Failure of microcausality in noncommutative field theories

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We revisit the question of microcausality violations in quantum field theory on noncommutative spacetime, taking $\mathcal{O}(x) = :\phi \star \phi:(x)$ as a sample observable. Using methods of the theory of distributions, we precisely describe the support properties of the commutator $[\mathcal{O}(x), \mathcal{O}(y)]$ and prove that, in the case of space-space noncommutativity, it does not vanish at spacelike separation in the noncommuting directions. However, the matrix elements of this commutator exhibit a rapid falloff along an arbitrary spacelike direction irrespective of the type of noncommutativity. We also consider the star commutator for this observable and show that it fails to vanish even at spacelike separation in the commuting directions and completely violates causality. We conclude with a brief discussion about the modified Wightman functions which are vacuum expectation values of the star products of fields at different spacetime points.

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I. INTRODUCTION

In recent years, considerable attention has been given to the construction of quantum field theories (QFTs) on noncommutative spacetimes (see, e.g., [1] for a review). The question of causality is a basic one in the development of the corresponding conceptual framework. A noncommutative deformation of the d-dimensional spacetime is formally defined by replacing the coordinates x^μ of $\mathbb{R}^d$ by operators $\hat{x}^\mu$ satisfying the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \qquad (1)$$

where $\theta^{\mu\nu}$ is a real antisymmetric $d \times d$-matrix, constant in the simplest case. However, this deformation can be combined with the basic principles of quantum theory in a variety of fashions. In particular, the issue of causality cannot be discussed in isolation from that of the implementation of spacetime symmetries. The relations (1) are not covariant under the Lorentz transformations, and noncommutative QFT is usually treated as a specific form of field theory with a nonlocal interaction breaking the Lorentz symmetry to a subgroup. In the Lagrangian formalism, the theory is defined by replacing the ordinary product of fields in the interaction terms of the actions with the Moyal $\star$-product given by

$$(\phi_1 \star \phi_2)(x) = \phi_1(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu\right) \phi_2(x). \qquad (2)$$

The star product commutation relation $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$ is identified with (1) via the Weyl correspondence between operators and their symbols. There is an essential distinction between the cases of space-space and time-space noncommutativity. If the time coordinate is involved in noncommutativity, then a string theoretical interpretation of the field theory comes up against the problem of nonunitarity [2] and inconsistency with the conventional Hamiltonian evolution [3]. A consistent Hamiltonian framework for the scalar field theory with time-space noncommutativity has been proposed in [4]. The definition given there leads to a perturbatively unitary S-matrix and is interesting by itself, even though its relationship with string theory is unclear. Field theories with only space noncommutativity (that is $\theta^{0\nu} = 0$) avoid the problems with unitarity, and models of this form attract the most notice because they describe a low energy limit of string theory in certain backgrounds. However, its causal structure is different from that of the standard QFT because the light cone is changed to a light wedge respecting the residual Lorentz symmetry [5–7]. The main object of the present paper is to analyze rigorously this modification of the causal structure.

At present, much consideration is being given to quantization of noncommutative theories with the use of a "twisted" version of the Poincaré covariance. These efforts are aimed at restoring the spacetime symmetries broken by noncommutativity and developing a covariant formulation even though the matrix $\theta^{\mu\nu}$ in (1) and (2) is constant. Within this setting, the issues of locality and causality were discussed, e.g., in [8–10], but up to now there is no consensus regarding the implementation of the twisted covariance in QFT and its physical consequences. Another way of looking at noncommutative spacetime was proposed in [11], where an infinite family of fields labeled by different noncommutativity parameters was considered and their relative localization properties were investigated.

In [12], locality and causality violations caused by noncommutativity were illustrated by a star product analogue of the normal ordered square $:\phi^2:$ of a free scalar field ϕ. Specifically, Chaichian *et al.* considered $\mathcal{O}(x) = :\phi \star \phi:(x)$ as a sample observable and found that the matrix element

$$\langle 0|[\mathcal{O}(x)), \mathcal{O}(y)]|_{x^0=y^0}|p_1, p_2\rangle \qquad (3)$$

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is nonzero only when $\theta^{0\nu} \neq 0$. More recently, Greenberg [13] considered the commutator $[\mathcal{O}(x), \partial_\nu \mathcal{O}(y)]$ with the derivatives of $\mathcal{O}$ and has shown that it fails to vanish at equal times even in the case in which $\theta^{0\nu} = 0$. As stated in [13], this result holds generally when there are time derivatives in the observables. A similar conclusion was reached in [14], where also a commutator involving time derivatives was treated, but with the use of a generalization of the Bogoliubov-Shirkov causality criterion.

In this paper, we analyze the commutator $[\mathcal{O}(x), \mathcal{O}(y)]$ more closely, using the techniques of the theory of distributions, which allows describing its support properties completely. We first consider the case of space-space noncommutativity, taking for definiteness $d = 4$ and $\theta^{12} = -\theta^{21} \neq 0$, with the other values of the θ-matrix equal to zero. In Sec. III, we show that then the commutator vanishes in the spacelike wedge $|x^0 - y^0| \leq |x^3 - y^3|$. In Sec. III, we prove that $[\mathcal{O}(x), \mathcal{O}(y)] \neq 0$ everywhere outside this wedge. This result demonstrates that the space-space noncommutativity violates the usual $SO(1,3)$ microcausality even if there are no time derivatives in the observables. In Sec. IV, we show that nevertheless the matrix elements of the commutator decrease rapidly in the whole cone $(x-y)^2 \leq 0$ and behave like $\exp(-|x-y|^2/|\theta|)$ at large spacelike separation. This is true without regard to the type of noncommutativity, in both space-space and time-space cases, and manifests itself after averaging the observable $\mathcal{O}(x)$ with sufficiently smooth and rapidly decreasing test functions. The best suitable class of test functions has been found and investigated in [15,16]. A slightly different class was independently proposed in [17]. In Sec. V, we examine the modified commutator $[\mathcal{O}(x), \mathcal{O}(y)]_\star = \mathcal{O}(x) \star \mathcal{O}(y) - \mathcal{O}(y) \star \mathcal{O}(x)$, where $\star$ denotes now a star multiplication of field operators at different spacetime points. Such a modification was also discussed in the literature. We prove that, contrary to expectations, the star commutator fails to vanish even in the spacelike wedge and completely violates causality. Our study shows, in particular, that the seemingly natural definition of the star product of fields at different spacetime points, as an operation dual to the corresponding operation on test functions, brings the causality principle and the spectral condition into conflict. Section VI contains concluding remarks.

II. A LIGHT WEDGE INSTEAD OF THE LIGHT CONE

Let ϕ be a free neutral scalar field of mass m on a spacetime of d dimensions and let $:\phi^2:(x) = \lim_{x_1,x_2\to x} :\phi(x_1)\phi(x_2):$. By the Wick theorem for normal ordered products, it follows that

$$\langle 0| :\phi^2:(x) :\phi^2:(y) :\phi(z_1)\phi(z_2): |0\rangle = 4w(x-y)w(x-z_1)w(y-z_2) + (z_1 \leftrightarrow z_2), \quad (4)$$

where w is the two-point function of ϕ, i.e.,

$$w(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{(2\pi)^{d-1}} \int dk \vartheta(k^0)\delta(k^2 - m^2) e^{-ik\cdot(x-y)}. \quad (5)$$

As a consequence, we have

$$\langle 0|[:\phi^2:(x), :\phi^2:(y)] :\phi(z_1)\phi(z_2): |0\rangle = 4i\Delta(x-y)w(x-z_1)w(y-z_2) + (z_1 \leftrightarrow z_2), \quad (6)$$

where $\Delta(x-y) = \frac{1}{i(2\pi)^{d-1}} \int dk \epsilon(k^0)\delta(k^2 - m^2) e^{-ik\cdot(x-y)}$ is the Pauli-Jordan function. Let us now consider the normal ordered expression

$$\begin{aligned}\mathcal{O}(x) &= :\phi \star \phi:(x) \\ &= \lim_{x_1,x_2\to x} :\phi(x_1) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu\right) \phi(x_2): \\ &= :\phi^2:(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1\nu_1} \ldots \theta^{\mu_n\nu_n} \\ &\quad \times :\partial_{\mu_1} \ldots \partial_{\mu_n} \phi(x) \partial_{\nu_1} \ldots \partial_{\nu_n} \phi(x):. \end{aligned} \quad (7)$$

Every term in the expansion (7) is well defined as a Wick monomial in derivatives of ϕ, see [18] or [19]. The technique developed in [20] allows us to define rigorously their sum as an operator-valued generalized function acting in the Hilbert space of ϕ, but we will not dwell on this point and now restrict our consideration to the vacuum expectation value

$$\mathcal{W}(x, y; z_1, z_2) = \langle 0|\mathcal{O}(x)\mathcal{O}(y) :\phi(z_1)\phi(z_2): |0\rangle, \quad (8)$$

which is an analogue of (4). Applying the Wick theorem again and using the formula

$$e^{ik\cdot x} \star e^{ip\cdot x} = e^{-i[k,p]} e^{i(k+p)\cdot x}, \quad (9)$$

where

$$[k, p] \stackrel{\text{def}}{=} (1/2) k_\mu \theta^{\mu\nu} p_\nu,$$

one can readily see that

$$\begin{aligned}\mathcal{W}(x, y; z_1, z_2) = 4 \int dk dp_1 dp_2 \tilde{w}(k) &\times e^{-ik\cdot(x-y) - ip_1\cdot(x-z_1) - ip_2\cdot(y-z_2)} \\ &\times \prod_{i=1,2} \tilde{w}(p_i) \cos[k, p_i] + (z_1 \leftrightarrow z_2),\end{aligned} \quad (10)$$

where $\tilde{w}(k) = (\mathcal{F}w)(k) = \int d\xi e^{ik\cdot\xi} w(\xi)$. More explicitly, the Fourier transform of (9) has the form

$$\tilde{\mathcal{W}}(k_1, k_2; p_1, p_2) = 4(2\pi)^{d+3}\delta(k_1 + k_2 + p_1 + p_2) \times \vartheta(k_1^0 + p_1^0)\delta((k_1 + p_1)^2 - m^2) \times \prod_{i=1,2} \vartheta(-p_i^0)\delta(p_i^2 - m^2)\cos[k_i, p_i] + (p_1 \leftrightarrow p_2), \quad (11)$$

where k_1, k_2 and p_1, p_2 are the momentum-space variables conjugate to x, y and z_1, z_2, respectively. The function

$$\mu = \cos[k_1, p_1]\cos[k_2, p_2] \quad (12)$$

is a multiplier of the Schwartz space $\mathcal{S}(\mathbb{R}^{4d})$ and hence the expression on the right-hand side of (11) and the vacuum expectation value (8) are well defined as tempered distributions.

From Eq. (10), it follows that

$$\langle 0|[\mathcal{O}(x), \mathcal{O}(y)]{:}\phi(z_1)\phi(z_2){:}|0\rangle = 4i\int dk dp_1 dp_2 \tilde{\Delta}(k) e^{-ik\cdot(x-y)-ip_1\cdot(x-z_1)-ip_2\cdot(y-z_2)} \times \prod_{i=1,2} \tilde{w}(p)\cos[k, p_i] + (z_1 \leftrightarrow z_2), \quad (13)$$

which agrees with formulas for the matrix element $\langle 0|[\mathcal{O}(x), \mathcal{O}(y)]|p_1, p_2\rangle$ in [12,13]. The Fourier transform of distribution (13) is obtainable by multiplying that of (6) by the multiplier (12). The distribution (6) is zero everywhere in the cone $(x-y)^2 < 0$, but this is not to say that the distribution (13) obeys microcausality. Let us turn to the case of space-space noncommutativity, assuming that $\theta^{12} = -\theta^{21} = \theta \neq 0$ and the other elements of the matrix $\theta^{\mu\nu}$ are equal to zero. It is easily seen that then the distribution (13) satisfies a weakened version of microcausality and vanishes in the wedge defined by

$$|x^0 - y^0| < |x^3 - y^3|. \quad (14)$$

In fact, the Fourier transformation converts multiplication into convolution[1] and hence the value of distribution (13) at a test function f coincides with the value of (6) at the test function $(2\pi)^{-4d}\tilde{\mu} * f$. Under our assumptions about the θ-matrix, the multiplier (12) does not depend on the variables conjugate to x^0, y^0, x^3, y^3 and its Fourier transform $\tilde{\mu}$ is the tensor product of $\delta(x^0)\delta(y^0)\delta(x^3)\delta(y^3)$ and a distribution in the other variables. Therefore, if $\mathrm{supp} f$ is contained in the wedge (14), then $\mathrm{supp}(\tilde{\mu} * f)$ also lies in this wedge and does not intersect the support of distribution (6). It follows that the distribution (13) vanishes for such test functions.

[1] To be more precise, we use the relations $(u, \tilde{g}) = (\tilde{u}, g)$ and $\widetilde{\mu g} = (2\pi)^{-d}\tilde{\mu} * \tilde{g}$, which hold for any $g \in \mathcal{S}(\mathbb{R}^d)$, $u \in \mathcal{S}'(\mathbb{R}^d)$ and for each multiplier μ.

III. VIOLATIONS OF MICROCAUSALITY

Now we intend to show that in the case of space-space noncommutativity, the commutator $[\mathcal{O}(x), \mathcal{O}(y)]$ does not vanish outside the wedge (14) and hence the observable $\mathcal{O}(x)$ defined by (7) does not satisfy the standard microcausality condition.

Theorem 1. Let $d = 4$ and let $\theta^{12} = -\theta^{21} = \theta \neq 0$, with the other elements of the matrix $\theta^{\mu\nu}$ equal to zero. Suppose that points $\bar{x}, \bar{y} \in \mathbb{R}^4$ satisfy the inequalities $(\bar{x} - \bar{y})^2 < 0$ and $|\bar{x}^0 - \bar{y}^0| > |\bar{x}^3 - \bar{y}^3|$. Then there is a state Φ such that $(\bar{x}, \bar{y})$ belongs to the support of

$$\mathcal{M}_\Phi(x, y) \stackrel{\text{def}}{=} \langle 0|[\mathcal{O}(x), \mathcal{O}(y)]|\Phi\rangle. \quad (15)$$

Proof. We take a state of the form

$$|\Phi\rangle = \int dz_1 dz_2 {:}\phi(z_1)\phi(z_2){:}h(z_1)h(z_2)|0\rangle = \phi^-(h)\phi^-(h)|0\rangle, \quad (16)$$

where $h \in \mathcal{S}(\mathbb{R}^4)$. Then $\mathcal{M}_\Phi$ is clearly a tempered distribution and by (13) we have

$$(\mathcal{M}_\Phi, f \otimes g) = \langle 0|[\mathcal{O}(f), \mathcal{O}(g)]|\Phi\rangle = 8i\int dp_1 dp_2 \prod_{i=1,2} \tilde{w}(p_i)\tilde{h}(p_i) \times \int dk \tilde{\Delta}(k)\cos[k, p_1]\cos[k, p_2] \times \tilde{f}(-k - p_1)\tilde{g}(k - p_2) \quad (17)$$

for any test functions $f, g \in \mathcal{S}(\mathbb{R}^4)$. This order of integration is permissible by the Fubini theorem because integrating over k^0, p_1^0, p_2^0 gives an integrable function on $\mathbb{R}^9$. The function

$$\psi_{p_1,p_2}(k) = \tilde{f}(-k - p_1)\tilde{g}(k - p_2)$$

belongs to the space $\mathcal{S}(\mathbb{R}^4)$ for any $p_1, p_2 \in \mathbb{R}^4$, and the function

$$\mu_{p_1,p_2}(k) = \cos[k, p_1]\cos[k, p_2]$$

is a multiplier of $\mathcal{S}(\mathbb{R}^4)$. Therefore the integral over k in (22) can be written as

$$(\tilde{\Delta}, \mu_{p_1,p_2} \cdot \psi_{p_1,p_2}) = \frac{1}{(2\pi)^4}(\Delta, \tilde{\mu}_{p_1,p_2} * \tilde{\psi}_{p_1,p_2}). \quad (18)$$

Let Θ be the linear map defined by $(\Theta p)^\mu = \frac{1}{2}\theta^{\mu\nu}p_\nu$. Then

$$\tilde{\mu}_{p_1,p_2}(\xi) = \int dk e^{ik\cdot\xi}\mu_{p_1,p_2}(k) = \frac{(2\pi)^4}{4}[\delta(\xi - \Theta(p_1 + p_2)) + \delta(\xi + \Theta(p_1 + p_2)) + \delta(\xi - \Theta(p_1 - p_2)) + \delta(\xi + \Theta(p_1 - p_2))].$$

Furthermore, we have

$$\tilde{\psi}_{p_1,p_2}(\xi) = \int dk e^{ik\cdot\xi}\psi_{p_1,p_2}(k)$$
$$= \int dk dx dy e^{ik\cdot\xi+i(-k-p_1)\cdot x+i(k-p_2)\cdot y} f(x)g(y)$$
$$= (2\pi)^4 \int dx dy e^{-ip_1\cdot x - ip_2\cdot y} \times \delta(\xi - x + y) f(x)g(y)$$
$$= (2\pi)^4 e^{-i(p_1-p_2)\cdot\xi/2}\varphi_{p_1,p_2}(\xi),$$

where

$$\varphi_{p_1,p_2}(\xi) = \int dX e^{-i(p_1+p_2)\cdot X} f(X+\xi/2) g(X-\xi/2). \tag{19}$$

In what follows, we set $\bar{y} = -\bar{x}$ and $\bar{x}^3 = \bar{y}^3 = 0$. This does not result in any loss of generality because the distribution (13) is invariant under translations and under boosts in the x^3-direction. If $\mathrm{supp} f$ is contained in the ε-neighborhood of $\bar{x}$ and $\mathrm{supp} g$ is contained in the ε-neighborhood of $-\bar{x}$, then only points X with $\| X \| \leq \varepsilon$ contribute in the integral in (19) and the functions φ_{p_1,p_2}, $\tilde{\psi}_{p_1,p_2}$ have support in the 2ε-neighborhood of the point $2\bar{x}$. We also note that the operation consisting in convolution with $\tilde{\mu}_{p_1,p_2}$ displaces $\mathrm{supp}\tilde{\psi}_{p_1,p_2}(\xi)$ by the vectors $\pm\Theta(p_1 \pm p_2)$. Now we specify the choice of h, setting

$$\bar{p}^{\,1} = 2\bar{x}^2/\theta, \qquad \bar{p}^2 = -2\bar{x}^1/\theta, \qquad \bar{p}^3 = 0,$$
$$\bar{p}^0 = \sqrt{m^2 + (\bar{p}^1)^2 + (\bar{p}^2)^2},$$

so that $\Theta\bar{p} = (0, \bar{x}^1, \bar{x}^2, 0)$. We take $\tilde{h}(p)$ to be a nonnegative function supported in a neighborhood U of $\bar{p}$ and such that $\tilde{h}(\bar{p}) > 0$. We choose U so small that the set of points $2\bar{x} \pm \Theta(p_1 - p_2)$, where p_1 and p_2 run through U, is separated from the cone $\bar{\mathbb{V}} = \{\xi \in \mathbb{R}^4 : \xi^2 \geq 0\}$ by a positive distance. Then for any $p_1, p_2 \in U$, one of the four functions obtained from $\tilde{\psi}_{p_1,p_2}$ by convolution with $\tilde{\mu}_{p_1,p_2}$ has support in a neighborhood of $\bar{\xi} = (2\bar{x}^0, 0, 0, 0)$, whereas the other three of them are supported in the spacelike region and do not contribute in the right-hand side of (18) if ε is small enough. Inside the cone $\bar{\mathbb{V}}$, the distribution $\Delta(\xi)$ is a regular function and we have the well-known representation

$$\Delta(\xi) = \frac{m}{4\pi\sqrt{\xi^2}}\epsilon(\xi^0)J_1(m\sqrt{\xi^2}), \qquad \xi \in \mathbb{V}.$$

We first assume that $J_1(2m|\bar{x}^0|) \neq 0$ and impose two additional restrictions on $\mathrm{supp}\tilde{h}$. Namely, we choose U so small that $J_1(m\sqrt{\xi^2})$ has a constant sign on the set

$$\{\xi \in \mathbb{R}^4 : \xi = 2\bar{x} - \Theta(p_1 + p_2), \quad p_1, p_2 \in U\} \tag{20}$$

and furthermore the inequality

$$|(p_1 - p_2)\cdot\bar{x}| < \pi/4 \tag{21}$$

holds for all $p_1, p_2 \in U$. We put $f(x) = g(-x)$ and assume that $f(x) \geq 0$, $f(\bar{x}) \neq 0$. Then the function $\varphi_{p_1,p_2}(\xi)$ is real because the product $f(X+\xi/2)f(-X+\xi/2)$ is invariant under the reflection $X \to -X$. If ε is sufficiently small, then $\varphi_{p_1,p_2}(\xi)$ is nontrivial and nonnegative for all p_1, $p_2 \in U$. From (21), it follows that $\mathrm{Re}\tilde{\psi}_{p_1,p_2}(\xi)$ also has these properties. The support of the shifted function $\psi_{p_1,p_2}(\xi - \Theta(p_1 + p_2))$ lies in the 2ε-neighborhood of the set (20) and, if ε is sufficiently small, then $J_1(m\sqrt{\xi^2})$ has a constant sign on this support. Therefore the expression $\mathrm{Re}(\Delta, \tilde{\mu}_{p_1,p_2} * \tilde{\psi}_{p_1,p_2})$ has a constant sign for all p_1, $p_2 \in \mathrm{supp}\tilde{h}$. We conclude that for arbitrarily small neighborhoods of the points $\bar{x}$ and $\bar{y}$, there exist test functions f and g supported in these neighborhoods and such that $\langle 0|[\mathcal{O}(f), \mathcal{O}(g)]|\Phi\rangle \neq 0$. This amounts to saying that $(\bar{x}, \bar{y})$ belongs to $\mathrm{supp}\mathcal{M}_\Phi$. If $J_1(2m|\bar{x}^0|) = 0$ and U is small enough, then the function $J_1(m\sqrt{\xi^2})$ has a constant sign on the set (20) except for $\xi = 2\bar{x}^0$, and we arrive at the same conclusion with a different choice of f. Namely, we can take f to be a nonnegative function supported in the $\varepsilon/2$-neighborhood of the point $(\bar{x}^0 \pm \varepsilon/2, \bar{x}^1, \bar{x}^2, \bar{x}^3)$, where the minus sign corresponds to positive $\bar{x}^0$ and the plus sign corresponds to negative $\bar{x}^0$. This completes the proof of Theorem 1.

Remark 1. It is worth noting that this theorem also holds for $\bar{x}^0 = \bar{y}^0$, $\bar{x}^3 = \bar{y}^3$, $(\bar{x} - \bar{y})^2 < 0$. In other words, the support of the commutator under study contains even the equal-time points which lie outside the wedge (14). The proof proceeds along the same lines, but in this case f should be chosen so that its support is contained in the $\varepsilon/2$-neighborhood of the point $(\varepsilon/2, \bar{x}^1, \bar{x}^2, 0)$.

Remark 2. Theorem 1 implies, in particular, that the power series expansion of the distribution (13) in θ does not converge in the topology of the space $\mathcal{S}'$ of tempered distributions. In fact, every term of this expansion is obtainable from (6) by applying a finite-order differential operator and hence is zero everywhere in the region $(x - y)^2 < 0$. If the expansion were convergent in $\mathcal{S}'$, its limit should also vanish in this region. A weaker topology, in which the expansion in powers of θ converges, is indicated in [16].

IV. θ-LOCALITY

We now show that the distribution $\mathcal{M}_\Phi(x, y)$, if smoothed properly, has a rapid decrease in the whole cone $(x - y)^2 < 0$ for all Φ ranging a dense set in the subspace of two-particle states. More precisely, it behaves like $\exp(-|x - y|^2/|\theta|)$ at large spacelike separation of the arguments.[2] This is true irrespectively of the form of the

[2]Here and in the sequel we use the notation $|\theta| = \sum_{\mu<\nu}|\theta^{\mu\nu}|$.

matrix $\theta^{\mu\nu}$ and, in particular, for both space-space and time-space noncommutativity.

A simple and well-known way of describing the behavior of a distribution at infinity is by considering its convolution with test functions decreasing sufficiently fast. In order to reveal the indicated decrease of $\mathcal{M}_\Phi$, it is natural to use test functions satisfying the inequalities

$$|\partial^\kappa f(x)| \leq C_\kappa e^{-|x/A|^2}, \tag{22}$$

where A is small in comparison to $\sqrt{|\theta|}$. In our case, however, the test functions should also be sufficiently smooth, as it is argued in [15,16]. The distribution (13) is obtained from the distribution (6) by applying the infinite-order differential operator

$$D_\theta = \cos\left(\frac{1}{2}\partial_x\theta\partial_{z_1}\right)\cos\left(\frac{1}{2}\partial_y\theta\partial_{z_2}\right), \tag{23}$$

where

$$\partial_x\theta\partial_z \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\mu}\theta^{\mu\nu}\frac{\partial}{\partial z^\nu}.$$

The function space defined by (22) is not invariant under the action of the basic Moyal operator defining the $\star$-product and under the action of D_θ. In other words, these operators spoil in general the behavior of its elements at infinity. Theorem 2 of [15] characterizes those subspaces of the Schwartz space that are invariant under the Moyal operator and shows that the smoothness properties of their elements should be matched with the decrease properties to ensure this invariance. A special role is played by the space denoted in [21] by $S^{1/2}_{1/2}$, which consists of the infinitely differentiable functions satisfying

$$|\partial^\kappa f(x)| \leq CB^{|\kappa|}\kappa^{\kappa/2}e^{-|x/A|^2}, \tag{24}$$

where C, B, A are positive constants depending on f and the usual multi-index notation is used. This space is the union of the Banach spaces $S^{1/2,B}_{1/2,A}$ with the norms

$$\| f \|_{A,B} = \sup_{\kappa,x} e^{|x/A|^2}\frac{|\partial^\kappa f(x)|}{B^{|\kappa|}\kappa^{\kappa/2}}, \tag{25}$$

and a sequence f_n is said to be convergent to zero in $S^{1/2}_{1/2}$ if there are A and B such that $f_n \in S^{1/2,B}_{1/2,A}$ and $\| f_n \|_{A,B} \to 0$ as $n \to \infty$. The space $S^{1/2}_{1/2}$ is invariant under both the Fourier operator and the Moyal operator and these operators are continuous in its topology.

Theorem 2. Let ϕ be a free scalar field on $\mathbb{R}^d$ and let $\mathcal{O}(x) = :\phi \star \phi:(x)$, with the $\star$-product defined by an arbitrary real antisymmetric matrix $\theta^{\mu\nu}$. Let $f, g, h_1, h_2 \in S^{1/2,B}_{1/2,A}$, where $A > 0$ and $0 < B < 1/\sqrt{e|\theta|}$. Suppose that a is a spacelike vector in $\mathbb{R}^d$ separated from the cone $\bar{\mathbb{V}}$ by an angular distance $\gamma = \inf_{\xi^2 \geq 0}|\xi - a/|a||$. Then the matrix element

$$(\mathcal{M}_\Phi, f_a \otimes g_{-a}) = \langle 0|[\mathcal{O}(f_a), \mathcal{O}(g_{-a})]|\Phi\rangle, \tag{26}$$

where $f_a(x) = f(x-a)$ and $|\Phi\rangle = \phi^-(h_1)\phi^-(h_2)|0\rangle$, satisfies the estimate

$$|(\mathcal{M}_\Phi, f_a \otimes g_{-a})| \leq C_{\Phi,A'} \| f \|_{A,B} \| g \|_{A,B}\, e^{-2|\gamma a/A'|^2} \tag{27}$$

for each $A' > A$.

Proof. We denote the vacuum expectation value (6) by M and set $\varphi = f \otimes g \otimes h_1 \otimes h_2$, $\varphi_a = f_a \otimes g_{-a} \otimes h_1 \otimes h_2$. Then

$$(\mathcal{M}_\Phi, f_a \otimes g_{-a}) = (M, D_\theta\varphi_a). \tag{28}$$

Theorem 1 of [16] shows that under the condition $B < 1/\sqrt{e|\theta|}$ the operator D_θ maps the space $S^{1/2,B}_{1/2,A}$ continuously into the space $S^{1/2,B'}_{1/2,A}$, where $B' = B\sqrt{2}$. In particular, $\| D_\theta\varphi \|_{A,B'} \leq C \| \varphi \|_{A,B}$, which gives the inequality

$$|\partial^\kappa(D_\theta\varphi)(x-a, y+a, z_1, z_2)| \leq C \| \varphi \|_{A,B}\, B'^{|\kappa|}\kappa^{\kappa/2}e^{-(|x-a|^2+|y+a|^2+|z_1|^2+|z_2|^2)/A^2}. \tag{29}$$

Clearly, M is a tempered distribution supported in the cone

$$\bar{\mathbb{V}} \times \mathbb{R}^{3d} = \{(x, y, z_1, z_2) \in \mathbb{R}^{4d} : (x-y)^2 \geq 0\}.$$

Therefore there exists an integer N and a constant C' such that, for each test function $\psi \in \mathcal{S}(\mathbb{R}^{4d})$, we have

$$|(M, \psi)| \leq C' \| \psi \|_{N,\bar{\mathbb{V}}\times\mathbb{R}^{3d}}, \tag{30}$$

where

$$\| \psi \|_{N,\bar{\mathbb{V}}\times\mathbb{R}^{3d}} = \sup_{|\kappa|\leq N}\sup_{\bar{\mathbb{V}}\times\mathbb{R}^{3d}}(1 + |x| + |y| + |z_1| + |z_2|)^N \times |\partial^\kappa\psi(x, y, z_1, z_2)|. \tag{31}$$

We put $\psi = D_\theta\varphi_a$ and denote $x - y$ by ξ. Combining (28)–(31), we obtain

$$\begin{aligned}|(\mathcal{M}_\Phi, f_a \otimes g_{-a})| &\leq C'' \| \varphi \|_{A,B}\sup_{\bar{\mathbb{V}}\times\mathbb{R}^{3d}}(1 + |x| + |y| + |z_1| + |z_2|)^N \\ &\quad\times e^{-(|x-a|^2+|y+a|^2+|z_1|^2+|z_2|^2)/A^2} \\ &\leq C_{h_1,h_2} \| f \otimes g \|_{A,B}\sup_{\xi\in\bar{\mathbb{V}}}(1 + |\xi|)^N e^{-|\xi-2a|^2/(2A^2)}.\end{aligned} \tag{32}$$

To complete the proof it suffices to observe that

$$|\xi - 2a| \geq 2\gamma|a|, \qquad |\xi - 2a| \geq \gamma|\xi| \quad \text{for all } \xi \in \bar{\mathbb{V}}.$$

The obtained estimate (27) is the stronger, the smaller A. However the space $S^{1/2,B}_{1/2,A}$ becomes trivial if AB is too small. For the readers' convenience, a proof of this simple fact is given in the appendix. If $AB > 2/\sqrt{e}$, then $S^{1/2,B}_{1/2,A}$ is nontrivial and, in particular, contains the Gaussian function $e^{-2|x/A|^2}$. Because of the restriction $B < 1/\sqrt{e|\theta|}$ in the assumptions of Theorem 2, the best result is at $A \sim 2\sqrt{|\theta|}$. It can be interpreted as demonstrating that the matrix element (26) decreases like $e^{-|\gamma a|^2/(2|\theta|)}$ at large spacelike separation of the test functions along the direction a, which refines the statement made at the beginning of this section.

V. THE STAR COMMUTATOR

In Refs. [8,22], a framework for noncommutative QFTs was formulated in terms of the vacuum expectation values of $\star$-products of field operators at different spacetime points. This product is formally written as

$$\phi(x_1)\star\ldots\star\phi(x_n)=\prod_{a<b}e^{(i/2)\partial_{x_a}\theta\partial_{x_b}}\phi(x_1)\cdots\phi(x_n). \tag{33}$$

It is generally agreed that a mathematically rigorous theory of quantum fields on noncommutative spacetime shall adopt the basic assumption of the traditional axiomatic approach [18,19] that quantum fields are operator-valued distributions. In other words, it is customary to assume that in this case, too, there is a linear mapping of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ (or another suitable test function space) into the operators of the Hilbert space of states: $f\longrightarrow\phi(f)$. This raises the question of a rigorous definition of the formal expression (33) in agreement with this assumption. First of all, we note that there is a multilinear mapping $\underbrace{\mathcal{S}(\mathbb{R}^d)\times\cdots\times\mathcal{S}(\mathbb{R}^d)}_{n}\longrightarrow\mathcal{S}(\mathbb{R}^{nd})$ associated naturally with the Moyal $\star$-product $(f_1,\ldots,f_n)\longrightarrow f_1\star\cdots\star f_n$. It is defined by

$$\begin{aligned}(f_1,\ldots,f_n)&\longrightarrow f_1(x_1)\star\cdots\star f_n(x_n)\\&=\frac{1}{(2\pi)^{dn}}\int dk_1\ldots dk_n\tilde f_1(k_1)\cdots\tilde f_n(k_n)\\&\quad\times e^{-i\sum_a k_a\cdot x_a}\prod_{a<b}e^{-(i/2)k_{a\mu}\theta^{\mu\nu}k_{b\nu}}.\end{aligned} \tag{34}$$

The notation $f_1(x_1)\star\cdots\star f_n(x_n)$ is accepted in the literature, though it seems reasonable to denote the function (34) by $f_1\otimes_\star\ldots\otimes_\star f_n$. The ordinary product of n functions $f_1,\ldots,f_n$ is obtained from $(f_1\otimes\ldots\otimes f_n)(x_1,\ldots,x_n)$ by the identification $x_1=\ldots=x_n$, and their Moyal product is obtained from (34) in the same fashion. Sometimes we will write $f_1\otimes_\star\ldots\otimes_\star f_n$ instead of $f_1(x_1)\star\cdots\star f_n(x_n)$ to avoid confusion and for short. If the test functions are sufficiently smooth, then (34) can be rewritten as

$$f_1(x_1)\star\cdots\star f_n(x_n)=\prod_{a<b}e^{(i/2)\partial_{x_a}\theta\partial_{x_b}}f_1(x_1)\cdots f_n(x_n). \tag{35}$$

In particular, the power series expansion of the expression on the right-hand side of (35) in θ converges to the function (34) in the space $\mathcal{S}^{1/2}(\mathbb{R}^{dn})$ whose elements satisfy the inequalities (22) for each $A>0$ (with a constant C_κ depending on f and A). The topology of $\mathcal{S}^{1/2}$ is defined by the system of norms corresponding to these inequalities. As shown in [15,16], $\mathcal{S}^{1/2}$ is the largest subspace of the Schwartz space with such a convergence property. The operation $(f_1,\ldots,f_n)\longrightarrow f_1\otimes_\star\cdots\otimes_\star f_n$ generates a dual operation over the distributions $u_j\in\mathcal{S}'(\mathbb{R}^d)$, which is equivalent to multiplication of $\tilde u_1\otimes\ldots\otimes\tilde u_n$ by the multiplier

$$\mu_n=\prod_{1\le a<b\le n}e^{-(i/2)k_{a\mu}\theta^{\mu\nu}k_{b\nu}}. \tag{36}$$

In particular, in the case of two distributions we have

$$\begin{aligned}(u\otimes_\star v)(x,y)&\equiv u(x)\star v(y)\\&=\frac{1}{(2\pi)^{2d}}\int dkdq e^{-ik\cdot x-iq\cdot y-i[k,q]}\tilde u(k)\tilde v(q),\end{aligned} \tag{37}$$

with the above notation $[k,q]=(1/2)k_\mu\theta^{\mu\nu}q_\nu$. This operation over distributions can also be considered as an extension of the operation (35) over test functions by continuity. The extension is unique, because $\mathcal{S}^{1/2}$ is dense in $\mathcal{S}'$.

Now let ϕ be an operator-valued tempered distribution defined on a dense invariant domain D in the Hilbert space $\mathcal{H}$, with the vacuum vector $\Psi_0\in D$. By the standard arguments [18] based on the Schwartz kernel theorem, the vector

$$\Phi_n(f)=\int dx_1\ldots dx_n\phi(x_1)\cdots\phi(x_n)f(x_1,\ldots,x_n)\Psi_0 \tag{38}$$

and the operator $\int dx_1\ldots dx_n\phi(x_1)\cdots\phi(x_n)f(x_1,\ldots,x_n)$ are well defined for each $f\in\mathcal{S}(\mathbb{R}^{dn})$. In particular, the operator

$$\int dx_1\ldots dx_n\phi(x_1)\cdots\phi(x_n)f_1(x_1)\star\cdots\star f_n(x_n), \tag{39}$$

is uniquely defined for any system of functions $f_a\in\mathcal{S}(\mathbb{R}^d)$, $a=1,\ldots,n$. An analogous statement holds in the case when $\mathcal{S}(\mathbb{R}^d)$ is replaced by another nuclear space which is a topological algebra under the $\star$-product, for instance, by the space $\mathcal{S}^{1/2}(\mathbb{R}^d)$. If we hold to the basic principle of the calculus of generalized functions and define the action of the differential operator in (33) by duality, then

$$\begin{aligned}&\int dx_1\ldots dx_n\phi(x_1)\star\cdots\star\phi(x_n)f_1(x_1)\cdots f_n(x_n)\\&\quad=\int dx_1\ldots dx_n\phi(x_1)\cdots\phi(x_n)f_1(x_1)\star\ldots\star f_n(x_n).\end{aligned} \tag{40}$$

As a consequence, we obtain the relation

$$\int dx_1 \dots dx_n W_\star^{(n)}(x_1, \dots, x_n) f_1(x_1) \cdots f_n(x_n) = \int dx_1 \dots dx_n W^{(n)}(x_1, \cdots, x_n) f_1(x_1) \star \dots \star f_n(x_n), \tag{41}$$

where $W^{(n)}(x_1, \dots, x_n) = \langle \Psi_0, \phi(x_1) \cdots \phi(x_n) \Psi_0 \rangle$ is the usual Wightman function and

$$W_\star^{(n)}(x_1, \dots, x_n) \overset{\text{def}}{=} \langle \Psi_0, \phi(x_1) \star \cdots \star \phi(x_n) \Psi_0 \rangle. \tag{42}$$

We note that (41) can also be written as

$$(W_\star^{(n)}, f_1 \otimes \cdots \otimes f_n) = (W^{(n)}, f_1 \otimes_\star \cdots \otimes_\star f_n).$$

Clearly, the two-point function $W(x, y)$ coincides with $W_\star(x, y)$ due to the translation invariance. Indeed, writing $W(x, y) = w(x - y)$, we have

$$\begin{aligned}\int dxdy w(x-y) f(x) \star g(y) &= \frac{1}{(2\pi)^d} \int dxdydkdq \tilde{w}(k) \delta(k+q) e^{-ik\cdot x - iq\cdot y} f(x) \star g(y) \\ &= \frac{1}{(2\pi)^d} \int dkdq \tilde{w}(k) \delta(k+q) \tilde{f}(-k) \tilde{g}(-q) e^{-i[k,q]} \\ &= \int dxdy w(x-y) f(x) g(y),\end{aligned}$$

because $e^{-i[k,q]} = 1$ for $k = -q$. But for $n > 2$, the distributions $W^{(n)}$ and $W_\star^{(n)}$ differ from one another.

Let us consider the definition (40) more closely, taking a free field ϕ as a simplest example. Let $w(x - y)$ be its two-point function. It is easy to see that if the product $\phi(x) \star \phi(y)$ is defined by (40), then

$$\langle 0 | \phi(x) \star \phi(y) {:} \phi(z_1) \phi(z_2) {:} | 0 \rangle = w(x - z_1) \star w(y - z_2) + w(x - z_2) \star w(y - z_1). \tag{43}$$

Indeed, by the Wick theorem we have

$$\langle 0 | \phi(x) \phi(y) {:} \phi(z_1) \phi(z_2) {:} | 0 \rangle = w(x - z_1) w(y - z_2) + w(x - z_2) w(y - z_1). \tag{44}$$

Let f, g, h_1, h_2 be functions in the Schwartz space. Using (37), (40), and (44), we obtain

$$\begin{aligned}&\int dxdydz_1dz_2 \langle 0 | \phi(x) \star \phi(y) {:} \phi(z_1) \phi(z_2) {:} | 0 \rangle f(x) g(y) h_1(z_1) h_2(z_2) \\ &= \int dz_1 dz_2 \int dxdy [w(x - z_1) w(y - z_2) + w(x - z_2) w(y - z_1)] f(x) \star g(y) h_1(z_1) h_2(z_2) \\ &= \int dz_1 dz_2 \int \frac{dkdq}{(2\pi)^{2d}} [e^{ik\cdot z_1 + iq\cdot z_2} + e^{ik\cdot z_2 + iq\cdot z_1}] \tilde{w}(k) \tilde{w}(q) e^{-i[k,q]} \tilde{f}(-k) \tilde{g}(-q) h_1(z_1) h_2(z_2) \\ &= \int dxdydz_1dz_2 [w(x - z_1) \star w(y - z_2) + w(x - z_2) \star w(y - z_1)] f(x) g(y) h_1(z_1) h_2(z_2),\end{aligned} \tag{45}$$

which proves our claim. The formula (43) is also obtainable by applying formally the operator $e^{(i/2)\partial_x \theta \partial_y}$ to (44). In momentum space, the distribution (43) takes the form

$$(2\pi)^{2d} \tilde{w}(k) \tilde{w}(q) e^{-i[k,q]} [\delta(k + p_1) \delta(q + p_2) + \delta(k + p_2) \delta(q + p_1)], \tag{46}$$

where the variables k, q, p_1, p_2 are, respectively, conjugate to the coordinate-space variables x, y, z_1, z_2. We note that (46) differs from the Fourier transform of the distribution (44) only by the factor $e^{-i[k,q]}$.

In [17,22], it was assumed that in the case of space-space noncommutativity, the star commutator $[\phi(x), \phi(y)]_* = \phi(x) \star \phi(y) - \phi(y) \star \phi(x)$ obeys microcausality with respect to the commuting coordinates (x^0, x^3), i.e., $[\phi(x), \phi(y)]_* = 0$ everywhere in the wedge $\{(x, y) \in \mathbb{R}^{2d} : |x^0 - y^0| < |x^3 - y^3|\}$. We shall show that this assumption contradicts the spectral condition if the product (33) is defined by duality, as indicated above.

Theorem 3. Let ϕ be a free neutral scalar field on $\mathbb{R}^d$ and let Φ be a two-particle state of the form (16). If the product $\phi(x) \star \phi(y)$ is defined by (40), then the distribution

$$\langle 0 | [\phi(x), \phi(y)]_\star | \Phi \rangle \tag{47}$$

does not vanish on any open set and so its support coincides with the whole space $\mathbb{R}^{2d}$.

Proof. From (46), it follows that the Fourier transform of $\langle 0 | \phi(x) \star \phi(y) | \Phi \rangle$ is of the form

$$2 \tilde{w}(k) \tilde{w}(q) e^{-i[k,q]} \tilde{h}(k) \tilde{h}(q). \tag{48}$$

The Fourier transform of $\langle 0 | \phi(y) \star \phi(x) | \Phi \rangle$ is obtained from (48) by interchanging k and q. Hence that of the matrix element (47) has the form

$$- 4i\tilde{w}(k)\tilde{w}(q)\sin[k, q]\tilde{h}(k)\tilde{h}(q)$$

and differs from $\tilde{w}\otimes\tilde{w}$ only by the factor $-4i\sin[k,q]\tilde{h}(k)\tilde{h}(q)$ which is a multiplier of the Schwartz space and does not vanish on $\mathrm{supp}(\tilde{w}\otimes\tilde{w})$ if $\Phi \neq 0$. The support of $\tilde{w}\otimes\tilde{w}$ is contained in the properly convex cone $\mathbb{V}_+\times\mathbb{V}_+$. Therefore, the distribution (47) is the boundary value of a function analytic in the tubular domain $\mathbb{R}^{2d}+i(\mathbb{V}_-\times\mathbb{V}_-)$ (see, e.g., [19], Theorem B.7). Applying the generalized uniqueness theorem (ibid., Theorem B.10), we conclude that this distribution does not vanish on any nonempty open set, because otherwise it would be identically zero on $\mathbb{R}^{2d}$. Theorem 3 is proved.

Now we return to the sample observable $\mathcal{O}(x)$ defined by (7) and consider the star commutator

$$[\mathcal{O}(x), \mathcal{O}(y)]_\star = \mathcal{O}(x)\star\mathcal{O}(y) - \mathcal{O}(y)\star\mathcal{O}(x). \tag{49}$$

Theorem 4. Let, as in Theorem 1, $d=4$, $\theta^{12}=-\theta^{21}\neq 0$, and the other elements of the matrix $\theta^{\mu\nu}$ be equal to zero. Then the star commutator (49) does not vanish in the wedge defined by (14).

Proof. Let $(\bar{x},\bar{y})$ be contained in the wedge (14) together with a neighborhood $U\times V$. In what follows we set $U=U_c\times U_{nc}$, $V=V_c\times V_{nc}$, where the labels c and nc indicate, respectively, sets in the planes (x^0,x^3) and (x^1,x^2). For definiteness, we assume that $|\bar{x}^0-\bar{y}^0|<\bar{x}^3-\bar{y}^3$ and

$$U_c - V_c \subset \mathbb{V}_R, \tag{50}$$

where $\mathbb{V}_R$ is the right component of the spacelike cone in $\mathbb{R}^2$. We shall show that there exist test functions f, g supported in U, V and a state Φ of the form (16) such that the matrix element

$$\langle 0|\int dxdy[\mathcal{O}(x),\mathcal{O}(y)]_\star f(x)g(y)|\Phi\rangle \tag{51}$$

is different from zero. Applying the operator $e^{(i/2)\partial_x\theta\partial_y}$ to (10), we obtain

$$\begin{aligned}\langle 0|\mathcal{O}(x)\star\mathcal{O}(y){:}\phi(z_1)\phi(z_2){:}|0\rangle &= 4\int dkdp_1dp_2\tilde{w}(k)e^{-ik\cdot(x-y)-ip_1\cdot(x-z_1)-ip_2\cdot(y-z_2)}\\ &\quad\times e^{-i[k,p_1+p_2]-i[p_1,p_2]}\prod_{i=1,2}\tilde{w}(p_i)\cos[k,p_i]\\ &\quad+(z_1\leftrightarrow z_2).\end{aligned} \tag{52}$$

On the other hand, applying $e^{(i/2)\partial_y\theta\partial_x}$ to $\mathcal{W}(y,x;z_1,z_2)$ gives

$$\begin{aligned}\langle 0|\mathcal{O}(y)\star\mathcal{O}(x){:}\phi(z_1)\phi(z_2){:}|0\rangle &= 4\int dkdp_1dp_2\tilde{w}(k)e^{-ik\cdot(y-x)-ip_1\cdot(y-z_1)-ip_2\cdot(x-z_2)}\\ &\quad\times e^{-i[k,p_1+p_2]-i[p_1,p_2]}\prod_{i=1,2}\tilde{w}(p_i)\cos[k,p_i]\\ &\quad+(z_1\leftrightarrow z_2).\end{aligned} \tag{53}$$

From (52) and (53), it follows that

$$\begin{aligned}\langle 0|[\mathcal{O}(x),\mathcal{O}(y)]_\star{:}\phi(z_1)\phi(z_2){:}|0\rangle &= 8i\int dkdp_1dp_2\tilde{\Delta}(k)e^{-ik\cdot(x-y)-ip_1\cdot(x-z_1)-ip_2\cdot(y-z_2)}\\ &\quad\times\cos([k,p_1+p_2]+[p_1,p_2])\prod_{i=1,2}\tilde{w}(p_i)\cos[k,p_i]\\ &\quad-8i\int dkdp_1dp_2\tilde{\Delta}_1(k)e^{-ik\cdot(x-y)-ip_1\cdot(x-z_1)-ip_2\cdot(y-z_2)}\\ &\quad\times\sin([k,p_1+p_2]+[p_1,p_2])\prod_{i=1,2}\tilde{w}(p_i)\cos[k,p_i]\\ &\quad+(z_1\leftrightarrow z_2),\end{aligned} \tag{54}$$

where $\tilde{\Delta}_1(k)=\tilde{w}(k)+\tilde{w}(-k)=2\pi\delta(k^2-m^2)$. The distribution defined by the first integral on the right-hand side of (54) and that obtained from it by the transposition $z_1\leftrightarrow z_2$ vanish in the wedge (14) by the argument that was used at the end of Sec. II. Now we consider the distribution

$$\begin{aligned}\mathcal{W}(x-y,x-z_1,y-z_2) &= -8i\int dkdp_1dp_2\tilde{w}(k)e^{-ik\cdot(x-y)-ip_1\cdot(x-z_1)-ip_2\cdot(y-z_2)}\\ &\quad\times\sin([k,p_1+p_2]+[p_1,p_2])\prod_{i=1,2}\tilde{w}(p_i)\cos[k,p_i].\end{aligned} \tag{55}$$

Clearly, it is not identically zero because there are points k, $p\in\mathrm{supp}\tilde{w}$ such that $\cos[k,p]\neq 0$ and $\sin[k,2p]\neq 0$. Moreover, there exists a function $h\in\mathcal{S}(\mathbb{R}^4)$ such that the distribution

$$T(x,y)=\int dz_1dz_2\mathcal{W}(x-y,x-z_1,y-z_2)h(z_1)h(z_2) \tag{56}$$

is also nonzero. The spectral condition $\mathrm{supp}\tilde{w}\subset\bar{\mathbb{V}}_+$ implies that T is the boundary value of a function analytic in the tubular domain defined by

$$\mathrm{Im}\,(x-y)\in\mathbb{V}_-,\qquad \mathrm{Im}\,x\in\mathbb{V}_-,\qquad \mathrm{Im}\,y\in\mathbb{V}_-. \tag{57}$$

By the generalized uniqueness theorem, T does not vanish on any open subset of $\mathbb{R}^8$ and, in particular, on $U\times V$. Every test function supported in $U\times V$ can be approximated by linear combinations of functions of the form $f\otimes g$, where $\mathrm{supp}f\subset U$, $\mathrm{supp}g\subset V$. Therefore, there are f, g supported in these neighborhoods and such that $(T,f\otimes g)\neq 0$. The matrix element (51) is written

$$\begin{aligned}&\int dxdydz_1dz_2(\mathcal{W}(x-y,x-z_1,y-z_2)\\ &\quad+\mathcal{W}(y-x,x-z_1,y-z_2))f(x)g(y)h(z_1)h(z_2)\end{aligned} \tag{58}$$

and, to complete the proof, it suffices to show that the expression (58) is equal to $2(T,f\otimes g)$. Clearly, we can assume that each of the functions f, g, h is the product of functions of the commuting coordinates and of the noncommuting coordinates which are, respectively, labeled c and nc below. Let

$$\mathcal{W}_c(x^c - y^c, x^c - z_1^c, y^c - z_2^c) \stackrel{\text{def}}{=} \int dx^{nc} dy^{nc} dz_1^{nc} dz_2^{nc} \mathcal{W}(x - y, x - z_1, y - z_2) \times f_{nc}(x^{nc}) g_{nc}(y^{nc}) h_{nc}(z_1^{nc}) h_{nc}(z_2^{nc}), \tag{59}$$

and let $\xi = x^c - y^c$, $\zeta_1 = x^c - z_1^c$, $\zeta_2 = y^c - z_2^c$. If $\chi \in \mathcal{S}(\mathbb{R}^4)$, then by the spectral condition the distribution $T_\chi(\xi) = \int \mathcal{W}_c(\xi, \zeta_1, \zeta_2)\chi(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2$ is the boundary value of a function analytic in the tubular domain whose base is the lower cone $\{\xi \in \mathbb{R}^2 : \xi^2 > 0, \xi^0 < 0\}$. This analytic function is invariant under Lorentz boosts and, by the simplest two-dimensional version of the Bargman-Hall-Wightman theorem [18,19], admits analytic continuation to an extended domain which contains all spacelike points of $\mathbb{R}^2$. The analytic extension is invariant under the reflection $\xi \to -\xi$ and hence $\int d\xi T_\chi(\xi)\psi(\xi) = \int d\xi T_\chi(-\xi)\psi(\xi)$ for every $\psi \in \mathcal{S}(\mathbb{R}^2)$ whose support is contained in $\mathbb{V}_R$. It follows that

$$\int d\xi d\zeta_1 d\zeta_2 \mathcal{W}_c(\xi, \zeta_1, \zeta_2)\varphi(\xi, \zeta_1 \zeta_2) = \int d\xi d\zeta_1 d\zeta_2 \mathcal{W}_c(-\xi, \zeta_1, \zeta_2)\varphi(\xi, \zeta_1 \zeta_2)$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^6)$ whose support is contained in $\mathbb{V}_R \times \mathbb{R}^2 \times \mathbb{R}^2$. The function

$$\varphi(\xi, \zeta_1, \zeta_2) = \int f_c(X) g_c(X - \xi) h_c(X - \zeta_1) \times h_c(X - \xi - \zeta_2) dX \tag{60}$$

has support in this wedge by construction and, for this function, we have

$$(\mathcal{W}_c, \varphi) = \int dx dy dz_1 dz_2 \mathcal{W}(x - y, x - z_1, y - z_2) \times f(x) g(y) h(z_1) h(z_2).$$

Thus the expression (58) is indeed equal to $2(T, f \otimes g)$. Theorem 4 is proved.

VI. CONCLUDING REMARKS

At the present time, there is no agreement regarding the physical interpretation of the $\star$-product of quantum fields at different spacetime points. In this connection we shall make some remarks about the proposals to formulate a framework for quantum field theories on noncommutative spacetime in terms of the n-point vacuum expectation values of such products (see, e.g., Refs. [8,22]). Let ϕ be a scalar field with test functions in $\mathcal{S}(\mathbb{R}^d)$ and with an invariant dense domain D in the Hilbert space $\mathcal{H}$, containing the vacuum state Ψ_0. If the star-modified Wightman functions $W_\star^{(n)}$ of ϕ are defined by (42) and (40), then we always can construct a field ϕ_θ such that

$$\langle \Psi_0, \phi_\theta(x_1)\phi_\theta(x_2) \cdots \phi_\theta(x_n)\Psi_0 \rangle = W_\star^{(n)}(x_1, x_2, \ldots, x_n). \tag{61}$$

Indeed, let $g \in \mathcal{S}(\mathbb{R}^d)$, $f \in \mathcal{S}(\mathbb{R}^{dn})$, and let $\Phi_n(f)$ be a vector of the form (38). We set

$$\phi_\theta(g)\Psi_0 = \phi(g)\Psi_0, \quad \phi_\theta(g)\Phi_n(f) = \Phi_{n+1}(g \otimes_\star f), \qquad n \geq 1, \tag{62}$$

where

$$(g \otimes_\star f)(y, x_1, \ldots, x_n) \stackrel{\text{def}}{=} \prod_{a=1}^n e^{(i/2)\partial_y \theta \partial_{x_a}} g(y) f(x_1, \ldots x_n). \tag{63}$$

Then (61) is satisfied. Clearly, $W_\star^{(n)} \in S'(\mathbb{R}^{dn})$ and, for each $\tilde{f} \in S(\mathbb{R}^{dn})$,

$$(\tilde{W}_\star^{(n)}, \tilde{f}) = (\tilde{W}^{(n)}, \mu_n \cdot \tilde{f}),$$

where $W^{(n)}$ is the ordinary n-point Wightman function of ϕ and μ_n is given by (36). The distributions $W_\star^{(n)}$, i.e., the vacuum expectation values of the ordinary products of the operators ϕ_θ, have the same spectral properties as $W^{(n)}$ because the multiplication by μ_n leaves these properties unchanged. If the field ϕ is Hermitian, then so are ϕ_θ. Indeed, for any $g \in \mathcal{S}(\mathbb{R}^d)$, $f \in \mathcal{S}(\mathbb{R}^{dn})$, $h \in \mathcal{S}(\mathbb{R}^{dm})$, we have

$$\langle \Phi_m(h), \Phi_{n+1}(g \otimes_\star f) \rangle = \langle \Phi_{m+1}(\bar{g} \otimes_\star h), \Phi_n(f) \rangle, \tag{64}$$

where the bar over g denotes the complex conjugation. This identity is easily verified by using the antisymmetry of $\theta^{\mu\nu}$ and going to the momentum-space representation because $\tilde{W}^{(m+n+1)}(q_1, \ldots, q_m, k, p_1, \ldots, p_n)$ contains the factor $\delta(k + \sum_1^m q_b + \sum_1^n p_a)$ due to the translation invariance. For the same reason the modified Wightman functions satisfy the ordinary positive definiteness conditions

$$\sum_{m,n=1}^N (W_\star^{(m+n)}, f_m^\dagger \otimes f_n) \geq 0, \tag{65}$$

where f_n are arbitrary elements of $\mathcal{S}(\mathbb{R}^{dn})$ and $f^\dagger(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \overline{f(x_n, \ldots, x_1)}$. To prove (65), it is enough to observe that $(W^{(m+n)}, g \otimes f) = (W^{(m+n)}, g \otimes_\star f)$ for any $g \in \mathcal{S}(\mathbb{R}^{dm})$ and $f \in \mathcal{S}(\mathbb{R}^{dn})$, where $g \otimes_\star f$ is defined by

$$(g \otimes_\star f)(y_1, \ldots, y_m, x_1, \ldots, x_n) = \prod_{b=1}^m \prod_{a=1}^n e^{(i/2)\partial_{y_b} \theta \partial_{x_a}} g(y_1, \ldots, y_m) f(x_1, \ldots x_n). \tag{66}$$

However the transformation and local properties of ϕ_θ differ radically from those of ϕ.

For the case of a free field ϕ, an alternate description of its associated field ϕ_θ is given in [11]. Namely, the creation and annihilation operators of these fields are related by

$$a_\theta(p) = e^{(i/2)p\theta P}a(p), \qquad a_\theta^\dagger(p) = e^{-(i/2)p\theta P}a^\dagger(p),$$

where P is the energy-momentum operator. The operators $a_\theta(p)$, $a_\theta^\dagger(p)$ satisfy the deformed commutation relations

$$\begin{aligned} a_\theta(p)a_\theta(p') &= e^{-ip\theta p'}a_\theta(p')a_\theta(p), \\ a_\theta^\dagger(p)a_\theta^\dagger(p') &= e^{-ip\theta p'}a_\theta^\dagger(p')a_\theta^\dagger(p), \\ a_\theta(p)a_\theta^\dagger(p') &= e^{ip\theta p'}a_\theta^\dagger(p')a_\theta(p) + 2\omega_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}'). \end{aligned}$$

As discussed above, the field ϕ_θ is essentially nonlocal, but fields with different θ have interesting relative localization properties found by Grosse and Lechner [11]. On the other hand, Fiore and Wess [8] argued that the twisted Poincaré covariance can be implemented in the theory of a free field on noncommutative spacetime in a manner compatible with microcausality only if the canonical commutation relations (CCR) of creation and annihilation operators are suitably deformed. This deformation compensates the spacetime noncommutativity and the vacuum expectation values of the $\star$-products defined in [8] coincide with the usual Wightman functions of a free field on commutative spacetime. In other words, we obtain a mathematically self-consistent formulation, but without a new physics and in fact even without noncommutativity. The same disappointing conclusion has been drawn in [8] for interacting fields treated perturbatively. The axiomatic scheme proposed for the star-modified Wightman functions in [22] differs from that of [8] and does not include a deformation of the CCR algebra. But then the spectral condition comes into conflict with causality as is evident from the foregoing.

It is quite possible that microcausality should be replaced by a weaker condition in order to develop a satisfactory framework for quantum field theory on noncommutative spacetime. We believe that the above-stated θ-locality condition is a possible candidate for this role because it precisely describes the nonlocal character of the Moyal $\star$-product. Conditions of this kind were previously used in nonlocal QFT and, together with the relativistic covariance and the spectral condition, they ensure the existence of CPT symmetry as well as the usual spin-statistics relation for nonlocal fields, see [23]. An extension of these results to the noncommutative setting is discussed in [24], where analogous theorems are proved for the case of a charge scalar field and space-space noncommutativity with the residual $SO_0(1,1)\times SO(2)$-symmetry.

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APPENDIX

Lemma. If $AB > 2/\sqrt{e}$, then the space $S^{1/2,B}_{1/2,A}$ is nontrivial. If $AB < \sqrt{2/e}$, then it contains only the trivial function which is identically zero.

Proof. If $f \in S^{1/2,B}_{1/2,A}$, then the function $f_\lambda(x) = f(\lambda x)$, where $\lambda > 0$, belongs to the space $S^{1/2,\lambda B}_{1/2,A/\lambda}$. Therefore, if a pair (A_0, B_0) is admissible, i.e., defines a nontrivial space, then all the pairs (A, B) for which $AB = A_0B_0$ are also admissible. Let us show that $e^{-|x|^2}$ belongs to any space $S^{1/2,B}_{1/2,A}$ with $A = \sqrt{2}$ and $B > \sqrt{2/e}$. Because $e^{-|x|^2} = \prod_j e^{-x_j^2}$, it suffices to consider the one-variable case. By the Cauchy inequality,

$$|\partial^\kappa e^{-x^2}| \le \frac{\kappa!}{r^\kappa}\max_{|\zeta-x|=r} e^{-\mathrm{Re}\,\zeta^2} \tag{A1}$$

for each $r > 0$. Setting $\zeta = x + re^{i\alpha}$ and using the elementary relation $\cos 2\alpha = 2\cos^2\alpha - 1$, we get

$$\mathrm{Re}\,\zeta^2 = x^2 + 2xr\cos\alpha + r^2\cos 2\alpha \ge \frac{x^2}{2} - r^2.$$

Therefore,

$$|\partial^\kappa e^{-x^2}| \le \kappa! e^{-x^2/2}\inf_{r>0}\frac{e^{r^2}}{r^\kappa} = \kappa! e^{-x^2/2}\left(\frac{2e}{\kappa}\right)^{\kappa/2}. \tag{A2}$$

By the Stirling formula, we have $\kappa! \le C_\epsilon(1/e+\epsilon)^\kappa\kappa^\kappa$ for any $\epsilon > 0$. The first statement of Lemma is thus proved.

Now we prove the second statement. Here again, it is sufficient to consider the one-variable case. From the inequalities (24), it follows that every $f \in S^{1/2,B}_{1/2,A}(\mathbb{R})$ is an entire analytic function and hence

$$f(\xi) = \sum_\kappa \frac{1}{\kappa!}(\xi - x)^\kappa\partial^\kappa f(x) \tag{A3}$$

for any $x, \xi \in \mathbb{R}$. Using (24) and (A3), and choosing $\bar{B} > B$, we estimate $f(\xi)$ in the following way:

$$\begin{aligned} |f(\xi)| &\le Ce^{-|x/A|^2}\sum_\kappa \frac{1}{\kappa!}B^\kappa\kappa^{\kappa/2}|\xi - x|^\kappa \\ &\le C'e^{-|x/A|^2}\sup_\kappa \frac{1}{\kappa!}\bar{B}^\kappa\kappa^{\kappa/2}|\xi - x|^\kappa, \end{aligned} \tag{A4}$$

where $C' = C\sum_\kappa(B/\bar{B})^\kappa < \infty$. Using the inequality $\kappa! \ge (\kappa/e)^\kappa$, we can replace the upper bound in (A4) by the function $M(e\bar{B}|\xi - x|)$, where

$$M(r) = \sup_{\kappa>0}\frac{r^\kappa}{\kappa^{\kappa/2}} = e^{r^2/(2e)}.$$

Because the point x can be taken arbitrarily, (A4) implies that $f(\xi) \equiv 0$ if $1/A^2 > e\bar{B}^2/2$. This completes the proof.

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