinates, the covariant derivative of n is

$$\nabla_\mu n^\nu = \partial_\mu n^\nu + \frac{\Omega}{2l^2}(x_\mu n^\nu - x^\nu n_\mu + (n\cdot x)\delta^\nu_\mu), \quad (13)$$

so our flat space condition implies that $\partial_\mu n^\nu = 0$ in this coordinate.

Let us now try to find the corresponding isometry subgroup. A general Killing field in de Sitter space has the following expression:

$$k^\mu = \lambda^\mu{}_\nu x^\nu + a^\mu l\left(1 + \frac{x^2}{4l^2}\right) - \frac{a\cdot x}{2l}x^\mu, \quad (14)$$

where λ is an infinitesimal (four-dimensional) Lorentz transformation and a an infinitesimal 4 vector. If it commutes with n then

$$\begin{aligned}[n,k]^\mu &= n^\nu \partial_\nu k^\mu = 0 \\ &= \lambda^\mu{}_\nu n^\nu + a^\mu \frac{n\cdot x}{2l} - x^\mu \frac{n\cdot a}{2l} - n^\mu \frac{a\cdot x}{2l}. \quad (15)\end{aligned}$$

This expression is a polynomial of degree 1 in x, so both terms must be zero

$$a^\mu n_\nu - \delta^\mu_\nu (n\cdot a) - n^\mu a_\nu = 0 \to a\cdot n = 0, \quad (16)$$

so we should have $a \propto n$. On the other hand,

$$\lambda^\mu{}_\nu n^\nu = 0, \quad (17)$$

so λ is a generator from $E(2)$.

This subgroup is precisely $E(2) \uplus \{P_+\}$, and it is contained in one of our generalizations of the VSR group ΛVSR. Improving this subgroup to full ΛVSR would require us to redefine our notions of local symmetry.

Assuming that the divergence of n vanishes, the equations of motion from the Lagrangian (12) read in stereographic coordinates

$$\begin{aligned} 2\frac{\not{\partial}\nu}{\Omega} + \frac{3}{l^2}\not{x}\nu - m\Omega\not{n}\chi - m\psi &= 0 \\ 2\partial_n\psi - \frac{\Omega}{l^2}(\not{x}\not{n} - x\cdot n)\psi + m\Omega\nu &= 0 \\ 2\partial_n\chi - \frac{\Omega}{l^2}(\not{x}\not{n} - x\cdot n)\chi + m\nu &= 0. \end{aligned} \quad (18)$$

Let us now perform a formal expansion of all variables in powers of the radius of the de Sitter space $l \equiv \frac{1}{M_\lambda}$, $\phi = \sum_i \phi^i \frac{1}{l^i}$, including the vector n.

At zero order in l, we recover the flat space equations

$$\begin{aligned} 2\not{\partial}\nu^0 - m\psi^0 - m\not{n}^0\chi^0 = 0 \qquad 2\partial_{n^0}\chi^0 + m\nu^0 = 0 \\ \partial_{n^0}\psi^0 + m\not{n}^0\nu^0 = 0. \end{aligned} \quad (19)$$

At *first* order in l, we have nontrivial equations

$$\begin{aligned} 2\not{\partial}\nu^1 - m\psi^1 - m\not{n}^0\chi^1 - m\not{n}^1\chi^0 &= 0 \\ \partial_{n^0}\chi^1 + \partial_{n^1}\chi^0 + m\nu^1 &= 0 \\ \partial_{n^0}\psi^1 + \partial_{n^1}\psi^0 + m\not{n}^0\nu^1 + m\not{n}^1\nu^0 &= 0. \end{aligned}$$

The fields that were auxiliary (i.e., with algebraic equations of motion) in the flat case do propagate now, so that the correct procedure would be to use the Feynman rules to compute any given process.

Nevertheless, we can formally eliminate the auxiliary fields, just to get an idea of the differences to the flat case. The equations of motion to first order then read

$$\begin{aligned} &\left(2\not{\partial} + \frac{m^2}{2}\left\{\not{n}^0, \frac{1}{\partial_{n^0}}\right\}\right)\nu^1 \\ &+ \frac{m^2}{2}\left(\{\not{n}^1, \partial_{n^0}\} - \left\{\frac{1}{\partial_{n^0}}\partial_{n^1}\frac{1}{\partial_{n^0}}, \not{n}^0\right\}\right)\nu^0 = 0. \end{aligned} \quad (20)$$

The flat space limit ($l \to \infty$) is then plainly as above.

IV. CONCLUSIONS

In the present paper, we have generalized VSR to the physical situation in which a cosmological constant is present. From the group theory approach, two different ways of breaking de Sitter invariance arise, focusing in both of them in subgroups of the de Sitter group.

The first, dubbed ΛVSR, in which we start from $SIM(2)$ and incorporate as many translations as possible (namely, only one). The remaining symmetry group is a five-dimensional subgroup of the Poincaré group. The second approach does *not* include $SIM(2)$ and is based upon $SIM(3)$, the subgroup that leaves a null vector proportional to itself. The symmetry group is in this case is seven-dimensional, that is, only one generator short of the ordinary VSR when the cosmological constant vanishes.

A Lagrangian that provides the equations of motion for a fermion suggested by Cohen and Glashow [2] can be easily found. Generalizing it to the de Sitter space through minimal coupling is straightforward. We examined the propagation of such a fermion, the (dim) possibility of detecting curvature effects in propagators, and the natural generalization of the VSR group [at least, the $E(2)$ plus translations subgroup of the Poincaré group] from dynamical equations.

It seems that only in the case that, for some unknown reason, the neutrino mass is of order of the Hubble scale, $m_\nu \sim M_\lambda$, could these effects be (marginally) measurable. The effects of the cosmological constant would be measured by experimentalists as Lorentz violating, position dependent, mass terms. If there is in addition a breaking of de Sitter symmetry, there are corrections near the endpoint of the beta decay spectrum proportional to the ratio of the square of the mass, divided by the product of its momentum times its energy $\frac{m^2}{Ep}$ similar to the ones uncovered previously by Cohen and Glashow [2] in the flat case.

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APPENDIX: FERMIONS IN DE SITTER

Let us consider physics in a local free-falling frame [4–6] defined by a tetrad $e_a{}^\mu$. Given a Dirac spinor with Lorentz transformation properties

$$\psi'^\alpha = M_s(\Lambda)^\alpha{}_\beta \psi^\beta; \qquad \text{(A1)}$$

where the $(1/2, 0) \oplus (0, 1/2)$ (spinorial) representation of the Lorentz transformation Λ is given by

$$M_s(\Lambda) = e^{(1/4)\sigma^{ab}\gamma_{ab}}, \qquad \text{(A2)}$$

where $\gamma_{ab} \equiv \gamma_a\gamma_b - \gamma_b\gamma_a$. This field is assumed to be an scalar under diff transformations $\delta_D\psi = \xi^\alpha\partial_\alpha\psi$.

The covariant derivative of the spinor field is given by

$$\nabla_\mu\psi = (\partial_\mu + \Gamma_\mu)\psi = (\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\psi, \qquad \text{(A3)}$$

where the spin connection transforms as

$$\delta_L\omega_\mu^{ab} = \partial_\mu\sigma^{ab} - [\omega_\mu, \sigma]^{ab}. \qquad \text{(A4)}$$

A particular solution is given in terms of the Ricci rotation coefficients

$$\Omega_{ab}{}^c = e_a{}^\mu e_b{}^\nu \partial_{[\mu} e^c_{\nu]},$$

$$\omega_{abc} = \Omega_{bca} - \Omega_{abc} - \Omega_{cab}, \qquad \omega_{\mu ab} \equiv e^c{}_\mu \omega_{cab}. \qquad \text{(A5)}$$

This guarantees that under a Lorentz transformation $\sigma^{ab}(x)$, $\delta_L\nabla_\mu\psi = \frac{1}{4}\sigma^{ab}\gamma_{ab}\nabla_\mu\psi$, so that the term $\Phi(x) \equiv \bar\psi\gamma^\mu\nabla_\mu\psi$ transform as a worldsheet scalar, and the full action

$$S \equiv \int d^4x\sqrt{|g|}\bar\psi\gamma^\mu\nabla_\mu\psi \qquad \text{(A6)}$$

is diff-invariant. This is true for a generic metric tensor.

When the spacetime enjoys isometries (as is the case for de Sitter space) then, for those particular diffeomorphisms generated by the Killing vectors, k, the action is invariant under $\delta_I\Phi = k^\alpha\nabla_\alpha\Phi$ (owing to $\nabla_\alpha k^\alpha = 0$). The scalar $\Phi(x)$ can now be rewritten as

$$\Phi = \bar\psi\gamma^a e_a^\mu\nabla_\mu\psi, \qquad \text{(A7)}$$

which conveys the invariance of the action under the transformations $\delta_I\psi = k^\alpha\partial_\alpha\psi$ *provided* the tetrad is chosen in such a way that $\delta_I e_a^\mu = 0$.

When Riemann stereographic coordinates are used, i.e.,

$$ds^2 = \Omega^2\eta_{\mu\nu}dx^\mu dx^\nu, \qquad \text{(A8)}$$

with $\Omega = \frac{1}{1-\frac{x^2}{4l^2}}$, $l = R_H$. The simplest choice for the tetrad one form is plainly $e_a^\mu = \Omega^{-1}\delta_a^\mu$. The Dirac action with the former tetrad reads

$$S = \int d^4x\Omega^4 i\bar\psi\left(\frac{\partial\!\!\!/}{\Omega} + \frac{3}{2l^2}x\!\!\!/\right)\psi. \qquad \text{(A9)}$$

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