Dualities among one-time field theories with spin, emerging from a unifying two-time field theory

Itzhak Bars and Guillaume Quélin

Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089-0484, USA

(Received 17 February 2008; published 18 June 2008)

The relation between two-time physics (2T-physics) and the ordinary one-time formulation of physics (1T-physics) is similar to the relation between a 3-dimensional object moving in a room and its multiple shadows moving on walls when projected from different perspectives. The multiple shadows as seen by observers stuck on the wall are analogous to the effects of the 2T-universe as experienced in ordinary 1T spacetime. In this paper we develop some of the quantitative aspects of this 2T to 1T relationship in the context of field theory. We discuss 2T field theory in $d + 2$ dimensions and its shadows in the form of 1T field theories when the theory contains Klein-Gordon, Dirac and Yang-Mills fields, such as the standard model of particles and forces. We show that the shadow 1T field theories must have hidden relations among themselves. These relations take the form of dualities and hidden spacetime symmetries. A subset of the shadows are 1T field theories in different gravitational backgrounds (different space-times) such as the flat Minkowski spacetime, the Robertson-Walker expanding universe, $\mathrm{AdS}_{d-k} \times S^k$, and others, including singular ones. We explicitly construct the duality transformations among this conformally flat subset, and build the generators of their hidden $\mathrm{SO}(d, 2)$ symmetry. The existence of such hidden relations among 1T field theories, which can be tested by both theory and experiment in 1T-physics, is part of the evidence for the underlying $d + 2$ dimensional spacetime and the unifying 2T-physics structure.

DOI: 10.1103/PhysRevD.77.125019 PACS numbers: 12.60.−i, 11.30.Ly, 14.80.Bn, 14.80.Mz

I. ALLEGORY ON THE RELATION BETWEEN 1T AND 2T PHYSICS

The physical content of 2T-physics and its relation to 1T-physics may be described with an allegory. The allegory is to consider a 3-dimensional object moving in a room and the relationships among different shadows of the same object when projected on 2-dimensional walls by shining light on it from different perspectives. To observers that live only on the walls (similar to living only in 3 + 1 dimensions) the different shadows appear as different "beasts" (like different 1T-physics systems). But with hard work, observers on the wall will discover enough relationships among the shadows to reconstruct the 3 dimensional object.

The allegory above applies because, due to a richer set of gauge symmetry constraints, 2T-physics in 4 + 2 dimensions with 2 times effectively behaves like 1T-physics in 3 + 1 dimensions with 1 time, but with previously unsuspected relationships in 1T-physics that are not apparent in the ordinary formulation of physics. Hidden relations among 1T-physics systems, predicted by 2T-physics, provide the observable clues and evidence of the underlying 4 + 2 nature of spacetime.

In the present paper we discuss some such relationships in the context of field theory and provide simple examples of the type of phenomena described above. These are dualities among 1T field theories in different gravitational backgrounds (different 1T spacetimes).

The Weyl and general coordinate transformations that relate the field theories discussed here are familiar transformations and the techniques are buried in old literature. But these transformations were not previously presented as duality transformations, nor were they understood to be part of gauge symmetries that unite the 1T-shadows into a single higher dimensional structure described by a parent 2T theory. We emphasize that the specific physics examples discussed explicitly here were not all familiar as being related by dualities.

We also stress that these simple examples form only a subset of a much larger set of shadows that obey more complicated duality transformations (not just Weyl and general coordinate) which were not known to exist until discovered through 2T-physics.

In this context, the usual standard model of particles and forces (SM) in 3 + 1 flat spacetime is regarded as one of the shadows of a parent field theory in 4 + 2 dimensions. According to our arguments it is dual to a variety of shadows, some of which are obtained by a series of Weyl and general coordinate transformations. It may be significant that one of the dual shadows is the SM in the Robertson-Walker expanding universe.

II. 2T PHYSICS

While theories with extra spacelike dimensions have been discussed extensively, theories with more than one timelike dimension have been largely left aside. M-theory itself as well as its extensions have provided various signals through supersymmetry structures and dualities that extra timelike dimensions could be relevant for an eventual understanding of fundamental physics [1–5]. However, it is not an easy step to construct a theory with full fledged extra timelike dimensions due to interpretational issues and most importantly because of the systematic presence of ghosts in the quantum theory. Even the first timelike dimension potentially introduces ghosts in relativistic quantum field or string theories. Experience over half a century shows that the cure, to remove the ghosts due to the first timelike dimension, lies in having the right mix of gauge symmetries to arrive at a unitary and physical theory.

Similarly, two-time physics, in d space and 2 time dimensions, is a general framework for a unitary and physical theory, which is achieved precisely by having the right mix of gauge symmetries. The key element of 2T-physics is the presence of a world line $Sp(2, R)$ gauge symmetry which acts in *phase space* (X^M, P_M) and makes position and momentum indistinguishable at any world line instant [6]. This $Sp(2, R)$ is an upgrade of world line reparametrization to a higher gauge symmetry. It yields nontrivial physical content only if the target spacetime includes two-time dimensions, and plays a crucial role to remove all unphysical degrees of freedom in a 2T spacetime, just as world line reparametrization removes unphysical degrees of freedom in a 1T spacetime.

In the case of spinning particles the world line gauge symmetry is extended to $OSp(n|2)$ [7,8] while adding fermionic spin degrees of freedom ψ^M in $d+2$ dimensions beyond phase space (X^M, P_M). Similarly, for more complicated systems, such as supersymmetric particles and others [9–16], as more degrees of freedom with potential ghosts in $d+2$ dimensions are added, the corresponding world line gauge symmetry is also larger, to insure the unitarity of the theory in the 2T-physics formulation. All extensions of the world line gauge symmetry must include the key ingredient $Sp(2, R)$, and hence is required to have 2 times.

2T-physics is elevated from the world line formulation to field theory through the process of covariant quantization. The spin 0, $\frac{1}{2}$, 1 fields, $\Phi(X)$, $\Psi_\alpha(X)$, $A_M(X)$, are then identified with the first quantized wavefunctions that obey the gauge symmetry constraints, implying that these fields describe the ghost-free *gauge invariant* sector of the world line theory, as long as the constraints are satisfied as on-shell equations of motion. 2T field theory is based on an action principle that generates these constraints as equations of motion, and furthermore extends them with interactions.

The 2T-physics field theory formalism has some features that differ from 1T-field theory formalism, such as a delta function in the volume element $\delta(X^2)d^{d+2}X$ and other properties [17], as outlined below. Thanks to these properties, minimizing the 2T field theory action leads to field equations that reproduce the $Sp(2, R)$ or other gauge symmetry constraints of the underlying world line action, thus insuring the unitarity of the theory.

In this 2T field theory setup, it has been shown that the usual 1T-physics standard model of particles and forces in $3+1$ dimensions is reproduced as one of the shadows of a 2T-physics field theory in $4+2$ dimensions [17]. The *emergent 1T standard model*, being a $3+1$ shadow of the $4+2$ theory with more symmetry, comes with some additional restrictions that are not present in the usual 1T formulation, but nevertheless agrees with all known physics. The differences occur only in hitherto unmeasured parts of the standard model, in particular, the axion and Higgs sectors, so they are of phenomenological as well as theoretical significance, and may provide tests at the LHC or in cosmology to distinguish 2T-physics from previous approaches.

There are more ways to test 2T-physics at all scales of physics by exploring the multiple 1T-physics shadows and the predicted relationships among them as well as their hidden symmetries that give information on the higher dimensions. Previous work in the context of the world line formalism displayed many examples of these shadows [18–20]. A graphical display of some of these examples can be found at [21]. In our recent paper [22] most of the known shadows were tabulated and useful mathematical formulas that describe them were summarized (see tables I, II and III and related discussion in [22]).

This avenue of investigation is still in its infancy. The purpose of our paper is to develop some techniques and concepts along this path by elucidating the dualities and hidden symmetries among a subset of these shadows. This subset is represented by 1T field theories in different gravitational backgrounds which are all conformally flat. In our recent paper [22] the dualities and hidden symmetries of a 1T scalar field theory in such backgrounds was discussed. In the present paper we further elaborate on these properties with fermionic fields that carry spin $1/2$, and Yang-Mills gauge fields that carry spin 1. It is then possible to discuss a subset of the dualities and hidden symmetries for the standard model. We expect that these dualities, together with the future extension of our results to other types of shadows, to be potentially useful for nonperturbative analysis of the standard model.

III. 2T FIELD THEORY

2T field theory has been fully formulated at the action level for fields of spins 0, $\frac{1}{2}$, 1 [17], and to the field equation of motion level for spin 2 [8] and beyond [23], and has also been supersymmetrized [24]. The scalar field was discussed extensively in our recent paper [22]. In the current paper, we will focus on the spin-$\frac{1}{2}$ and spin-1 cases.

A. Spin-1 fields

The 2T action for spin-1 Yang-Mills fields is

$$S(A) = Z \int d^{(d+2)}X \delta(X^2) \times \left(-\frac{1}{4} \Phi^{(2(d-4)/d-2)} \operatorname{Tr}(F_{MN}F^{MN}) \right) \quad (3.1)$$

where Z is an overall normalization constant that will be determined. The dilaton Φ, which drops out when $d = 4$ in the above expression, is necessary when $d \neq 4$ for consistency of constraints or 2T gauge symmetries (see [17]). The action for the dilaton $S(\Phi)$ and its duality properties have already been discussed in our previous paper [22] that described any scalar, including the dilaton. Turning to the matrix valued Yang-Mills gauge field A_M in the adjoint representation of the gauge group G, the field strength F_{MN} is defined as usual

$$F_{MN} \equiv \partial_M A_N - \partial_N A_M - ig[A_M, A_N]. \tag{3.2}$$

Varying the action with respect to the matrix A_N results in the expression

$$\delta S(A) = Z \int d^{(d+2)}X \,\mathrm{Tr}\{\delta A_N[\delta(X^2) D_M(\Phi^{(2(d-4)/d-2)} F^{MN} + 2\delta'(X^2)\Phi^{(2(d-4)/d-2)} X_M F^{MN})]\} \tag{3.3}$$

where $\delta'(X^2)$ emerges from an integration by parts. Since the delta function $\delta(X^2)$ and its derivative $\delta'(X^2)$ are linearly independent distributions, minimizing the action $\delta S(A) = 0$ for general δA_N gives two separate equations of motion[1] for A_M

$$[X^N F_{MN}]_{X^2=0} = 0, [D_M(\Phi^{(2(d-4)/d-2)} F^{MN})]_{X^2=0} = 0. \tag{3.4}$$

The two conditions $X^2 = 0$ and $[X^N F_{MN}]_{X^2=0} = 0$ have been called "kinematical" constraints [17] that parallel two of the world line Sp$(2, R)$ constraints $X^2 = X \cdot P = 0$ (applied on states P is a derivative). The remaining "dynamical" equation of motion that contains two derivatives parallels the third Sp$(2, R)$ world line constraint $P^2 = 0$. The field theoretic version of these Sp$(2, R)$ constraints[2] evidently include field interactions that are consistent with the familiar Yang-Mills gauge symmetry.

The delta function $\delta(X^2)$ that appears in the action[3] invites an expansion of every field in powers of X^2. For the gauge field one can write

$$A_M = A_M^0 + X^2 \tilde{A}_M \tag{3.5}$$

where we define $A_M^0 \equiv [A_M]_{X^2=0}$ while $X^2 \tilde{A}_M = A_M - A_M^0$ is the remainder that includes all higher powers of X^2. As shown in [17], the action $S(A)$ has also a "2T-gauge symmetry" under the variation

$$\delta_\Lambda A_N = \Phi^{(-2(d-4)/d-2)} X^2 \Lambda_N(X) \tag{3.6}$$

which can be verified (with some restrictions[4] on the local gauge parameter $\Lambda_M(X)$) by inserting $\delta_\Lambda A_N$ into Eq. (3.3) instead of the general δA_N. This gauge symmetry can be used to thin out the degrees of freedom in $A_M(X)$. In [17] it was argued that there is just enough "2T-gauge symmetry" to remove the remainder $\tilde{A}_M(X)$ if so desired, and reduce it to the physical field A_M^0, thus showing that the gauge fixed fields become independent of X^2. This amounts to eliminating one spacetime coordinate among the X^M.

The strategy to descend to 1T-physics from 2T-physics is then to make gauge choices and solve the two kinematic constraints $X^2 = 0$, $[X^N F_{MN}]_{X^2=0} = 0$. Upon inserting the solution into the dynamical field equation or into the original action, one realizes that the remaining dynamics is in one less space and one less time dimensions precisely as in 1T-physics field theory, but in a variety of spacetimes. This is then how we obtain many 1T shadows of the 2T field theory.

The interesting phenomena are that there are many Yang-Mills 1T shadows in different emerging 1T spacetimes that materialize from different solutions of the kine-

[1]Strictly speaking, the equation of motion of the form $\delta(X^2)F(X) + \delta'(X^2)G(X) = 0$ yields the two equations $[F(X) + \tilde{G}(X)]_{X^2} = 0$ and $[G(X)]_{X^2=0} = 0$, where $\tilde{G}(X)$ is the "remainder" of the field as defined by the second term in the expansion of $G(X) = [G(X)]_{X^2=0} + X^2[\tilde{G}(X)]_{X^2=0} + \cdots$ in powers of X^2. In our case the extra term $[\tilde{G}(X)]_{X^2=0}$ can be dropped due to the properties of the "remainders" of the fields Φ and A_M as given in Eq. (3.5) and discussed in footnote 4 and Ref. [17].

[2]Taking into consideration the spin degrees of freedom carried by the vector field $A_M(X)$, the full set of constraints is actually OSp(2|2), where OSp(2|2) is the gauge symmetry of the world line theory for a spin 1 particle [7,8].

[3]Delta functions in the action also appeared in some of the related work to 2T-physics that followed Dirac's approach to conformal symmetry ([25–36]). But what we call here "kinematic constraints" on the fields were imposed as external constraints rather than being derived from the action principle. This was the approach in 2T-physics independently arrived at some time ago [8]. Our treatment of the delta functions $\delta(X^2)$, $\delta'(X^2)$ in the current paper is technically different and follows the discussion in [17,37]. In this way the kinematic constraints on the physical field A_M^0 in Eq. (3.5) follow from our action and its symmetries.

[4]In [17] the 2T gauge symmetry was discussed under the assumption that the *remainder* $\tilde{A}_M(X)$ in Eq. (3.5) *a priori* satisfied a homogeneity condition $(X \cdot D + 3)\tilde{A}_M = 0$ (but unrestricted physical field A_M^0). This condition on the remainder $\tilde{A}_M(X)$ was a partial gauge choice for a larger gauge symmetry, and therefore the gauge parameter $\Lambda_M(X)$ was also restricted by a corresponding homogeneity condition $(X \cdot D + 3)\Lambda_M = 0$. A homogeneous remainder $\tilde{A}_M(X)$ made it easier to derive the two separate equations in (3.4) as the unique outcome of minimizing the action. This assumption for $\tilde{A}_M(X)$ can be dropped at the expense of a more elaborate discussion of the larger 2T gauge symmetry, as will be further elucidated in a separate paper. With this, one arrives again at the same on-shell equations of motion (3.4). Either way, the conclusions of the present paper remain unchanged.

matic equations $X^2 = 0$, $[X^N F_{MN}]_{X^2=0} = 0$, and that the emergent 1T field theories may come with some symmetry restrictions that are not anticipated with only 1T field theory methods. For example, in the case of the standard model [17] the latter restrictions lead to new concepts on the generation of mass.

As already mentioned, among the many possible solutions, in the next section we will concentrate on an easier subset of solutions that correspond to conformally flat spacetimes and then explore the dualities among the resulting field theories.

B. Spin-$\frac{1}{2}$ fields

The 2T free field action for spinor fields is given by [17]

$$S(\Psi) = \frac{i}{2} Z \int (d^{d+2}X)\delta(X^2)(\bar{\Psi} \not{X} \bar{\not{\partial}} \Psi + \bar{\Psi} \overleftarrow{\bar{\not{\partial}}} \not{X} \Psi) \quad (3.7)$$

where Z is the same normalization constant as in (3.1), and $\not{X} \equiv \Gamma^M X_M$ and $\bar{\not{\partial}} \equiv \bar{\Gamma}^M \partial_M$, using the $\mathrm{SO}(d, 2)$ gamma matrices Γ^M, $\bar{\Gamma}^M$ in the footnote.[5] Varying the action gives

$$\delta S(\Psi) = iZ \int (d^{d+2}X)\delta(X^2)\delta\bar{\Psi}\left[\not{X}\bar{\not{\partial}}\Psi - \left(X \cdot \partial + \frac{d}{2}\right)\Psi\right] + \mathrm{H.c.} \quad (3.8)$$

where the second term emerges from integration by parts and using $X^2\delta'(X^2) = -\delta(X^2)$. As was shown in [17], the two terms in the bracket actually need to vanish separately when we require $\delta S(\Psi) = 0$ for general $\delta\bar{\Psi}$. So the equations of motion are[6]

$$\left[\left(X \cdot \partial + \frac{d}{2}\right)\Psi\right]_{X^2=0} = 0, \qquad [\not{X}\bar{\not{\partial}}\Psi]_{X^2=0} = 0. \quad (3.9)$$

It should be noted that the action $S(\Psi)$ is invariant under the following "2T gauge transformation"

$$\delta_\zeta \Psi = X^2 \zeta_1 + \not{X}\zeta_2, \qquad \delta_\zeta \bar{\Psi} = X^2 \bar{\zeta}_1 + \bar{\zeta}_2 \not{X}. \quad (3.10)$$

[5] We use the following explicit $\mathrm{SO}(d, 2)$ gamma matrices

$$\Gamma^{+'} = \begin{pmatrix} 0 & -i\sqrt{2} \\ 0 & 0 \end{pmatrix}, \qquad \Gamma^{-'} = \begin{pmatrix} 0 & 0 \\ -i\sqrt{2} & 0 \end{pmatrix},$$
$$\Gamma^{\mu} = \begin{pmatrix} \bar{\gamma}^{\mu} & 0 \\ 0 & -\gamma^{\mu} \end{pmatrix} \qquad \bar{\Gamma}^{+'} = \begin{pmatrix} 0 & -i\sqrt{2} \\ 0 & 0 \end{pmatrix},$$
$$\bar{\Gamma}^{-'} = \begin{pmatrix} 0 & 0 \\ -i\sqrt{2} & 0 \end{pmatrix}, \qquad \bar{\Gamma}^{\mu} = \begin{pmatrix} \gamma^{\mu} & 0 \\ 0 & -\bar{\gamma}^{\mu} \end{pmatrix}$$

where $\gamma_\mu = (1, \gamma_i)$, $\bar{\gamma}_\mu = (-1, \gamma_i)$, or $\gamma^\mu = (-1, \gamma^i)$, $\bar{\gamma}^\mu = (1, \gamma^i)$ are $\mathrm{SO}(d-1, 1)$ gamma matrices. For further detailed properties of these gamma matrices see the appendix of [24].

[6] These equations of motion amount to $\mathrm{OSp}(1|2)$ constraints [17], where $\mathrm{OSp}(1|2)$ is the gauge symmetry [8] of the underlying world line theory [7] (see Appendix A). Imposing $\mathrm{OSp}(1|2)$ constraints is the requirement that the physical configurations of the field $\Psi(X)$ be gauge invariant under the $\mathrm{OSp}(1|2)$ gauge symmetry.

This is verified (see [17]) by inserting $\delta_\zeta \bar{\Psi}$ in (3.8) instead of the general $\delta\bar{\Psi}$. The role of the gauge spinors ζ_1, ζ_2 are as follows. Because of the delta function we are invited to expand the field in powers of X^2, thus $\Psi = \Psi^0 + X^2\tilde{\Psi}$, where we define $\Psi^0 \equiv [\Psi]_{X^2=0}$ while $X^2\tilde{\Psi} = \Psi - \Psi^0$ is the remainder that includes all higher powers of X^2. In [17] it was shown that the gauge parameter ζ_1 that appears in Eq. (3.10) can be used to remove the remainder $\tilde{\Psi}$ if so desired. The remaining $\Psi^0_\alpha(X)$ is then independent of X^2, however compared to 2 less dimensions it has double the number of spinor components. With the gauge symmetry ζ_2 one can show [17] that half of the degrees of freedom in $\Psi^0_\alpha(X)$ are gauge degrees of freedom while the remaining half are physical. In this role, the ζ_2 transformation is similar to kappa-type local supersymmetry, and it can be used to eliminate half of the spinor components, if so desired.

Interactions of fermions with the gauge fields are obtained by simply replacing all derivatives by the covariant derivative $\partial_M \to D_M = \partial_M - igA_M$. The Yukawa interaction with a scalar $H(X)$ takes the form [17] $H(\bar{\Psi}\not{X}\Psi)\Phi^{(2(d-4)/(d-2))}$, where Φ is the dilaton that does not appear if $d + 2 = 6$. The fermionic gauge symmetry of Eq. (3.10) remains as a valid symmetry in the presence of these interactions, and it will be used to obtain the proper spin-$\frac{1}{2}$ degrees of freedom in the lower dimensional actions.

The strategy to descend to 1T-physics from 2T-physics for fermions is then to make gauge choices by using ζ_1, ζ_2 and solve the two kinematic constraints $X^2 = 0$ and $((X \cdot D + \frac{d}{2})\Psi)_{X^2=0} = 0$. Upon inserting the solution into the original action (including interactions) it is seen that the remaining dynamics has precisely the familiar form of 1T field theory.

As in the case of gauge fields above, various 1T spacetimes materialize from different solutions of the kinematic equations. These emerging 1T field theories in $(d-1)+1$ dimensions, that include scalars, fermions, and Yang-Mills bosons, are then dual to each other. This duality will be illustrated below for a subset of the solutions.

IV. EMERGENT $(d-1)+1$ FIELD THEORY

The strategy described in the previous section to reduce 2T field theory to 1T field theory will be implemented in this section by solving the kinematic equations

$$X^2 = 0, \qquad \left[\left(X \cdot D + \frac{d}{2}\right)\Psi\right]_{X^2=0} = 0, \qquad [X^N F_{MN}]_{X^2=0} = 0. \quad (4.1)$$

The result, which will involve fields in 2 less spacetime variables, will be inserted in the original action to yield the "shadows" in the form of 1T field theories. To solve these equations we follow the footsteps for solving the corresponding constraints $X^2 = X \cdot P = 0$ in the underlying

TABLE I. A sample of 1T physics shadows that emerge from the flat $(d+2)$ 2T theory.

The massless relativistic particle in d flat Minkowski space. (cf)
The massive relativistic particle in d flat Minkowski space.
The nonrelativistic free massive particle in $d-1$ space dimensions.
The nonrelativistic hydrogen atom (i.e. $1/r$ potential) in $d-1$ space dimensions.
The harmonic oscillator in $d-2$ space dimensions, with its mass $\Leftrightarrow$ an extra dimension.
The particle on AdS_d, or on dS_d. (cf)
The particle on $\mathrm{AdS}_{d-k}\times S^k$ for $k=1,2,\cdots,d-1$. (cf)
The particle on the Robertson-Walker spacetime (open or closed universes). (cf)
The particle on any maximally symmetric space of positive or negative curvature. (cf)
The particle on any of the above spaces modified by any conformal factor.
A related family of other particle systems, including some singular backgrounds. (cf)

world line theory. This involved making some gauge choices for phase space $[X^M(\tau), P_M(\tau)]$ by using the world line local $\mathrm{Sp}(2,R)$ gauge symmetry.

In this way the 1T systems listed in Table I emerge as shadows from the 2T theory in flat $d+2$ dimensions. In this table the cases marked as (cf) correspond to conformally flat curved spaces, on which we concentrate in this paper. The details of the world line gauge choices for (X^M, P_M) was summarized in tables I,II,III in [22]. Those tables provide details for a variety of embeddings of $(d-1)+1$ dimensions into $d+2$ dimensions, with distinct forms of "time" and "Hamiltonian" as interpreted in the lower dimension (i.e. the 1T shadows).

In 2T field theory, we cannot choose a gauge[7] for X^M like we do for the world line theory $X^M(\tau)$. Instead, we *parameterize* X^M as in e.g. Eq. (4.3), which is a form that is parallel to a subset of gauge choices of the world line theory (compare to Appendix A). We start by choosing an embedding of the 1T spacetime x^μ into the 2T spacetime X^M. To do so, it is useful to distinguish one space and one-time dimensions $X^{0'}$, $X^{1'}$, to define a lightcone-type basis, $M=(+',-',m)$, with $X^{\pm'}\equiv\frac{1}{\sqrt{2}}(X^{0'}\pm X^{1'})$, so that the flat metric η_{MN} in $d+2$ dimensions takes the form

$$ds^2 = dX^M dX^N \eta_{MN} = -2dX^{+'}dX^{-'} + dX^m dX^n \eta_{mn} \tag{4.2}$$

where η_{mn} is the flat Minkowski metric in d dimensions including 1 time dimension. Next we choose the embedding by the following *parametrization* of X^M in terms of the 1T spacetime x^μ and two other dimensions κ, λ

$$X^{+'} = \kappa e^{\sigma(x)}, \qquad X^{-'} = \lambda\kappa e^{\sigma(x)}, \qquad X^m = \kappa e^{\sigma(x)} q^m(x), \tag{4.3}$$

where the functions $\sigma(x)$ and $q^m(x)$ remain unspecified. Solving for κ, λ, and $q^m(x)$, in terms of $X^{\pm'}$, X^m we get the inverse parametrization

$$\kappa = e^{-\sigma(x)}X^{+'}, \qquad \lambda = \frac{X^{-'}}{X^{+'}}, \qquad q^m(x) = \frac{X^m}{X^{+'}}. \tag{4.4}$$

From $q^m(x)=\frac{X^m}{X^{+'}}$ we solve in principle for $x^\mu = f^\mu(\frac{X^m}{X^{+'}})$, where $f^\mu(q^m)$ is the inverse map of $q^m(x^\mu)$. This inverse map is inserted in $\sigma(x)=\sigma(f^\mu(\frac{X^m}{X^{+'}}))$ in Eq. (4.4) to complete the full solution of $\kappa = X^{+'}\exp(-\sigma(f^\mu(\frac{X^m}{X^{+'}})))$ in terms of $X^{\pm'}$, X^m.

Such parametrizations of X^M, combined with gauge choices for Yang-Mills gauge symmetry and 2T gauge symmetries (3.6) and (3.10), lead to the solutions of Eqs. (4.1) as will be shown below.

The physics of the emerging 1T shadows as field theories is anticipated from the corresponding shadows in the classical world line theory. The improvements in field theory include (i) an automatic resolution of ordering ambiguities of nonlinear terms in the quantization of the world line theory (see Appendix A), (ii) the inclusion of interactions and (iii) dualities among interacting field theories which may be used as a new tool for investigating 1T field theory.

To implement the 2T $\rightarrow$ 1T reduction for spin-$\frac{1}{2}$ and spin-1 fields we solve Eqs. (4.1). We follow the methods of our previous investigation of scalar fields [22] which focused on conformally flat 1T-spacetimes that emerged through the 2T $\rightarrow$ 1T embeddings described by Eq. (4.3). The conformally flat backgrounds, which is only a subset of the shadows listed in Table I, are those marked as (cf) including the flat massless Minkowski spacetime, $\mathrm{AdS}_{d-k}\times S^k$, AdS_d, dS_d, Robertson-Walker, maximally symmetric spaces, and some singular spaces. The other interesting cases listed in Table I, such as the massive particle(s), hydrogen atom, and harmonic oscillator are not conformally flat backgrounds. To describe those other nonconformally flat shadows, which are also solutions of Eqs. (4.1), a parametrization of the embedding of the

[7] This is because $\mathrm{Sp}(2,R)$ is not a gauge symmetry of the *field theory* action $S(\Phi, A, \Psi)$, but rather the action generates on-shell equations of motion that reproduce the $\mathrm{Sp}(2,R)$ constraints of the *world line theory*, as explained in the previous section. These fields which satisfy the $\mathrm{Sp}(2,R)$ constraints are then the $\mathrm{Sp}(2,R)$ gauge invariant physical configurations.

1T spacetime into the 2T spacetime that is rather different than Eq. (4.3) is required.[8]

The fields $\Phi(X) = \Phi(\kappa, \lambda, x^\mu)$, $\Psi(X) = \Psi(\kappa, \lambda, x^\mu)$ and $A_M(X) = A_M(\kappa, \lambda, x^\mu)$ are now considered functions of κ, λ, x^μ. The $(d-1)+1$ spacetime x^μ has been embedded in $d+2$ dimensions in different forms that vary as the functions $q^m(x)$ and $\sigma(x)$ change.

To obtain the kinetic energy terms for the fields $\Phi(X)$, $A_M(X)$, $\Psi_\alpha(X)$ we use the chain rule to compute the partial derivatives $\frac{\partial}{\partial X^M}$ in terms of $\frac{\partial}{\partial\kappa}, \frac{\partial}{\partial\lambda}, \frac{\partial}{\partial x^\mu}$, consistent with the parametrization (4.3). The result is

$$\frac{\partial}{\partial X^{-'}} = \frac{1}{\kappa} e^{-\sigma} \frac{\partial}{\partial\lambda} \tag{4.5}$$

$$\frac{\partial}{\partial X^m} = \frac{1}{\kappa}\left(-e_m^\mu \partial_\mu \sigma \kappa \frac{\partial}{\partial\kappa} + e_m^\mu \partial_\mu\right) \tag{4.6}$$

$$\frac{\partial}{\partial X^{+'}} = \frac{1}{\kappa}\Big([e^{-\sigma} + q^m e_m^\mu \partial_\mu \sigma]\kappa \frac{\partial}{\partial\kappa} - e^{-\sigma}\lambda\frac{\partial}{\partial\lambda} - q^m e_m^\mu \partial_\mu\Big) \tag{4.7}$$

Here $e_m^\mu(x)$ is the inverse of the vielbein. The vielbein itself in the reduced spacetime is defined as

$$e_\mu^m(x) = e^{\sigma(x)} \frac{\partial q^m(x)}{\partial x^\mu}, \tag{4.8}$$

Then the inverse $e_m^\mu(x)$ can also be written as $e_m^\mu(x) = e^{-\sigma(x)} \frac{\partial x^\mu}{\partial q^m} = e^{-\sigma(x)} \frac{\partial f^\mu(q)}{\partial q^m}(x)$, where $x^\mu = f^\mu(q)$ is the inverse map discussed following Eq. (4.4). This is verified by using the chain rule $e_\nu{}^m(x) e_m^\mu(x) = e^{\sigma(x)} \frac{\partial q^m}{\partial x^\nu} e^{-\sigma(x)} \frac{\partial x^\mu}{\partial q^m} = \frac{\partial x^\mu}{\partial x^\nu} = \delta_\nu{}^\mu$. We note, in particular, that the dimension operator $X\cdot\partial$ that we will need to solve the kinematic equations (4.1) takes a simple form $\kappa\frac{\partial}{\partial\kappa}$

$$X\cdot\partial = +X^{+'}\frac{\partial}{\partial X^+} + X^{-'}\frac{\partial}{\partial X^-} + X^m \frac{\partial}{\partial X^m} = \kappa\frac{\partial}{\partial\kappa}. \tag{4.9}$$

[8]The special case treated in this paper in Eq. (4.3) embeds d dimensional space x^μ space into $d+2$ dimensional space X^M, from which we can figure out the embedding of momentum [derivatives applied on fields as in Eqs. (4.5), (4.6), and (4.7)]. The more general case embeds not only space x^μ, but all of all of phase space (x^μ, p_μ) in d dimensions into phase space (X^M, P_M) in $d+2$ dimensions. Consequently, the emergent spacetimes are not only conformally flat, but much more interesting. Some such examples include the massive particle(s), the hydrogen atom and harmonic oscillator listed in Table I. In these later cases the parametrization of X^M involves momenta in addition to positions (see tables I,II,III in [22]). While this is straightforward to implement in the world line formalism, it is more challenging in the context of field theory, since momenta are replaced by derivatives. For this reason, the field theoretic investigation of this more complicated type of "shadow" is left to future work.

With this parametrization we see that the volume element takes the form

$$X^2 = -2\kappa^2 e^{2\sigma}(\lambda - \tfrac{1}{2}q^2(x)), \tag{4.10}$$

$$(d^{d+2}X)\delta(X^2) = \tfrac{1}{2}\kappa^{d-1}\det(e_\mu^m(x)) d\kappa d\lambda d^d x \delta(\lambda - \tfrac{1}{2}q^2(x)). \tag{4.11}$$

where we have taken into account the Jacobian for the change of variables

$$J\left(\frac{X^{+'}, X^{-'}, X^m}{\kappa, \lambda, x^\mu}\right) = \kappa^{d+1} e^{(d+2)\sigma}\det(\partial_\mu q^m) = \kappa^{d+1} e^{2\sigma}\det(e_\mu^m(x)). \tag{4.12}$$

It is also worth noticing that, after taking into account the delta function that imposes $\lambda = \frac{1}{2}q^2(x)$, the metric in $d+2$ dimensions in Eq. (4.2) collapses to the curved metric $g_{\mu\nu}(x)$ in $(d-1)+1$ dimensions

$$ds^2 = (dX\cdot dX)_{\lambda=q^2/2} = \kappa^2 g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{4.13}$$

Here $g_{\mu\nu}(x)$ is conformally flat since it has the form

$$g_{\mu\nu} = e_\mu^m e_\nu^n \eta_{mn} = e^{2\sigma(x)} \frac{\partial q^m(x)}{\partial x^\mu}\frac{\partial q^n(x)}{\partial x^\nu}\eta_{mn}. \tag{4.14}$$

Specific forms of $\sigma(x^\mu)$, $q^m(x^\mu)$ that produce all of the conformally flat examples included in Table I were given explicitly in [22]. As an illustration, we give here the Robertson-Walker case, where $a(t)$ is any function that represents the expanding size of an open universe

$$(ds^2)_{\lambda=q^2/2} = \frac{\kappa^2}{R_0^2}\left[-dt^2 + \frac{a^2(t)}{R_0^2}\left(\frac{R_0^2}{R_0^2+r^2}dr^2 + r^2 d\Omega^2\right)\right] \tag{4.15}$$

This is an example of Eq. (4.13) that is obtained by inserting the following explicit forms of $\sigma(x)$ and $q^m(x)$

Robertson-Walker expanding open universe $(r>0)$

$$e^{\sigma(x)} \equiv \frac{a(t)}{R_0}\exp\left(\pm\int^t \frac{dt'}{a(t')}\right),$$
$$\vec{q}(x) \equiv \pm\frac{\vec{r}}{R_0}\exp\left(\mp\int^t\frac{dt'}{a(t')}\right), \tag{4.16}$$

$$q^0(x) \equiv \mp\sqrt{1+\frac{r^2}{R_0^2}}\exp\left(\mp\int^t\frac{dt'}{a(t')}\right) \tag{4.17}$$

$$e_\mu{}^m = e^{\sigma(x)}\frac{\partial q^m}{\partial x^\mu} = \frac{1}{R_0}\begin{pmatrix} \sqrt{1+\frac{r^2}{R_0^2}} & -\frac{r^i}{R_0} \\ \mp\frac{a(t)}{R_0}\frac{r_i}{\sqrt{R_0^2+r^2}} & \pm\frac{a(t)}{R_0}\delta_i{}^j \end{pmatrix} \tag{4.18}$$

This parametrization of $X^M(\kappa, \lambda, x^\mu)$ given above for the Robertson-Walker spacetime is slightly different than the one given in [22], but is related to it by a simple redefinition of coordinates.

The discussion above is common to fields of any spin. The 2T → 1T reduction of the spin 0 case was discussed in [22], so we now focus on the spin-$\frac{1}{2}$ and spin-1 cases.

A. Spin-1 field

The kinematic equations (4.1) were first solved by Dirac [25] (see also related work in [26–36]) who did not have an action principle but only suggested equations of motion that were arrived at by a different set of arguments and the motiovation being an explanation of conformal symmetry $SO(d, 2)$ in flat Minkowski space in d dimensions. His solution yielded only one of the possible shadows, namely, the one in flat Minkowski space. The existence of all the other shadows in a variety of spacetimes, and the existence of moduli such as curvature, mass, interaction parameters, were discovered via methods of 2T-physics. In what follows we adapt the 2T-physics methods to discuss a subset of the shadows.

Taking advantage of the Yang-Mills gauge symmetry of the full action $S(\Phi, A, \Psi)$, we first choose the Yang-Mills axial gauge

$$X \cdot A = 0 \tag{4.19}$$

so that the nonlinear kinematic equation for the Yang-Mills field in (4.1) simplifies to a linear equation independent of interactions $0 = X^N F_{MN} = -(X \cdot \partial + 1)A_M(X)$. Using the parametrization for X^M in (4.3) that yields the dimension operator as in (4.9), the kinematic constraint takes the even simpler form

$$\left(\kappa \frac{\partial}{\partial \kappa} + 1\right) A_M(\kappa, \lambda, x) = 0. \tag{4.20}$$

This determines uniquely the κ dependence of the field as

$$A_M(\kappa, \lambda, x) = \frac{1}{\kappa} \hat{A}_M(\lambda, x). \tag{4.21}$$

In the axial gauge (4.19) there is still leftover Yang-Mills gauge symmetry with parameter $\Lambda(\lambda, x)$ that is independent of κ. Using this, we can fix the Yang-Mills gauge further, by taking a lightcone-type gauge

$$\hat{A}_{-'}(\lambda, x) = 0. \tag{4.22}$$

Inserting this in the axial gauge condition $0 = X \cdot \hat{A} = X^{-'} \hat{A}_{-'} + X^{+'} \hat{A}_{+'} + X^m A_m$, we also obtain $\hat{A}_{+'} = -\frac{X^m}{X^{+'}} \hat{A}_m$, which may be written as

$$\hat{A}_{+'}(\lambda, x) = -q^m(x) \hat{A}_m(\lambda, x). \tag{4.23}$$

Given the delta function $\delta(\lambda - \frac{1}{2} q^2(x))$ in the volume element (4.11) we are invited to expand the field in powers of $\lambda - \frac{1}{2} q^2(x)$. Now we take advantage of the 2T gauge symmetry of Eq. (3.6) which allows us to gauge fix the part of the gauge field $A_M = A_M^0 + X^2 \tilde{A}$ proportional to X^2. This means that in the expansion $\hat{A}_m(\lambda, x^\mu) = \bar{A}_m(x^\mu) + (\lambda - \frac{q^2}{2}) \tilde{A}_m(\lambda, x^\mu)$ we can choose the gauge $\tilde{A}_m(\lambda, x^\mu) = 0$. The remaining gauge field $\bar{A}_m(x^\mu)$ is now independent of both λ and κ. Without loss of generality we can write it in the form $\bar{A}_m(x^\mu) = e_m^\mu(x) A_\mu(x)$ where e_m^μ is the inverse vielbein discussed in the previous section.

To summarize, we have shown that, by taking advantage of both the Yang-Mills and 2T gauge symmetries, the general Yang-Mills field $A_M(X)$ can be gauge fixed to the following form

$$A_{-'} = 0, \qquad A_{+'} = -\frac{1}{\kappa} q^m(x) e_m^\mu(x) A_\mu(x), \qquad A_m = \frac{1}{\kappa} e_m^\mu(x) A_\mu(x) \tag{4.24}$$

where only $A_\mu(x)$ is the independent component. We now compute the field strength $F_{MN}(X)$ by using the chain rule given in Eqs. (4.5), (4.6), and (4.7). After some algebra we find[9]

$$F_{+'-'} = 0, \qquad F_{-'m} = 0, \qquad F_{+'m} = \frac{1}{\kappa^2} q^n(x) F_{mn}, \qquad F_{mn} = \frac{1}{\kappa^2} e_m^\mu e_n^\nu F_{\mu\nu}(x) \tag{4.25}$$

with

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{4.26}$$

The quantity $F_{MN} F^{MN}$ in flat $d + 2$ dimensions is then reduced to its shadow in curved d dimensions as follows

$$\begin{aligned} F_{MN} F^{MN} &= F_{mn} F^{mn} = F_{mn} F_{kl} \eta^{mk} \eta^{nl} \\ &= \frac{1}{\kappa^4} (e_m^\mu e_n^\nu F_{\mu\nu})(e_k^\kappa e_l^\lambda F_{\kappa\lambda}) \eta^{mk} \eta^{nl} \\ &= \frac{1}{\kappa^4} g^{\mu\kappa} g^{\nu\lambda} F_{\mu\nu} F_{\kappa\lambda} \equiv \frac{1}{\kappa^4} F_{\mu\nu} F^{\mu\nu} \end{aligned} \tag{4.27}$$

Recalling also the reduced form of the scalar field from [22]

$$\Phi(\kappa, \lambda, x^\mu) = \kappa^{-(d-2/2)} \phi(x^\mu), \tag{4.28}$$

we can rewrite the action in terms of the lower dimensional shadow fields in curved space (the subscript "red" indicates that the solution of the kinematic equations are inserted to obtain the reduced action)

$$S(A, \Phi)_{\text{red}} = Z \int d^{(d+2)} X \delta(X^2) \times \left(-\frac{1}{4} \Phi^{(2(d-4)/d-2)} \operatorname{Tr}(F_{MN} F^{MN})\right)_{\text{red}} \tag{4.29}$$

[9]The details for similar steps in flat space (i.e. when $e_\mu^m(x) = \delta_\mu^m$) are given in [17].

$$= Z\int d\kappa d\lambda d^d x \frac{1}{2}\kappa^{d-1}\det e^m_\mu \delta\Big(\lambda - \frac{q^2(x)}{2}\Big) - \frac{1}{4}\kappa^{4-d}\phi^{(2(d-4)/d-2)}\,\mathrm{Tr}\Big(\frac{1}{\kappa^4}F_{\mu\nu}F^{\mu\nu}\Big) = \int d^d x\sqrt{-g}\Big(-\frac{1}{4}\phi^{(2(d-4)/d-2)}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\Big). \tag{4.30}$$

In the last step we integrated λ, κ and absorbed an infinite constant by normalizing[10] Z as $Z\int \frac{d\kappa}{2\kappa} = 1$, thus arriving at an action expressed in terms of only the d dimensional shadow spacetime x^μ.

It must be emphasized that the reduction of the scalar field[11] discussed in [22] produced exactly the same overall normalization Z and gave the action for the conformal scalar $\phi(x)$ in the same background metric $g_{\mu\nu}(x)$. The conformal scalar action is given in Eq. (A14).

The resulting reduced action $S(A,\Phi)_{\mathrm{red}}$ is the action for a spin-1 gauge field in a variety of shadow curved spacetimes, all with conformally flat metrics of Eq. (4.14). Note that these shadows of the same $(d+2)$ theory change as the functions $\sigma(x)$, $q^m(x)$ are arbitrarily chosen. Hence these 1T field theories must be dual to each other and they must describe the same gauge invariants from the point of view of $d+2$ dimensions. The duality transformations among such 1T field theories will be discussed in the next section.

B. Spin-$\frac{1}{2}$ field

The 2T spin-$\frac{1}{2}$ action, including the gauge field $S(\Psi, A)$ is obtained from (3.7) as usual by replacing ∂_M by the covariant derivative $D_M = \partial_M - igA_M$. However, in the axial gauge (4.19) the gauge field drops out in the kinematic equation (4.1) since $X\cdot D = X\cdot\partial$. Hence, just like the cases of the scalar and Yang-Mills fields, the spinor kinematic equation is free from interactions, and simplifies greatly in the parametrization of Eq. (4.3). So, it takes the form

$$\Big(X\cdot\partial + \frac{d}{2}\Big)\Psi(X) = \Big(\kappa\frac{\partial}{\partial\kappa} + \frac{d}{2}\Big)\Psi(\kappa,\lambda,x^\mu) = 0, \tag{4.31}$$

which is solved generally by a homogeneous Ψ of degree $-\frac{d}{2}$

$$\Psi(\kappa,\lambda,x^\mu) = \kappa^{-(d/2)}\hat\Psi(\lambda,x^\mu). \tag{4.32}$$

Expanding Ψ in the form $\Psi = \Psi_0 + X^2\tilde\Psi$, we can write $\hat\Psi(\lambda,x^\mu) = \Psi_0(x^\mu) + (\lambda - \frac{q^2}{2})\tilde\Psi(\lambda,x^\mu)$. Using the $X^2\zeta_1$ part of the 2T gauge symmetry (3.10), we can choose the gauge that eliminates the remainder $\tilde\Psi(\lambda,x^\mu)=0$, leading to a λ-independent $\hat\Psi(\lambda,x^\mu) = \Psi_0(x^\mu)$. Therefore, $\Psi(X)$ takes the gauge fixed form $\Psi(\kappa,\lambda,x^\mu) = \kappa^{-(d/2)}\Psi_0(x^\mu)$. Using now the (kappa type) $\not X\zeta_2$ part of the gauge symmetry (3.10), we can remove half of the remaining degrees of freedom of Ψ_0. In particular, one can make the gauge choice

$$\Gamma^{+'}\Psi = 0. \tag{4.33}$$

With a choice of basis for the flat space SO$(d,2)$ gamma matrices Γ^M (see Appendix of [9]), the gauge condition $\Gamma^{+'}\Psi = 0$ forces the lower components of Ψ (equivalent to the first two components of $\bar\Psi$) to vanish. Therefore, the gauge fixed form of $\Psi(X)$ is

$$\Psi(\kappa,\lambda,x^\mu) = \kappa^{-(d/2)}e^{-(\sigma(x)/2)}\begin{pmatrix}\psi(x^\mu)\\ 0\end{pmatrix}, \quad \bar\Psi(\kappa,\lambda,x^\mu) = \kappa^{-(d/2)}e^{-(\sigma(x)/2)}\begin{pmatrix}0 & \bar\psi(x^\mu)\end{pmatrix} \tag{4.34}$$

where $\psi(x^\mu)$ is an SO$(d-1,1)$ spinor and $\bar\psi(x^\mu)$ is its antispinor. We have inserted the extra factor $e^{-(\sigma/2)}$ for later convenience in the interpretation of ψ.

We now focus on the term $(\bar\Psi \not X \bar{\not D}\Psi)$ in the action, where $\bar{\not D} = \bar\Gamma^M D_M = \bar\Gamma^{+'}D_{+'} + \bar\Gamma^{-'}D_{-'} + \bar\Gamma^m D_m$ includes the gauge field. With our gauge choices in Eqs. (4.24), (4.33), and (4.34) we can drop the terms, $\bar\Gamma^{+'}D_{+'}\Psi = 0$ and $\bar\Gamma^{-'}D_{-'}\Psi = \bar\Gamma^{-'}\frac{1}{\kappa}e^{-\sigma}\frac{\partial}{\partial\lambda}\Psi = 0$, and using explicitly our SO$(d,2)$ gamma matrices we get

$$\bar{\not D}\Psi = \bar\Gamma^m D_m\Psi = \begin{pmatrix}\gamma^m D_m & 0\\ 0 & -\bar\gamma^m D_m\end{pmatrix}\begin{pmatrix}\kappa^{-(d/2)}e^{-(\sigma/2)}\psi\\ 0\end{pmatrix} = \begin{pmatrix}\gamma^m D_m(\kappa^{-(d/2)}e^{-(\sigma/2)}\psi)\\ 0\end{pmatrix} \tag{4.35}$$

where γ^m are now the SO$(d-1,1)$ gamma matrices in *flat tangent space* labeled by m. Next, we apply $\not X = \Gamma^M X_M = -\Gamma^{+'}X^{-'} - \Gamma^{-'}X^{+'} + \Gamma^m X_m$ on $\bar{\not D}\Psi$. The first term $\Gamma^{+'}X^{-'}$ gives zero when acting on $\bar{\not D}\Psi$. The other two terms give

[10] Another way to interpret this procedure of absorbing the infinity into the constant in front of the action is to view it as a renormalization of the Planck constant $\hbar$ since only the combination of $S/\hbar$ appears in the path integral. The infinity is easily controlled by a cutoff and then renormalized away as we proposed. This method is preferred over some gauge fixing methods [29,30,35] since this preserves the symmetries of the action without making them intractable through gauge choices.

[11] There can of course be a variety of scalars in a full 2T theory, such as a Higgs boson doublet $H(X)$ in the standard model. But as required by 2T field theory, in particular, there is also a flavor-color singlet dilaton $\Phi(X)$ that gets reduced to $\phi(x)$ and which must couple as in Eqs. (4.29) and (4.30). This coupling of the dilaton disappears in $d=4$, but there can be additional couplings among the scalars, such as Higgs and dilaton, which can play a crucial role in driving the electroweak phase transition that generates masses, by linking it to other dilaton driven phase transitions, as explained in [17].

$$\begin{aligned}\not{X}\bar{\not{D}}\Psi &= \kappa e^{\sigma}\begin{pmatrix}\bar{\gamma}^m q_m & 0\\ 1 & -\gamma^m q_m\end{pmatrix}\\ &\quad\times\begin{pmatrix}\gamma^k D_m(\kappa^{-(d/2)}e^{-(\sigma/2)}\psi)\\ 0\end{pmatrix}\\ &= \kappa e^{\sigma}\begin{pmatrix}\bar{\gamma}^m q_m\gamma^k D_k(\kappa^{-(d/2)}e^{-(\sigma/2)}\psi)\\ \gamma^k D_k(\kappa^{-(d/2)}e^{-(\sigma/2)}\psi)\end{pmatrix}. \qquad (4.36)\end{aligned}$$

Therefore

$$\begin{aligned}\bar{\Psi}\not{X}\bar{\not{D}}\Psi &= \kappa e^{\sigma}e^{-(\sigma/2)}\kappa^{-(d/2)}(0\quad\bar{\psi})\\ &\quad\times\begin{pmatrix}\bar{\gamma}^m q_m\gamma^k D_k(\kappa^{-(d/2)}e^{-(\sigma/2)}\psi)\\ \gamma^k D_k(\kappa^{-(d/2)}e^{-(\sigma/2)}\psi)\end{pmatrix}\\ &= \kappa^{-d}\bar{\psi}\gamma^k e_k^{\mu}\left(D_\mu + \frac{d-1}{2}\partial_\mu\sigma\right)\psi. \qquad (4.37)\end{aligned}$$

As an additional step, we note the identity

$$\frac{d-1}{2}(\gamma^k e_k^{\mu})\partial_\mu\sigma = \frac{1}{4}(\gamma^k\gamma_{ij})e_k^{\mu}\omega_\mu^{ij} \qquad (4.38)$$

that expresses the term that contains $\partial_\mu\sigma$ as due to the following special spin connection $\omega_\mu^{ij}(x)$ for the SO$(d-1,1)$ in tangent space

$$\omega_\mu^{ij}(x) = (e_\mu^i e^{j\nu} - e_\mu^j e^{i\nu})\partial_\nu\sigma(x). \qquad (4.39)$$

But this $\omega_\mu^{ij}(x)$ is precisely the spin connection that is constructed from the vielbein in curved space as usual

$$\begin{aligned}\omega_\mu^{ij}(x) &= e^{i\lambda}e^{j\sigma}(c_{\mu\lambda\sigma} - c_{\lambda\sigma\mu} - c_{\sigma\mu\lambda}), \quad \text{with}\\ c_{\mu\lambda\sigma} &\equiv -\tfrac{1}{2}e_\mu^k(\partial_\lambda e_{\sigma k} - \partial_\sigma e_{\lambda k}).\end{aligned} \qquad (4.40)$$

If we insert our $e_\mu^i = e^{\sigma}\partial_\mu q^i(x)$ in this general expression, we recover precisely the special spin connection above. Therefore, the result in Eq. (4.37) can now be written as

$$(\bar{\Psi}\not{X}\bar{\not{D}}\Psi)_{\rm red} = \kappa^{-d}\bar{\psi}\gamma^k e_k^{\mu}\hat{D}_\mu\psi \qquad (4.41)$$

where the covariant derivative $\hat{D}_\mu$ includes both the Yang-Mills and the spin connection in $(d-1)+1$ dimensions.

$$\hat{D}_\mu = \partial_\mu - igA_\mu + \tfrac{1}{4}\omega_\mu^{ij}\gamma_{ij}. \qquad (4.42)$$

The reduced action in which the kinematic constraints are solved, now takes the form

$$\begin{aligned}S(\Psi,A)_{\rm red} &= \frac{i}{2}Z\int(d^{d+2}X)\delta(X^2)(\bar{\Psi}\not{X}\bar{\not{D}}\Psi + \text{H.c.})_{\rm red}\\ &= \frac{i}{2}Z\int d\kappa d\lambda d^d x\frac{1}{2}\kappa^{d-1}\det(e_\mu^m)\delta\left(\lambda - \frac{q^2(x)}{2}\right)\\ &\quad\times(\kappa^{-d}\bar{\psi}\gamma^k e_k^{\mu}\hat{D}_\mu\psi + \text{H.c.})\\ &= \frac{1}{2}\int d^d x\sqrt{-g}\bar{\psi}i\gamma^k\hat{D}_k\psi + \text{H.c} \qquad (4.43)\end{aligned}$$

where we have used again the volume element in (4.11) as well as the previous universal normalization $Z\int\frac{d\kappa}{2\kappa} = 1$.

As in the cases of the scalar and vector fields, the resulting spinor action in Eq. (4.43) is the standard 1T field theory action in a $(d-1)+1$ curved spacetime. The conformally flat metric $g^{\mu\nu}(x)$ is again the same one that describes the shadow spacetime for the other fields. As before, the shadows are different as we change the functions $\sigma(x^\mu)$ and $q^m(x^\mu)$. So the field theories with the different conformally flat backgrounds must be dual to each other since each shadow must describe the same gauge invariant content of the original 2T field theory in $d+2$ dimensions.

V. DUALITIES

We have shown above that the 2T field theory in $d+2$ dimensions leads to a family of 1T field theories corresponding to all possible conformally flat backgrounds in $(d-1)+1$ dimensions. We now show the relations that transform one shadow with a fixed spacetime metric $g_{\mu\nu}(x)$ (example, flat Minkowski spacetime) into another shadow with a different spacetime metric $\tilde{g}_{\mu\nu}$ (example, Robertson-Walker expanding universe). From the point of view of 1T physics, this is a transformation between two different theories with no *a priori* relation to each other. But from the point of view of 2T physics, from the derivation above, it is evident that such transformations among 1T field theories should be an actual symmetry among the shadows that does not change the physical content, and hence we call it a duality transformation in 1T-physics.

The duality transformations that we will discuss here take the following form

$$S_{\sigma,q_m}(\phi,A_\mu,\psi) = S_{\tilde{\sigma},\tilde{q}_m}(\tilde{\phi},\tilde{A}_\mu,\tilde{\psi}) \qquad (5.1)$$

On the left side $S_{\sigma,q_m}(\phi,A_\mu,\psi)$ represents the 1T field theory with scalars, vectors and spinors in a background geometry generated by the functions $\sigma(x)$, $q_m(x)$. On the right side the background geometry has been changed to a new one $\tilde{\sigma}(x)$, $\tilde{q}_m(x)$, and when the dynamical fields are transformed into new ones by a duality transformation $(\phi,A_\mu,\psi)\to(\tilde{\phi},\tilde{A}_\mu,\tilde{\psi})$, the actions can be shown to be equal. Hence, such a duality transformation is a symmetry of the system. Of course, this symmetry among the shadows is a simple consequence of the fact that either expression is merely a parametrization of the solutions of the kinematic constraints (4.1) of the same 2T action

$$S_{\sigma,q_m}(\phi,A_\mu,\psi) = S(\Phi,A_M,\Psi_\alpha)_{\rm red} = S_{\tilde{\sigma},\tilde{q}_m}(\tilde{\phi},\tilde{A}_\mu,\tilde{\psi}). \qquad (5.2)$$

In 1T physics we now verify directly that the sample cases given in Table I, indeed form a set of dual field theories.

We consider the following two types of *local* transformations of the background functions that relate a subset of the shadow spacetimes to one another.

(i) First consider replacing the functions $q^n(x)$ by new ones $\tilde{q}^m(x)$. This can be implemented by general coordinate transformation in q-space $q^m \to \tilde{q}^m(q)$, which yields $\tilde{q}^m(x) = \tilde{q}^m(q(x))$. Since general coordinate transformations in x-space $x^\mu \to y^\mu(x)$ have the same amount of freedom as q-space reparametrizations, the resulting function $\tilde{q}^m(x)$ can also be built through general x-reparametrizations. Thus we can write $\tilde{q}^m(x)$ in two ways $\tilde{q}^m(q^n(x)) = \tilde{q}^m(x) = q^m(y(x))$. To prove the duality in Eq. (5.1) we will treat q-reparametrization as general coordinate transformations in x-space. In that case the background functions $\sigma(x)$ and $q^m(x)$ are transformed like general coordinate scalars

$$\tilde{\sigma}(x) = \sigma(y(x)), \qquad \tilde{q}^m(x) = q^m(y(x)). \tag{5.3}$$

These induce general coordinate transformations on the background geometry $e^m_\mu(x) \to \tilde{e}^m_\mu(x) = e^{\tilde{\sigma}(x)}\partial_\mu \tilde{q}^m(x)$ which takes the form

$$\begin{aligned}\tilde{e}^m_\mu(x) &= \partial_\mu y^\lambda(x) e^m_\lambda(y(x)),\\ \tilde{g}_{\mu\nu}(x) &= \partial_\mu y^\lambda(x)\partial_\nu y^\sigma(x) g_{\lambda\sigma}(y(x)).\end{aligned} \tag{5.4}$$

(ii) Now consider changing $\sigma(x)$ to a new one, leaving $q^m(x)$ alone. This can be implemented as follows

$$\tilde{\sigma}(x) = \sigma(x) + \lambda(x), \qquad \tilde{q}^m(x) = q^m(x) \tag{5.5}$$

This induces a scale transformation on both the vielbein and metric

$$\tilde{e}^{\,m}_{\mu}(x) = e^{\lambda(x)} e^n_\mu(x), \qquad \tilde{g}_{\mu\nu}(x) = e^{2\lambda(x)} g_{\mu\nu}(x). \tag{5.6}$$

Hence the change $\sigma(x) \to \tilde{\sigma}(x)$ amounts to a Weyl transformation.

Since the reduced 1T action is formally invariant under general coordinate transformations, we can claim that the action with background $(\sigma, q_m)(x)$ will be equal to the action with background $(\tilde{\sigma}, \tilde{q}_m)(y(x))$ as in Eq. (5.1) provided the fields (ϕ, A_μ, ψ) are also transformed by the general coordinate transformations

$$\begin{aligned}\tilde{\phi}(x) = \phi(y(x)), \qquad \tilde{A}_\mu(x) &= \partial_\mu y^\lambda(x) A_\lambda(y(x)),\\ \tilde{\psi}(x) &= \psi(y(x)).\end{aligned} \tag{5.7}$$

This is then the duality transformation that relates actions with the two different backgrounds in Eq. (5.3).

A less obvious duality symmetry is Weyl transformations given by the transformation of the background geometry in Eqs. (5.5) and (5.6) and the following transformations of the dynamical fields

$$\begin{aligned}\tilde{\phi}(x) = e^{-(d-2/2)\lambda(x)}\phi(x), \qquad \tilde{A}_\mu(x) &= A_\mu(x),\\ \tilde{\psi}(x) &= e^{-(d-1/2)\lambda(x)}\psi(x).\end{aligned} \tag{5.8}$$

We will now prove that this is a duality symmetry as in Eq. (5.1).

For the spin-1 action in Eq. (4.30), note that

$$\begin{aligned}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} &= F_{\mu\nu}\tilde{g}^{\mu\lambda}\tilde{g}^{\nu\sigma}F_{\lambda\sigma}\\ &= e^{-4\lambda}F_{\mu\nu}g^{\mu\lambda}g^{\nu\sigma}F_{\lambda\sigma}\\ &= e^{-4\lambda}F_{\mu\nu}F^{\mu\nu}.\end{aligned} \tag{5.9}$$

Then, the transformed action is seen to be invariant

$$\tilde{S}(\tilde{A}, \tilde{\phi}) = \int d^d x (e^{\lambda d}\sqrt{-g})\left(-\frac{1}{4}(e^{-(d-2/2)\lambda}\phi)^{(2(d-4)/d-2)}\,\mathrm{Tr}(F_{\mu\nu}e^{-4\lambda}F^{\mu\nu})\right) \tag{5.10}$$

$$= \int d^d x e^{(d-(d-4)-4)\lambda}\sqrt{-g}\left(-\frac{1}{4}\phi^{(2(d-4)/d-2)}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\right) \tag{5.11}$$

$$= S(A, \phi) \tag{5.12}$$

which proves the duality symmetry when the background and dynamical fields are transformed according to (5.3) and (5.7).

For the spin-$\frac{1}{2}$ case in Eq. (4.43), it is faster to prove the duality if we use the version of the covariant derivative in Eq. (4.37). Then we have

$$\tilde{S}(\tilde{\psi}, \tilde{A}) = \frac{1}{2}\int d^d x\left\{(e^{\lambda d}\sqrt{-g})(e^{-(d-1/2)\lambda}\bar{\psi})i\gamma^k(e^{-\lambda}e^\mu_k)\left(D_\mu + \frac{d-1}{2}\partial_\mu(\sigma+\lambda)\right)(e^{-(d-1/2)\lambda}\psi)\right\} + \mathrm{H.c.} \tag{5.13}$$

$$= \frac{1}{2}\int d^d x\left\{e^{\lambda d}e^{-(d-1/2)\lambda}e^{-\lambda}e^{-(d-1/2)\lambda}\sqrt{-g}\bar{\psi}i\gamma^k e^\mu_k\left(D_\mu - \frac{d-1}{2}\partial_\mu\lambda + \frac{d-1}{2}\partial_\mu(\sigma+\lambda)\right)\psi\right\} + \mathrm{H.c.} \tag{5.14}$$

$$= S(\psi, A). \tag{5.15}$$

So it is invariant under Weyl transformations, which proves the duality symmetry when the background and dynamical fields are transformed according to (5.5) and (5.8).

Dualities under general coordinate transformations and Weyl transformations of the type above hold for all background metrics $g_{\mu\nu}$, not only for the conformally flat metrics, so is there something more special in the present case, and how would the general case be recovered in 2T-physics? The answer is found by recalling that we have investigated duality properties of the shadows of a specific 2T-theory.

First, we must emphasize that there are more shadows of the same theory that are not conformally flat field theories, but also participate in similar duality transformations. Those have not been discussed in our preliminary work in this paper as explained in footnote 8, as our main motivation here was to provide some simple examples of the dualities generated by 2T-field theory.

Second, the starting point can be various 2T-field theories, including curved backgrounds in $d+2$ dimensions rather than the flat background in (4.2) used in our present case. Curved backgrounds in $d+2$ dimensions will lead to shadows in more general backgrounds $g_{\mu\nu}$ that would not be necessarily conformally flat, but will satisfy the dualities generated by Weyl and general coordinate transformations as in the more general case.

Third, by starting from a specific 2T-theory we can generate only those shadows that capture the underlying properties of that theory. So, the conformally flat spacetimes represented by a subset of shadows in Table I, must have additional properties that reflect the properties of flat spacetime in $d+2$ dimensions. Specifically, these shadows must have a hidden $SO(d,2)$ *global* symmetry. This additional property of each shadow is discussed in the next section.

VI. SO(d, 2) GLOBAL SYMMETRY AND ITS GENERATORS

Similarly to the spin-0 case, the $SO(d,2)$ global symmetry of the original 2T theory must still be present after imposing the $SO(d,2)$ invariant kinematic constraints (4.1). In this section, we provide the explicit form of the $SO(d,2)$ generators J_{MN} as applied on each shadow.

To do so, we will use the same trick as in [22]. The generic field $\chi_{\mu_1\mu_2\cdots}$ is a shadow in curved space, with metric

$$g_{\mu\nu}(x) = e^{2\sigma(x)}\eta_{mn}\partial_\mu q^m(x)\partial_\nu q^n(x). \tag{6.1}$$

This can be related by dualities to the shadow field $\chi^0_{\mu_1\mu_2\cdots}$, in flat space with metric $\eta^{\mu\nu}$. The duality relation is a combination of Weyl and general coordinate transformations parametrized by $\lambda(x)$ and $y^\mu(x)$ as shown in Eqs. (5.3), (5.5), (5.7), and (5.8). The starting point we want to use for the duality transformation is the shadow with flat space Minkowski metric $\eta_{\mu\nu}$, hence we want to apply the combined transformations in Eqs. (5.3) and (5.5) as follows

$$\eta_{\mu\nu} \to g_{\mu\nu}(x) = e^{2\lambda(x)}\partial_\mu y^\rho(x)\partial_\nu y^\sigma(x)\eta_{\rho\sigma}. \tag{6.2}$$

Then, the parameters $\lambda(x)$, $y^\mu(x)$ that produce the general shadow spacetime of Eq. (6.1) are precisely $\lambda(x)=\sigma(x)$ and $y^\mu(x)=\delta^\mu_m q^m(x)$. Hence by these duality transformations, the generic field χ can be written in terms of the flat-space field χ^0 as in Eqs. (5.7) and (5.8)

$$\chi_{\mu_1\mu_2\cdots}(x) = e^{-w\sigma(x)}\left(\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\frac{\partial y^{\nu_2}}{\partial x^{\mu_2}}\cdots\right)\chi^0_{\nu_1\nu_2\cdots}(y(x)), \tag{6.3}$$

where w is the Weyl weight of the field χ as it appears in Eq. (5.8) for the relevant fields in this paper.

Now we begin to investigate the $SO(d,2)$ transformations. The starting point is the form of the $SO(d,2)$ generators in the $d+2$ theory, which is

$$J^{MN} = (X^M P^N - X^N P^M) + S^{MN}, \tag{6.4}$$

where $P_M = -i\partial/\partial X^M$ is a differential operator as applied on any 2T field, and S^{MN} is the representation of $SO(d,2)$ as applied on the spin indices of the fields

$$S^{MN}\Phi = 0, \qquad S^{MN}A_K = (\delta^M_K\eta^{NL} - \delta^N_K\eta^{ML})A_L,$$
$$S^{MN}\Psi_\alpha = \frac{1}{2i}(\Gamma^{MN})^\beta_\alpha\Psi_\beta. \tag{6.5}$$

Note that the kinematic conditions (4.1) are invariant under these transformations. Therefore, this form of J^{MN} implies corresponding transformations for the shadow fields in the lower dimension. For a particular parametrization, such as Eq. (4.3) for X^M and Eqs. (4.5), (4.6), and (4.7) for $P_M = -i\partial/\partial X^M$, the generator J^{MN} implements the infinitesimal $SO(d,2)$ transformation on the original fields Φ, A_M, Ψ_α but now as functions of κ, λ, x^μ. When we insert the solutions of the kinematic equations, all $\partial/\partial\lambda$ derivatives vanish on the solutions, any remaining explicit λ is replaced by $\lambda = q^2(x)/2$, and the κ dependence becomes trivial since it appears as overall factors. In particular, if we pick the parametrization that corresponds to shadows in Minkowski space [i.e. $\sigma=0$ and $q^m(x) = \delta^m_\mu x^\mu$ in Eq. (4.3)], then the shadow fields $\chi^0(q)$ transform under $SO(d,2)$ as

$$\text{flat}: \ \delta_\omega\chi^0(q) = \frac{i}{2}\omega_{MN}J_0^{MN}\chi^0. \tag{6.6}$$

where J_0^{MN} takes the form of the familiar conformal transformations [6] [contrast to the classical version in Eqs. (A17)–(A19) when specialized to $\sigma=0$ and $q^m(x) = \delta^m_\mu x^\mu$]

$$J_0^{mn} = (q^m p^n - q^n p^m) + S^{mn} \quad \text{(Lorentz transf.)} \tag{6.7}$$

$$J_0^{+'m} = p^m \quad \text{(translations)}; \qquad J_0^{+'-'} = q^m p_m - ik \quad \text{(dilatations)} \tag{6.8}$$

$$J_0^{-'m} = \tfrac{1}{2} q_l q^l p^m - q^m q^l p_l - q_l S^{ml} + ikq^m \quad \text{(conformal transf.)} \tag{6.9}$$

where p_m is understood as a differential operator

$$p_m \equiv -i\frac{\partial}{\partial q^m}, \tag{6.10}$$

S^{mn} is the appropriate spinor representation of $SO(d-1, 1)$, and k is the scaling dimension of the corresponding field

$$k_\phi = \frac{d-2}{2}, \qquad k_\psi = \frac{d-1}{2}, \qquad k_A = 1. \tag{6.11}$$

Now, by using the duality transformation (6.3), we derive the $SO(d, 2)$ transformation for the fields in the curved background as $\delta_\omega \chi_i(x) = e^{-w\sigma}\Lambda_i{}^j(\delta_\omega \chi_j^0(q(x)))$. We obtain [here the indices i and the symbol $\Lambda_i{}^j$ is short-hand notation for those that appear in Eq. (6.3)]

$$\delta_\omega \chi_i(x) = e^{-w\sigma}\Lambda_i{}^j(\delta_\omega \chi_j^0(q(x))) = \frac{i}{2}\omega_{MN} e^{-w\sigma}\Lambda_i{}^j J_0^{MN}\chi_j^0(q(x)) \tag{6.12}$$

$$= \frac{i}{2}\omega_{MN} e^{-w\sigma}\Lambda_i{}^j J_0^{MN}(e^{w\sigma}(\Lambda^{-1})_j{}^k \chi_k(x)) \equiv \frac{i}{2}\omega_{MN} J^{MN}\chi_i(x). \tag{6.13}$$

Hence the action of the J^{MN} defined by the last expression is given by the differential operators

$$J^{MN}\chi_i(x) = e^{-w\sigma}\Lambda_i{}^j J_0^{MN}[e^{w\sigma}(\Lambda^{-1})_j{}^k \chi_k(x)]. \tag{6.14}$$

Let us now specialize to the cases of spin-$\frac{1}{2}$ and spin-1 fields (spin-0 is given in [22]). Since $\psi(x) = e^{-(d-1/2)\sigma}\psi_\beta^{(0)}(q(x))$, the $SO(d, 2)$ generators for spin 1/2 fields in conformally flat curved space are given by

$$J^{MN}\psi(x) = [e^{-(d-1/2)\sigma(x)} J_0^{MN} e^{(d-1/2)\sigma(x)}]\psi(x). \tag{6.15}$$

Similarly, since $A_\mu(x) = e_\mu^m(x) A_m^{(0)}(q(x))$, the $SO(d, 2)$ generators for spin 1 fields in conformally flat curved space are given by

$$J^{MN} A_\mu(x) = [e_\mu^i(x) J_0^{MN} e_i^\nu(x)] A_\nu(x). \tag{6.16}$$

In these expressions to compute the action of p_m that appears in J_0^{MN} we just use the chain rule to apply p_m on any function of x as follows

$$p_m f(x) = -i\frac{\partial}{\partial q^m} f(x) = -i\frac{\partial x^\mu}{\partial q^m}\frac{\partial}{\partial x^\mu} f(x) = -ie^\sigma e_m^\mu \frac{\partial}{\partial x^\mu} f(x), \tag{6.17}$$

where we inserted $\frac{\partial x^\mu}{\partial q^m} = e^\sigma e_m^\mu$ as discussed before. Then the resulting expressions are the quantum ordered versions of the classical generators given in Eqs. (A17)–(A19).

We emphasize that the fixed background metric $g_{\mu\nu}(x)$ is unchanged by the global $SO(d, 2)$ transformation [this is seen easily from the construction of $g_{\mu\nu}$ in Eq. (4.13)]. Therefore, without reference to the flat theory, but only using the generator J^{MN} above, we see that this is a true invariance of the action with the fixed background (σ, q^i)

$$\delta_\omega S_{\sigma,\chi}(\phi, A_\mu, \psi_\alpha) = \frac{\partial S_{\sigma,q}}{\partial \phi}\delta_\omega \phi + \frac{\partial S_{\sigma,q}}{\partial A_\mu}\delta_\omega A_\mu + \frac{\partial S_{\sigma,q}}{\partial \psi_\alpha}\delta_\omega \psi_\alpha = 0. \tag{6.18}$$

This hidden global symmetry is nothing but the original global $SO(d, 2)$ symmetry of the action $S(\Phi, A, \Psi)$ in $d+2$ dimensions, and hence each shadow for any $(\sigma(x), q^i(x))$ must also be invariant.

It is straightforward to see that there is a symmetry as in Eq. (6.18) for each shadow, when presented as an outcome of the higher dimensional formulation, but this symmetry is not so easy to spot for specific backgrounds in 1T-physics field theory. For example, we claim that the emergent field theory in the Robertson-Walker expanding universe has this hidden $SO(d, 2)$ global symmetry, which was not noticed before. The resulting expressions for the hidden $SO(d, 2)$ generators J^{MN} given above are new.

The Robertson-Walker example, as well as all the others listed in Table I, show that 1T-physics is not equipped to predict the hidden symmetries or dualities. However, within 1T-physics field theory, with hard work and some guidance, one can find new properties, such as the dualities and hidden symmetries described above. Within 1T-physics these are the clues as well as the evidence of the higher dimensional nature of the underlying 2T spacetime, as predicted by 2T-physics.

VII. CONCLUSION

In this paper we generalized results for the Klein-Gordon field as reported in [22]. Here have shown that 1T field theories involving Dirac and Yang-Mills fields propagating in any conformally flat metric in $(d-1)+1$ dimensions can be obtained as the shadows of 2T field theory in flat $d+2$ dimensions. Similar work on some of the shadows for free fields was performed in [30] for the equations of motion rather than the action; modulo details, and absence of field interactions, we are in principle in agreement with those results for the equations of motion. in $(d-1)+1$ dimensions. Since the shadows belong to the same parent theory, there has to be hidden relationships among the emergent 1T field theories. We have displayed some of these hidden relationships in the form of dualities and also in the form of hidden symmetries, which were not previously known to exist for many of the specific examples listed in Table I.

This, of course can be applied to theories with several interacting fields of different spins as is the case of the standard model. Indeed the usual standard model in $3+1$ dimensions is already known to be the flat Minkowski shadow of a corresponding field theory in $4+2$ dimensions [17], and therefore our approach in this paper, which extends also to the shadows of the standard model, may find practical applications.

We should emphasize that the particular class of shadow spacetimes that we have discussed only constitutes a starting point. The infinity of possible gauge choices in the world line formalism suggests a similar richness in 2T field theory. In particular, we would like to extend the theory to allow gauge choices equivalent to those in the world line formalism which involve mixing of x and p (footnote 8). This may result in dualities between local and nonlocal field theories at least in some instances. It is to be noted that the appearance of mass, coupling and curvature, as moduli in the world line formalism was related to such gauge choices. Here we have seen examples of 1T field theory where curvature emerged as moduli in the reduction from 2T field theory. This suggests the possibility that mass in field theory might also come as a modulus in the embedding of $3+1$ dimensional phase space into $4+2$ dimensional phase space. This is a topic which is worth pursuing in more detail.

We also believe that the more general dualities provided by 2T-physics could provide new tools to investigate the properties of the standard model, including QCD. For instance, one could use one form of the 1T-physics action to learn some nonperturbative information about the other 1T-physics action. This suggests that we may be able to take advantage of the type of dualities discussed here, and their extensions (as suggested in footnote 8), to develop nonperturbative tools for analyzing the standard model itself as well as its dual versions.

So far, our discussion of field theory was purely classical. Another goal of our program is the quantization of our theory directly in the 2T formulation. This step is obviously necessary in order to fully express the standard model as a 2T theory at a quantum level. This is being pursued in the path integral formalism, taking into consideration the Faddeev-Popov formalism for gauge fixing the local symmetries of the 2T-field theory.[12]

Further research on these topics is warranted and is currently being pursued.

ACKNOWLEDGMENTS

We would like to thank S.-H. Chen, Y.-C. Kuo and B. Orcal for helpful discussions. This work was partially supported by the US Department of Energy, Grant No. DE-FG03-84ER40168

APPENDIX A: RELATING WORLD LINE 2T-PHYSICS TO 1T FIELD THEORY

There is another way to obtain 1T field theory from 2T-physics. This would start with the world line formalism in $d+2$ dimensions, gauge fix to $(d-1)+1$ dimensions to specify a shadow, and then do first quantization of that shadow. The first quantized wave function is the shadow field in 1T field theory. In this appendix we compare this procedure to the results obtained directly from 2T field theory, and in this way illustrate the greater power of the 2T field theory formalism.

For comparison purposes with 2T field theory, we concentrate in this appendix on deriving the Dirac equation in curved space through first quantization of the classical gauge fixed 2T world line theory. The OSp(1|2) gauge invariant action is [7]

$$S = \int d\tau \Big[P \cdot \dot{X} + \frac{i}{2} \Psi \cdot \dot{\Psi} - \frac{1}{2} A^{ij} X_i \cdot X_j + i F^i X_i \cdot \Psi \Big] \tag{A1}$$

where $X_1^M \equiv X^M$ and $X_2^M \equiv P^M$ is bosonic phase space and Ψ^M are fermionic degrees of freedom that represent spin. The OSp(1|2) gauge symmetry, with bosonic and fermionic gauge potentials A^{ij}, F^i, has 2 fermionic and three bosonic gauge parameters, which allow us to freely choose some gauges corresponding to these parameters. We use one of the fermionic gauge parameters to fix the fermion $\Psi^{+'}(\tau) = 0$ for all τ [using the basis $\Psi^M = (\Psi^{+'}, \Psi^{-'}, \Psi^m)$ in Eq. (4.2) for the flat SO$(d,2)$ metric]. Similarly we use two of the bosonic gauge parameters to fix $P^{+'}(\tau) = 0$ and $X^{+'}(\tau) = \exp(\sigma(X^m(\tau)))$ where $\sigma(X^m)$ is an arbitrary function of the other coordinates. Then we solve explicitly one out of the two fermionic constraints $X \cdot \Psi = 0$ and two out of the three bosonic constraints $X^2 = 0$, $X \cdot P = 0$. Then $X^M(\tau)$, $P^M(\tau)$, $\Psi^M(\tau)$, take the following gauge fixed forms

[12] Previous discussions of some aspects of quantization using different approaches appear in [28,32]. These point out some subtleties that may or may not be specific to the gauge choices made before quantization.

$$X^M(\tau) = e^{\sigma(x(\tau))}(\overset{+'}{1}, \tfrac{1}{2}\overset{-'}{q^2(x(\tau))}, \overset{m}{q^m(x(\tau))}) \quad \text{(A2)}$$

where we used $X^m = e^{\sigma(x)}q^m(x)$ as a parametrization of X^m in terms of $x^\mu(\tau)$ via d arbitrary functions $q^m(x)$ (i.e. not a gauge choice), and similarly,

$$P_M(\tau) = (\overset{+'}{0}, \overset{-'}{q^m(x(\tau))e^\mu_m(x(\tau))p_\mu}, \overset{m}{e^\mu_m(x(\tau))p_\mu(\tau)}) \quad \text{(A3)}$$

$$\Psi^M(\tau) = (\overset{+'}{0}, \overset{-'}{q(x(\tau))\cdot\zeta(\tau)}, \overset{m}{\zeta^m(\tau)}) \quad \text{(A4)}$$

where $e^\mu_m(x(\tau))$ is the inverse of the vielbein $e^m_\mu(x) = e^{\sigma(x)}\frac{\partial q^m(x)}{\partial x^\mu}$. The remaining so far unsolved constraints take the form $P^2 = g^{\mu\nu}p_\mu p_\nu = 0$, $\Psi\cdot P = \zeta^k e^\mu_k p_\mu = 0$. If we insert these forms into the action (A1) we obtain the spinning particle action in a curved background as follows.

$$S = \int d\tau\Big[p_\mu \dot{x}^\mu + \frac{i}{2}\zeta_m\dot{\zeta}^m - \frac{1}{2}A^{22}g^{\mu\nu}(x)p_\mu p_\nu + iF^2 e^\mu_m(x)\zeta^m p_\mu\Big]. \quad \text{(A5)}$$

Here e^μ_m, $g^{\mu\nu}$ are the inverses of the emergent vielbein and metric

$$e^m_\mu(x) = e^{\sigma(x)}\frac{\partial q^m(x)}{\partial x^\mu}, \qquad g_{\mu\nu}(x) = \eta_{mn}e^m_\mu(x)e^n_\nu(x) \quad \text{(A6)}$$

and they are in agreement with the corresponding expressions that emerge directly in field theoretic approach as given in Eq. (4.14).

The fact that this is indeed the metric can be confirmed by our derivation of the Dirac equation which is obtained in covariant first quantization by imposing the last fermionic OSp(1|2) constraint

$$\Psi\cdot P = \zeta^k e^\mu_k p_\mu = 0. \quad \text{(A7)}$$

We now quantize this equation, by representing the Clifford algebra among the ζ^k by the SO$(d-1,1)$ gamma matrices γ^k acting on a Dirac spinor $\psi_\alpha(x)$, as usual. We must also take into account quantum ordering issues for x, p in the nonlinear expression $\zeta^k e^\mu_k(x)p_\mu$ where p_μ is replaced by a derivative. This ordering ambiguity leads to the addition of some function $a_k(x)$ in the Dirac equation as shown below

$$\{i\gamma^k e^\mu_k(x)\partial_\mu + i\gamma^k a_k(x)\}\psi(x) = 0 \quad \text{(A8)}$$

The ambiguity $a_k(x)$ must be fixed by requiring that the SO$(d,2)$ global symmetry of the world line action (A1) must be preserved at the quantum level. As it turns out (as verified in the text) this criterion also matches with the requirement that Eq. (A8) should be compatible with the general form of the Dirac equation in curved space

$$i\gamma^k e^\mu_k(\partial_\mu + \tfrac{1}{4}\omega^{mn}_\mu\gamma_{mn})\psi(x) = 0. \quad \text{(A9)}$$

The spin connection ω^{mn}_μ is generally obtained from the vielbein e^m_μ via the well known formula in Eq. (4.40). For the vielbein of the form $e^m_\mu(x) = e^{\sigma(x)}\frac{\partial q^m(x)}{\partial x^\mu}$ that emerged above, the spin connection takes the form

$$\omega^{mn}_\mu = (e^m_\mu e^{n\nu} - e^n_\mu e^{m\nu})\partial_\nu\sigma(x). \quad \text{(A10)}$$

Inserting this into the Dirac equation above, we can finally calculate the spin connection term

$$\frac{i}{4}\gamma^k e^\mu_k\omega^{mn}_\mu\gamma_{mn} = \frac{i}{2}(d-1)\gamma^k e^\mu_k\partial_\mu\sigma. \quad \text{(A11)}$$

Comparing this with $i\gamma^k a_k(x)$ in Eq. (A8), we fix the ambiguity as

$$a_k(x) = \tfrac{1}{2}(d-1)e^\mu_k\partial_\mu\sigma, \quad \text{(A12)}$$

and obtain the Dirac equation:

$$i\gamma^k e^\mu_k(\partial_\mu + \tfrac{1}{2}(d-1)\partial_\mu\sigma)\psi(x) = 0. \quad \text{(A13)}$$

This is precisely in agreement with the result obtained from 2T field theory as seen in Eq. (4.37).

Similar treatments for fields of spin-0, 1 or higher would also be in agreement with 2T field theory, because the SO$(d,2)$ covariant 2T field theoretic approach automatically fixes the quantum ordering ambiguities for any gauge of the world line theory. In particular, we remind the reader of our result in [22] that when the ambiguity for the scalar field is fixed, the resulting scalar field is the conformal scalar in a curved background described by the action

$$S(\Phi)_{\text{red}} = \int d^dx\sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{d-2}{8(d-1)}R\phi^2\Big), \quad \text{(A14)}$$

where R is the curvature of the space, which in our conformally flat space is given by

$$R = (1-d)[dg^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + 2g^{\mu\nu}\partial_\mu\partial_\nu\sigma + 2e^{n\mu}\partial_\mu e^\nu_n\partial_\nu\sigma]. \quad \text{(A15)}$$

Another ordering ambiguity that occurs with the SO$(d,2)$ hidden global symmetry generators gets resolved as follows. The 2T world line action (A1) has a global

SO$(d,2)$ symmetry with generators

$$J^{MN} = X^M P^N - X^N P^M + S^{MN},$$
$$S^{MN} \equiv \frac{1}{2i}(\Psi^M \Psi^N - \Psi^N \Psi^M). \tag{A16}$$

The gauge fixed action (A5) must also have SO$(d,2)$ as a hidden symmetry because the generators J^{MN} above are gauge invariant since they commute with the OSp$(1|2)$ gauge generators X^2, P^2, $X \cdot P$, $X \cdot \Psi$, $P \cdot \Psi$. Inserting the gauge fixed forms of X^M, P^M, Ψ^M in the J^{MN} give the correct generators of the hidden symmetry of the world line action (A5)

$$J^{+'-'} = e^{\sigma(x)} q^m(x) e_m^\mu(x) p_\mu, \qquad J^{+'m} = e^{\sigma(x)} e^{m\mu}(x) p_\mu \tag{A17}$$

$$J^{mn} = e^{\sigma(x)}[q^m(x) e^{n\mu}(x) - q^n(x) e^{m\mu}(x)] p_\mu + S^{mn},$$
$$S^{mn} \equiv \frac{1}{2i}(\zeta^m \zeta^n - \zeta^n \zeta^m) \tag{A18}$$

$$J^{-'m} = e^{\sigma(x)}[\tfrac{1}{2} q^2(x) \eta^{mn} - q^m(x) q^n(x)] e_n^\mu(x) p_\mu - q_n(x) S^{mn} \tag{A19}$$

These far from obvious conserved hidden symmetry charges $J^{MN}(x,p)$ are used with Poisson brackets to generate the phase space transformations in the classical theory in the lower d dimensions. In particular they apply for the curved spacetimes in Table I marked as (cf).

$$\delta_\omega x^\mu = \frac{\omega_{MN}}{2}\{J^{MN}, x^\mu\}, \qquad \delta_\omega p_\mu = \frac{\omega_{MN}}{2}\{J^{MN}, p_\mu\},$$
$$\delta_\omega \zeta^m = \frac{\omega_{MN}}{2}\{J^{MN}, \zeta^m\}. \tag{A20}$$

In the quantum theory the factor ordering of the operators x^μ and p_μ must be resolved in these expressions such that they are Hermitian and correctly form the SO$(d,2)$ Lie algebra under quantum commutators. This difficult problem is automatically resolved in 2T field theory in Sec. VI where the quantum ordered version of these generators is provided.

[1] I. Bars, arXiv:hep-th/9604200; Phys. Rev. D **54**, 5203 (1996); **55**, 2373 (1997); arXiv:hep-th/9608061.
[2] C. Vafa, Nucl. Phys. **B469**, 403 (1996).
[3] C. M. Hull, J. High Energy Phys. 07 (1998) 021.
[4] I. Bars and C. Kounnas, Phys. Lett. B **402**, 25 (1997); Phys. Rev. D **56**, 3664 (1997).
[5] H. Nishino and E. Sezgin, Phys. Lett. B **388**, 569 (1996); E. Sezgin, Phys. Lett. B **403**, 265 (1997).
[6] I. Bars, C. Deliduman, and O. Andreev, Phys. Rev. D **58**, 066004 (1998); For reviews of subsequent work see: I. Bars, arXiv:hep-th/9809034; Classical Quantum Gravity **18**, 3113 (2001); in *New Developments in Fundamental Interaction Theories: 37th Karpacz Winter School of Theoretical Physics*, edited by J. Lukierski and J. Rembielinski, AIP Conf. Proc. No. 589 (AIP, New York, 2001), p. 18.
[7] I. Bars and C. Deliduman, Phys. Rev. D **58**, 106004 (1998).
[8] I. Bars, Phys. Rev. D **62**, 046007 (2000).
[9] I. Bars, C. Deliduman, and D. Minic, Phys. Rev. D **59**, 125004 (1999); Phys. Lett. B **457**, 275 (1999).
[10] I. Bars, Phys. Rev. D **70**, 104022 (2004).
[11] I. Bars, in *Fundamental Interactions and Twistor-like Methods: XIX Max Born Symposium*, edited by J. Lukierski and D. Sorokin, AIP Conf. Proc. No. 767 (AIP, New York, 2005), p. 3.
[12] I. Bars, Phys. Rev. D **66**, 105024 (2002); Phys. Rev. D **66**, 105023 (2002).
[13] I. Bars, Phys. Lett. B **483**, 248 (2000); Ref. [11]
[14] I. Bars and M. Picon, Phys. Rev. D **73**, 064002 (2006); **73**, 064033 (2006).
[15] I. Bars, Superstring Theory and M-theory, edited by J. X. Lu (to be published); Quantum Theory and Symmetries IV, edited by V. K. Dobrev (Heron Press, Sofia, Bulgaria, 2006), Vol. 2, p. 487.
[16] I. Bars and B. Orcal, Phys. Rev. D **75**, 104015 (2007).
[17] I. Bars, Phys. Rev. D **74**, 085019 (2006); arXiv:hep-th/0610187.
[18] I. Bars, Phys. Rev. D **58**, 066006 (1998). This paper covers the "free relativistic massless particle" and the "hydrogen atom" gauges.
[19] I. Bars, Phys. Rev. D **59**, 045019 (1999). This covers the "nonrelativistic particle", the "massive relativistic. particle" and the "particle on AdS$_{d-n} * S^n$" gauges (the AdS$_{d-n} * S^n$ case uses a slightly different parametrization).
[20] I. Bars and M. Picon, Phys. Rev. D **73**, 064002 (2006). The "sphere*time" gauge is presented in the context of 2T twistor theory.
[21] For a graphical display of gauge choices in [18–20] see http://physics.usc.edu/~bars/2Tphysics.pdf.
[22] I. Bars, S.-H. Chen, and G. Quélin, Phys. Rev. D **76**, 065016 (2007).
[23] I. Bars and C. Deliduman, Phys. Rev. D **64**, 045004 (2001); I. Bars, Phys. Rev. D **62**, 085015 (2000).
[24] I. Bars and Y.-C. Kuo, Phys. Rev. Lett. **99**, 041801 (2007); Phys. Rev. D **76**, 105028 (2007).
[25] P. A. M. Dirac, Ann. Math. **37**, 429 (1936).
[26] H. A. Kastrup, Phys. Rev. **150**, 1183 (1966).
[27] G. Mack and A. Salam, Ann. Phys. (N.Y.) **53**, 174 (1969).
[28] S. Adler, Phys. Rev. D **6**, 3445 (1972); **8**, 2400 (1973).

[29] P. Budini, P. Furlan, and R. Raczka, Nuovo Cimento B **52**, 191 (1979).

[30] R. Marnelius and B. Nilsson, Phys. Rev. D **22**, 830 (1980).

[31] P. Arvidsson and R. Marnelius, arXiv:hep-th/0612060.

[32] F. Bayen, M. Flato, C. Fronsdal, and A. Haidari, Phys. Rev. D **32**, 2673 (1985).

[33] W. Siegel, Int. J. Mod. Phys. A **3**, 2713 (1988); Int. J. Mod. Phys. A **4**, 2015 (1989).

[34] S. Ferrara, Nucl. Phys. **B77**, 73 (1974).

[35] C. R. Preitschopf and M. A. Vasiliev, Nucl. Phys. **B549**, 450 (1999).

[36] M. A. Vasiliev, J. High Energy Phys. 12 (2004) 046.

[37] I. Bars and Y.-C. Kuo, Phys. Rev. D **74**, 085020 (2006).