One loop superstring effective actions and $\mathcal{N} = 8$ supergravity

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In a previous article we have shown the existence of a new independent $\mathcal{R}^4$ term, at one loop, in the type IIA and heterotic effective actions, after reduction to four dimensions, besides the usual square of the Bel-Robinson tensor. It had been shown that such a term could not be directly supersymmetrized, but we showed that was possible after coupling to a scalar chiral multiplet. In this article, we study the extended ($\mathcal{N} = 8$) supersymmetrization of this term, where no other coupling can be taken. We show that such supersymmetrization cannot be achieved at the linearized level. This is in conflict with the theory one gets after toroidal compactification of type II superstrings being $\mathcal{N} = 8$ supersymmetric. We interpret this result in the face of the recent claim that perturbative supergravity cannot be decoupled from string theory in $d \geq 4$, and $\mathcal{N} = 8$, $d = 4$ supergravity is in the swampland.

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I. INTRODUCTION

String theories require higher order in α' corrections to their corresponding low energy supergravity effective actions. Among these corrections, at order α'^3, the $\mathcal{R}^4$ terms (the fourth power of the Riemann tensor) are present in type II [1,2] and heterotic [3] superstrings and in M theory [4]. These corrections need to be supersymmetric; the topic of their supersymmetrization has been object of research for a long time [5–8].

These corrections are also present in four-dimensional supergravity theories. Originally, they were looked at as candidate counterterms to these theories, which were believed to be divergent. From the string theory point of view, they are seen as compactified string corrections. In any case, these corrections must be supersymmetric. The number $\mathcal{N}$ of four-dimensional supersymmetries and different matter couplings depend crucially on the manifold where the compactification is taken.

The four-dimensional supersymmetrization of $\mathcal{R}^4$ terms has been considered both in simple [9–11] and in extended [12–15] supergravities. Although there are two independent $\mathcal{R}^4$ terms in $d = 4$, all these cases only studied one such term: the square of the Bel-Robinson. Indeed, in another article [16] it is shown that the other four-dimensional $\mathcal{R}^4$ term is part of a class of terms, which are not supersymmetrizable.

That term has not deserved any further attention until recently. In our previous paper [17], we computed the dimensional reduction, to four dimensions, on a torus, of the ten-dimensional $\mathcal{R}^4$ terms from type II and heterotic superstrings.[1] We have then shown that the other $\mathcal{R}^4$ term is part of the heterotic and type IIA effective actions, at one loop, when compactified to $d = 4$. Now, when compactified to $d = 4$ on a 6-torus $\mathbb{T}^6$, should be respectively $\mathcal{N} = 4$, 8 supersymmetric. Plus, $\mathbb{T}^6$ is the most basic manifold one can think of in order to compactify a ten-dimensional theory; all the terms one gets from this compactification are present when one rather takes a more complicated manifold. This means the new (or less known) $\mathcal{R}^4$ term is present in any compactification to $d = 4$ of type IIA and heterotic superstrings.

In our previous work [17], we focused on $\mathcal{N} = 1$ supergravity. By taking a coupling to a chiral multiplet, we were able to circumvent the argument of [16] and indeed include the less known $\mathcal{R}^4$ term in an $\mathcal{N} = 1$ supersymmetric Lagrangian.

In this work, we focus particularly on maximal $\mathcal{N} = 8$ supergravity, the most restrictive of all the $d = 4$ theories (its multiplet is unique, and there are no matter couplings to take), and one of the main reasons is precisely because this is the theory which results after compactifying type IIA supergravity on $\mathbb{T}^6$. Besides, the study of higher-order corrections in $\mathcal{N} = 8$ supergravity is particularly relevant considering the recent claims that this theory may actually be eight-loop finite [18,19] or even ultraviolet finite [20,21].

In Sec. II , we will review and summarize some of the results of [17], concerning $\mathcal{R}^4$ terms in $d = 10$ and their reduction to $d = 4$. In Sec. III, we briefly review linearized

[1]The $\mathcal{R}^4$ term from M theory, when reduced to $d = 10$ on $\mathbb{S}^1$, results in the one-loop $\mathcal{R}^4$ term in type IIA superstring. The results of [17], therefore, also include the toroidal compactification of M theory to $d = 4$.

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$d = 4$ extended supergravity in superspace and some known higher-order linearized extended superinvariants and the symmetries they should preserve. In Sec. IV, we proceed with trying to supersymmetrize in $\mathcal{N} = 8$ the other less known, but existing, $\mathcal{R}^4$ term using different possibilities.

II. $\mathcal{R}^4$ TERMS IN $d = 10$ AND $d = 4$

A. $\mathcal{R}^4$ terms in $d = 10$

The superstring α'^3 effective actions contain two independent bosonic terms I_X, I_Z, from which two separate superinvariants are built [5,22]. These terms are given, at linear order in the NS-NS (Neveu-Schwarz–Neveu-Schwarz) gauge field B_{mn}, by

$$\begin{aligned} I_X &= t_8 t_8 \mathcal{R}^4 + \tfrac{1}{2}\varepsilon_{10} t_8 B \mathcal{R}^4 =: X + \tfrac{1}{2}\varepsilon_{10} t_8 B \mathcal{R}^4, \\ I_Z &= -\varepsilon_{10}\varepsilon_{10}\mathcal{R}^4 + 4\varepsilon_{10} t_8 B\mathcal{R}^4 =: Z + 4\varepsilon_{10} t_8 B \mathcal{R}^4. \end{aligned} \tag{1}$$

For the heterotic string, another two independent terms Y_1 and Y_2 appear at order α'^3 [5,6,22]. Their parts which involve only the Weyl tensor are given, respectively, by

$$Y_1 := t_8(\mathrm{tr}\,\mathcal{W}^2)^2, \qquad Y_2 := t_8 \mathrm{tr}\,\mathcal{W}^4 = \frac{X}{24} + \frac{Y_1}{4}. \tag{2}$$

Each t_8 tensor has eight free spacetime indices. It acts in four two-index antisymmetric tensors, as defined in [1,2]. In our case,

$$\begin{aligned} t_8 t_8 \mathcal{R}^4 &= t^{mnpqrstu} t^{m'n'p'q'r's't'u'} \mathcal{R}_{mnm'n'} \mathcal{R}_{pqp'q'} \\ &\quad \times \mathcal{R}_{rsr's'} \mathcal{R}_{tut'u'}, \\ \varepsilon_{10} t_8 B \mathcal{R}^4 &= t^{mnpqrstu} \varepsilon^{vwm'n'p'q'r's't'u'} B_{vw} \mathcal{R}_{mnm'n'} \mathcal{R}_{pqp'q'} \\ &\quad \times \mathcal{R}_{rsr's'} \mathcal{R}_{tut'u'}, \\ \varepsilon_{10}\varepsilon_{10}\mathcal{R}^4 &= \varepsilon_{vw}{}^{mnpqrstu} \varepsilon^{vwm'n'p'q'r's't'u'} \mathcal{R}_{mnm'n'} \mathcal{R}_{pqp'q'} \\ &\quad \times \mathcal{R}_{rsr's'} \mathcal{R}_{tut'u'}. \end{aligned} \tag{3}$$

The effective action of type IIB theory must be written, because of its well known SL(2, $\mathbb{Z}$) invariance, as a product of a single linear combination of order α'^3 invariants and an overall function of the complexified coupling constant $\Omega = C^0 + ie^{-\phi}$, C^0 being the axion. The order α'^3 part of this effective action, which involves only the Weyl tensor, is given in the string frame by

$$\begin{aligned} \frac{1}{\sqrt{-g}} \mathcal{L}_{\mathrm{IIB}}\big|_{\alpha'^3} &= -e^{-2\phi}\alpha'^3 \frac{\zeta(3)}{3\times 2^{10}}\left(I_X - \frac{1}{8} I_Z\right) \\ &\quad - \alpha'^3 \frac{1}{3\times 2^{16}\pi^5}\left(I_X - \frac{1}{8} I_Z\right). \end{aligned} \tag{4}$$

The corresponding part of the action of type IIA superstrings has a relative " $-$" sign flip in the one-loop term [23]. This sign difference is because of the different chirality properties of type IIA and type IIB theories, which reflects on the relative Gliozzi-Scherk-Olive projection between the left and right movers:

$$\begin{aligned} \frac{1}{\sqrt{-g}} \mathcal{L}_{\mathrm{IIA}}\big|_{\alpha'^3} &= -e^{-2\phi}\alpha'^3 \frac{\zeta(3)}{3\times 2^{10}}\left(I_X - \frac{1}{8} I_Z\right) \\ &\quad - \alpha'^3 \frac{1}{3\times 2^{16}\pi^5}\left(I_X + \frac{1}{8} I_Z\right). \end{aligned} \tag{5}$$

Heterotic string theories in $d = 10$ have $\mathcal{N} = 1$ supersymmetry, which allows corrections already at order α', including $\mathcal{R}^2$ corrections. These corrections come both from three and four graviton scattering amplitudes and anomaly cancellation terms (the Green-Schwarz mechanism). Up to order α'^3, the terms from this effective action, which involve only the Weyl tensor, are given in the string frame by

$$\begin{aligned} \frac{1}{\sqrt{-g}} \mathcal{L}_{\mathrm{het}}\big|_{\alpha'+\alpha'^3} &= e^{-2\phi}\bigg[\frac{1}{16}\alpha' \mathrm{tr}\mathcal{R}^2 + \frac{1}{2^9}\alpha'^3 Y_1 \\ &\quad - \frac{\zeta(3)}{3\times 2^{10}}\alpha'^3\left(I_X - \frac{1}{8} I_Z\right)\bigg] \\ &\quad - \alpha'^3 \frac{1}{3\times 2^{14}\pi^5}(Y_1 + 4Y_2). \end{aligned} \tag{6}$$

In order to consider these terms in the context of supergravity, one should write them in the Einstein frame. To pass from the string to the Einstein frame, we redefine the metric in d (noncompact) dimensions through a conformal transformation involving the dilaton, given by

$$\begin{aligned} g_{\mu\nu} &\to \exp\left(\frac{4}{d-2}\phi\right) g_{\mu\nu}, \\ \mathcal{R}_{\mu\nu}{}^{\rho\sigma} &\to \exp\left(-\frac{4}{d-2}\phi\right)\tilde{\mathcal{R}}_{\mu\nu}{}^{\rho\sigma}, \end{aligned} \tag{7}$$

with $\tilde{\mathcal{R}}_{\mu\nu}{}^{\rho\sigma} = \mathcal{R}_{\mu\nu}{}^{\rho\sigma} - \delta_{[\mu}{}^{[\rho}\nabla_{\nu]}\nabla^{\sigma]}\phi$.

Let $I_i(\mathcal{R}, \mathcal{M})$ be an arbitrary term in the string frame Lagrangian. $I_i(\mathcal{R}, \mathcal{M})$ is a function, with conformal weight w_i, of any given order in α', of the Riemann tensor $\mathcal{R}$ and any other fields—gauge fields, scalars, and also fermions—which we generically designate by $\mathcal{M}$. The transformation above takes $I_i(\mathcal{R}, \mathcal{M})$ to $e^{(4/(d-2))w_i\phi} I_i(\tilde{\mathcal{R}}, \mathcal{M})$. After considering all the dilaton couplings and the effect of the conformal transformation on the metric determinant factor $\sqrt{-g}$, the string frame Lagrangian

$$\frac{1}{2}\sqrt{-g}e^{-2\phi}\left(-\mathcal{R} + 4(\partial^\mu\phi)\partial_\mu\phi + \sum_i I_i(\mathcal{R}, \mathcal{M})\right) \tag{8}$$

is converted into the Einstein frame Lagrangian

$$\begin{aligned} &\frac{1}{2}\sqrt{-g}\bigg(-\mathcal{R} - \frac{4}{d-2}(\partial^\mu\phi)\partial_\mu\phi \\ &\quad + \sum_i e^{(4/(d-2))(1+w_i)\phi} I_i(\tilde{\mathcal{R}}, \mathcal{M})\bigg). \end{aligned} \tag{9}$$

Next, we will take the terms we wrote above, but reduced to four dimensions, in the Einstein frame.

B. $\mathcal{R}^4$ terms in $d = 4$

In four dimensions, the Weyl tensor can be decomposed in its self-dual and antiself-dual parts[2]

$$\mathcal{W}_{\mu\nu\rho\sigma} = \mathcal{W}^+_{\mu\nu\rho\sigma} + \mathcal{W}^-_{\mu\nu\rho\sigma},$$
$$\mathcal{W}^{\mp}_{\mu\nu\rho\sigma} := \tfrac{1}{2}\Big(\mathcal{W}_{\mu\nu\rho\sigma} \pm \tfrac{i}{2}\varepsilon_{\mu\nu}{}^{\lambda\tau}\mathcal{W}_{\lambda\tau\rho\sigma}\Big). \tag{10}$$

The totally symmetric Bel-Robinson tensor is given in four dimensions by $\mathcal{W}^+_{\mu\rho\nu\sigma}\mathcal{W}^{-\rho}_{\tau}{}^{\sigma}_{\lambda}$. In the van der Warden notation, using spinorial indices [24], to $\mathcal{W}^+_{\mu\rho\nu\sigma}$, $\mathcal{W}^-_{\mu\rho\nu\sigma}$ correspond the totally symmetric $\mathcal{W}_{ABCD}$, $\mathcal{W}_{\dot A\dot B\dot C\dot D}$ being given by (in the notation of [11])

$$\mathcal{W}_{ABCD} := -\tfrac{1}{8}\mathcal{W}^+_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}},$$
$$\mathcal{W}_{\dot A\dot B\dot C\dot D} := -\tfrac{1}{8}\mathcal{W}^-_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{\dot A\dot B}}\sigma^{\rho\sigma}_{\underline{\dot C\dot D}}.$$

The decomposition (10) is written as

$$\mathcal{W}_{A\dot A B\dot B C\dot C D\dot D} = -2\varepsilon_{\dot A\dot B}\varepsilon_{\dot C\dot D}\mathcal{W}_{ABCD} - 2\varepsilon_{AB}\varepsilon_{CD}\mathcal{W}_{\dot A\dot B\dot C\dot D}. \tag{11}$$

The Bel-Robinson tensor is simply given by $\mathcal{W}_{ABCD}\mathcal{W}_{\dot A\dot B\dot C\dot D}$.

In four dimensions, there are only two independent real scalar polynomials made from four powers of the Weyl tensor [25], given by

$$\mathcal{W}^2_+\mathcal{W}^2_- = \mathcal{W}^{ABCD}\mathcal{W}_{ABCD}\mathcal{W}^{\dot A\dot B\dot C\dot D}\mathcal{W}_{\dot A\dot B\dot C\dot D}, \tag{12}$$

$$\mathcal{W}^4_+ + \mathcal{W}^4_- = (\mathcal{W}^{ABCD}\mathcal{W}_{ABCD})^2 + (\mathcal{W}^{\dot A\dot B\dot C\dot D}\mathcal{W}_{\dot A\dot B\dot C\dot D})^2. \tag{13}$$

In particular, the Weyl-dependent parts of the invariants I_X, I_Z, Y_1, Y_2, when computed directly in four dimensions (i.e. replacing the ten-dimensional indices $m, n, \ldots$ by the four-dimensional indices $\mu, \nu, \ldots$), should be expressed in terms of them. The details of the calculation can be seen in [17]; the resulting $\mathcal{W}^4$ terms are

$$I_X - \frac{1}{8}I_Z = 96\mathcal{W}^2_+\mathcal{W}^2_-,$$
$$X + \frac{1}{8}Z = 48(\mathcal{W}^4_+ + \mathcal{W}^4_-) + 672\mathcal{W}^2_+\mathcal{W}^2_-,$$
$$Y_1 = 8\mathcal{W}^2_+\mathcal{W}^2_-,$$
$$Y_1 + 4Y_2 = \frac{X}{6} + 2Y_1 = 80\mathcal{W}^2_+\mathcal{W}^2_- + 4(\mathcal{W}^4_+ + \mathcal{W}^4_-).$$

$I_X - \frac{1}{8}I_Z$ is the only combination of I_X and I_Z which in $d = 4$ does not contain (13), i.e. which contains only the square of the Bel-Robinson tensor (12). Interestingly, from (1) exactly this very same combination is the only one that does not depend on the ten-dimensional B^{mn} field and, therefore, due to its gauge invariance, is the only one that can appear in string theory at arbitrary loop order.

We should consider another possibility: could there be any four-dimensional $\mathcal{W}^4$ terms coming from the original ten-dimensional $I_X + \frac{1}{8}I_Z$ term in (1), but this time including the (four-dimensional) $B^{\mu\nu}$ field, as a scalar, after toroidal compactification and dualization (for a detailed treatment see [26])? Let us take

$$\partial^{[\mu}B^{\nu\rho]} = \epsilon^{\mu\nu\rho\sigma}\partial_\sigma D. \tag{14}$$

$B^{\mu\nu}$ is a pseudo 2-form under parity; after dualization in $d = 4$, D is a true scalar. This way, from the $\varepsilon_{10}t_8 B\mathcal{R}^4$ term in $d = 10$, one gets in $d = 4$, among other terms, derivatives of scalars and at most an $\mathcal{R}^2$ factor. (One also gets simply derivatives of scalars, without any Riemann tensor.) An $\mathcal{R}^4$ factor would only come, after dualization, from a higher-order term, always multiplied by derivatives of scalars. Therefore, we cannot get any $\mathcal{R}^4$ terms this way.

We then write the effective actions (4)–(6) in four dimensions, in the Einstein frame (considering only terms that are simply powers of the Weyl tensor, without any other fields except their couplings to the dilaton, and introducing the $d = 4$ gravitational coupling constant κ):

$$\frac{\kappa^2}{\sqrt{-g}}\mathcal{L}_{\mathrm{IIB}}|_{\mathcal{R}^4} = -\frac{\zeta(3)}{32}e^{-6\phi}\alpha'^3\mathcal{W}^2_+\mathcal{W}^2_- - \frac{1}{2^{11}\pi^5}e^{-4\phi}\alpha'^3\mathcal{W}^2_+\mathcal{W}^2_-, \tag{15}$$

$$\frac{\kappa^2}{\sqrt{-g}}\mathcal{L}_{\mathrm{IIA}}|_{\mathcal{R}^4} = -\frac{\zeta(3)}{32}e^{-6\phi}\alpha'^3\mathcal{W}^2_+\mathcal{W}^2_- - \frac{1}{2^{12}\pi^5}e^{-4\phi}\alpha'^3[(\mathcal{W}^4_+ + \mathcal{W}^4_-) + 224\mathcal{W}^2_+\mathcal{W}^2_-], \tag{16}$$

$$\frac{\kappa^2}{\sqrt{-g}}\mathcal{L}_{\mathrm{het}}|_{\mathcal{R}^2+\mathcal{R}^4} = -\frac{1}{16}e^{-2\phi}\alpha'(\mathcal{W}^2_+ + \mathcal{W}^2_-) + \frac{1}{64}(1 - 2\zeta(3))e^{-6\phi}\alpha'^3\mathcal{W}^2_+\mathcal{W}^2_- - \frac{1}{3\times 2^{12}\pi^5}e^{-4\phi}\alpha'^3[(\mathcal{W}^4_+ + \mathcal{W}^4_-) + 20\mathcal{W}^2_+\mathcal{W}^2_-]. \tag{17}$$

These are only the moduli-independent $\mathcal{R}^4$ terms. Strictly speaking, not even these terms are moduli-independent, since they are all multiplied by the volume of the compactification manifold, a factor we omitted for simplicity. But they are always present, no matter which compactification is taken. The complete action, for every different compactification manifold, includes many other moduli-dependent

[2] We used latin letters— $m, n, \ldots$—to represent ten-dimensional spacetime indices. From now on, we will be only working with four-dimensional spacetime indices which, to avoid any confusion, we represent by greek letters $\mu, \nu, \ldots$.

terms, which we do not consider here: we are mostly interested in a $\mathbb{T}^6$ compactification.

C. $\mathcal{R}^4$ terms and $d = 4$ supersymmetry

We are interested in the full supersymmetric completion of $\mathcal{R}^4$ terms in $d = 4$. In general, each superinvariant consists of a leading bosonic term and its supersymmetric completion, given by a series of terms with fermions.

The supersymmetrization of the square of the Bel-Robinson tensor $\mathcal{W}_+^2 \mathcal{W}_-^2$ has been known for a long time, in simple [9,10] and extended [12,15] four-dimensional supergravity. For the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$, there is a "no-go theorem," which goes as follows [16]: for a polynomial $I(\mathcal{W})$ of the Weyl tensor to be supersymmetrizable, each one of its terms must contain equal powers of $\mathcal{W}^+_{\mu\nu\rho\sigma}$ and $\mathcal{W}^-_{\mu\nu\rho\sigma}$. The whole polynomial must then vanish when either $\mathcal{W}^+_{\mu\nu\rho\sigma}$ or $\mathcal{W}^-_{\mu\nu\rho\sigma}$ do.

The derivation of this result is based on $\mathcal{N} = 1$ chirality arguments, which require equal powers of the different chiralities of the gravitino in each term of a superinvariant. The rest follows from the supersymmetric completion. That is why the only exception to this result is $\mathcal{W}^2 = \mathcal{W}_+^2 + \mathcal{W}_-^2$: in $d = 4$, this term is part of the Gauss-Bonnet topological invariant (it can be made equal to it with suitable field redefinitions). This term plays no role in the dynamics, and it is automatically supersymmetric; its supersymmetric completion is 0 and therefore does not involve the gravitino.

The derivation of [16] has been obtained using $\mathcal{N} = 1$ supergravity, whose supersymmetry algebra is a subalgebra of $\mathcal{N} > 1$. Therefore, it should remain valid for extended supergravity too. But one must keep in mind the assumptions that were made, namely, the preservation by the supersymmetry transformations of R symmetry which, for $\mathcal{N} = 1$, corresponds to U(1) and is equivalent to chirality. In extended supergravity theories, R symmetry is a global internal U($\mathcal{N}$) symmetry, which generalizes (and contains) U(1) from $\mathcal{N} = 1$.

Preservation of chirality is true for pure $\mathcal{N} = 1$ supergravity, but to this theory and to most of the extended supergravity theories, one may add matter couplings and extra terms that violate U(1) R symmetry and yet can be made supersymmetric, inducing corrections to the supersymmetry transformation laws that do not preserve U(1) R symmetry.

That was the procedure taken in [17], where the $\mathcal{N} = 1$ supersymmetrization of (14) was achieved by coupling this term to a chiral multiplet. A similar procedure may be taken in $\mathcal{N} = 2$ supergravity, since there exist $\mathcal{N} = 2$ chiral superfields which must be Lorentz and SU(2) scalars but can have an arbitrary U(1) weight, allowing for supersymmetric U(1) breaking couplings.

Such a result should be more difficult to achieve for $\mathcal{N} \geq 3$, because there are no generic chiral multiplets. But for $3 \leq \mathcal{N} \leq 6$ there are still matter multiplets which one can couple to the Weyl multiplet. Those couplings could eventually (but not necessarily) break U(1) R symmetry and lead to the supersymmetrization of (13).

An even more complicated problem is the $\mathcal{N} = 8$ supersymmetrization of (13). The reason is the much more restrictive character of $\mathcal{N} = 8$ supergravity, compared with lower $\mathcal{N}$. Besides, its multiplet is unique, which means there are no extra matter couplings one can take in this theory. Plus, in this case, the R-symmetry group is SU(8) and not U(8): the extra U(1) factor, which in $\mathcal{N} = 2$ could be identified with the remnant $\mathcal{N} = 1$ R symmetry and, if broken, eventually turn the supersymmetrization of (13) possible, does not exist. Apparently, there is no way to circumvent in $\mathcal{N} = 8$ the result from [16]. In order to supersymmetrize (13) in this case, one should then explore the different possibilities that were not considered in [16]. Since that article only deals with the term (13) by itself, one can consider extra couplings to it and only then try to supersymmetrize. This procedure is very natural, taking into account the scalar couplings that multiply (13) in the actions (16) and (17).

We now proceed with trying to supersymmetrize (13) but, first, we review the superspace formulation of $\mathcal{N} \geq 4$ supergravities and also some known higher-order superinvariants in these theories.

III. LINEARIZED SUPERINVARIANTS IN $d = 4$ SUPERSPACE

In this section, we review the superspace formulation of pure $\mathcal{N} \geq 4$ linearized supergravity theories and some of the known higher-order superinvariants, including a little discussion on the symmetries they should preserve. We will only be working at the linearized level, for simplicity.

One typically decomposes the U($\mathcal{N}$) R symmetry into SU($\mathcal{N}$) ⊗ U(1) and considers only SU($\mathcal{N}$) for the superspace geometry. U(1) is still present, but not in the superspace coordinate indices. The only exception is for $\mathcal{N} = 8$; the more restrictive supersymmetry algebra requires in this case the R-symmetry group to be SU(8), and there is no U(1) to begin with. We always work therefore in this section in conventional extended superspace with structure group SL(2; $\mathbb{C}$) ⊗ SU($\mathcal{N}$).

A. Linearized $\mathcal{N} \geq 4$, $d = 4$ supergravity in superspace

The field content of $\mathcal{N} \geq 4$ supergravity is essentially described by a superfield W^{abcd} [27,28], totally antisymmetric in its SU($\mathcal{N}$) indices, its complex conjugate $\bar{W}_{abcd}$ and their derivatives.

Still at the linearized level, one has the differential relations

$$\nabla_{Aa} W^{bcde} = -8\delta_a^{[b} W_A^{cde]}, \qquad \nabla_{Aa} W_B^{bcd} = 6\delta_a^{[b} W_{AB}^{cd]},$$
$$\nabla_{Aa} W_{BC}^{bc} = -4\delta_a^{[b} W_{ABC}^{c]}, \qquad \nabla_{Aa} W_{BCD}^{b} = -\delta_a^{b} W_{ABCD},$$
$$\nabla_{Aa} W_{BCDE} = 0, \tag{18}$$

and

$$\nabla^a_{\dot{A}} W_{BCDE} = 2i\nabla_{\underline{B}\dot{A}} W^a_{\underline{C}\underline{D}\underline{E}}, \qquad \nabla^a_{\dot{A}} W^b_{BCD} = i\nabla_{\underline{B}\dot{A}} W^{ab}_{\underline{C}\underline{D}},$$
$$\nabla^a_{\dot{A}} W^{bc}_{BC} = -i\nabla_{\underline{B}\dot{A}} W^{abc}_{\underline{C}}, \qquad \nabla^a_{\dot{A}} W^{bcd}_{B} = iN^{abcd}_{B\dot{A}}. \tag{19}$$

This last relation defines the superfield $N^{abcd}_{A\dot{A}}$ which, therefore, also satisfies

$$N^{abcd}_{A\dot{A}} = \nabla_{A\dot{A}} W^{abcd}, \tag{20}$$

$$\nabla_{Aa} N^{bcde}_{B\dot{B}} = -8\delta_a^{[b}\nabla_{\underline{A}\dot{B}} W^{cde]}_{\underline{B}}. \tag{21}$$

Here we should notice that these relations are valid for $N^{abcd}_{A\dot{A}}$, but not for its complex conjugate $\bar{N}_{A\dot{A}abcd}$. In other words, $\nabla_{Aa}\bar{N}_{B\dot{B}bcde}$ is another independent relation, like its Hermitian conjugate $\nabla_{\dot{A}a} N^{bcde}_{B\dot{B}}$, as we will see below [27,28].

The spinorial indices in the differential relations (18) are completely symmetrized. Indeed, at the linearized level, the corresponding terms with contracted indices vanish, through the Bianchi identities

$$\nabla^A{}_{\dot{A}} W^a_{ABC} = 0, \tag{22}$$

$$\nabla^A{}_{\dot{A}} W_{ABCD} = 0, \tag{23}$$

$$\nabla_{\underline{A}}{}^{\dot{B}} N^{bcde}_{\underline{B}\dot{B}} = 0. \tag{24}$$

For $\mathcal{N} \leq 6$, W^{abcd} is a complex superfield, which together with $\bar{W}_{abcd}$, describes at $\theta = 0$ the $2\binom{\mathcal{N}}{4}$ real scalars of the theory. In $\mathcal{N} = 8$ supergravity, the superfield W^{abcd} represents at $\theta = 0$ the $\binom{8}{4} = 70$ scalars of the full nonlinear theory. On shell, it satisfies the reality condition [27,28]

$$W^{abcd} = \frac{1}{4!}\varepsilon^{abcdefgh}\bar{W}_{efgh}. \tag{25}$$

Since $N^{abcd}_{A\dot{A}} = \nabla_{A\dot{A}} W^{abcd}$, from the previous relation one also has *on shell*, in linearized $\mathcal{N} = 8$ supergravity

$$N^{abcd}_{A\dot{A}} = \frac{1}{4!}\varepsilon^{abcdefgh}\bar{N}_{A\dot{A}efgh}. \tag{26}$$

Among the derivatives of W^{abcd}, there is the superfield W_{ABCD}, which from the differential relations (18) is related to W^{abcd} at the linearized level by $W_{ABCD} \propto \nabla_{Aa}\nabla_{Bb}\nabla_{Cc}\nabla_{Dd} W^{abcd} + \ldots$ The Weyl tensor appears as the $\theta = 0$ component of W_{ABCD}:

$$W_{ABCD}| = \mathcal{W}_{ABCD}. \tag{27}$$

Also, $W^b_{BCD}|$ is the Weyl tensor of the $\mathcal{N}$ gravitinos, $W^{bc}_{BC}|$ is the field strength of $\binom{\mathcal{N}}{2}$ vector fields and $W^{bcd}_{B}|$ are the $\binom{\mathcal{N}}{3}$ Weyl spinors.

In $\mathcal{N} = 6, 7$ supergravity, there exist extra $\binom{\mathcal{N}}{6}$ vector fields, described by $\bar{W}_{BCbcdefg}|$. In $\mathcal{N} = 5, 6, 7$ supergravity there also exist additional $\binom{\mathcal{N}}{5}$ Weyl spinors, described by $\bar{W}_{Bbcdef}|$.[3] In $\mathcal{N} = 8$ supergravity, these superfields do not represent new physical degrees of freedom, because then we have the following relations:

$$\bar{W}_{Bbcdef} = \tfrac{1}{2}\varepsilon_{bcdefgha} W^{gha}_{B},$$
$$\bar{W}_{BCbcdefg} = \tfrac{1}{6}\varepsilon_{bcdefgha} W^{ha}_{BC}. \tag{28}$$

The differential relations satisfied by these superfields can be derived, in $\mathcal{N} = 8$, from (28) and the previous relations (18) and (19). For $\mathcal{N} \leq 6$ supergravities, which are truncations of $\mathcal{N} = 8$, these relations are obtained from the $\mathcal{N} = 8$ corresponding ones, but considering that (25), (26), and (28) are not valid anymore (i.e. by considering W^{abcd} and $\bar{W}_{abcd}$ as independent superfields). This is the way one can derive the differential relations that are missing in (18) and (19), like $\nabla_{Aa}\bar{W}_{bcde} = -\frac{2}{3}\bar{W}_{Aabcde}$, and so on.

Again, for $4 \leq \mathcal{N} \leq 8$, on shell (which in linearized supergravity is equivalent to setting the SU($\mathcal{N}$) curvatures to zero), one has among others the field equations

$$\nabla^{A\dot{A}} W^{ab}_{AB} = 0, \tag{29}$$

$$\nabla^{A\dot{A}} N^{abcd}_{A\dot{A}} = 0. \tag{30}$$

At the component level, at $\theta = 0$ (30) represents the field equation for the scalars in linearized supergravity. Eqs. (25), (26), and (30) are only valid on shell, and are logically subjected to α' corrections. Plus, most of the equations in this section include nonlinear terms that we did not include here, but which can be seen in [27,28].

B. Higher-order superinvariants in superspace and their symmetries

Next, we will be analyzing linearized higher-order superinvariants in superspace.

There are known cases in the recent literature of apparent linearized $\mathcal{R}^4$ superinvariants in ten-dimensional type IIB supergravity that did not become true superinvariants [29,30]. One may therefore wonder if that could not happen in our case. But in $d = 4$, the structure of the transformation laws and the invariances of the supermultiplets are relatively easier and better understood than in $d = 10$, which guarantees us that the existence of the full superinvariants from the linearized ones is not in jeopardy,

[3] In $\mathcal{N} = 7$ supergravity, there also exists an additional ($\binom{\mathcal{N}}{7} = 1$) gravitino. Indeed, the $\mathcal{N} = 7$ and $\mathcal{N} = 8$ multiplets are identical.

although they may not fully preserve their symmetries. We summarize here the explanation that can be found in [14].

For $\mathcal{N} \leq 3$, one can get a full nonlinear superspace invariant from a linearized one simply by inserting a factor of E, the determinant of the supervielbein. This is also true for $\mathcal{N} \geq 4$, but here some remarks are necessary, as fields that transform nonlinearly may be present. In these cases, the classical equations of motion of the theory are invariant under some global symmetry group G. The theory also has a local H invariance, H being the maximal compact subgroup of G. The supergravity multiplet includes a set of Abelian vector fields with a local U(1) invariance. Because of this invariance, the U(1) potentials corresponding to the vector fields cannot then transform under H and must be representations of G.

In all these cases in the full nonlinear theory the scalar fields, represented in superspace by W^{abcd}, are elements of the coset space G/H. They do not transform linearly under G, but they still transform linearly under H. One can use the local H invariance to remove the nonphysical degrees of freedom by a suitable gauge choice. In order for this gauge to be preserved, nonlinear G transformations must be compensated by a suitable local H transformation depending on the scalar fields. Because of this, linearized superinvariants can then indeed be generalized to the nonlinear case by inserting a factor of E, the determinant of the supervielbein, but they will not have the full G symmetry of the original equations of motion. If we want the nonlinear superinvariants to keep this symmetry, we must restrict ourselves to superfields that also transform linearly, like those that occur directly in the superspace torsions.

In full nonlinear $\mathcal{N} = 8$ supergravity [31] $G = \mathrm{E}_{7(7)}$, a real noncompact form of E_7 whose maximal subgroup is $\mathrm{SL}(2;\mathbb{R}) \otimes \mathrm{O}(6,6)$ but whose maximal compact subgroup is $H = \mathrm{SU}(8)$. The 70 scalars are elements of the coset space $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$. Nonperturbative quantum corrections break $\mathrm{E}_{7(7)}$ to a discrete subgroup $\mathrm{E}_7(\mathbb{Z})$, which implies breaking the maximal subgroup $\mathrm{SL}(2;\mathbb{R}) \otimes \mathrm{O}(6,6)$ to $\mathrm{SL}(2;\mathbb{Z}) \otimes \mathrm{O}(6,6;\mathbb{Z})$. $\mathrm{O}(6,6;\mathbb{Z})$ is the T-duality group of a superstring compactified on a six-dimensional torus; $\mathrm{SL}(2;\mathbb{Z})$ extends to the full superstring theory as an S-duality group. In [32], evidence is given that $\mathrm{E}_7(\mathbb{Z})$ extends to the full superstring theory as a U-duality group. It is this U duality that requires (from a string theory point of view) that all the 70 scalars of the $\mathbb{T}^6$ compactification of superstring theory are on the same footing, even if originally, in the $d = 10$ theory, the dilaton is special.

Analogously, for $\mathcal{N} = 4$ supergravity coupled to m vector multiplets, we have $G = \mathrm{SL}(2;\mathbb{R}) \otimes \mathrm{O}(6,m)$, $H = \mathrm{U}(1) \otimes \mathrm{O}(6) \otimes \mathrm{O}(m)$. The conjectured full duality group for the corresponding toroidally compactified heterotic string, with $m = 16$, is $\mathrm{SL}(2;\mathbb{Z}) \otimes \mathrm{O}(6,22;\mathbb{Z})$.

The four-dimensional supergravity theories we have been considering can be seen as low energy effective field theories of toroidal compactifications of type II or heterotic superstring theories. The true moduli space of these string theories is the moduli space of the torus factored out by the discrete T-duality group Γ_T. For the case where the left-moving modes of the string are compactified on a p torus $\mathbb{T}^p$ and the right-moving modes on a q torus $\mathbb{T}^q$ [33], the moduli space is

$$\frac{\mathrm{SO}(p,q)}{\mathrm{SO}(p) \otimes \mathrm{SO}(q)} \Big/ \Gamma_T,$$

with $\Gamma_T = \mathrm{SO}(p,q;\mathbb{Z})$.

In particular, for type II theories compactified on $\mathbb{T}^6$, the moduli space is

$$\frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6) \otimes \mathrm{SO}(6)} \Big/ \Gamma_T, \tag{31}$$

with $\Gamma_T = \mathrm{SO}(6,6;\mathbb{Z})$.

For heterotic theories, left-moving modes are compactified on $\mathbb{T}^6$ and right-moving modes on $\mathbb{T}^{22}$, resulting for the moduli space

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,22)}{\mathrm{SO}(6) \otimes \mathrm{SO}(22)} \Big/ \Gamma_T,$$

with $\Gamma_T = \mathrm{SO}(6,22;\mathbb{Z})$. The factor $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$ is a separated component of moduli space spanned by a complex scalar including the dilaton, which lies in the gravitational multiplet and does not mix with the other toroidal moduli, lying in the 22 Abelian vector multiplets.

C. Some known linearized higher-order superinvariants

In Ref. [34], a general (for all $\mathcal{N}$) formalism for constructing four-dimensional superinvariants by integrating over even-dimensional submanifolds of superspace (''superactions'') was developed. Using this formalism, we will review some known linearized higher-order Riemann superinvariants. We will mostly be concerned with $\mathcal{N} = 8$ superinvariants, although the results can be easily extended to $4 \leq \mathcal{N} \leq 8$. For a more detailed treatment, see [34,35].

We will start by considering $\mathcal{W}^2 = \mathcal{W}^2_+ + \mathcal{W}^2_-$, the leading α' correction in the heterotic string effective action. Its $\mathcal{N} = 8$ supersymmetrization at the linearized level is given, up to numerical factors, by

$$\begin{aligned}\int \bar{W}_{efgh} W^{efgh} \mathrm{d}^8\theta + \mathrm{H.c.} \\ \propto \nabla^a_A \cdots \nabla^d_D \nabla^A_a \cdots \nabla^D_d (W^{efgh} \bar{W}_{efgh}) + \mathrm{H.c.} \\ \propto W^{ABCD} W_{ABCD} + \mathrm{H.c.}.\end{aligned} \tag{32}$$

The spinorial derivatives should be antisymmetrized, but from (18), one realizes that indeed happens, since all spinor indices are symmetrized and all SU(8) indices are antisymmetrized. In order to understand why $\mathcal{W}^2$, and no other dynamical terms, indeed result from (32), some preliminary basic calculations are necessary. From the differential relations (18), one can see that, at the linearized level, each W^{abcd} present cannot be acted by more than four (either dotted or undotted) spinorial derivatives

$$\begin{aligned} \nabla_{Aa}\nabla_{Bb}\nabla_{Cc}\nabla_{Dd}\nabla_{Ee}W^{fghi} &= 0, \\ \nabla^{a}_{\dot{A}}\nabla^{b}_{\dot{B}}\nabla^{c}_{\dot{C}}\nabla^{d}_{\dot{D}}\nabla^{e}_{\dot{E}}\bar{W}_{fghi} &= 0. \end{aligned} \tag{33}$$

From the same relations, one also easily derives

$$\begin{aligned} \nabla^{a}_{A}\nabla^{b}_{B}\nabla^{c}_{C}\nabla^{d}_{D}(W^{efgh}\bar{W}_{efgh}) = -384W^{abcd}W_{ABCD} \\ + \text{vector and fermion terms} \\ + \ldots \end{aligned} \tag{34}$$

In (32), one acts on (34) with four more spinorial derivatives, which from (18) do not act on W_{ABCD}: they act exclusively on W^{abcd} and in the fermion and vector terms, in such a way that by the end each W^{efgh}, $\bar{W}_{efgh}$ is acted by four and only four derivatives, such that the only possible final result is given by $\mathcal{W}^2 = \mathcal{W}^2_+ + \mathcal{W}^2_-$. From (33), any other possibility would vanish.

Because of the integration measure $d^8\theta$, (32) is not even an integral over half superspace; yet, this expression is indeed $\mathcal{N} = 8$ supersymmetric (and so are its $\mathcal{N} < 8$ truncations). To verify that we recall that at $\theta = 0$ the spinorial superderivatives equal the supersymmetry transformations

$$\nabla_{Bb}| = Q_{Bb}|, \qquad \nabla^{b}_{\dot{B}}| = Q^{b}_{\dot{B}}|.$$

That $\nabla^{b}_{\dot{B}}\nabla^{\dot{A}_1}_{a_1}\cdots\nabla^{\dot{A}_4}_{a_4}\nabla^{a_1}_{\dot{A}_1}\cdots\nabla^{a_4}_{\dot{A}_4}W^{efgh}\bar{W}_{efgh} = 0$ is obvious from (33). This way the supersymmetry variation of (32) is proportional to

$$\begin{aligned} &\nabla^{b}_{\dot{B}}[\nabla^{A_1}_{a_1}\cdots\nabla^{A_4}_{a_4}\nabla^{a_1}_{A_1}\cdots\nabla^{a_4}_{A_4}W^{efgh}\bar{W}_{efgh}] \\ &\quad \propto \nabla^{b}_{\dot{B}}(W^{ABCD}W_{ABCD}) \\ &\quad = 4iW^{ABCD}\nabla_{A\dot{B}}W^{b}_{BCD} \\ &\quad = 4i\nabla_{A\dot{B}}W^{ABCD}W^{b}_{BCD}, \end{aligned} \tag{35}$$

where in the last line we have used (23). This means (32) is indeed supersymmetric, as it transforms as a spacetime derivative. We notice that $\mathcal{W}^2_+ + \mathcal{W}^2_-$ is, by itself, supersymmetric [the completion is zero, as we noticed: no other terms result from (32)]. This is no surprise since, up to nondynamical Ricci terms, $\mathcal{W}^2$ is a topological invariant in $d = 4$.

The method of [34] was also used to obtain the $\mathcal{N} = 8$ supersymmetrization of $\mathcal{W}^2_+\mathcal{W}^2_-$ at the linearized level, which from (34) and its conjugate is given by [13]

$$\begin{aligned} &\int (\overline{W}_{a_1a_2a_3a_4}W^{a_1a_2a_3a_4})^2 d^8\theta d^8\overline{\theta} \\ &\quad \propto \int [\nabla^{\dot{A}_1}_{a_1}\cdots\nabla^{\dot{A}_4}_{a_4}\nabla^{a_1}_{\dot{A}_1}\cdots\nabla^{a_4}_{\dot{A}_4}\overline{W}_{b_1b_2b_3b_4}W^{b_1b_2b_3b_4}] \\ &\qquad \times [\nabla^{c_1}_{A_1}\cdots\nabla^{c_4}_{A_4}\nabla^{A_1}_{c_1}\cdots\nabla^{A_4}_{c_4}\overline{W}_{d_1d_2d_3d_4}W^{d_1d_2d_3d_4}] + \ldots \\ &\quad \propto W^{A_1A_2A_3A_4}W_{A_1A_2A_3A_4}W^{\dot{A}_1\dot{A}_2\dot{A}_3\dot{A}_4}W_{\dot{A}_1\dot{A}_2\dot{A}_3\dot{A}_4} + \ldots \end{aligned} \tag{36}$$

The " ..." represent extra terms at the linearized level resulting when the dotted and undotted derivatives act together in the same scalar superfield. Because of all these extra terms the $\mathcal{N} = 8$ supersymmetry of (36) is not so obvious, but it has been shown to be true [35].

IV. $\mathcal{W}^4_+ + \mathcal{W}^4_-$ AND EXTENDED SUPERSYMMETRY

In this section, we turn our attention to the new higher-order term $\mathcal{W}^4_+ + \mathcal{W}^4_-$ and try to supersymmetrize it at the linearized level using different methods.

We will only be working at the linearized level, for simplicity. Therefore, we will not be particularly concerned with the string loop effects considered in the discussion on the string effective actions, because of their dilaton couplings, which are necessarily highly nonlinear. We will be mainly concerned with the new $\mathcal{R}^4$ term in linearized supergravity, not worrying about the dilatonic factor in front of it to begin with (later this factor will be considered).

A. Superfield expression of $\mathcal{W}^4_+ + \mathcal{W}^4_-$

In the same way as $\mathcal{W}^4 = (\mathcal{W}^2_+ + \mathcal{W}^2_-)^2 = \mathcal{W}^4_+ + \mathcal{W}^4_- + 2\mathcal{W}^2_+\mathcal{W}^2_-$, the way of writing $\mathcal{W}^4$ as $\theta = 0$ components of superfields can also be seen—at the linearized level—as the "square" of the superfield expression of $\mathcal{W}^2 = \mathcal{W}^2_+ + \mathcal{W}^2_-$, given by (32). This way, by "taking the square" of (32), one obtains (36) and

$$\begin{aligned} &[\nabla^{c_1}_{A_1}\cdots\nabla^{c_4}_{A_4}\nabla^{A_1}_{c_1}\cdots\nabla^{A_4}_{c_4}\bar{W}_{d_1d_2d_3d_4}W^{d_1d_2d_3d_4}]^2 + \text{H.c.} \\ &\quad \propto (W^{A_1A_2A_3A_4}W_{A_1A_2A_3A_4})^2 + \text{H.c.} \end{aligned} \tag{37}$$

From (33), one sees that, in order for (37) not to vanish, each W^{abcd} must be acted by four and only four spinorial derivatives. This way, by the same arguments we used for (32) we see that from (37) one gets only a sum of products of four W_{ABCD} terms, eventually with different index contractions. Because of the uniqueness of $\mathcal{W}^4$ terms we mentioned—only (12) and (13)—the final result must be $\mathcal{W}^4_+ + \mathcal{W}^4_-$.

Therefore, (37) represents the expression of $\mathcal{W}^4_+ + \mathcal{W}^4_-$ in terms of superfields, up to some numerical factor. The fact that one can write this or any other term as a superfield component does not necessarily mean that it can be made supersymmetric; for that one has to show how to get it from a superspace invariant. In the present case, for (37), the most obvious candidate for such a superinvariant is

$$\int (W^{abcd}\bar{W}_{abcd})^2 \mathrm{d}^{16}\theta + \mathrm{H.c.} \tag{38}$$

By its structure (it requires integration over 16 θs), one can see that (38) is only valid for $\mathcal{N} = 8$ supergravity. One can write a similar expression but which is also valid for lower $\mathcal{N}$ by replacing W^{abcd} by some of its spinorial derivatives, while correspondingly lowering the number of θs in the measure. An expression that is equivalent (at the linearized level) to (37) but that is valid for $4 \leq \mathcal{N} \leq 8$ is

$$\int W^{abcd}\bar{W}_{abcd}W^{ABCD}W_{ABCD}\mathrm{d}^8\theta + \mathrm{H.c.}. \tag{39}$$

Here we notice that although both (38) and (39) are equivalent in $\mathcal{N} = 8$ as linearized component expansions (up to some different numerical factor), they represent two distinct expressions at the nonlinear level. Using (18), (19), and (23), one can compute the supersymmetry variation of the result of the θ integrations, which from (33) in both cases is uniquely given by (37). This variation, at the linearized level, is

$$\begin{aligned}\nabla^a_{\dot{A}}[(W^{BCDE}W_{BCDE})^2 + (W^{\dot{B}\dot{C}\dot{D}\dot{E}}W_{\dot{B}\dot{C}\dot{D}\dot{E}})^2] \\ = -8i\nabla_{B\dot{A}}(W^{FGHI}W_{FGHI}W^{BCDE}W^a_{CDE}) \\ + 16iW^{FGHI}\nabla_{B\dot{A}}(W_{FGHI}W^{BCDE}W^a_{CDE}).\end{aligned} \tag{40}$$

This supersymmetry transformation is not a total derivative and cannot be transformed into one. Therefore, neither (38) nor (39) represent a valid linearized superinvariant. This result is expected: it is just the confirmation of the prediction from [16] in $\mathcal{N} = 8$ which, as we said, is not easy to circumvent. Also, as we saw at the linearized level, (38) only gives $\mathcal{W}^4_+ + \mathcal{W}^4_-$ and no other terms. If (38) were supersymmetric, this would mean $\mathcal{W}^4_+ + \mathcal{W}^4_-$ was supersymmetric by itself, which does not make sense since it does not represent a topological invariant in $d = 4$, like $\mathcal{W}^2$ does. Therefore, the supersymmetrization of $\mathcal{W}^4_+ + \mathcal{W}^4_-$, if it exists, must come in a different way.

B. Attempts of supersymmetrization without modification of the linearized Bianchi identities

We now try to find out possible ways of supersymmetrizing $\mathcal{W}^4_+ + \mathcal{W}^4_-$ at the linearized level in $\mathcal{N} \geq 4$, $d = 4$ supergravity in superspace. The known solution to the superspace Bianchi identities [27,28] (equivalent to the x-space supersymmetry transformations) is only valid on shell for pure supergravity (without any kind of string corrections).

In principle, in order to supersymmetrize a higher-order term in the Lagrangian, one needs higher-order corrections to the superspace Bianchi identities (so one does to the x-space supersymmetry transformation laws), which should be of the same order in α'. In this section, we attempt to supersymmetrize (13) assuming that the solution to the Bianchi identities for pure supergravity remains valid. This a matter of simplicity: the complete solution to the Bianchi identities involves, even without any α' corrections, many nonlinear terms that we have not considered [27,28]. The α' corrections to the supersymmetry transformations are necessarily nonlinear and should affect and generate only nonlinear terms; it does not make sense to consider them if we are looking only for linearized superinvariants.

First, we check if it is possible to make some change in (39) in order to make it supersymmetric. We notice that the result in (40) only tells us that (39) is not supersymmetric by itself; it does not mean that it is not part of some superinvariant. In fact, maybe there exists some counterterm Φ that can be added to (39) in order to cancel the supersymmetry variation (40), so that the sum of (39) and Φ is indeed supersymmetric. In order for Φ to exist, it must then satisfy, for some $\Phi^e_{A\dot{A}\dot{E}}$,

$$\nabla^e_{\dot{E}}[((W^{ABCD}W_{ABCD})^2 + \mathrm{H.c.}) + \Phi] = \nabla^{A\dot{A}}\Phi^e_{A\dot{A}\dot{E}}. \tag{41}$$

Together with (40) this is a very difficult differential equation, to which we did not find any solution in terms of known fields, both for Φ and $\Phi^e_{A\dot{A}\dot{E}}$.

The second possibility in order to try to cancel the supersymmetry variation (40) is to multiply (39) by some factors Φ, $\bar{\Phi}$, such that the product is supersymmetric. In this case, Φ, $\bar{\Phi}$ must satisfy, for some $\Phi^e_{A\dot{A}\dot{E}}$,

$$\nabla^e_{\dot{E}}[\bar{\Phi}(W^{ABCD}W_{ABCD})^2 + \mathrm{H.c.}] = \nabla^{A\dot{A}}\Phi^e_{A\dot{A}\dot{E}}. \tag{42}$$

In this case, the factors Φ, $\bar{\Phi}$ must satisfy some restrictions, both by dimensional analysis (we want an α'^3 term) and by component analysis [we want to supersymmetrize $\mathcal{W}^4_+ + \mathcal{W}^4_-$ in the Einstein frame (16) and (17), with a factor of $\exp(-4\phi)$ and at most some other scalar couplings resulting from the compactification from $d = 10$]. Therefore, the only acceptable (and actually very natural) factors Φ, $\bar{\Phi}$ are simply functions of W^{abcd}, $\bar{W}_{abcd}$.

In any case, again (42) is a very difficult differential equation, which we tried to solve in terms of each of the different known fields. We were not able to find any solution, both for Φ, $\bar{\Phi}$ and $\Phi^e_{A\dot{A}\dot{E}}$, as one can see by considering (40), which cannot be canceled simply by taking factors of W^{abcd}, $\bar{W}_{abcd}$.

Therefore, one cannot supersymmetrize (13) using only the linearized (on-shell) solution to the Bianchi identities in pure supergravity. This result is not so expected and is not a confirmation of the prediction from [16] in $\mathcal{N} = 8$, which applies to (13) by itself and not when it is multiplied by a scalar factor. In the following subsection, we will use the full nonlinear solution to the Bianchi identities, but still at $\alpha' = 0$.

C. Attempts of supersymmetrization with nonlinear $\alpha' = 0$ Bianchi identities

The generic effective action (9) has a series of terms that we designated by $I_i(\tilde{\mathcal{R}}, \mathcal{M})$. Some of these terms can be directly supersymmetrized: they constitute the "leading terms," each one of them corresponding to an independent superinvariant. The remaining terms are part of the supersymmetric completion of the leading ones.

In general, it is very hard to determine the number of independent superinvariants. This problem becomes even more difficult in the presence of α' correction terms, because one single superinvariant includes terms at different orders in α'. For the complete supersymmetrization of a given higher-derivative term of a certain order in α', typically an infinite series of terms of arbitrarily high order in α' shows up. This series may be truncated to the order in α' in which one is working, but when supersymmetrizing the terms of higher order in α' the contributions from the lower-order terms must be considered. The reason is, of course, the α' dependence of the supersymmetry transformations. This has been explicitly shown for (12) and for $\mathcal{N} = 1, 2$ in [10,15]. At any given order in α', therefore, there are new leading terms (i.e. new superinvariants), and other terms that are part of superinvariants at the same order and at lower order.

Each time the supersymmetry transformation laws of single fields include linear terms, it should be possible to determine how to supersymmetrize an expression written only in terms of these fields already at the linearized level. A leading term of an independent superinvariant should then be invariant already at the linearized level. If this linearized supersymmetrization cannot be found for the term in question, but it still has to be made supersymmetric, it cannot be a leading term, and must emerge only at the nonlinear level, as part of the supersymmetric completion of some other term. That must be the case of (13), which we have tried to supersymmetrize directly at the linearized level, and we did not succeed. For the remainder of this section, we will examine that possibility.

Since the α' corrections necessarily introduce nonlinear terms in the supersymmetry transformations, and since one should not consider any higher-order term before considering all the corresponding lower-order terms, before looking for higher-order corrections to the supersymmetry transformations, one should first look at their nonlinear $\alpha' = 0$ terms. Here, we will only be concerned with the nonlinear terms of the on-shell relations, i.e. of those relations that will probably acquire α' corrections (25), (26), and (30).

The first two linearized equations, (25) and (26), refer to the 70 scalar fields of $\mathcal{N} = 8$ supergravity. As we mentioned, in the nonlinear theory, these fields are given by the coset space $E_{7(7)}/SU(8)$; they transform nonlinearly under $E_{7(7)}$, but they still transform linearly under SU(8) [31]. On shell, in superspace, at order $\alpha' = 0$, going from the linearized to the full nonlinear theory corresponds to replacing the constraint "SU(8) curvature=0" by "$E_{7(7)}$ curvature $= 0$". A complete treatment can be found in [27,28].

The superspace field Eq. (30) reflects the linearized field equation of the scalar fields in $4 \leq \mathcal{N} \leq 8$ supergravity, including the dilaton. For the action (9), the complete dilaton equation is given by

$$\nabla^2\phi - \frac{1}{2}\sum_i e^{(4/(2-d))(1+w_i)\phi} I_i(\tilde{\mathcal{R}}, \mathcal{M}) = 0. \tag{43}$$

At order $\alpha' = 0$, among the terms $I_i(\tilde{\mathcal{R}}, \mathcal{M})$, there should be those which contain field strengths corresponding to each of the vector fields present in the theory. Plus, still at order $\alpha' = 0$ there are couplings of the scalars to fermions, which we never considered explicitly but must be reflected in their field equations. In that order in α', the $\mathcal{N} = 8$ nonlinear version of (30), the field equation for the scalars, is given by [27,28]

$$\begin{aligned}\nabla^{A\dot{A}} N^{abcd}_{A\dot{A}} &= W^{\dot{A}\dot{B}}_{ef} W^{abcdef}_{\dot{A}\dot{B}} + 12 W^{AB[ab} W^{cd]}_{AB} \\ &\quad - \frac{3}{2} i W^{\dot{A}}_{efg} W^{Ae[ab} N^{cd]fg}_{A\dot{A}} \\ &\quad - \frac{2}{3} i W^{\dot{A}}_{efg} W^{A[abc} N^{d]efg}_{A\dot{A}} \\ &\quad + \frac{i}{12} W^{\dot{A}}_{efg} W^{Aefg} N^{abcd}_{A\dot{A}} + \text{4-fermion terms.}\end{aligned} \tag{44}$$

As one can see, this expression does not contain any nonlinear term that is exclusively dependent on the Weyl tensor. As one can confirm in [27,28], the same is true for each of the differential relations considered in (18) and (19). Therefore, we cannot expect (13) to emerge from the nonlinear completion of some (necessarily α'^3) linearized superinvariant. One must really understand the α' corrections to the Bianchi identities. Since these corrections are necessarily nonlinear, this means one cannot supersymmetrize (13) at the linearized level at all. Here, one must notice that never happened for the previously known higher-order terms, which all had its linearized superinvariant.

D. Corrections to the solution of the linearized Bianchi identities in $\mathcal{N} \geq 4$, $d = 4$ superspace: Some considerations

In each of the three effective actions (15)–(17), only the $\mathcal{W}^2_+ \mathcal{W}^2_-$ term contains the transcendental coefficient $\zeta(3)$. This term must then have its own superinvariant, as no other term has such a coefficient. Therefore, the changes in the supersymmetry transformation laws the other terms generate do not have such a coefficient and could not, by themselves, cancel the supersymmetry variation of (12).

Since the numerical coefficient in front of (13) in the $d = 4$ effective actions (16) and (17) is not transcendental, this term may eventually not need its own superinvariant and be part of some other superinvariant, with a different leading bosonic term, maybe even of a lower order in α', being related to (13) by an α'-dependent supersymmetry transformation. But even if such relation is valid in $d = 4$, that does not mean at all it should keep being valid in $d = 10$.

One can try to generate a higher-order (in α') term from a lower-order, higher-derivative superinvariant; maybe the higher-order term would lie on the orbit of its supersymmetry transformations. But in order to generate the higher-order term this way, one obviously needs to know the α'-corrected supersymmetry transformation laws.

One possibility would be to see if (13) could be obtained from the supersymmetrization of the $\mathcal{W}^2$ term in (32), of order α'. But this term does not come from type II theories, which only admit α'^3 corrections and higher; it only comes from the heterotic theories. Therefore, a $\mathcal{W}^2$ term must only be present as a correction to $\mathcal{N} = 4$ supergravity: it can also be written as an $\mathcal{N} = 8$ invariant, given by (32), but in this case, its stringy origin is not so obvious. Indeed, $\mathcal{R}^2$ terms show up from the $\mathcal{R}^4$ terms we are considering when we compactify string theory on a Calabi-Yau manifold [23], but for the moment we are only considering toroidal compactifications with maximal $d = 4$ supersymmetry.

There are other different terms one can consider. For instance, when going from the string frame (8) to the Einstein frame (9) with the transformation (7), one gets from a polynomial of the Riemann tensor a dilaton coupling and powers of derivatives of ϕ. The α'^3 effective action should contain, besides (12) and (13), the terms $((\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi))^2$, $(\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^2\phi)^2$ and $(\nabla^2\phi)^4$.

Taking as an example the α'^3 term $(\nabla^2\phi)^4$, it can be represented in superspace as part of $[(\nabla^{A\dot{A}}N^{abcd}_{A\dot{A}}) \times (\nabla_{B\dot{B}}\bar{N}^{B\dot{B}}_{abcd})]^2|$, which can indeed be supersymmetrized: from (18) and (19), this term should come from (36) by acting in each W^{abcd} with two undotted and two dotted spinorial derivatives (the same for $\bar{W}_{abcd}$). This should then be one of the terms represented by the dots in (36).

One therefore may expect the supersymmetrization of the higher-derivative term $I(\mathcal{R})$ (which in the case we are interested includes $\mathcal{W}^4_+ + \mathcal{W}^4_-$) to lie in the orbit of some power of $\nabla^2\phi$ or some other superinvariant of lower order in α', so that one term may result from the other via an α' dependent supersymmetry transformation. If that is the case, one needs to find the α' corrections to the (on-shell) solution of the superspace Bianchi identities, namely, to the nonlinear versions of (25) and (26), and especially (30).

Let us take, for example, the nonlinear dilaton field equation. Considering the pure gravitational α' corrections expressed in the effective actions (15)–(17), we are able to "guess" the expected corrections to (44), knowing the field content of W^{abcd} and its derivatives. Neglecting for the moment the numerical coefficients, one can see that some of the expected corrections to (44) (only the purely gravitational ones, i.e. those depending only on the Weyl tensor) are of the form

$$\begin{aligned}\nabla^{A\dot{A}}N^{abcd}_{A\dot{A}}|_{\alpha'+\alpha'^3} \propto\ & \alpha' W^{abcd}[W^{ABCD}W_{ABCD} + \text{H.c.}] \\ & + \alpha'^3 W^{abcd}[((W^{ABCD}W_{ABCD})^2 + \text{H.c.}) \\ & + (W^{ABCD}W_{ABCD})(W^{\dot{A}\dot{B}\dot{C}\dot{D}}W_{\dot{A}\dot{B}\dot{C}\dot{D}})] \\ & + \ldots \qquad (45)\end{aligned}$$

Of course, this equation must be completed with other contributions, which may be derived, including the numerical coefficients, from (16) and (17), once they are completed with the other leading α' corrections that do not depend only on the Riemann tensor.

It remains to be seen how are these corrections compatible with the superspace Bianchi identities. This would allow us to determine the α' corrections one needs to introduce in the other superspace field equations in order to the superspace Bianchi identities remain valid to this order in α'. This is a technically very complicated problem, which we are not addressing in the present work.

E. $\mathcal{W}^4_+ + \mathcal{W}^4_-$, U duality, and $\mathcal{N} = 8$ supergravity

As we mentioned before, the no-go theorem for the supersymmetrization of (13) given in [16] is based on $\mathcal{N} = 1$ chirality arguments. In order to circumvent these arguments, a reasonable possibility is to try to construct a superinvariant that violates the U(1) symmetry or (for $\mathcal{N} > 1$) some of the R symmetry. But the superfield expression corresponding to (13) given by (37) is even U(1) symmetric, as W_{ABCD} is U(1) invariant. (This is more clearly derived in $\mathcal{N} = 1$ superspace [17], but it is easily understood if one thinks that from (27) $W_{ABCD}|$ is a component of the Riemann tensor.) The best one can aim at is to break U(1) or part of the SU($\mathcal{N}$) by taking a different integration measure, as suggested in [34] and as we tried with (39). In $\mathcal{N} = 8$ superspace, one can keep trying extra couplings of the scalar superfields W^{abcd} combined with different nonstandard integration measures. But it is easier if we are allowed to consider other multiplets than the gravitational, whose couplings automatically violate U(1). That is not possible in $\mathcal{N} = 8$ supergravity, both because there are no other multiplets than the gravitational to consider, and because the extra U(1) symmetry does not exist. We recall that $\mathcal{N} \leq 6$ theories have a U($\mathcal{N}$) symmetry, which is split into SU($\mathcal{N}$) ⊗ U(1), but the more restrictive $\mathcal{N} = 8$ theory has originally only an SU(8) symmetry. This may be part of the origin of all the difficulties we faced when trying to supersymmetrize (13) in $\mathcal{N} = 8$.

But the main obstruction to this supersymmetrization is that, opposite to $\mathcal{W}_+^2 \mathcal{W}_-^2$, the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ does not seem to be compatible with the full R-symmetry group SU(8). In Ref. [36], a complete study has been made of all possible higher-order terms in $\mathcal{N} = 8$ supergravity, necessarily compatible with SU(8), and (13) does not appear in the list of possible terms.

Indeed, as we saw in the discussion of Sec. III B, only the local symmetry group of the moduli space of compactified string theories should be preserved by the four-dimensional perturbative string corrections. As we saw in (31), for $\mathbb{T}^6$ compactifications of type II superstrings, this group is given by $SO(6) \otimes SO(6) \sim SU(4) \otimes SU(4)$, which is a subgroup of SU(8). Most probably the perturbative string correction term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ only has this $SU(4) \otimes SU(4)$ symmetry. If that is the case, in order to supersymmetrize this term besides the supergravity multiplet, one must also consider U-duality multiplets [37], with massive string states and nonperturbative states. These would be the contributions we were missing.

But in conventional extended superspace, one cannot simply write down a superinvariant that does not preserve the $SU(\mathcal{N})$ R symmetry, which is part of the structure group. One can only consider higher-order corrections to the Bianchi identities that preserve $SU(\mathcal{N})$, like the ones from (45), but these corrections would not be able to supersymmetrize (13). $\mathcal{N} = 8$ supersymmetrization of this term would then be impossible; the only possible supersymmetrizations would be at lower $\mathcal{N}$, eventually consider U-duality multiplets.

The fact that one cannot supersymmetrize in $\mathcal{N} = 8$ a term that string theory requires to be supersymmetric, together with the fact that one needs to consider nonperturbative states (from U-duality multiplets) in order to understand a perturbative contribution may be seen as indirect evidence that $\mathcal{N} = 8$ supergravity is indeed in the swampland, as proposed in [38]. We believe that topic deserves further study.

V. CONCLUSIONS

We had shown in [17] that type IIA and heterotic string theories predict the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ to show up at one loop when compactified to four dimensions. Nonetheless, an older article [16] stated that this term, by itself, simply could not be made supersymmetric in $d = 4$. In [17], we worked out its $\mathcal{N} = 1$ supersymmetrization, by coupling it to a chiral multiplet. In this article, we considered the more complicated problem of its $\mathcal{N} = 8$ supersymmetrization. We obtained the superfield expression of that term, given by (37), and we have shown that expression indeed was not part of a superinvariant.

Since that term in $d = 10$ should come coupled to a dilaton, and it may acquire other scalar couplings after compactification to $d = 4$, in order to try to circumvent the argument of [16], we tried to construct a superinvariant that included this term, together with a proper scalar coupling, in general $4 \leq \mathcal{N} \leq 8$ superspace. We concluded that the supersymmetrization of this term at the linearized level, by itself, cannot be achieved, something that was always possible for the previously known higher-derivative string corrections.

We proposed some changes to the on-shell solution to the superspace Bianchi identities in order to include the lowest order α' corrections. We did not present the whole set of possible α' corrections to the Bianchi identities, nor did we try to solve them in order to check the consistency of these corrections and to determine their coefficients. In $\mathcal{N} = 8$ superspace, one can only have SU(8) invariant terms, and we argued $\mathcal{W}_+^4 + \mathcal{W}_-^4$ should be only $SU(4) \otimes SU(4)$ invariant. If that is the case, in order to supersymmetrize this term besides the supergravity multiplet, one must introduce U-duality multiplets, with massive string states and nonperturbative states. $\mathcal{N} = 8$ supersymmetrization of (13) may not be possible at all, which may be another argument favoring the hypothesis that $\mathcal{N} = 8$ supergravity is in the swampland [38]. This is a very fundamental topic of study, together with the recent claims of possible finiteness of $\mathcal{N} = 8$ supergravity. Plus, as we concluded from our analysis of the dimensional reduction of order α'^3 gravitational effective actions, the new $\mathcal{R}^4$ term (13) has its origin in the dimensional reduction of the corresponding term in M theory, a theory of which there is still a lot to be understood. We believe therefore that the complete study of this term and its supersymmetrization deserves further attention in the future.

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APPENDIX: SUPERSPACE CONVENTIONS

The superspace conventions for index manipulations and complex conjugations are essentially the same as in [15]. Underlined (resp. in square brackets) indices are symmetrized (resp. antisymmetrized) with weight one, i.e.

$$X_{\underline{AB}} = \frac{X_{AB} + X_{BA}}{2}, \qquad X_{[ab]} = \frac{X_{ab} - X_{ba}}{2}.$$

At the linearized level, when interchanging superspace covariant derivatives, we take all the supertorsions/curvatures to zero with the exception of

$$T_{Aa\dot{B}}{}^{bm} = -2i\delta_a^b \sigma^m_{A\dot{B}}. \quad \text{(A1)}$$

For a complete treatment of superspace supergravity at the nonlinear level, including the solution to the superspace Bianchi identities, we refer the reader to [27,28]. In the paper, we just summarize the results we need.

[1] D. J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
[2] M. T. Grisaru, A. E. M. van de Ven, and D. Zanon, Phys. Lett. B **173**, 423 (1986).
[3] D. J. Gross and J. H. Sloan, Nucl. Phys. **B291**, 41 (1987).
[4] M. B. Green, M. Gutperle, and P. Vanhove, Phys. Lett. B **409**, 177 (1997).
[5] K. Peeters, P. Vanhove, and A. Westerberg, Classical Quantum Gravity **18**, 843 (2001).
[6] M. de Roo, H. Suelmann, and A. Wiedemann, Nucl. Phys. **B405**, 326 (1993).
[7] Y. Hyakutake and S. Ogushi, Phys. Rev. D **74**, 025022 (2006).
[8] Y. Hyakutake and S. Ogushi, J. High Energy Phys. 02 (2006) 068.
[9] S. Deser, J. H. Kay, and K. S. Stelle, Phys. Rev. Lett. **38**, 527 (1977).
[10] F. Moura, J. High Energy Phys. 09 (2001) 026.
[11] F. Moura, J. High Energy Phys. 08 (2002) 038.
[12] S. Deser and J. H. Kay, Phys. Lett. **76B**, 400 (1978).
[13] R. E. Kallosh, Phys. Lett. **99B**, 122 (1981).
[14] P. S. Howe and U. Lindstrom, Nucl. Phys. **B181**, 487 (1981).
[15] F. Moura, J. High Energy Phys. 07 (2003) 057.
[16] S. M. Christensen, S. Deser, M. J. Duff, and M. T. Grisaru, Phys. Lett. **84B**, 411 (1979).
[17] F. Moura, J. High Energy Phys. 06 (2007) 052.
[18] M. B. Green, J. G. Russo, and P. Vanhove, J. High Energy Phys. 02 (2007) 099.
[19] M. B. Green, J. G. Russo, and P. Vanhove, Phys. Rev. Lett. **98**, 131602 (2007).
[20] Z. Bern, L. J. Dixon, and R. Roiban, Phys. Lett. B **644**, 265 (2007).
[21] Z. Bern *et al.*, Phys. Rev. Lett. **98**, 161303 (2007).
[22] A. A. Tseytlin, Nucl. Phys. **B467**, 383 (1996).
[23] I. Antoniadis, S. Ferrara, R. Minasian, and K. S. Narain, Nucl. Phys. **B507**, 571 (1997).
[24] R. Penrose and W. Rindler, *Spinors and Space-time—1—Two-spinor Calculus and Relativistic Fields* (Cambridge University Press, Cambridge, England, 1984).
[25] S. A. Fulling, R. C. King, B. G. Wybourne, and C. J. Cummins, Classical Quantum Gravity **9**, 1151 (1992).
[26] E. Cremmer, B. Julia, H. Lu, and C. N. Pope, Nucl. Phys. **B523**, 73 (1998).
[27] P. S. Howe, Nucl. Phys. **B199**, 309 (1982).
[28] M. Muller, *Consistent Classical Supergravity Theories*, Lect. Notes Phys, vol. 336 (Springer-Verlag, Berlin, 1989).
[29] S. de Haro, A. Sinkovics, and K. Skenderis, Phys. Rev. D **67**, 084010 (2003).
[30] A. Rajaraman, Phys. Rev. D **72**, 125008 (2005).
[31] L. Brink and P. S. Howe, Phys. Lett. **88B**, 268 (1979).
[32] C. M. Hull and P. K. Townsend, Nucl. Phys. **B438**, 109 (1995).
[33] K. S. Narain, Phys. Lett. **169B**, 41 (1986).
[34] P. S. Howe, K. S. Stelle, and P. K. Townsend, Nucl. Phys. **B191**, 445 (1981).
[35] P. S. Howe, arXiv:hep-th/0408177.
[36] J. M. Drummond, P. J. Heslop, P. S. Howe, and S. F. Kerstan, J. High Energy Phys. 08 (2003) 016.
[37] I. Bars and S. Yankielowicz, Phys. Rev. D **53**, 4489 (1996).
[38] M. B. Green, H. Ooguri, and J. H. Schwarz, Phys. Rev. Lett. **99**, 041601 (2007).