Negative radiation pressure exerted on kinks

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The interaction of a kink and a monochromatic plane wave in one dimensional scalar field theories is studied. It is shown that in a large class of models the radiation pressure exerted on the kink is negative, i.e. the kink is *pulled* towards the source of the radiation. This effect has been observed by numerical simulations in the ϕ^4 model, and it is explained by a perturbative calculation assuming that the amplitude of the incoming wave is small. Quite importantly the effect is shown to be robust against small perturbations of the ϕ^4 model. In the sine-Gordon (SG) model the time-averaged radiation pressure acting on the kink turns out to be zero. The results of the perturbative computations in the SG model are shown to be in full agreement with an analytical solution corresponding to the superposition of a SG kink with a cnoidal wave. It is also demonstrated that the acceleration of the kink satisfies Newton's law.

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I. INTRODUCTION

It is by now universally accepted that spatially localized solutions of nonlinear equations (solitary waves, particle-like objects) are of great importance in several areas of physics; we refer to some of the recent monographs [1–3]. The study of localized solutions in various systems in a single spatial dimension (i.e. in $1 + 1$ dimensional space-time) has proved to be quite fruitful in hydrodynamics, condensed matter physics, and also for particle physics as a theoretical laboratory. A special class of nonlinear equations admits "genuine" soliton solutions that retain their shape even after interactions and this feature distinguishes them from more generic particlelike objects. One can mention the sine-Gordon (SG) equation as a prototype example admitting genuine soliton solutions. It is of considerable interest both from a theoretical point of view and for physical applications to study the interactions of these spatially localized objects (not only of genuine solitons). There is a well-developed framework for soliton perturbations in integrable and in near-integrable systems [4,5], where many powerful methods of integrable systems are available. In generic models, however, one has to resort to various perturbation techniques and in general also to numerical simulations. The motion of a localized object when subjected to a force has been shown to be governed by Newton's law for nonrelativistic velocities [4]. It has turned out that in some cases, such as the SG model, deformation effects are also important and can lead to deviations from Newton's law [6,7].

In this paper we study the motion of particlelike solutions in $1 + 1$ dimensional scalar field theories (these objects are commonly referred to as "kinks") under the influence of an incident wave. In one of the most studied scalar models, in the "ϕ^4" theory, we have found that the kink starts to accelerate in *the direction* of the incoming wave. This effect has been first observed in Ref. [8]; in the present work we study it in much more detail both numerically and analytically. The unusual behavior of the kink can be interpreted as being caused by a "negative radiation pressure" exerted on it.

We have computed the force exerted on the kink by the radiation in a generic model in perturbation theory, assuming that the amplitude A of the incoming radiation is small. The leading order force exerted on a kink by a wave coming from the right (i.e. from $x = \infty$) is then given as $F = -A^2q^2|\mathbf{R}|^2$, where q is the wave number of the incident wave and $\mathbf{R}$ is the reflection coefficient. This means that the kink is pushed back by the radiation as expected. Now in a class of theories containing among others the ϕ^4 and the SG models, the leading order reflection coefficient is zero, $\mathbf{R} = 0$. In such cases the force is determined by higher order terms, in fact in the next order the force is $F \sim \mathcal{O}(A^4)$. The basic physical effect responsible for the negative radiation pressure can then be understood as follows. For small enough amplitudes of the incoming radiation, the kink of the ϕ^4 theory is *transparent* to the waves to first order in A. This is due to the reflectionless nature of the effective potential in the kink background. Because of the nonlinearities, during the interaction higher frequency waves are also generated. Some of the energy of the incoming wave is transferred to higher frequency (mostly double frequency) waves. These higher frequency waves carry more momentum than the incoming one. In the case of the ϕ^4-type models the amplitude of the transmitted waves with double frequency is larger than that of the reflected ones. This way a momentum surplus is created behind the kink, thus pushing it forward. For large enough amplitudes the radiation pressure becomes positive.

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Comparing the perturbative result with numerical simulations, reasonably good agreement has been found for both the nonrelativistic acceleration of the kink and for the force acting on it. Also for small enough amplitudes of the incident wave the kink starts to accelerate according to Newton's law.

There is no such negative radiation pressure exerted on the kink in the SG model, although the SG kink is also transparent to the waves. As it turns out the SG kink is transparent to all orders in A (this special feature of the SG model is due to its integrability). In fact there is an analytical solution found in Ref. [9] corresponding to the nonlinear superposition of a kink and a traveling wave, which corresponds precisely to the problem we study. We have compared our perturbative results for the SG kink with this analytical solution and the perfect agreement found serves as a good test of our approach.

In our context the negative radiation pressure appears because the kink is transparent to small amplitude incoming waves to first order in perturbation theory. Since this happens only in rather special cases, the structural stability of the effect should be addressed. We have demonstrated the robustness of the negative radiation pressure on the example of a generic perturbation of the ϕ^4 model. For an arbitrarily small perturbation of the ϕ^4 theory, to first order in the amplitude of the incoming wave, the radiation pressure exerted on the kink becomes positive, just as expected in a generic theory. The higher order contributions tend to compensate this however, and we have found that even for noninfinitesimal perturbations of the ϕ^4 model there exists a critical amplitude of the incoming wave, above which the radiation pressure becomes negative again. This robustness of the effect makes it worthwhile for further studies.

The organization of the paper is as follows. In Sec. II we introduce the models and present the first order calculation of the radiation pressure exerted by an incoming wave on the kink. In Sec. III we discuss the higher order perturbative calculations of the force and of the kink's acceleration. In Sec. IV these results are applied to the ϕ^4 and to the SG model, where an analytic formula for the force exerted on the kink is derived, and we also elucidate the physical reasons for the negative radiation pressure. In Sec. V the basic setup for the numerical simulations used is given and the analytical results are compared to those obtained by the numerical simulations. In Sec. VI the structural stability of the negative radiation pressure with respect to perturbations of the Lagrangian is studied.

Most of the computational details are relegated to three Appendices. In Appendix A and B some details of the higher order perturbative computations are given. In Appendix C we perform a suitable expansion of the analytical solution of the SG kink on a cnoidal wave pertinent to our problem.

II. FORMULATION OF THE PROBLEM

A. Models considered

We consider the following class of scalar theories in 1 + 1 space-time dimensions specified by the Lagrangian[1]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi), \tag{1}$$

where the self-interaction potential $U(\phi)$ is assumed to possess at least two degenerate minima (vacua), denoted by $U(\phi_{\rm vac})$. The equation of motion obtained from the Lagrangian (1) can be written as

$$\ddot{\phi} - \phi'' + U'(\phi) = 0. \tag{2}$$

The class of theories (1) admits static, finite energy solutions of the field equations (2). These solutions interpolate between two vacua of $U(\phi)$ and are commonly referred to as a kink or antikink. A kink (or antikink) $\phi_s(x)$ is given explicitly by the following formula:

$$x - x_0 = \pm\int_{\phi(x_0)}^{\phi(x)}\frac{d\phi}{\sqrt{2U(\phi)}}. \tag{3}$$

In this paper we shall mostly work with the ϕ^4 where $U(\phi) = (\phi^2-1)^2/2$, respectively with the sine-Gordon (SG) where $U(\phi) = 1-\cos\phi$ models. Note that in the case of the ϕ^4 model, $\phi_{\rm vac} = \phi(\pm\infty) = \pm 1$, and $\phi_{\rm vac} = \phi(\pm\infty) = 2\pi, 0$ for the SG equation.

The corresponding kink solutions are well-known:

$$\phi_s = \tanh x, \text{ in the } \phi^4, \text{ respectively, } \phi_s = 4\arctan\exp(x), \text{ in the SG model.} \tag{4}$$

These kinks are well (exponentially) localized and their position x_0 is conveniently defined by $\phi_s(x_0) = 0$ in the ϕ^4 and $\phi_s(x_0) = \pi$ in the sine-Gordon models, respectively, which also corresponds to the maximum of their energy density.

The energy-momentum tensor of the scalar field theory (1) is given as

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}, \tag{5}$$

whose components are spelled out explicitly for later convenience:

$$T_{00} = \mathcal{E} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi), \tag{6}$$

$$T_{01} = T_{10} = -\mathcal{P} = \phi'\dot{\phi}, \tag{7}$$

$$T_{11} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 - U(\phi), \tag{8}$$

where $\mathcal{E}$ and $\mathcal{P}$ are the energy and momentum densities,

[1]Our conventions are: $g_{\mu\nu} = {\rm diag}(1,-1)$, $\partial_0 f = \partial_t f = \dot{f}$, $\partial_1 f = \partial_x f = f'$.

respectively. The energy-momentum conservation laws $\partial_\mu T^\mu_\nu = 0$ can be written as

$$\partial_t \mathcal{E} = \partial_x(\phi' \dot{\phi}), \tag{9}$$

$$\partial_t \mathcal{P} = -\frac{1}{2}\partial_x(\dot{\phi}^2 + \phi'^2 - 2U(\phi)). \tag{10}$$

B. Interaction of the kink with radiation

The physical problem we wish to study is the interaction of a kink with an incoming (scalar) radiation. We assume that at least in the case when the radiation can be considered as a small perturbation, it is a reasonable approximation to treat the kink as a particle accelerating under the force coming from the radiation pressure exerted by the radiation. More precisely we consider the problem that a monochromatic wave $\xi(t, x)$ coming from the right (from $x = \infty$) is incident on an initially static kink at $x = 0$. The incident wave $\xi(t, x)$ itself reduces asymptotically to a plane wave, i.e.

$$\xi(t, x \to \infty) \to A\mathrm{Re}\{e^{i(\omega t + qx)}\}, \tag{11}$$

where A is the asymptotic amplitude. The dispersion relation between the frequency ω and the wave number q is easily read off from Eq. (2):

$$-\omega^2 + q^2 + U''(\phi_{\mathrm{vac}}) = 0. \tag{12}$$

In order to allow for a perturbative solution of the time-dependent problem, we shall assume that the amplitude of the incoming wave A is sufficiently small and expand the solution of the nonlinear wave equation (2) in power series of A:

$$\phi = \phi_s + \xi = \phi_s + A\xi^{(1)} + A^2\xi^{(2)} + \cdots. \tag{13}$$

To first order in A, the "radiation" $\xi^{(1)}$ satisfies a linear wave equation in the background of the kink,

$$\ddot{\xi}^{(1)} - \xi^{(1)''} + U''(\phi_s(x))\xi^{(1)} \equiv \ddot{\xi}^{(1)} + \hat{\mathbf{L}}\xi^{(1)} = 0, \tag{14}$$

where $U''(\phi_s(x))$ corresponds to the potential of the Schrödinger-type operator $\hat{\mathbf{L}} = -d^2/dx^2 + U''(\phi_s(x))$. This potential can be written explicitly for the ϕ^4 and for the SG models as

$$\begin{aligned} U''(\phi_s) &= 4 - \frac{6}{\cosh^2 x} \text{ in the } \phi^4, \text{ respectively,} \\ U''(\phi_s) &= 1 - \frac{2}{\cosh^2 x} \text{ in the SG model.} \end{aligned} \tag{15}$$

Introducing

$$\xi^{(1)} = \frac{1}{2}(e^{i\omega t}\eta_q(x) + e^{-i\omega t}\eta_{-q}(x)) := \mathrm{Re}\{\eta_q(x)e^{i(\omega t)}\}, \tag{16}$$

Equation (14) can be separated, where the η_q are eigenfunctions of the Schrödinger operator $\hat{\mathbf{L}}$

$$\hat{\mathbf{L}}\eta_q := \left(-\frac{d^2}{dx^2} + U''(\phi_s(x))\right)\eta_q(x) = \omega^2\eta_q(x). \tag{17}$$

As it is well-known from elementary quantum mechanics, for potentials tending to zero for $|x| \to \infty$, the asymptotic forms of the scattering eigenfunctions are

$$\eta_q(x \to +\infty) = e^{iqx} + \mathbf{R}e^{-iqx}, \tag{18}$$

$$\eta_q(x \to -\infty) = \mathbf{T}e^{iqx}, \tag{19}$$

where $\mathbf{R}$ and $\mathbf{T}$ are the reflection and transition coefficients. The first order solution $\xi^{(1)}$ corresponds to the incoming radiation field, which reduces asymptotically to a monochromatic plane wave coming from the right to the kink. Let us remind the reader at this point that the kink in any translationally invariant theory possesses a discrete eigenfunction with eigenvalue $\omega = 0$ (a zero mode), which is called the translational mode. In some cases the Schrödinger operator $\hat{\mathbf{L}}$ also possesses other discrete eigenstates for $\omega > 0$. There is one such discrete (or internal) mode in the ϕ^4 model for $\omega_d = \sqrt{3}$, and there is none in SG model.

From the energy conservation law, Eq. (9) it is easily seen that to first order the change of the total energy of the kink + radiation system in a box of size $2L$ is

$$\partial_t E = \int_{-L}^{L} dx \partial_t \mathcal{E} = A^2 \xi^{(1)'}\dot{\xi}^{(1)}|_{-L}^{L}. \tag{20}$$

Averaging in time over a period $T = 2\pi/\omega$ one finds

$$\langle \partial_t E|_{-L}^{L}\rangle_T = A^2 q\omega(1 - |\mathbf{T}|^2 - |\mathbf{R}|^2)/2,$$

where $\langle F\rangle_T$ denotes the average of the quantity F in time (over a period T).

Assuming that the size of the box is sufficiently large and that the radiation of the kink itself can be neglected, the energy contained in the box is conserved, and then it follows that

$$|\mathbf{R}|^2 + |\mathbf{T}|^2 = 1. \tag{21}$$

The rate of change of the total momentum in the box $[-L, L]$ can be identified with the total force exerted on the system of kink + radiation in its inside. To linear order in perturbation theory from Eq. (10) this force is found to be

$$\begin{aligned} \langle \partial_t P|_{-L}^{L}\rangle_T = F^{(2)} &= \frac{1}{2}A^2 q^2(-1 - |\mathbf{R}|^2 + |\mathbf{T}|^2) \\ &= -A^2 q^2 |\mathbf{R}|^2. \end{aligned} \tag{22}$$

We now show that for not too long times the kink obeys Newton's law under the action of the force given by (22). In order to define the acceleration of the kink we make the (usual) assumption that for small enough velocities one can neglect deformation and other effects and approximate the moving kink by

$$\phi(x, t) = \phi_s(x - X(t)). \quad (23)$$

This approximation corresponds to introducing simply a collective coordinate $X(t)$ for the position of the kink. Therefore for small enough displacements, $\phi(x, t) \approx \phi_s(x) - X(t)\phi_s'(x)$. In fact $\eta_t := \phi_s'(x)$ is nothing but the translational zero mode of the kink, which is orthogonal to all other linearized (internal and radiation) modes of the operator $\hat{\mathbf{L}}$. The leading order displacement of the kink is then easily obtained:

$$X(t) = -\frac{(\eta_t|\xi)}{(\eta_t|\eta_t)}, \quad (24)$$

where $(f|g)$ denotes the natural Hilbert space scalar product. The acceleration a to leading order can then be calculated as follows:

$$a^{(n)} = \ddot{X}(t) = -\frac{(\eta_t|\ddot{\xi}^{(n)})}{(\eta_t|\eta_t)} = -\frac{(\eta_t|\ddot{\xi}^{(n)} + \hat{\mathbf{L}}\xi^{(n)})}{(\eta_t|\eta_t)}, \quad (25)$$

where $\xi^{(n)}$ is the lowest order approximation, i.e. the smallest n for which the acceleration is nonzero. [Note that in Eq. (25) $\hat{\mathbf{L}}\eta_t = 0$ has been used.] To compute higher order corrections to the acceleration is nontrivial, since in our approximation the time dependence of the kink has been encoded in the single collective coordinate $X(t)$, whereas one has to take into account distortion, radiation, etc., effects. In the generic case discussed above $n = 2$. Therefore to lowest order in perturbation theory (PT) the force (22) is quadratic in the amplitude A and clearly according to our definition (25) the acceleration of the kink is also $\mathcal{O}(A^2)$. Then the second order perturbative solution $\xi^{(2)}$ is needed. A not too difficult computation [see Eqs. (A7) and (A11) in Appendix A] yields

$$\begin{aligned} m_s a^{(2)} &= -A^2(\eta_t|\ddot{\xi}_0^{(2)}) = \frac{A^2}{4}(U'''(\phi_s)\eta_q\eta_{-q}|\eta_t) \\ &= -A^2 q^2 |\mathbf{R}|^2, \end{aligned} \quad (26)$$

where the relation $(\eta_t|\eta_t) = m_s$ has been used, with m_s denoting the mass of the static kink. The method to project unto the translational mode to compute the kink's acceleration was also used in Ref. [10] where the dynamics of a kink in the ϕ^4 model with a perturbed potential was studied. One sees that to leading order in PT the acceleration of the kink $a^{(2)}$ is indeed given by Newton's law $F^{(2)} = m_s a^{(2)}$. In particular one can see that the kink is pushed back under the action of the force coming from the radiation pressure as expected. Therefore it is consistent to identify the time-averaged momentum flow in the box Eq. (22) with the *total force* acting *on the kink* to leading order. When the radiation field can be treated as a small perturbation the effect of the momentum flow on the radiation field itself can be neglected as a first approximation. One would then expect that for small enough amplitudes of the incident radiation, it is a reasonably good approximation to the solution of Eq. (2), that an initially static kink starts to accelerate as a nonrelativistic particle of mass m_s. It is natural to expect that other effects, such as the radiation by the kink, its distortion, etc., show up only in higher orders. We remind the reader that identifying the force exerted by the radiation on the kink with the total momentum flow in the box is valid only to leading order (and after time averaging), and also for relatively short time intervals. The force (22) is quadratic both in the amplitude of the incoming wave and in the reflection coefficient, in complete analogy to the well-known radiation pressure in classical electrodynamics.

It is quite illuminating to compare the prediction for the acceleration of the kink (2) by solving numerically the nonlinear wave equation (2) for a few common one dimensional field theories. On Fig. 1 we have depicted the positions of the kinks in the ϕ^4, sine-Gordon, and ϕ^8 (where $U(\phi) = \frac{1}{4}(\phi^2 - 1)^4$) models interacting with an incoming radiation from $+\infty$. It is somewhat surprising that the kink has been pushed by the radiation pressure only in the very last example, in agreement with Eq. (22). As it can be seen on Fig. 1 the time average of the acceleration of the kink in the SG model is zero; the kink is steadily oscillating around its initial position. Most remarkably the kink in the ϕ^4 model accelerates towards the source of radiation and it is this interesting effect that we interpret as *negative radiation pressure*. Taking into account other collective coordinates such as the shape mode would not substantially influence our main results, therefore we have chosen to ignore them. Note that the acceleration of the kink in the ϕ^8 model is noticeably larger as compared to the ϕ^4 one, for the same amplitude of the incoming wave ($A = 0.14$). In fact, while according to Eq. (22) the acceleration of the kink is *quadratic* in the amplitude A, this is only true for the ϕ^8 model. The

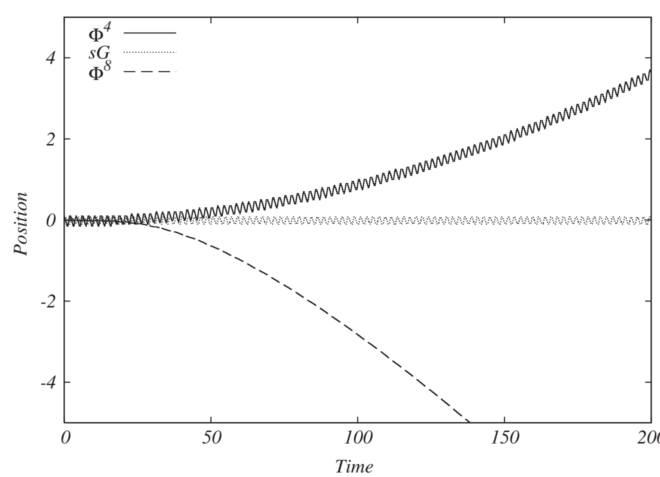

FIG. 1. Position of the kinks in the ϕ^4, sine-Gordon, and ϕ^8 models as a function of time. The position of the ϕ^8 kink was scaled by 0.1.

acceleration of the ϕ^4 kink turns out to be proportional to A^4.

Motivated by the unexpected results shown on Fig. 1 we shall investigate the interaction of a kink with radiation in more detail in order to give an explanation of the negative radiation pressure. It is immediately clear that the derivation of the force obtained in Eq. (22) is only valid if its leading contribution comes from the linear approximation. In the case when $|\mathbf{R}| \ll 1$ the validity of this assumption is questionable since then the higher order terms may contribute in an important way; therefore the first order result in Eq. (22) is not necessarily correct. As a matter of fact it is rather well-known that in both the ϕ^4 and in the SG models the potentials (15) are reflectionless, i.e. $\mathbf{R} = 0$, for all frequencies. Therefore in such models the dynamics of the kinks are determined by higher order terms.

III. COMPUTATION OF THE FORCE ON THE KINK UP TO $\mathcal{O}(A^4)$

In this section we shall outline the computation of the force acting on the kink as well as its acceleration to higher orders of perturbation theory. As we shall show below the next nontrivial correction to the force turns out to be of $\mathcal{O}(A^4)$. Fortunately one does not have to go up to fourth order in perturbation theory.

By expanding the equation of motion of the kink (2) in power series of the amplitude of the incoming wave A, corresponding to (13), the nth order solution of the equation of motion is determined by the inhomogeneous linear equations

$$\ddot{\xi}^{(n)} + \hat{\mathbf{L}}\xi^{(n)} = f^{(n)}, \tag{27}$$

where the source terms $f^{(n)}$ can be calculated from the lower order terms in the perturbation series; see Appendix A for details. To define the solution of the perturbative equations (27) uniquely, we impose in each order that there are no incoming waves from the left ($x < 0$), as boundary conditions. These correspond to the physical problem of an incoming wave from the right-hand side of the kink. We remark that to avoid resonances in higher orders, the frequency ω has also to be expanded as

$$\omega = \omega^{(0)} + A\omega^{(1)} + A^2\omega^{(2)} + \cdots. \tag{28}$$

The total force acting on the system kink + radiation inside the segment $[-L, L]$ can be calculated to higher orders in A from the energy and momentum conservation laws similarly to the leading order computation. We compute the rate of change of the total momentum inside the box $[-L, L]$ to obtain the total force acting on the system, which after averaging in time can be consistently identified to leading order with the force *acting on the kink* just as in the calculation in the previous section.

Using $\phi(x, t) = \phi_s + \xi$ in Eq. (10) and integrating over the interval $[-L, L]$, one finds

$$\partial_t P = -\frac{1}{2}\Big(\dot{\xi}^2 + \xi'^2 - U''(\phi)\xi^2 - \frac{1}{3}U'''(\phi)\xi^3 - \cdots\Big)\Big|_{-L}^{L}, \tag{29}$$

where we have omitted terms which are exponentially small for $L \gg 1$. Choosing L to be sufficiently large, one can use the asymptotic form of the higher order solutions to find the rate of change of the averaged momentum inside the segment $[-L, L]$ just as for the lowest order case. Assuming that $\partial_t P = 0$ to $\mathcal{O}(A^2)$ in Eq. (29) (reflectionless case) one finds that all terms of order $\mathcal{O}(A^3)$ also drop out, i.e. there is no momentum flow to this order into the segment (for more details see Appendix A). Therefore the first nontrivial contribution to Eq. (29) is of order $\mathcal{O}(A^4)$, implying that one would also need $\xi^{(3)}$, i.e. one should compute up to third order in perturbation theory. This complicates considerably the problem, even if only the asymptotic forms of the $\xi^{(k)}$ are needed. For some computational details of the higher order calculations we refer to Appendix A. Remarkably though one can in fact eliminate the contributions coming from the third order terms from the momentum balance in the segment $[-L, L]$ by exploiting the law of energy conservation. As it will be shown below all the information needed to calculate the force acting on the kink is actually encoded in the asymptotic form of the time-dependent part of $\xi^{(2)}$.

Denoting by $\xi_m^{(n)}$ the mth coefficient in the Fourier expansion of ξ in the $\mathcal{O}(A^n)$ order, the $\mathcal{O}(A^2)$ order solutions can be written as

$$\xi^{(2)} = e^{2i\omega t}\xi_{+2}^{(2)} + \xi_0^{(2)} + e^{-2i\omega t}\xi_{-2}^{(2)}. \tag{30}$$

We also note that for reflectionless potentials $\xi_{+2}^{(2)*} = \xi_{-2}^{(2)}$. The asymptotic form of the time-dependent part of the second order term $\xi^{(2)}$ has the form

$$\xi_{+2}^{(2)}(x \to \pm\infty) = \frac{U'''(\phi_{\mathrm{vac}})}{24U''(\phi_{\mathrm{vac}})}\eta_{\mp q}^2 + \alpha_{22,\pm k}(q)\eta_{\mp k}, \tag{31}$$

where

$$\alpha_{22,k}(q) = -\frac{1}{8W}\int_{-\infty}^{\infty} dx'\,\eta_k\eta_q^2 U'''(\phi_s), \tag{32}$$

$k = \sqrt{4\omega^2 - U''(\phi_{\mathrm{vac}})}$ is the wave number corresponding to 2ω, and $W = \eta_k\eta'_{-k} - \eta'_k\eta_{-k} = -2ik$ is the Wronskian. In a way $\alpha_{22,k}(q)$ encodes the reflection and transition coefficients due to the nonlinear effects. The details of the computation of Eq. (29) up to $\mathcal{O}(A^4)$ can be found in Appendix A, leading to the result

$$\begin{aligned}\langle\partial_t P\rangle_T &= F^{(4)} \\ &= -A^4[2k^2(|\alpha_{22,+k}^2| - |\alpha_{22,-k}^2|) \\ &\quad - 2q^2\mathrm{Re}(\alpha_{31,-q})],\end{aligned} \tag{33}$$

where the pertinent contribution from the third order terms is encoded in a single coefficient $\alpha_{31,-q}$, defined in Eq. (A25). In Eq. (33) we have identified the time average of the overall momentum $\langle\partial_t P\rangle_T$ flowing into the segment $[-L, L]$, with the force F exerted by the incoming radiation on the kink.

In the case of reflectionless potentials the first nonvanishing contribution to the acceleration comes from the $\mathcal{O}(A^4)$ terms. Considering for simplicity such theories where the kink is spatially antisymmetric (this includes both ϕ^4 and the SG models) a straightforward computation yields

$$\begin{aligned} m_s a^{(4)} &= -A^4(\ddot{\xi}_0^{(4)}|\eta_t) \\ &= \mathrm{Re}(U'''(\phi_s)[\xi_1^{(3)}\eta_{-q} + \xi_2^{(2)}\xi_{-2}^{(2)}] \\ &\quad + U^{(\mathrm{iv})}(\phi_s)\xi_2^{(2)}\eta_{-q}^2/4|\eta_t), \end{aligned} \tag{34}$$

i.e. it is sufficient to compute the second and third order solutions (see Appendix A for more details). By a direct computation we have checked that with the definition of the force in Eq. (33) acting on the kink within our perturbative framework, Newton's law

$$F^{(4)} = m_s a^{(4)} \tag{35}$$

still holds, at least up to the fourth order $\mathcal{O}(A^4)$. In our view this result lends strong support to identify the time average of the momentum flow in the segment $[-L, L]$ with the force acting on the kink to leading order in PT.

We give here a derivation of Newton's law, which also indicates the limits of its validity. Using the following simple identity

$$\ddot{\xi}\phi_s' = \partial_t(\dot{\phi}\phi' - \dot{\xi}\xi') = -\partial_t(\mathcal{P} + \dot{\xi}\xi'), \tag{36}$$

by integrating over a segment $[-L, L]$ one easily obtains the relation

$$\int_{-L}^{L} dx\ddot{\xi}\eta_t = -\dot{P}_L - \int_{-L}^{L} dx\partial_t(\dot{\xi}\xi'), \tag{37}$$

where $\dot{P}_L$ denotes the momentum change inside the segment $[-L, L]$. We shall approximate the left-hand side of Eq. (37) simply by $(\ddot{\xi}|\eta_t) := -m_s a$ since the difference between them is exponentially small in L. After averaging in time we obtain

$$F_L = m_s a - \int_{-L}^{L} dx\langle\partial_t(\dot{\xi}\xi')\rangle_T, \tag{38}$$

where F_L denotes the total force acting on the box. Now in perturbation theory the solution ξ can be decomposed as

$$\xi = \xi_p(t, x) + \xi_0(x) - \frac{1}{2}at^2\xi_t(x, t), \tag{39}$$

where $\xi_p(t, x)$ is periodic in time and it is at least of order $\mathcal{O}(A)$, ξ_t corresponds to the "accelerating part" (with initially constant acceleration), and $\xi_0(x)$ is the time-independent part. This holds to order $\mathcal{O}(A^2)$ when $a^{(2)} \neq 0$, respectively to order $\mathcal{O}(A^4)$ when $a^{(2)} = 0$,

$$\begin{aligned} \xi_p &= A\xi^{(1)} + A^2\xi^{(2)} + \mathcal{O}(A^3), \\ \xi_0(x) &= A^2\xi_0^{(2)}(x) + \mathcal{O}(A^4), \\ \xi_t &= A^n\eta_t(x) + \mathcal{O}(A^{n+2}), \end{aligned} \tag{40}$$

where $n = 2$ if $a^{(2)} \neq 0$, and $n = 4$ if $a^{(2)} = 0$. The leading order correction is $\propto aA^{n+1}$, which depends on time averaging and indicates the limits of the validity of our simple approach.

Let us note here that the mass of the kink gets renormalized due to its interaction with the radiation field. A standard calculation for the first correction to the kink mass gives $m^* = m_s + A^2\delta m^{(2)}\cdots$, i.e. to lowest nontrivial order δm is proportional to A^2, therefore to $\mathcal{O}(A^4)$ it does not show up in Eq. (35). Nevertheless the numerical simulations (see Sec. V) indicate that the effective mass of the kink is quite close to the renormalized mass m^*.

The computation of the rate of change of the energy inside the segment is completely analogous to the previous momentum balance calculation and we find

$$\langle\partial_t E\rangle = -A^4[4\omega k(|\alpha_{22,k}^2| + |\alpha_{22,-k}^2|) + 2\omega q\mathrm{Re}(\alpha_{31,-q})]. \tag{41}$$

Assuming that after averaging, at least for some initial time the kink can be considered as a rigidly accelerating particle, if it was initially at rest, i.e. $v(t = 0) = 0$ then obviously

$$\langle\partial_t E\rangle|_{t=0} = mv\dot{v}|_{t=0} = 0. \tag{42}$$

This equation together with Eq. (41) can be now used to eliminate the coefficient $\alpha_{31,-q}$ from Eq. (33), and then one obtains a remarkably simple formula determining the force acting on the kink:

$$F^{(4)} = 2A^4k[(k - 2q)|\alpha_{22,-k}^2| - (k + 2q)|\alpha_{22,k}^2|]. \tag{43}$$

By a direct computation of the energy we have verified that $\langle\partial_t E\rangle|_{t=0}$ is indeed zero up to $\mathcal{O}(A^6)$, which shows the validity of Eq. (42).

To conclude this section, we have calculated the force exerted by an incoming wave on the kink to the first nontrivial order in perturbation theory in the class of models where the linearization around the kink yields a reflectionless potential. It is important to emphasize that for the class of models where the effective potential is reflectionless, the force exerted on the kink turns out to be proportional to $F \sim \mathcal{O}(A^4)$. This is to be contrasted to more generic models where the effective potential is reflective, in which case $F \sim \mathcal{O}(A^2)$.

IV. NEGATIVE RADIATION PRESSURE IN THE ϕ^4 MODEL

In this section we apply the previously obtained general results to compute explicitly the force exerted on the kink by an incoming wave in the ϕ^4 and in the SG models.

In the ϕ^4 model the full nonlinear equation for the "radiation" ξ is

$$\ddot{\xi} + \hat{\mathbf{L}}\xi + 6\phi_s\xi^2 + 2\xi^3 = 0. \tag{44}$$

The first order solution $\xi^{(1)}$ in Eq. (16) can be explicitly given both in the ϕ^4 and in the SG models as

$$\eta_q^{\phi^4} = \frac{3\tanh^2 x - 1 - q^2 - 3iq\tanh x}{\sqrt{(q^2+1)(q^2+4)}} e^{iqx}, \quad \text{where } q^2 + 4 = \omega^2, \tag{45}$$

and

$$\eta_q^{\rm sG} = \frac{iq - \tanh x}{\sqrt{q^2+1}} e^{iqx}, \quad \text{where } q^2 + 1 = \omega^2. \tag{46}$$

The second order solution $[\mathcal{O}(A^2)]$ given by Eq. (30) contains the zero frequency term $\xi_0^{(2)}$, which in general depends both on t and x. In the present case it is consistent to assume that $\xi_0^{(2)}$ is time-independent (see Appendix A for a proof), moreover it can be written explicitly. The second order "transition and reflection" coefficients $\alpha_{22,k}(q)$ in Eq. (32) can also be calculated analytically and the result is

$$\alpha_{22,k}(q) = -\frac{3}{2}\pi\frac{q^2+4}{q^2+1}\sqrt{\frac{q^2+4}{k^2+1}}\frac{1}{k\sinh(\frac{2q+k}{2}\pi)}, \tag{47}$$

where $q^2 = \omega^2 - 4$, $k^2 = 4(\omega^2 - 1)$. Using Eq. (47) it is now easy to find the averaged force (43) exerted on the ϕ^4 kink by an incident wave of frequency ω; we obtain

$$F^{(4)} = \frac{9\pi^2 A^4\omega^6}{k(4\omega^2-3)(\omega^2-3)^2}\left[\frac{\omega_-}{\sinh^2\pi\omega_-} - \frac{\omega_+}{\sinh^2\pi\omega_+}\right] \quad \text{with } \omega_\pm := \sqrt{\omega^2-1} \pm \sqrt{\omega^2-4}. \tag{48}$$

Introducing $f^{(4)} = F^{(4)}/A^4$, the behavior of the function $f^{(4)}(\omega)$ for $\omega \to 2$ (i.e. small values of q) is given as $f^{(4)}(\omega \to 2) \approx 0.3749\sqrt{\omega - 2}$ while $f^{(4)}(\omega) \to 3/4$ for large values of ω. Quite interestingly the force acting on the kink is *positive*, therefore it accelerates towards the source of radiation. This is the effect we refer to as *negative radiation pressure*. The origin of the negative radiation pressure can be understood by noticing that for all frequencies ω the amplitude of the (nonlinearly) reflected wave $|\alpha_{22,+k}|$ is smaller than the amplitude of the transmitted wave $|\alpha_{22,-k}|$, i.e. $\alpha_{22,k}^2 \ll \alpha_{22,-k}^2$. In first order perturbation theory, such an effect would not be possible because of the identity (21) expressing energy conservation at the linear level.

The surprising effect of negative radiation pressure on the kink exists only because of the presence of nonlinearities. In the linear approximation the kink is transparent to the incident wave, therefore it does not accelerate. Because of the nonlinear terms, part of the energy of the incoming wave is transformed into a wave whose frequency is twice that of the original one. This double frequency wave has a larger ratio of momentum to energy density than the incident wave with smaller frequency, hence it carries more momentum than the originally incident one. This way a surplus of momentum is created behind the kink, which pushes it towards the direction of the incoming wave. The above is of course only an intuitive explanation of the negative radiation pressure on the kink. The effect of negative radiation pressure has been clearly observed in our numerical simulations of the ϕ^4 model (see Sec. V).

Next we compute the acceleration of the ϕ^4 kink using Eq. (34) derived in the previous section. We need to compute in fact the projection of the second and third order solutions on the translational mode η_t. We have computed them numerically by two different methods, using the integral representation based on the explicitly known Green's function and also by direct numerical integration of the corresponding equations (27). We show separately the three projections in (34) on Fig. 2 (divided by $(\eta_t|\eta_t) = 4/3$). It is worthwhile to point out that all the three projections are positive.

Let us now turn to the sine-Gordon kink and evaluate the force of order $\mathcal{O}(A^4)$. In the case of the sine-Gordon model our numerical results did not show any net radiation pressure, the SG kink was oscillating around its initial position (cf. Figure 1). Interestingly in the SG model the second order waveform can be calculated in closed form

$$\xi_{+2}^{(2)} = \frac{-iq + \tanh(x)}{16(1+q^2)\cosh(x)} e^{2iqx}, \tag{49}$$

from which one can immediately see that the coefficients

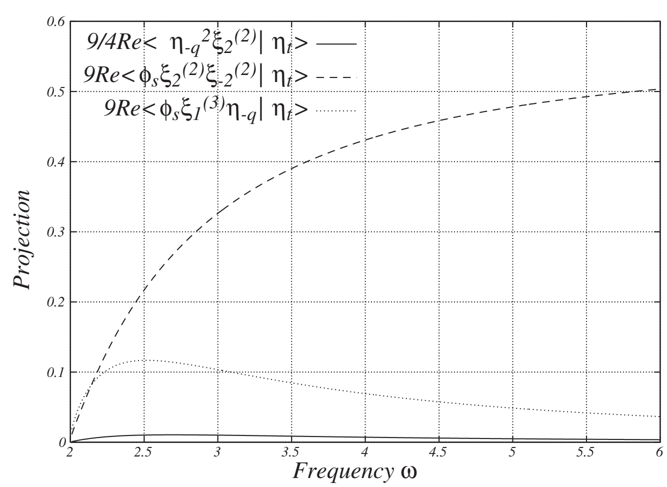

FIG. 2. Three projections obtained in second and third order in PT [used in Eq. (34)].

$\alpha_{22,\pm k}$ determined by the asymptotic behavior of $\xi^{(2)}$ in Eq. (31) are zero. This implies that the radiation exerts no force at all on SG kink at least up to this order. Clearly this interesting fact should be related to the special feature of the SG model, namely, its integrability. As a matter of fact there is a remarkable analytic solution corresponding to the nonlinear superposition of a kink with a travelling (cnoidal) wave in the SG model obtained by Shin [9] using the Darboux transformation method. In Appendix C we give a short review of Shin's solution which is somewhat complicated, and we demonstrate that one can expand it in a parameter which can be identified with the asymptotic amplitude of the incoming wave. This way we have verified that both the first and the second order solutions obtained by our perturbative calculations agree perfectly with the Taylor expansion of the analytical solution. This comparison has also served as a test of the validity of our perturbative method.

V. NUMERICAL SIMULATION

In the present section we outline the numerical method used to solve the nonlinear wave equation (2) describing the interaction of a kink with an incident wave. We shall present the results of the numerical simulations for the ϕ^4 theory in the form of figures and tables.

We have discretized Eq. (2) in the spatial variable x as $\phi(nh, t) := \phi_n(t)$. The second derivative was approximated using the following five point scheme:

$$\phi'' \equiv D\phi_n = \frac{1}{12h^2}(-\phi_{n-2} + 16\phi_{n-1} - 30\phi_n + 16\phi_{n+1} - \phi_{n+2}) + \mathcal{O}(h^4). \tag{50}$$

This way Eq. (2) reduced to a system of ordinary differential equations:

$$\ddot{\phi}_n = D\phi_n - U'(\phi_n). \tag{51}$$

We have simply put this coupled infinite system into a finite box of size $2L$, which was then solved using a standard fourth order Runge-Kutta method.

Our initial conditions have been chosen to correspond to a kink together with a first order travelling wave

$$\phi(x, t = 0) = \phi_s(x) + \frac{1}{2}A\eta_q(x) + \text{c.c.}, \tag{52}$$

$$\dot{\phi}(x, t = 0) = \frac{1}{2}i\omega A\eta_q(x) + \text{c.c.}, \tag{53}$$

and we have fixed the boundary values of $\phi(x, t)$ at $x = \pm L$ as

$$\phi(x = \pm L, t) = \pm 1. \tag{54}$$

The evolution time of the system was restricted to be smaller than L, to avoid the unphysical influence of the reflected waves from the artificial boundaries at $x = \pm L$ on the kink's motion. The position of a static kink in the ϕ^4 theory can be quite unambiguously identified with the location of its zero which coincides with the maximum of its energy density. It is less clear how to define the position of an interacting kink. In our case one has to separate first the field of the kink from that of the radiation, which can already be problematic and the position of the kink is not very well defined; in general the maximum of the energy density and the zero of $\phi(x, t)$ do not coincide. For small enough amplitudes the kink is only slightly perturbed and therefore its topological zero is still a rather satisfactory definition as the position of the kink and we have used this definition in our work.

We have plotted the position of the zero of $\phi(x, t)$ as a function of time for the frequency $\omega = 3.0$ and for the amplitude of the wave $A = 0.12$ on Fig. 3. On this figure one can clearly see that the trajectory of the zero of $\phi(x, t)$ is quite close to a parabola, corresponding to the expected nonrelativistic acceleration of the kink. A numerical fit confirms that a parabola of the form $at^2/2$ to the trajectory is a good approximation indeed. The fitted value of the acceleration in this case was $a_{\text{num}} = 9.72 \times 10^{-5}$ which is not that far from the result of our analytical calculations in Eq. (34), giving $a_{\text{theor}} = 9.02 \times 10^{-5}$. Taking into account that in the analytical calculation only the leading terms have been used, this agreement appears to us satisfying. Next we have checked if the measured acceleration is indeed $\mathcal{O}(A^4)$ as predicted by the leading order perturbative result (34). On Fig. 4 we have plotted the fitted acceleration for $\omega = 3.0$ divided by A^4 for the amplitudes of incoming wave varying between $0.1 \le A \le 0.3$. As one can see for $0.1 \le A \le 0.22$ the curve is close to being flat implying that the dominant term is indeed proportional to A^4. This proportionality breaks down when the value of the amplitude increases to about $A \approx 0.24$. For $A = 0.25$ even the sign of the fitted acceleration changes. In Table I we have

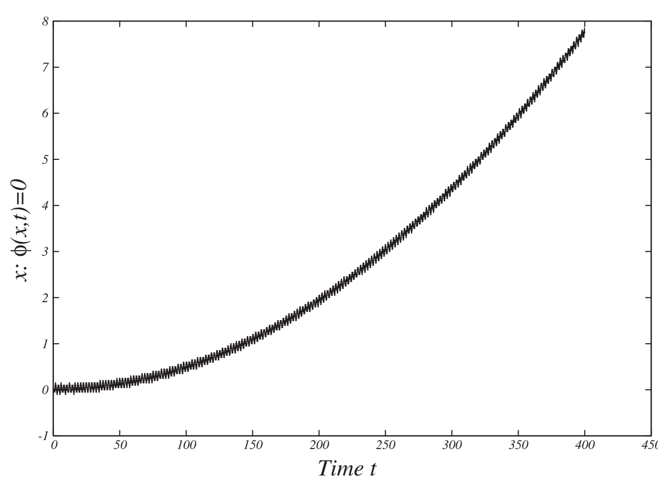

FIG. 3. The position of the zero of $\phi(x, t)$ in ϕ^4 theory as a function of time for $A = 0.12$, $\omega = 3.0$.

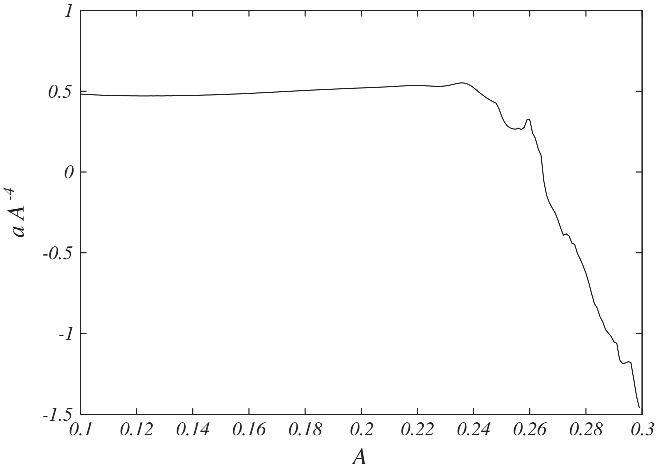

FIG. 4. Fitted acceleration divided by A^4 for $\omega = 3.0$ as a function of A.

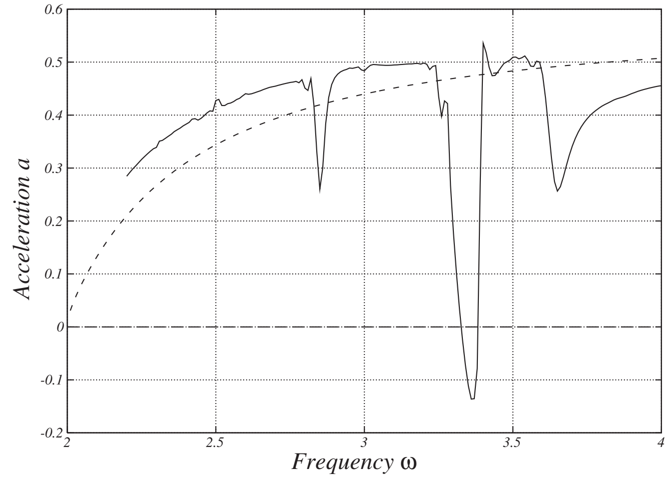

FIG. 5. Fitted acceleration divided by A^4 for $A = 0.16$ (solid line) and the acceleration calculated analytically (dashed line).

compared the numerically obtained values of the acceleration to the theoretical ones for a range of amplitudes.

From Table I one can see that the agreement between the calculated and the fitted values of the acceleration is reasonably good up to values of $A < 0.22$. These results confirm that for amplitudes of the incoming wave in the range $0.1 \leq A \leq 0.22$ the kink accelerates nonrelativistically, and also that its acceleration scales as A^4. Next we exhibit the numerically obtained acceleration in function of the frequency ω on Fig. 5, together with the theoretical curve, and some results are given in Table II.

The first thing one might notice on Fig. 5 is the presence of three resonancelike structures completely absent from the theoretical curve which is a monotonously increasing function of ω. The largest resonance is not very far from $2\omega_d \approx 3.46$ which indicates that it is likely to be related to the coupling between internal (or shape) mode of the kink and radiation. A plausible explanation of the important change in the acceleration at frequencies when the shape mode couples strongly to the incoming wave is the following. At such a "resonance" frequency the shape mode accumulates a substantial amount of energy, which is then radiated symmetrically in both directions. Far from the kink this produces the same effect as a reflected wave, thus at such resonant frequencies the kink is not transparent.

On Fig. 6 the path of the kink is plotted for a value of the frequency near the resonance. As one can see the motion of the kink is somewhat irregular there. All in all the discrepancy between the results of the perturbative computations to leading nontrivial order and those of the numerical simulation does not exceed 10% for a large range of frequencies with the exception of three resonances. It seems to us that this agreement is satisfactory in view of the approximations used.

As it can be seen from Table I, the numerically found acceleration is systematically larger than the leading order theoretical one. Clearly higher order effects could play a role here, and the simplest one to be taken into account is the renormalization of the kink mass due to the radiation field. The lowest order $\mathcal{O}(A^2)$ contribution to the mass $\delta m^{(2)}$ is negative for all frequencies, i.e. the effective mass $m^* = m + A^2\delta m^{(2)} < m$ which goes into the right direction. It is quite difficult to obtain a sufficiently precise numerical value for the effective mass; nevertheless we have obtained some indicative results. The numerically

TABLE I. Fitted and theoretical values of the acceleration for $\omega = 3.0$.

Amplitude A	Fitted acceleration	Theoretical value
0.10	0.000 048 2	0.000 043 54
0.12	0.000 097 7	0.000 090 29
0.16	0.000 318 8	0.000 285 37
0.18	0.000 529 6	0.000 457 10
0.20	0.000 832 5	0.000 696 70
0.22	0.001 254 1	0.001 020 05
0.24	0.001 730 0	0.001 444 69
0.26	0.001 480 6	0.001 989 86

TABLE II. Fitted acceleration divided by A^4 for $A = 0.16$.

Frequency ω	Fitted acceleration aA^{-4}	Theoretical value
2.50	0.4285	0.339 743
2.70	0.4472	0.388 489
2.90	0.4397	0.421 959
3.10	0.4865	0.446 130
3.30	0.3510	0.464 260
3.50	0.4761	0.478 267
3.70	0.3310	0.489 349
3.90	0.4322	0.498 290
4.30	0.4672	0.505 624

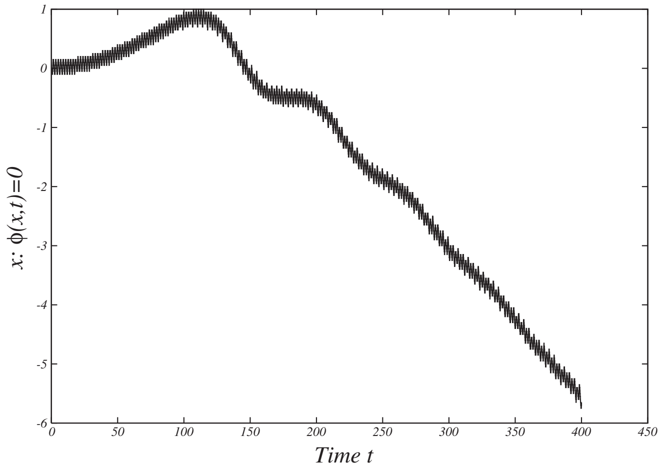

FIG. 6. The position of the zero of ϕ as a function of time for $\omega = 3.38$ and $A = 0.14$.

computed momentum balance (force) inside a box $[-20, 20]$ is presented on Fig. 7. Dividing this force by the measured acceleration we obtain the effective mass m^*. The mass obtained in this case is $m^*_{\rm num} = 0.954 m_s$. An analytical calculation yields $-\delta m^{(2)} = 3A^2(\omega^2 - 2)/(\omega^2 - 3)$, which gives $m^*_{\rm theor} = 0.962 m_s$, so there is a reasonable degree of agreement between the two.

In conclusion the numerical results show that our perturbative calculations are quite reliable for amplitudes $A < 0.2$ and for frequencies far from resonance points. Finally on Fig. 8 the behavior of the acceleration divided by A^4 of the ϕ^4 kink for a range of amplitudes and frequencies of the incoming wave is depicted. From this figure it can be seen that the acceleration stays positive over an impressively large portion of the (A, ω) plane.

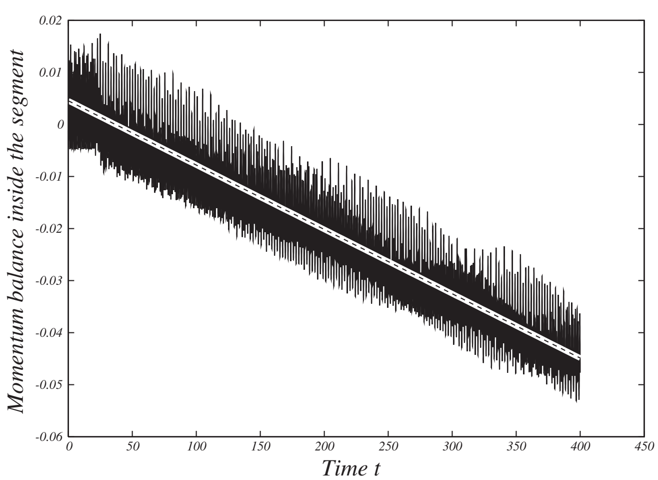

FIG. 7. Momentum balance inside the segment $[-20, 20]$ for $\omega = 3.0$ and $A = 0.12$. The slope of the fitted straight line (corresponding to a sort of averaging) is -1.24×10^{-5}, corresponding to an effective mass $m^* = 0.954m$.

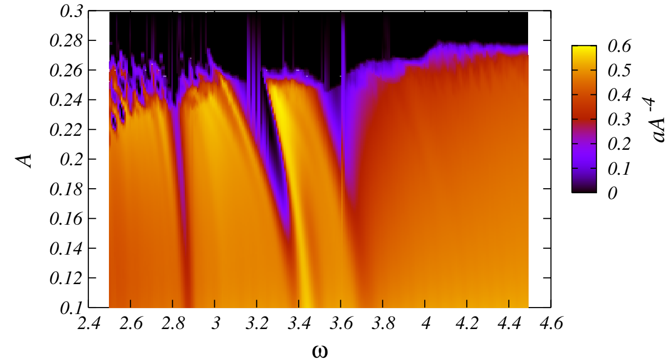

FIG. 8 (color online). Acceleration aA^{-4} as a function of the amplitude and of the frequency of incoming wave.

VI. STABILITY OF THE EFFECT UNDER PERTURBATIONS

In this section we shall demonstrate that the effect of negative radiation pressure has a certain degree of robustness with respect to perturbations of the original ϕ^4 model. This fact makes the effect, which is in itself interesting, much more relevant for physical applications. At first sight it is not so obvious that this effect could survive a small perturbation of the model at all, since a generic perturbation, no matter how small it be, destroys the reflectionless nature of the potential in Eq. (14). This way a first order perturbative contribution (in the amplitude A) is generated. Therefore the leading term in the expression for the force changes under the influence of a generic perturbation from being of order $\mathcal{O}(A^4)$ in Eq. (43) to $\mathcal{O}(A^2)$ as in Eq. (22). We shall show that although a small but generic perturbation of the ϕ^4 model changes the leading term for the force to being of order $\mathcal{O}(A^2)$; indeed, for an important frequency range still the $\mathcal{O}(A^4)$ term will dominate if the amplitude is larger than a critical (minimal) value $A > A_{\rm crit}$. We shall consider a concrete example of perturbation which illustrates that the critical amplitude $A_{\rm crit}$ can turn out to be small and that the negative radiation pressure stays practically unaffected.

To start with we shall consider a generic perturbation of the field equation (2) of the form

$$\ddot{\phi} - \phi'' + U'(\phi) + \epsilon \delta U'(\phi) = 0, \qquad (55)$$

where ϵ is a small parameter and $\delta U'(\phi)$ is the perturbation. We look for the solution of the perturbed Eq. (55) again expanded in a power series in the amplitude of the incoming wave,

$$\phi = \phi_s(x) + A\xi^{(1)}(x, t) + \cdots, \qquad (56)$$

where $\phi_s(x)$ denotes the static kink solution of the perturbed field equation (55) and $\xi^{(1)}(x, t)$ satisfies the field equation linearized around $\phi_s(x)$,

$$\ddot{\xi}^{(1)} - \xi^{(1)\prime\prime} + U''(\phi_s)\xi^{(1)} + \epsilon \delta U''(\phi_s)\xi^{(1)} = 0. \qquad (57)$$

Now we also expand the solution of Eq. (55) in ϵ determining the perturbation of the original theory

$$\phi_s(x) = \phi_s^{(0)} + \epsilon\phi_s^{(1)} + \cdots, \quad \xi^{(1)} = \xi^{(10)} + \epsilon\xi^{(11)} + \cdots, \tag{58}$$

where $\phi_s^{(0)}(x)$ is the static kink of the unperturbed Eq. (2), and $\xi^{(10)}(x, t)$ is a solution of the linearization of the unperturbed field equation around $\phi_s^{(0)}(x)$, i.e. $\ddot{\xi}^{(10)} + \hat{\mathbf{L}}\xi^{(10)} = 0$. The equations for the first order corrections in the perturbative parameter ϵ are

$$-\phi_s^{(1)\prime\prime} + U''(\phi_s^{(0)})\phi_s^{(1)} + \delta U'(\phi_s^{(0)}) = 0, \tag{59}$$

$$\ddot{\xi}^{(11)} - \xi^{(11)\prime\prime} + U''(\phi_s^{(0)})\xi^{(11)} + U'''(\phi_s^{(0)})\phi_s^{(1)}\xi^{(10)} + \delta U''(\phi_s^{(0)})\xi^{(10)} = 0. \tag{60}$$

Let us recall that $\xi^{(10)}(x, t) = \frac{1}{2}e^{i\omega t}\eta_q^{(0)} + \text{c.c.}$, and look for the solution as $\xi^{(11)} = \frac{1}{2}e^{i\omega t}\eta_q^{(1)} + \text{c.c.}$, where $\eta_q^{(1)}(x)$ satisfies the following equation:

$$(\hat{\mathbf{L}} - \omega^2)\eta_q^{(1)} + [\delta U''(\phi_s^{(0)}) + U'''(\phi_s^{(0)})\phi_s^{(1)}]\eta_q^{(0)} = 0. \tag{61}$$

Since Eq. (61) is an inhomogeneous equation of the form of Eq. (A13) its general solution can be obtained from Eq. (A14). In order to obtain the force we need the reflection coefficient $\mathbf{R}$, therefore it is sufficient to compute the asymptotic behavior of $\eta_q^{(1)}$ for large $|x|$, which is found to be given as

$$\eta_q^{(1)}(x \to +\infty) = -\frac{\eta_{-q}^{(0)}}{W}\int_{-\infty}^{\infty} dx[\delta U''(\phi_s^{(0)}) + U'''(\phi_s^{(0)})\phi_s^{(1)}]\eta_q^{(0)^2} := \beta_R^{(1)}\eta_{-q}^{(0)}, \tag{62}$$

$$\eta_q^{(1)}(x \to -\infty) = -\frac{\eta_{+q}^{(0)}}{W}\int_{-\infty}^{\infty} dx[\delta U''(\phi_s^{(0)}) + U'''(\phi_s^{(0)})\phi_s^{(1)}]\eta_q^{(0)}\eta_{-q}^{(0)} := \beta_T^{(1)}\eta_q^{(0)}. \tag{63}$$

The expressions $\epsilon\beta_R^{(1)}$ and $\epsilon\beta_T^{(1)}$ are the first nontrivial corrections to the reflection ($\mathbf{R}$) and transition ($\mathbf{T}$) coefficients. Recall that in the reflectionless case, such as in the ϕ^4 model, $\mathbf{R} = 0$ and $|\mathbf{T}| = 1$. From Eq. (62) it immediately follows that in the perturbed field equation (55) the kink is not transparent anymore. The leading contribution to the force is then determined by the first order linear term in the amplitude A. As found in Eq. (22) the dominant part of the force acting on the kink $F^{(2)}$ is proportional to the square of the reflection coefficient, i.e.

$$F^{(2)} := A^2 f^{(2)} = -q^2|\mathbf{R}^2|A^2 \approx -q^2\epsilon^2|\beta_R^{(1)}|^2 A^2. \tag{64}$$

The above equation holds if $|\epsilon\beta_R^{(1)}|, |\epsilon\beta_R^{(1)}| \ll 1$, which is true for sufficiently small values of ϵ and for a certain range of q. The first perturbative correction in ϵ to the force of order $\mathcal{O}(A^4)$ in the amplitude $F^{(4)} \equiv A^4 f^{(4)}$ [cf. Eq. (43)] will be at least of order $\mathcal{O}(A^4\epsilon^2)$ therefore this term can be neglected in the following. The leading contribution to the force acting on the kink due to a perturbation of a theory with reflectionless potential is given as

$$F = A^2(-q^2\epsilon^2|\beta_R^{(1)}|^2 + A^2 f^{(4)}). \tag{65}$$

Assuming that $f^{(4)} > 0$ (i.e. that the radiation pressure is negative in the unperturbed model) it follows from Eq. (65) that the amplitude of the incoming wave must be larger than a critical value $A > A_{\text{crit}}$ for a fixed value of ϵ to ensure $F > 0$, i.e. that the effect of the negative radiation pressure be present. The value of the critical amplitude is determined by the condition $F^{(2)} + F^{(4)} = 0$, leading to

$$A_{\text{crit}} = \frac{\epsilon q|\beta_R^{(1)}|}{\sqrt{f^{(4)}}}. \tag{66}$$

Clearly the result in (66) is meaningful in our perturbative framework, only if $A_{\text{crit}} \ll 1$.

We now apply the above general results to the ϕ^4 theory [where $U'(\phi) = 2\phi(\phi^2 - 1)$]. We have chosen the following perturbation for $\delta U'(\phi)$:

$$\delta U'(\phi) = \phi(\phi^2 - 1)^2. \tag{67}$$

Recall that $\phi_s^{(0)}(x)$ is nothing but the static kink in the unperturbed ϕ^4 theory, so

$$\phi_s^{(0)}(x) = \tanh x. \tag{68}$$

The first order correction in ϵ to the static kink is determined by Eq. (59), which can be analytically solved

$$\phi_s^{(1)}(x) = -\frac{1}{6}\frac{\tanh x}{\cosh^2 x}. \tag{69}$$

One can also evaluate the integral (62) analytically, and the first order result in $\mathcal{O}(\epsilon)$ for the reflection coefficient is given as

$$\beta_R^{(1)} = \frac{2\pi i(4 + q^2)}{15\sinh\pi q}. \tag{70}$$

Evaluating the critical amplitude in Eq. (66) for some values of the perturbation parameter ϵ, one finds for $\epsilon = 0.1$ and $q = 1.2$ (corresponding to $\omega \approx 2.33$) $A_{\text{crit}} \approx 0.021$ and $A_{\text{crit}} \approx 0.041$ for $\epsilon = 0.2$. For increasing values of the frequency ω, $\beta_R^{(1)} \to 0$ exponentially fast therefore A_{crit} becomes very small.

In order to confirm the existence of the critical amplitude and compare its magnitude with the one found in Eq. (66), we have performed some numerical simulations on the perturbed ϕ^4 model (67). It is not an easy numerical task to measure the critical amplitude since the acceleration tends to be very small for $A \sim A_{\text{crit}}$ and therefore long-time evolution with large spatial resolution is needed. The value of the perturbation parameter ϵ cannot be chosen to be too small either since then $A_{\rm crit}$ becomes so small that we cannot measure it. Also the initial conditions used in the simulations of the perturbed ϕ^4 theory gave some nonzero contribution to the initial velocity of the kink. The measured values of the accelerations were as small as 10^{-9}, 10^{-8}. For $\epsilon = 0.1$ and $\omega = 2.33$ we have found that $A_{\rm crit} = 0.0168 \pm 0.0001$, and $A_{\rm crit} = 0.0292 \pm 0.0002$ for $\epsilon = 0.2$. The measured values for $A_{\rm crit}$ do not agree very precisely with the prediction of Eq. (66), the discrepancies being about 20% and 30% for $\epsilon = 0.1$ and $\epsilon = 0.2$ respectively. Nevertheless we consider the numerically found values to be consistent with Eq. (66) for the following reasons. $A_{\rm crit}/\epsilon$ still varies considerably (by $\sim$13%) for the two considered values of ϵ, indicating that higher order corrections are still important here. For the smaller value of ϵ the discrepancy between the measured value and the result of Eq. (66) is also smaller. In any case we have been able to demonstrate the existence of a critical amplitude in the perturbed ϕ^4 model above which the radiation pressure becomes negative. The measured value of $A_{\rm crit}$ is consistent with the theoretical estimate in Eq. (66). Moreover the negative radiation pressure persists for rather large values of the perturbation parameter ϵ. For example in our numerical simulations for $A = 0.1$ and $\omega = 3.0$ the radiation pressure became positive only for either $\epsilon > 1.3$ or $\epsilon < -1.2$. We have shown that for a rather large range in the magnitude of a generic perturbation of the ϕ^4 model, the phenomenon of negative radiation pressure persists.

Finally coming to the perturbation of SG model, we have shown that to order $\mathcal{O}(A^4)$ the force is zero. In fact using the analytic solution of Shin [9] we conclude that the force is zero to all orders. Therefore our derivation for the critical amplitude does not apply to the particular case of the SG model. We have observed that the kink only oscillates around its initial position, and the average of its velocity is zero. This implies that even a small perturbation of the SG model may change this qualitative behavior and determine the motion of the kink.

VII. CONCLUSIONS

We have studied the interaction of a kink in 1 + 1 dimensional scalar models with an incoming wave in perturbation theory. We have shown that in a certain class of theories (such as the ϕ^4 model), the kink is *pulled* towards the direction of the incident radiation, instead of being pushed back. This interesting phenomenon constitutes an interesting example of negative radiation pressure, which in this case is due to the nonlinearities and to higher order effects. Comparing the results of the perturbative calculations to numerical simulations in various field theoretical models (mostly in ϕ^4 and in SG) a rather good agreement was found for not too large values of the amplitude of the incoming wave (up to $A \approx 0.2$). We have also addressed the important problem of structural stability of the negative radiation pressure with respect to generic perturbations of the theory. In models where the reflection coefficient is small we have established the existence of a critical amplitude above which the kink experiences negative radiation pressure. This is closely related to the robustness of the effect which has been demonstrated on the example of a perturbation of the ϕ^4 model. We have found in fact that even for large perturbation there exists a critical amplitude of the incoming wave above which the radiation pressure becomes negative.

In the SG model the radiation pressure turns out to be zero. In this model there is an analytical solution corresponding to the superposition of a kink and an incoming (cnoidal) wave [9], and we could confirm the correctness of our perturbative results by comparing them to the expansion of the exact solution.

We have also shown that under the action of the averaged force exerted by the radiation the kinks accelerate in all these models according to Newton's law.

It is clearly an important open question if the effect of negative radiation pressure is also present in other, in particular, higher dimensional theories. Our preliminary results suggest that this phenomenon is also present at least in two other models of quite some physical interest: in the 2 + 1 dimensional complex ϕ^4 theory (Goldstone's model) and in the Abelian-Higgs model admitting vortices. This suggests that this effect might not be so rare as one could have expected at first sight.

The phenomenon of negative radiation pressure is relevant not only for the interaction of a single kink with radiation but also for a system of many kinks. Since the interaction of well-localized kinks is rather weak (e.g. for two ϕ^4 kinks at a distance L the force between them is $F \sim e^{-\alpha L}$), the radiation pressure can play an important role in many kink systems. One would expect, for example, that in a dilute many-kink system the negative radiation pressure might lead to attractive interactions which can dominate for certain separations.

It would clearly be interesting to investigate this effect also for domain walls (or higher dimensional branes).

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APPENDIX A: DETAILS OF THE CALCULATION OF HIGHER ORDER PERTURBATIONS

In this appendix we shall present some of the details of the second and third order perturbative calculations necessary to find the force acting on the kink and its acceleration.

All equations arising in perturbation theory are second order linear inhomogeneous partial differential equations of the form

$$\ddot{\xi}^{(n)} + \hat{\mathbf{L}}\xi^{(n)} = f^{(n)}(U^{(k)}(\phi_s), \xi^{(l)}), \qquad \text{(A1)}$$

where the inhomogeneous term must be computed from the solutions obtained in order lower than $\mathcal{O}(A^n)$. To avoid presenting too complicated general formulae, we start by writing out explicitly the second order equations. They are obtained by simply substituting the perturbative expansion for the field $\phi(x, t)$ Eq. (13) and for the frequency (28) into the equation of motion (2)

$$\ddot{\xi}^{(2)} - 2\omega^{(0)}\omega^{(1)}\xi^{(1)} + \hat{\mathbf{L}}\xi^{(2)} = -\frac{1}{8}U'''(\phi_s)(\eta_q^2 e^{2i\omega t} - 2\eta_q\eta_{-q} - \eta_{-q}^2 e^{-2i\omega t}). \qquad \text{(A2)}$$

On the right-hand side of Eq. (A2) there are two source terms oscillating with frequency $\pm 2\omega$ and a time-independent term. Therefore we can seek the solutions of Eq. (A2) in the form

$$\xi^{(2)}(x, t) = \xi_{+2}^{(2)}(x)e^{2i\omega t} + \xi_{-2}^{(2)}(x)e^{-2i\omega t} + \xi_0^{(2)}(x, t). \qquad \text{(A3)}$$

Denoting the mth coefficient in Fourier's expansion of ξ of order $\mathcal{O}(A^n)$ by $\xi_m^{(n)}$, we obtain the following equations:

$$-2\omega^{(0)}\omega^{(1)}\xi^{(1)} = 0, \qquad \text{(A4)}$$

$$\ddot{\xi}_0^{(2)} + \hat{\mathbf{L}}\xi_0^{(2)} = -\frac{1}{4}U'''(\phi_s)\eta_q\eta_{-q}, \qquad \text{(A5)}$$

$$(\hat{\mathbf{L}} - 4\omega^{(0)^2})\xi_{\pm 2}^{(2)} = -\frac{1}{8}U'''(\phi_s)\eta_{\pm q}^2. \qquad \text{(A6)}$$

The first equation gives immediately $\omega^{(1)} = 0$, i.e. there is no correction to frequency in the first order. Projecting Eq. (A5) onto the translational mode η_t, and using the identity $(\hat{\mathbf{L}}\ddot{\xi}_0^{(2)}|\eta_t) = (\hat{\mathbf{L}}\eta_t|\ddot{\xi}_0^{(2)})^* = 0$, we obtain

$$(\ddot{\xi}_0^{(2)}|\eta_t) = -\frac{1}{4}(U'''(\phi_s)\eta_q\eta_{-q}|\eta_t). \qquad \text{(A7)}$$

As we shall show now the right-hand side of Eq. (A7) vanishes precisely for reflectionless potentials. To prove this we shall take the derivative of the eigenvalue problem of the linear operator $\hat{\mathbf{L}}$

$$\left(-\frac{d^2}{dx^2} + U''(\phi_s(x))\right)\eta_q(x) = \omega^2\eta_q(x), \qquad \text{(A8)}$$

then multiply it with η_{-q} and integrating leads to

$$\int dx\eta_{-q}(\hat{\mathbf{L}} - \omega^2)\eta_q' = -(U'''(\phi_s(x))\eta_q\eta_{-q}|\phi_s'). \qquad \text{(A9)}$$

Integration by parts (over some interval $(-L, L)$) of the left-hand side of the above equation and using Eq. (A8) satisfied by η_{-q} gives

$$(\eta_q'\eta_{-q}' - \eta_{-q}\eta_q'')|_{-L}^{L} = -(U'''(\phi_s)\eta_q\eta_{-q}|\eta_t), \qquad \text{(A10)}$$

where we have also used that the translational mode $\eta_t = \phi_s'$. Finally it is easy to calculate the boundary values in (A10) by using the asymptotic form for η_q (18) and (19) leading to the interesting identity

$$4q^2|\mathbf{R}|^2 = -(U'''(\phi_s)\eta_q\eta_{-q}|\eta_t). \qquad \text{(A11)}$$

This demonstrates that for reflectionless potentials, i.e. $\mathbf{R} \equiv 0$, $(U'''(\phi_s)\eta_q\eta_{-q}|\eta_t) = 0$. Therefore it is fully consistent to assume $\ddot{\xi}_0^{(2)} = 0$ in Eq. (A5) when $\mathbf{R} \equiv 0$. It is not difficult to obtain the asymptotic form of $\xi_0^{(2)}(x)$ since it has a limit for $x \to \infty$:

$$\xi_0^2(x \to \pm\infty) = -\frac{U'''(\phi_{\text{vac}})}{4U''(\phi_{\text{vac}})}. \qquad \text{(A12)}$$

To obtain the asymptotic form of the solution of Eq. (A6) we make use of the Green's function for the inhomogeneous problem. Since all inhomogeneous equations we have to solve can be written as

$$\hat{\mathbf{L}}\xi_m^{(n)} = f_m^{(n)}(x), \qquad \text{(A13)}$$

we give their solution satisfying our boundary conditions in a general form:

$$\xi_m^{(n)}(x) = -\frac{\eta_{-\kappa}(x)}{W}\int_{-\infty}^{x} dx'\eta_\kappa(x')f_m^{(n)}(x') - \frac{\eta_\kappa(x)}{W}\int_{x}^{\infty} dx'\eta_{-\kappa}(x')f_m^{(n)}(x'), \qquad \text{(A14)}$$

where η_κ are the solutions of the homogeneous equation (eigenfunctions of $\hat{\mathbf{L}}$), $\kappa = \sqrt{m^2\omega^2 - U''(\phi_{\text{vac}})}$ is the wave number corresponding to the frequency $m\omega$ and $W = \eta_\kappa\eta_{-\kappa}' - \eta_\kappa'\eta_{-\kappa} = -2i\kappa$ is the Wronskian.[2] To obtain the asymptotic form of the solutions we write the integrals as a difference, e.g.

$$\int_{-\infty}^{x} dx'\eta_\kappa(x')f_m^{(n)}(x') = \left(\int_{-\infty}^{\infty} - \int_{x}^{\infty}\right)dx'\eta_\kappa(x')f_m^{(n)}(x'). \qquad \text{(A15)}$$

The integral over the real line can be calculated using the method of residua while the second one can be calculated using the asymptotic form of η_κ and $f_m^{(n)}$. This way we obtain the asymptotic form of the solution of Eq. (A6)

[2]Note that as there is no first order space derivative in the equations of motion, the Wronskian is independent of x, and thus can be calculated from the asymptotic form of η_q and η_{-q}.

$$\xi^{(2)}_{+2}(x\to\pm\infty)=\frac{U'''(\phi_{\rm vac})}{24U''(\phi_{\rm vac})}\eta^2_{+q}+\alpha_{22,\pm k}(q)\eta_{\mp k}, \tag{A16}$$

where k is the wave number corresponding to the frequency 2ω, and

$$\alpha_{22,k}(q)=-\frac{1}{8W}\int_{-\infty}^{\infty}dx'\eta_k\eta_q^2U'''(\phi_s). \tag{A17}$$

The solution for the negative frequency can be found as a complex conjugation of the above solution ($\xi^{(2)}_{-2}=\xi^{(2)^*}_{+2}$).

All the coefficients of the homogeneous part will be denoted as

$$\alpha_{mn,\kappa}(q)=-\frac{1}{W}\int_{-\infty}^{\infty}dx'\eta_\kappa(x')f_m^{(n)}(x'). \tag{A18}$$

The third order equations have the following forms:

$$(\hat{\mathbf{L}}-9\omega^{(0)^2})\xi^{(3)}_{\pm3}=-\frac{1}{6}U^{(\rm iv)}(\phi_s)\xi^{(1)^3}_{\pm1}-U'''(\phi_s)\xi^{(1)}_{\pm1}\xi^{(2)}_{\pm2}, \tag{A19}$$

$$(\hat{\mathbf{L}}-\omega^{(0)^2})\xi^{(3)}_{\pm1}=-U'''(\phi_{\rm s})(\xi^{(1)}_{\pm1}\xi^{(2)}_{0}+\xi^{(1)}_{\mp1}\xi^{(2)}_{\pm2})-\frac{1}{2}U^{(\rm iv)}(\phi_s)\xi^{(1)^2}_{\pm1}\xi^{(1)}_{\mp1}+2\omega^{(0)}\omega^{(2)}\xi^{(1)}_{\pm1}. \tag{A20}$$

Taking into account the asymptotic forms of $\xi^{(n)}_m$ Eq. (A20) can be rewritten as

$$(\hat{\mathbf{L}}-\omega^{(0)^2})\xi^{(3)}_{1}(x\to\pm\infty)=-\frac{1}{2}U'''\alpha_{22,\pm k}(q)\eta_{-q}\eta_{\mp k}+\eta_{+q}\left(\frac{5U'''^2}{48U''}-\frac{U^{(\rm iv)}}{16}+\omega^{(0)}\omega^{(2)}\right), \tag{A21}$$

where derivatives of U has to be taken at $\phi=\phi_{\rm vac}$. Note that the left-hand side of the above equation is the same as for harmonic oscillator with frequency q. On the right-hand side there is a source term which oscillates with the resonant frequency. Therefore the following condition must be fulfilled to cancel this resonance term:

$$\omega^{(2)}=-\frac{1}{\omega^{(0)}}\left(\frac{5U'''^2}{48U''}-\frac{U^{(\rm iv)}}{16}\right). \tag{A22}$$

This gives the first correction to the frequency. (Note that the values of the derivatives of the potential must be taken at vacuum.) Having this we can write the asymptotic form of the Eq. (A20) in much simpler form

$$(\hat{\mathbf{L}}-\omega^{(0)^2})\xi^{(3)}_{+1}(x\to\pm\infty)=-\frac{1}{2}U'''\alpha_{22,\pm k}(q)\eta_{-q}\eta_{\mp k} \tag{A23}$$

which leads to the solution

$$\xi^{(3)}_{1}(x\to\pm\infty)=-\frac{\alpha_{22,k}U'''}{p_\pm^2+U''-\omega^{(0)2}}\eta_{\mp k}\eta_{-q}+\alpha_{31,\pm q}\eta_{\mp q}, \tag{A24}$$

where $p_\pm=\mp k-q$. A computation similar to the above yields

$$\xi^{(3)}_{3}(x\to\pm\infty)=\frac{1}{384U''}(U''U^{(\rm iv)}+U'''^2)\eta_q^3\mp\frac{1}{2}\frac{U'''\alpha_{22,k}}{p_\pm^2-9\omega^{(0)2}+U''}\eta_{\mp k}\eta_q+\alpha_{33,\pm s}\eta_{\mp s}, \tag{A25}$$

where s is the wave number corresponding to the frequency 3ω. As one can see in the third order solution there are only terms which oscillate with time, therefore its projection onto the translational mode gives no contribution (after averaging in time) to the time-independent part of the acceleration. The next time-independent term appears at fourth order, $\mathcal{O}(A^4)$. Then, the equation for $\xi^{(4)}_0$ has the form

$$\ddot{\xi}^{(4)}_0+\hat{\mathbf{L}}\xi^{(4)}_0=-\frac{1}{2}U'''(\phi_s)(2\xi^{(1)}_1\xi^{(3)}_{-1}+2\xi^{(1)}_{-1}\xi^{(3)}_{1}+2\xi^{(2)}_{2}\xi^{(2)}_{-2}+\xi^{(2)^2}_{0})-\frac{1}{2}U^{(\rm iv)}(\phi_s)(\xi^{(1)^2}_{1}\xi^{(2)}_{-2}+2\xi^{(1)}_1\xi^{(1)}_{-1}\xi^{(2)}_{0}+\xi^{(1)^2}_{-1}\xi^{(2)}_{2})-\frac{1}{4}U^{(\rm v)}(\phi_s)\xi^{(2)^2}_{1}\xi^{(2)^2}_{-1}. \tag{A26}$$

Now computing the projection of $\xi^{(4)}_0$ to the translational mode η_t, exploiting the obvious reflection symmetries some of the terms in Eq. (A26) do not contribute, and we are led to the following result:

$$(\ddot{\xi}^{(4)}_0|\eta_t)=-{\rm Re}(U'''(\phi_s)(\xi^{(3)}_1\eta_{-q}+\xi^{(2)}_2\xi^{(2)}_{-2})+\frac{1}{4}U^{(\rm iv)}(\phi_s)\xi^{(2)}_2\eta^2_{-q}|\eta_t). \tag{A27}$$

APPENDIX B: NEWTON'S LAW

In this appendix we show by explicit computation that Newton's law holds to leading order in PT. We start by calculating the projection of the second time derivative of the perturbation $\ddot{\xi}$ onto the translational mode of the kink. Computing the second order acceleration in PT, by projecting $\ddot{\xi}$ on the translational mode, one obtains

$$(ma)_2=\int_{-L}^{L}\phi'_sU'''(\phi_s)\xi^{(1)}_1\xi^{(1)}_{-1}dx=U''(\phi_s)\xi^{(1)}_1\xi^{(1)}_{-1}|^L_{-L}-\int_{-L}^{L}U''(\phi_s)(\xi^{(1)'}_1\xi^{(1)}_{-1}+\xi^{(1)}_1\xi^{(1)'}_{-1}). \tag{B1}$$

Next looking at the relevant component of the stressenergy tensor,

$$(-T_{11})_2 = \xi_0^{(2)\prime} U'(\phi_s) + U''(\phi_s)\xi_1^{(1)}\xi_{-1}^{(1)} - \xi_1^{(1)\prime}\xi_{-1}^{(1)\prime} - \dot{\xi}_1^{(1)}\dot{\xi}_{-1}^{(1)} + \phi_s'\xi_0^{(2)\prime}, \tag{B2}$$

one sees that the first and the last terms give negligible contributions $(-T_{11})|_{-L}^{L}$, if L is sufficiently large. The second term is exactly the boundary term in (B1). To prove the equality, let us put

$$-\xi_1^{(1)\prime}\xi_{-1}^{(1)\prime}|_{-L}^{L} = -\int_{-L}^{L}(\xi_1^{(1)\prime\prime}\xi_{-1}^{(1)\prime} + \xi_1^{(1)\prime}\xi_{-1}^{(1)\prime\prime}),$$

and note that $\dot{\xi}_1^{(1)} = i\omega\xi_1^{(1)}$. Therefore, the remaining terms in $(-T_{11})_2|_{-L}^{L}$ are equal to

$$\int_{-L}^{L}(\xi_1^{(1)\prime}(\ddot{\xi}_{-1}^{(1)} - \xi_{-1}^{(1)\prime\prime}) + \xi_{-1}^{(1)\prime}(\ddot{\xi}_1^{(1)} - \xi_1^{(1)\prime\prime})) = -\int_{-L}^{L}(\xi_1^{(1)\prime}U''(\phi_s)\xi_{-1}^{(1)} + \xi_{-1}^{(1)\prime}U''(\phi_s)\xi_1^{(1)}),$$

where we used the equation of motion $(\hat{\mathbf{L}} - \omega^2)\xi_{\pm 1}^{(1)} = 0$. Comparing this with the last term in (B1) completes the proof.

If the acceleration is vanishing to second order, the first nontrivial contribution to it can only come form the fourth order. This is because neither ma nor T_{11} has a zero frequency part in the third order. The fourth order acceleration is given by $(m_s a)_4 = -\langle\ddot{\xi}_0^{(4)}\rangle$ where $\ddot{\xi}_0^{(4)}$ can be replaced by its source term [see Eq. (A26)]. This should be equal to $(-T_{11})_4|_{-L}^{L}$, where

$$\begin{aligned}(-T_{11})_4 = {} & U''(\phi_s)\left\{\frac{1}{2}\xi_0^{(2)^2} + \xi_2^{(2)}\xi_{-2}^{(2)} + \xi_{-1}^{(1)}\xi_1^{(3)} + \xi_1^{(1)}\xi_{-1}^{(3)}\right\} \\ & + U'''(\phi_s)\left\{\xi_1^{(1)}\xi_{-1}^{(1)}\xi_0^{(2)} + \frac{1}{2}\xi_{-1}^{(1)2}\xi_2^{(2)} + \frac{1}{2}\xi_1^{(1)2}\xi_{-2}^{(2)}\right\} \\ & - \frac{1}{2}\dot{\xi}_0^{(2)^2} - \dot{\xi}_2^{(2)}\dot{\xi}_{-2}^{(2)} - \dot{\xi}_{-1}^{(1)}\dot{\xi}_1^{(3)} - \dot{\xi}_1^{(1)}\dot{\xi}_{-1}^{(3)} \\ & - \frac{1}{2}\xi_0^{\prime(2)^2} - \xi_2^{\prime(2)}\xi_{-2}^{\prime(2)} - \xi_{-1}^{\prime(1)}\xi_1^{\prime(3)} - \xi_1^{\prime(1)}\xi_{-1}^{\prime(3)} \\ & + U^{(\mathrm{iv})}(\phi_s)\frac{1}{4}\xi_1^{(1)^2}\xi_{-1}^{(1)^2} + 4\omega\omega^{(2)}\xi_1^{(1)}\xi_{-1}^{(1)}.\end{aligned} \tag{B3}$$

We remark that to remove the resonance terms from the source of $\xi_{\pm 1}^{(3)}$, the value of $\omega^{(2)}$ is nonzero. This term induces a fourth order correction to $(T_{11})_2$. Explicitly it is given by

$$(-T_{11})_{4\mathrm{res}} = 4\omega\omega^{(2)}\xi_1^{(1)}\xi_{-1}^{(1)}|_{-L}^{L}.$$

To establish Newton's law to this order proceeds essentially the same way as in the second order case. Performing the partial integrations in the integral $(ma)_4$ gives all the terms in the energy-momentum tensor containing the potential as boundary terms. The integral terms after the partial integration in $(ma)_4$ are

$$\begin{aligned}& -\int_{-L}^{L}U''(\phi_s)\Big\{(\xi_2^{(2)}\xi_{-2}^{(2)})' + \frac{1}{2}(\xi_0^{(2)}\xi_0^{(2)})' + (\xi_{-1}^{(1)}\xi_1^{(3)})' \\ & + (\xi_1^{(1)}\xi_{-1}^{(3)})'\Big\} - \int_{-L}^{L}U'''(\phi_s)\Big\{\frac{1}{2}(\xi_{-1}^{(1)}\xi_{-1}^{(1)}\xi_2^{(2)})' \\ & + \frac{1}{2}(\xi_1^{(1)}\xi_1^{(1)}\xi_{-2}^{(2)})' + (\xi_1^{(1)}\xi_{-1}^{(1)}\xi_0^{(2)})'\Big\} \\ & - \int_{-L}^{L}\frac{1}{4}U^{(\mathrm{iv})}(\phi_s)(\xi_1^{(1)}\xi_1^{(1)}\xi_{-1}^{(1)2})'.\end{aligned} \tag{B4}$$

Note, that the terms in the integrands are the source terms of equations of motion (14), (A5), (A6), and (A20). Replacing them with the right-hand side of these equations of motion, and reorganizing the terms, one gets total derivatives, the integrals of which are the remaining terms of $(-T_{11})|_{-L}^{L}$.

Equation (B3) can be used to obtain the momentum in the segment $[-L, L]$ to order $\mathcal{O}(A^4)$. The rate of change of the momentum is given by Eq. (10) and in fact we need its integrated form Eq. (29). Direct substitution of the series (13) and expansion into powers of A show that for reflectionless potentials the first nonvanishing term is of order $\mathcal{O}(A^4)$, therefore $\partial_t P = A^4\partial_t P^{(4)}$. We need to calculate the energy-momentum tensor at the boundaries of the segment T_{11} proportional to A^4, $-T_{11}^{(4)}|_{-L}^{L}$. It is sufficient to use the asymptotic form of the solutions (A12), (A16), (A24), and (A25) together with the frequency correction (A22). The calculation of the boundary term leads to a complicated expression, which after averaging in time gives the leading term for the force

$$F = -A^4[2k^2(|\alpha_{22,+k}^2| - |\alpha_{22,-k}^2|) - 4q^2\,\mathrm{Re}\,\alpha_{31,-q}]. \tag{B5}$$

At the end of this appendix, we will show that to leading order the energy of the accelerating kink is $\frac{1}{2}mv^2$ indeed, as used in Sec. III. Writing $\phi = \phi_s + \xi$, and using Eq. (7) we obtain

$$\epsilon = \frac{1}{2}\phi_s'^2 + U(\phi_s) + \phi_s'\xi' + U'(\phi_s)\xi + \frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\xi'^2 + \frac{1}{2}U''(\phi_s)\xi^2 + \frac{1}{6}U'''(\phi_s)\xi^3 + \frac{1}{24}U^{(\mathrm{iv})}(\phi_s)\xi^4. \tag{B6}$$

The linear terms in ξ give no contribution because ϕ_s solves the static equation of motion. The only time-dependent nonperiodic terms (up to the fourth order) are those containing $\xi_0^{(4)} = -\frac{1}{2}at^2\phi_s' + \cdots$. In the fourth order, there are no such terms. In the sixth order, the quadratic terms in ξ give a contribution that is proportional to the equation of motion of $\xi_0^{(2)}$ multiplied by $\xi_0^{(4)}$ (in one term one has to perform a partial integration). The term $\frac{1}{6}U'''(\phi_s)\xi^3$ gives a term proportional to $\xi_1^{(1)}\xi_{-1}^{(1)}\xi_0^{(4)}\phi_s$, which is antisymmetric, therefore its integral is 0. The last term contains no sixth order contribution proportional to $\xi_0^{(4)}$. Finally in the eighth order, the quadratic terms give the contribution $\frac{1}{2}mv^2$.

APPENDIX C: SUPERPOSITION OF THE SG KINK WITH A CNOIDAL WAVE

In this appendix we review first the remarkable analytic solution corresponding to the nonlinear superposition of a SG kink and a traveling wave. This solution has been obtained by H. J. Shin by Darboux transformation methods [9]. In Ref. [9] light-cone coordinates $(z, \bar{z})$ are used, which are related to (t, x) used in this paper as

$$z = \frac{-x + t}{4\sqrt{\beta}}, \qquad \bar{z} = \frac{-x - t}{4\sqrt{\beta}}. \tag{C1}$$

The sine-Gordon equation is written in Ref. [9] as

$$\partial_{\bar{z}}\partial_z\tilde{\phi} = 2\beta\sin 2\tilde{\phi}, \tag{C2}$$

with the scale parameter β kept for bookkeeping purposes. Solutions of the SG equation using our conventions and those of Ref. [9] are related by $\phi(x, t) = 2\tilde{\phi}(z, \bar{z})$. In light-cone coordinates the static kink solution of Eq. (C2) is given by

$$\tilde{\phi}_s = 2\arctan e^{-2\sqrt{\beta}(z+\bar{z})}. \tag{C3}$$

The solution of interest for our purposes is called type 2 in Ref. [9], and it can be written in the following way:

$$\partial_z\tilde{\phi}(z, \bar{z}) = 2k\sqrt{\frac{\beta}{V}}\mathrm{cn}(\zeta, k^2) + 4\sqrt{\frac{\beta}{V}}\frac{\mathrm{cn}(u, k^2)}{\mathrm{sn}(u, k^2)\,\mathrm{dn}(u, k^2)}\frac{S}{S^2+1}, \tag{C4}$$

where the function S is defined as

$$S = -\frac{ak\,\mathrm{sn}(u, k^2)\,\mathrm{cn}(\zeta - u, k^2)\mathcal{X} + b\,\mathrm{dn}(u, k^2)\mathcal{Y}}{bk\,\mathrm{sn}(u, k^2)\,\mathrm{cn}(\zeta + u, k^2)\mathcal{Y} + a\,\mathrm{dn}(u, k^2)\mathcal{X}}, \tag{C5}$$

with

$$\begin{aligned}\mathcal{X} &= \exp(M\bar{z} + kN\zeta)\Theta_t(\zeta - u),\\ \mathcal{Y} &= \exp(-M\bar{z} - kN\zeta)\Theta_t(\zeta + u),\end{aligned} \tag{C6}$$

$$\begin{aligned}M &= \sqrt{V\beta}\left[\frac{\mathrm{cn}(u, k^2)}{\mathrm{sn}(u, k^2)\,\mathrm{dn}(u, k^2)} + \frac{\mathrm{dn}(u, k^2)\,\mathrm{sn}(u, k^2)}{\mathrm{cn}(u, k^2)} - 2k^2\frac{\mathrm{sn}(u, k^2)\,\mathrm{cn}(u, k^2)}{\mathrm{dn}(u, k^2)}\right],\\ N &= \frac{\Theta_t'(u)}{k\Theta_t(u)} + \frac{\mathrm{cn}(u, k^2)}{2k\,\mathrm{dn}(u, k^2)\,\mathrm{sn}(u, k^2)} - k\frac{\mathrm{sn}(u, k^2)\,\mathrm{cn}(u, k^2)}{\mathrm{dn}(u, k^2)},\end{aligned} \tag{C7}$$

and

$$\zeta = 2\sqrt{\frac{\beta}{V}}(z - V\bar{z}). \tag{C8}$$

In Eqs. (C4)–(C7), a, b, k, u, and V are constants, sn, cn and dn denote Jacobi's elliptic functions, $\Theta_t(u)$ is given by

$$\Theta_t(u) = \vartheta_4\left(\frac{\pi u}{2(K - iK')}, \hat{q}\right) = 1 + 2\sum_{n=1}^{\infty}(-)^n\hat{q}^{n^2}\cos\left(\frac{n\pi u}{K - iK'}\right), \tag{C9}$$

where $K = K(k^2)$ is the complete elliptic integral $K' = K(1 - k^2)$ and the nome $\hat{q}$ is given as $\hat{q} = \exp[-\pi K'/(K - iK')]$. The notations and conventions of the special functions are those of Abramowitz and Stegun [11], and due to this, there are some notational differences with Ref. [9]. (The nonstandard notation used here for the nome $\hat{q}$ is to avoid confusion with the wave number q.)

Remarkably $\sin 2\tilde{\phi}$ can also be expressed in a simple form,

$$\sin 2\tilde{\phi} = 4SU[1 - 2k^2\,\mathrm{sn}^2(\zeta, k^2)] - 2\tilde{U}k\,\mathrm{sn}(\zeta, k^2)\,\mathrm{dn}(\zeta, k^2), \tag{C10}$$

where $U = (S^2 - 1)/(S^2 + 1)^2$ and $\tilde{U} = 1 - 8S^2/(S^2 + 1)^2$.

The solution (C4) [or equivalently (C10)] depends essentially on four parameters: u, k, V, and b/a. We have to point out here that the parameter k in Eq. (C4) should not be confused with the wave number, also denoted by k in the previous sections of this paper. In fact as it turns out it can be related to the amplitude as $k = A/2$, therefore it is a suitable expansion parameter to study the small amplitude limit. In this appendix we shall carry out the small k expansion of the solution (C4) in order to compare it to the perturbative one. The solution should be transformed first to a form that is more suitable to be expanded for small k. As it stands $\Theta_t(u)$ is not well-suited for the small k expansion since $\hat{q} \to -1$ for $k \to 0$. It is more convenient to transform it by a modular transformation to the following form:

$$\Theta_t(u) = e^{i\pi/4}\sqrt{1 - iK'/K}\exp\left(\frac{\pi u^2}{4K(K' + iK)}\right)\vartheta_4\left(\frac{\pi u}{2K}, \hat{q}\right), \tag{C11}$$

where the nome is now given by $\hat{q} = \exp(i\pi K'/K)$. It is now straightforward to perform the $k \to 0$ expansion. Since S depends only on $\mathcal{X}/\mathcal{Y}$, the common singular parts can be dropped and the remaining functions are analytic in k. Thus in the solution (C4) $\mathcal{X}$, $\mathcal{Y}$, and kN can be replaced by

$$\mathcal{X}' = \exp(M\bar{z} + kN'\zeta)\Theta(\zeta - u), \tag{C12}$$

$$\mathcal{Y}' = \exp(-M\bar{z} - kN'\zeta)\Theta(\zeta + u), \quad \text{(C13)}$$

and

$$\begin{aligned} kN' &= kN - \frac{\pi u}{2K(K' + iK)} \\ &= \frac{\Theta'(u)}{\Theta(u)} + \frac{\mathrm{cn}(u, k^2)}{2\,\mathrm{dn}(u, k^2)\,\mathrm{sn}(u, k^2)} \\ &\quad - k^2 \frac{\mathrm{sn}(u, k^2)\,\mathrm{cn}(u, k^2)}{\mathrm{dn}(u, k^2)}, \end{aligned} \quad \text{(C14)}$$

where $\Theta(u)$ is given as

$$\Theta(u) = \vartheta_4\left(\frac{\pi u}{2K}, \hat{q}\right). \quad \text{(C15)}$$

The leading (zeroth) order expansion (i.e. $k = 0$) should clearly yield an isolated kink. An easy computation gives

$$M_0 = \sqrt{V\beta}(\cot u + \tan u), \qquad (kN')_0 = \frac{1}{2}\cot u, \quad \text{(C16)}$$

together with

$$\mathcal{X}'_0 = \exp\left(\bar{z}\sqrt{V\beta}\tan u + z\sqrt{\frac{\beta}{V}}\cot u\right) = e^X, \quad \text{(C17)}$$

$$\mathcal{Y}'_0 = \exp\left(-\bar{z}\sqrt{V\beta}\tan u - z\sqrt{\frac{\beta}{V}}\cot u\right) = e^{-X}, \quad \text{(C18)}$$

$$S_0 = -\frac{b\mathcal{Y}'_0}{a\mathcal{X}'_0} = -\frac{b}{a}\exp(-2X). \quad \text{(C19)}$$

Putting the above together we find

$$\begin{aligned} \partial_z\tilde{\phi}_0 &= -4\sqrt{\frac{\beta}{V}}\cot u \frac{e^{-2X}b/a}{1 + e^{-4X}(b/a)^2}, \\ \tilde{\phi}_0 &= 2\arctan\exp\left(-2X + \log\frac{b}{a}\right). \end{aligned} \quad \text{(C20)}$$

Comparing this result to a Lorentz boosted kink at x_0,

$$\phi_v = 4\arctan\exp\left(\frac{x - x_0 - vt}{\sqrt{1 - v^2}}\right),$$

we find that

$$\log\frac{b}{a} = \frac{-x_0}{\sqrt{1 - v^2}}, \qquad \sqrt{V}\tan u = \frac{1 + v}{\sqrt{1 - v^2}}. \quad \text{(C21)}$$

In what follows, we will look at the case in which the kink is at rest in the origin, thus $x_0 = 0$, $v = 0$ and therefore we set $a = b = 1$ and $\sqrt{V}\tan u = 1$. Note, that to leading order $X = -x/2$.

Proceeding to the first order expansion in k one obtains

$$\tilde{\phi}_1 = -\cos 2u \sin\zeta - \sin 2u \tanh x \cos\zeta. \quad \text{(C22)}$$

Therefore the first order correction to the solution can be written as

$$\tilde{\phi}_1 = (\eta_q e^{i\omega t} + \eta_{-q} e^{-i\omega t}), \quad \text{(C23)}$$

where $\omega + q = \sqrt{V}$, $\omega - q = 1/\sqrt{V}$. Comparing $\tilde{\phi}_1$ to Eq. (16) one can see that the amplitude of the wave is given by $A = 2k$, indeed.

The expansion to second order in k is somewhat more complicated but straightforward. Here we just present the final result:

$$\begin{aligned} \tilde{\phi}_2 = &-t\frac{4u + \sin 4u}{8\sin 2u}\frac{1}{\cosh x} + e^x\left(\frac{2}{\cosh^2 x} - 1\right) \\ &\times \frac{(2e^x x\cosh x + \cos(\omega t - qx)\sin(2u)(e^{2x}\sin(\omega t - qx - 2u) + \sin(\omega t - qx + 2u)))}{1 - 6e^{2x} + e^{4x}}. \end{aligned} \quad \text{(C24)}$$

From the first term in Eq. (C24) one can see that to this order, the kink now moves with a constant velocity

$$v_2 = \frac{4u + \sin 4u}{8\sin 2u}.$$

In order to obtain a static kink, v should be cancelled by the addition of suitable, second order correction to the parameter $\sqrt{V}\tan u$, i.e. $\sqrt{V}\tan u = 1 - k^2 v_2$. $\tilde{\phi}_2$ can be further simplified by introducing the wave number q instead of u and ω:

$$\begin{aligned} \xi^{(2)} &= \xi_2^{(2)} e^{2i\omega t} + \xi_{-2}^{(2)} e^{-2i\omega t} + \xi_0^{(2)} \\ &= \frac{e^{i(2\omega t + 2qx)}(-iq + \tanh x)}{16\cosh x(1 + q^2)} + \frac{e^{-i(2\omega t + 2qx)}(iq + \tanh x)}{16\cosh x(1 + q^2)} \\ &\quad + \frac{1}{8\cosh x}\left(\frac{\tanh x}{1 + q^2} - 2x\right), \end{aligned} \quad \text{(C25)}$$

which coincides precisely with the solution obtained using the perturbative method in Eq. (49).

Let us finally note that the time-averaged (i.e. zero frequency part) motion of the zero of the kink can be also calculated in a closed form. The condition is $\phi = 0$ and implies

$$M\bar{z} + kN'\zeta = 0, \quad \text{(C26)}$$

leading to

$$\frac{v + 1}{v - 1} = \frac{z}{\bar{z}} = -\frac{M - kN'2\sqrt{V\beta}}{2kN'\sqrt{\beta/V}}. \quad \text{(C27)}$$

This means that the velocity of the kink v is constant, which is a free parameter of the solution (C10).

[1] A. C. Scott, *Nonlinear Science* (Oxford University, New York, 1999).
[2] N. S. Manton and P. Sutcliffe, *Topological Solitons* (Cambridge University Press, Cambridge, England, 2004).
[3] T. Dauxois and M. Peyrard, *Physics of Solitons* (Cambridge University Press, Cambridge, England, 2006).
[4] M. B. Fogel, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, Phys. Rev. B **15**, 1578 (1977).
[5] Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
[6] G. Reinish and J. C. Fernandez, Phys. Rev. B **24**, 835 (1981).
[7] M. J. Rice, Phys. Rev. B **28**, 3587 (1983).
[8] T. Romańczukiewicz, Acta Phys. Pol. B **35**, 523 (2004).
[9] H. J. Shin, arXiv:nlin/0410065.
[10] V. G. Kiselev and Ya. M. Shnir, Phys. Rev. D **57**, 5174 (1998).
[11] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).