Domain walls with non-Abelian clouds

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(Received 4 March 2008; published 5 June 2008)

Domain walls in $U(N)$ gauge theories, coupled to Higgs scalar fields with degenerate masses, are shown to possess normalizable non-Abelian Nambu-Goldstone (NG) modes, which we call non-Abelian clouds. We construct the moduli space metric and its Kähler potential of the effective field theory on the domain walls by focusing on two models: a $U(1)$ gauge theory with several charged Higgs fields, and a $U(N)$ gauge theory with $2N$ Higgs fields in the fundamental representation. We find that non-Abelian clouds spread between two domain walls and that their rotation induces a long-range repulsive force, in contrast to a $U(1)$ mode in models with fully nondegenerate masses which gives a short-range force. We also construct a bound state of dyonic domain walls by introducing the imaginary part of the Higgs masses. In the latter model we find that when all walls coincide, $SU(N)_{\rm L} \times SU(N)_{\rm R} \times U(1)$ symmetry is broken down to $SU(N)_{\rm V}$, and $U(N)_{\rm A}$ NG modes and the same number of quasi-NG modes are localized on the wall. When n walls separate, off-diagonal elements of $U(n)$ NG modes have wave functions spreading between two separated walls (non-Abelian clouds), whereas some quasi-NG modes turn to NG bosons as a result of further symmetry breaking $U(n)_{\rm V} \to U(1)^n_{\rm V}$. In the case of $4+1$-dimensional bulk, we can dualize the effective theory to the supersymmetric Freedman-Townsend model of non-Abelian 2-form fields.

DOI: 10.1103/PhysRevD.77.125008 PACS numbers: 11.27.+d, 11.25.−w, 11.30.Pb, 12.10.−g

I. INTRODUCTION

The moduli space of solitons provides an elegant description of their classical and quantum dynamics [1]. If a global symmetry of the theory is spontaneously broken by the presence of solitons, a part of the moduli space is parametrized by Nambu-Goldstone (NG) modes associated with that broken symmetry. The broken symmetry acts on the moduli space metric as an isometry, which sometimes makes it an interesting object useful to determine the metric. In the case of symmetry spontaneously broken in vacua, the low energy effective action of corresponding NG modes can be constructed from only the information of a symmetry breaking pattern, by using the nonlinear realization method [2]. In some cases the moduli space metric of solitons can be determined thoroughly by symmetry alone. For instance in the case of Yang-Mills instantons, a single instanton solution in $SU(N)$ gauge theory can be obtained by embedding the minimal solution $A_\mu^{\rm BPST}$ of $SU(2)$ gauge theory found by Belavin *et al.* (BPST) with the position x_0 and the size ρ [3] as

$$A_\mu = U\begin{pmatrix} A_\mu^{\rm BPST}(x_0,\rho) & 0 \\ 0 & \mathbf{0}_{N-2}\end{pmatrix}U^\dagger, \qquad U \in \frac{SU(N)}{SU(N-2)\times U(1)}. \tag{1.1}$$

Here U brings a solution to another solution with degenerate masses or tension, so it gives a coset space of the NG modes. The moduli space in this case can be written as $\mathcal{I}_N^{k=1} \simeq \mathbf{C}^2 \times \mathbf{R}^+ \times \frac{SU(N)}{SU(N-2)\times U(1)}$ with $\mathbf{C}^2$ and $\mathbf{R}^+$ parametrized by x_0 and ρ, respectively [4].[1] The cone singularity of this moduli space corresponds to a small instanton configuration and is blown up in the case of the noncommutative $\mathbf{R}^4$ with the noncommutativity parameter θ [5], to yield

$$\mathcal{I}_{N,\theta}^{k=1} \simeq \mathbf{C}^2 \times T^*\mathbf{C}P^{N-1}. \tag{1.2}$$

The moduli space of separated multiple instantons is a symmetric product of $\mathcal{I}_N^{k=1}$'s (or $\mathcal{I}_{N,\theta}^{k=1}$'s). The orbifold singularities of it are resolved by the Hilbert scheme resulting in the full moduli space (which is smooth for $\mathcal{I}_{N,\theta}^{k=1}$ but still contains small instanton singularities for $\mathcal{I}_N^{k=1}$).

A similar structure has been recently found in the case of vortices in certain non-Abelian gauge theories [6,7]. A

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[1]As a result the moduli space $\mathcal{I}_N^{k=1}$ is a cone over a tri-Sasakian manifold.

$U(N)$ gauge theory with N Higgs fields in the fundamental representation, denoted by an N by N matrix H, admits a minimal vortex solution

$$H = U\begin{pmatrix} H^{\rm ANO}(z-z_0) & 0 \\ 0 & \sqrt{c}\mathbf{1}_{N-1}\end{pmatrix}U^\dagger,$$
$$F_{12} = U\begin{pmatrix} F_{12}^{\rm ANO}(z-z_0) & 0 \\ 0 & \mathbf{0}_{N-1}\end{pmatrix}U^\dagger, \qquad (1.3)$$
$$U \in \frac{SU(N)}{SU(N-1)\times U(1)} \simeq \mathbf{C}P^{N-1}$$

where the Abrikosov-Nielsen-Olesen (ANO) [8] vortex solution ($H^{\rm ANO}$, $F_{12}^{\rm ANO}$) in the Abelian-Higgs model is embedded into the uppermost and leftmost components of the N by N matrices of the Higgs fields H and the gauge (magnetic) fields F_{12} in the x^1– x^2 plane. Here $z = x^1 + ix^2$ is a co-dimensional coordinate of vortices. A remarkable point is that the vortex solution (1.3) contains non-Abelian orientational moduli $\mathbf{C}P^{N-1}$ as in the instanton solution (1.1), in addition to the translational moduli $z_0 \in \mathbf{C}$; The moduli space is [6,7]

$$\mathcal{V}_N^{k=1} \simeq \mathbf{C} \times \mathbf{C}P^{N-1}. \qquad (1.4)$$

Again the moduli space of separate multiple vortices is a symmetric product of $\mathcal{V}_N^{k=1}$'s. The full moduli space was constructed in [9] in which it turns out to be smooth with resolving orbifold singularities similarly to instantons. These vortices are called "non-Abelian vortices" because the unbroken symmetry of the vacuum is non-Abelian. In general, when solitons exist in a symmetry breaking $G \to H$ with non-Abelian group H, they are called non-Abelian solitons irrespective of whether H is a gauge (local) or global symmetry.[2] Then non-Abelian solitons are usually accompanied by non-Abelian orientational moduli. See [10–12] for a review. It was observed by Hanany and Tong [6] that the moduli space of non-Abelian vortices is a certain middle-dimensional submanifold of the moduli space of noncommutative instantons. In fact, the moduli space $\mathcal{V}_N^{k=1}$ in (1.4) of the single vortex solution (1.3) is a special Lagrangian submanifold of the moduli space $\mathcal{I}_{N,\theta}^{k=1}$ in (1.2) of the single noncommutative instanton. Physically this correspondence may be understood by the fact that instantons become vortices (sigma model instantons) if they lie inside a vortex [13,14].

Similarly to instantons and vortices, a correspondence between "Abelian" monopoles and Abelian domain walls was found by Hanany and Tong [15]. The 't Hooft-Polyakov monopoles [16] are called Abelian because they occur when a gauge symmetry G is broken to an Abelian subgroup H of G. Typically it is $G = SU(2) \to H = U(1)$. In this case each monopole carries the moduli

$$\mathcal{M}^{k=1} \simeq \mathbf{R}^3 \times S^1, \qquad (1.5)$$

where $\mathbf{R}^3$ corresponds to the position and S^1 to the phase of the internal space.

In this paper we study domain walls in supersymmetric gauge theories (and corresponding nonlinear sigma models) with eight supercharges. So far, domain walls with eight supercharges have been mostly considered in gauge theories with $U(1)$ gauge field [17–20] or $U(N)$ gauge fields [21–25] coupled to Higgs scalar fields with *non-degenerate* masses except for [26,27]. In the case of non-degenerate Higgs masses, the flavor symmetry is Abelian: $U(1)^{N_{\rm F}-1}$ and the symmetry of the vacua is also Abelian. As a result each domain wall carries a $U(1)$ orientational modulus [17,21].The moduli space of a single domain wall is

$$\mathcal{W}^{k=1} \simeq \mathbf{R} \times S^1. \qquad (1.6)$$

From this viewpoint, these domain walls should be called Abelian domain walls even when the gauge symmetry of the Lagrangian is non-Abelian [21–25].[3]

The moduli space (1.6) of a single domain wall is a middle-dimensional submanifold of the moduli space (1.5) of a single Abelian monopole as discussed in [15]. The moduli space of multiple domain walls was constructed in $U(N_{\rm C})$ gauge theory coupled to Higgs fields with non-degenerate masses [21,22], and the correspondence to the multimonopole moduli space was studied in [15] as noted above. Similarly to the correspondence between instantons and vortices, this correspondence may be understood by noting that monopoles become domain walls inside a vortex [13,28,29].

Non-Abelian monopoles appear when gauge symmetry G is broken down to non-Abelian subgroup H [30–33] which is the case that some vacuum expectation values (VEV) of adjoint Higgs fields are degenerate. As a result non-Abelian zero modes appear around the non-Abelian monopoles. Some of these zero modes are normalizable modes which are easy to deal with; when we turn on a small difference in VEVs of adjoint Higgs fields with degenerate VEVs, one non-Abelian monopole is split into two Abelian monopoles, the light monopole with the mass corresponding to the small difference between the VEVs and the one with almost the same mass of the original non-Abelian monopole. When the difference between the VEVs decreases, the light monopole grows with the size bounded from the above by the distance to the other monopole. This mode was called the *non-Abelian cloud* by Eric Weinberg [30]. However, the other modes around non-Abelian monopoles are nonnormalizable and cannot be considered as moduli of monopoles themselves.

[2]In the case of instantons H is G itself because the symmetry G is not broken.

[3]In our early papers [21–25] we called these solutions non-Abelian domain walls because of the non-Abelian gauge symmetry, but this is not appropriate in the current definition of non-Abelian solitons.

The latter makes the study of non-Abelian monopoles difficult, which is in fact a notorious problem. Non-Abelian monopoles are important ingredients for a non-Abelian extension of duality in supersymmetric gauge theories [32,33].

Our concern in this paper is about domain walls with non-Abelian orientational moduli, which may be called *non-Abelian domain walls*. One expects that the relation found by Hanany-Tong [15] between Abelian monopoles and Abelian domain walls can be extended to the one between *non-Abelian* monopoles and *non-Abelian* domain walls. One motivation to study non-Abelian domain walls is to obtain a hint to understand non-Abelian monopoles through this correspondence. Unlike instantons or vortices, Higgs fields need masses for domain walls to exist. Once the Higgs masses are (partially) *degenerate*, the model exhibits a non-Abelian flavor symmetry G and the vacua break G into its non-Abelian subgroup H. Then the domain wall solutions further break the non-Abelian symmetry H of vacua and are expected to acquire non-Abelian orientational moduli associated with the breaking of H, resulting in non-Abelian domain walls. In fact $U(2)$ moduli were already found by Shifman and Yung [26,27] in the $U(2)$ gauge theory coupled to four charged Higgs fields with the common $U(1)$ charge and the mass matrix $M = \mathrm{diag}(m, m, -m, -m)$.

In this paper we study zero modes of non-Abelian domain walls and their properties in two different models. The first model is a $U(1)$ gauge theory with N_F Higgs fields with an N_F by N_F mass matrix $M = \mathrm{diag}(m_1, 0, \cdots, 0, -m_2)$. The second model is a $U(N)$ gauge theory with $N_F = 2N$ Higgs fields in the fundamental representation, with half of the Higgs masses being $-m$ and the rest being m. We call the latter the generalized Shifman-Yung (GSY) model because the case of $N = 2$ was discussed by Shifman and Yung [26,27]. Solitons preserving a part of supersymmetry in supersymmetric theories are called Bogomol'nyi-Prasad-Sommerfield (BPS) solitons. Recently we have worked out a method to obtain an effective Lagrangian on the BPS solitons [34]. By using this method, we construct the Kähler potential and the metric of the effective Lagrangian of normalizable zero modes (moduli) of domain walls in these two models. It is a supersymmetric nonlinear sigma model with the moduli space of domain walls as its target space. We find that the target space of the first model is $\mathbf{C}^* \times \mathbf{C}^{N_F-2}$, equipped with a nonflat metric for the latter, on which the isometry $\mathbf{C}^* \times U(N_F - 2)$ acts. The target space of the second (GSY) model turns out to be $GL(N, \mathbf{C}) \simeq \mathbf{C}^* \times SL(N, \mathbf{C})$ on which the isometry $\mathbf{C}^* \times SU(N)_L \times SU(N)_R$ acts. We find the following. When positions of all domain walls coincide, $SU(N)_V$ symmetry is preserved and the massless Nambu-Goldstone modes $[SU(N)_L \times SU(N)_R]/SU(N)_V \simeq SU(N)_A$, associated with the non-Abelian flavor symmetry breaking $SU(N)_L \times SU(N)_R \to SU(N)_V$, are localized at the coincident wall. When n (among N) domain walls are separated, however, the $SU(n)_V$ subgroup of $SU(N)_V$ is further broken down to $U(1)_V^{n-1}$. Consequently only the diagonal $U(1)_A^{n-1}$ Nambu-Goldstone modes in $SU(n)_A \times [\subset SU(N)_A]$ are localized on each individual wall and the off-diagonal Nambu-Goldstone modes in $SU(n)_A$ have wave functions spreading between a set of two separated walls. The latter can be called *non-Abelian clouds* because corresponding modes have been introduced in the context of non-Abelian monopoles [30]. We find that these non-Abelian clouds remain massless in the GSY model when domain walls are separated.[4]

In the above we see that the number of NG modes can change depending on the positions of the walls. A question is whether the number of massless modes or dimensionality of the moduli space changes or not. The answer is no; the total number of massless modes is preserved. Key ingredients to understanding this phenomenon are so-called quasi-Nambu-Goldstone modes which do not directly correspond to underlying spontaneously broken global symmetry but are required from unbroken supersymmetry [35,36].[5] When all the domain walls coincide there exist quasi-NG modes, as many as $SU(N)_A$ NG modes. Among them diagonal $N - 1$ modes represent positions of the walls. When n walls separate, some quasi-NG modes turn to the NG modes $SU(n)_V/U(1)_V^{n-1}$ for the further symmetry breaking $SU(n)_V \to U(1)_V^{n-1}$. Therefore, the quasi-NG modes and NG modes can change to each other with the total number of massless modes unchanged. All of these states with different symmetry breaking patterns are degenerate, which was originally found by G. Shore [36] in the context of supersymmetric nonlinear realizations.

We also construct the Lagrangian in a dual description by 2-form fields on the domain wall world volume, when domain walls (with 3 + 1-dimensional world volume) exist in $d = 4 + 1$ dimensions. This is in contrast to the 2 + 1-dimensional world volume, where vector fields in a dual description have been obtained only for free field part without interactions [26]. In the case of the GSY model, we can obtain the supersymmetric extension [38] of the so-called Freedman-Townsend model [39] of non-Abelian 2-forms with nontrivial interaction.

Although we have emphasized the importance of the relation to non-Abelian monopoles in this introduction,

[4]In Ref. [26], the authors argued that these modes spreading between walls become massive, contrary to our results.

[5]These massless bosons are considered in the context of the preon models and the nonlinear realization of spontaneously broken global symmetries with preserving supersymmetry. The additional massless fermions to constitute the chiral multiplets are called quasi-Nambu-Goldstone fermions [37]. The presence of these massless non-Abelian clouds is a distinguishing feature of the walls with non-Abelian flavor symmetry at the classical level, in contrast to the open string modes becoming massive when D-branes are separated.

this work may have some impacts on the brane-world scenario [40] too. Our model can be made in dimension $d = 4 + 1$ so that we have domain walls as branes with $3 + 1$-dimensional world volume and $\mathcal{N} = 1$ supersymmetry. The non-Abelian clouds found in this paper have a wave function spreading between two branes. One brane has an interaction from another brane mediated by these interbrane modes.

This paper is organized as follows. In Sec. II, we define our model and review methods to obtain the effective Lagrangian on walls. In Sec. II C the local structure of the moduli space is investigated. In Sec. II D we understand it by means of the kinky D-brane configurations.

In Sec. III we study the $U(1)$ gauge theory with $N_{\rm F}$ charged Higgs scalar fields as the simplest model of domain walls with degenerate Higgs masses. First of all in Sec. III A we study the simplest case of $N_{\rm F} = 4$ to show the behavior of non-Abelian clouds. To this end we introduce a small mass spiting ϵ in degenerate masses so that we have a domain wall with a small tension (proportional to ϵ) between the two usual domain walls. As the mass splitting decreases $\epsilon \to 0$, the width of such a domain wall (proportional to ϵ^{-1}) grows. In the end the domain wall profile is bounded by the positions of the neighboring two domain walls and fills between them. Thus it becomes a non-Abelian cloud. This technique was used by E. Weinberg to study non-Abelian clouds in non-Abelian monopoles [30]. In Sec. III B we construct the Kähler potential of the effective action for the moduli of the domain walls with general $N_{\rm F}$. The moduli space is $\mathbf{C}^* \times \mathbf{C}^{N_{\rm F}-2}$ with the isometry $\mathbf{C}^* \times U(N_{\rm F} - 2)$. In order to study the dynamics of the domain walls, we construct the conserved charges of the isometry $U(2)$ in the case of $N_{\rm F} = 4$ for degenerate masses. We find that the two kinds of repulsive forces exist between the two domain walls with distance R; one comes from the $U(1)$ part of the isometry and its potential exponentially approaches a constant as the distance R becomes large, and the other comes from the $SU(2)$ part of the isometry and its potential behaves as $1/R$. The former has been known in the model with nondegenerate masses, which is mediated by massive modes between the two walls. The latter is new and is mediated by the non-Abelian clouds which are massless modes propagating between the two walls. In Sec. III C we construct a bound state of domain walls by introducing additional masses in the imaginary parts of the Higgs fields. The additional masses introduce an attraction between the walls and then balance with the repulsion by the charges of the non-Abelian clouds. The bound state is a dyonic domain wall of a new kind.

In Sec. IV we work out the generalized Shifman-Yung model. After presenting the vacua in Sec. IV A we construct domain wall solutions in Sec. IV B. In Sec. IV C we study the symmetry structure of the moduli space of the domain walls in the GSY model. If the positions of all the domain walls coincide, $SU(N)_{\rm L} \times SU(N)_{\rm R}[\times U(1)_{\rm A}]$ is spontaneously broken to $SU(N)_{\rm V}$ in the presence of the domain walls. A part of moduli space is parametrized by the Nambu-Goldstone modes (we may call them pions in analogy with the chiral symmetry breaking in hadron physics) associated with this breaking. The rest is parametrized by quasi-Nambu-Goldstone modes which are required by unbroken supersymmetry. Some of them correspond to the positions of domain walls. When the walls are separated, the symmetry $SU(N)_{\rm V}$ is further broken down to its subgroup and hence there are more Nambu-Goldstone modes. These Nambu-Goldstone modes at finite wall separation become the quasi-Nambu-Goldstone modes in the limit of coincident walls. In Sec. IV D we construct the Kähler potential of the effective Lagrangian of domain walls with arbitrary gauge coupling constant. The moduli space turns out to be $GL(N, \mathbf{C}) \simeq \mathbf{C}^* \times SL(N, \mathbf{C})[\simeq U(N)^{\mathbf{C}} \simeq T^* U(N)]$ on which $SU(N)_{\rm L} \times SU(N)_{\rm R} \times U(1)_{\rm A}$ acts as the isometry. In Sec. IV E we study wave functions of the modes by taking the strong gauge coupling limit. We find that non-Abelian clouds are spread between domain walls. Finally in Sec. IV F we expand the effective Lagrangian around the configurations in the cases of coincident walls and well-separated walls. In the former case we obtain the chiral Lagrangian as expected. We then study the effect of imaginary masses of the Higgs fields, to obtain a pion mass term.

In Sec. V the duality transformation is performed for the massless particles to obtain the non-Abelian tensor multiplets.

In Sec. VI we apply our results to non-Abelian monopoles confined by non-Abelian vortices in the Higgs phase. We briefly discuss a monopole-monopole bound state.

Section VII is devoted to the conclusion and a discussion.

II. NON-ABELIAN WALLS WITH DEGENERATE HIGGS MASSES

A. Models, symmetry, and vacua

We consider $U(N_{\rm C})$ gauge theory in space-time dimension from $d = 1 + 1$ to $d = 4 + 1$ with (at least one) real scalar field Σ in the adjoint representation and $N_{\rm F}(>N_{\rm C})$ flavors of massive Higgs scalar fields in the fundamental representation, denoted as an $N_{\rm C} \times N_{\rm F}$ matrix H. Choosing the minimal kinetic term, we obtain

$$\mathcal{L} = \mathcal{L}_{\rm kin} - V, \tag{2.1}$$

$$\mathcal{L}_{\rm kin} = {\rm Tr}\Big(-\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{g^2}\mathcal{D}_\mu \Sigma \mathcal{D}^\mu \Sigma + \mathcal{D}^\mu H (\mathcal{D}_\mu H)^\dagger \Big), \tag{2.2}$$

where the covariant derivatives and field strengths are defined as $\mathcal{D}_\mu \Sigma = \partial_\mu \Sigma + i[W_\mu, \Sigma]$, $\mathcal{D}_\mu H = (\partial_\mu + iW_\mu)H$, $F_{\mu\nu} = -i[\mathcal{D}_\mu, \mathcal{D}_\nu]$. Our convention for the space-time metric is $\eta_{\mu\nu} = \mathrm{diag}(+, -, \cdots, -)$. The scalar potential V is given in terms of a diagonal mass matrix M and a real parameter c as

$$V = \mathrm{Tr}\left[\frac{g^2}{4}(c\mathbf{1} - HH^\dagger)^2 + (\Sigma H - HM)(\Sigma H - HM)^\dagger\right]. \tag{2.3}$$

This Lagrangian can be made supersymmetric by adding another scalar in the fundamental representation ($H^1 \equiv H$, $H^2 = 0$) and fermionic partners of all these bosons. The resulting theory has eight supercharges. We have chosen for simplicity the gauge couplings for $U(1)$ and $SU(N_\mathrm{C})$ to be identical to obtain simple solutions classically, even though they are independent. The real positive parameter c is called the Fayet-Iliopoulos (FI) parameter, which can appear in supersymmetric $U(1)$ gauge theories [41].

Next let us discuss the vacuum structure of this model. In the case of massless Higgs fields ($M = 0$), the Lagrangian enjoys a flavor symmetry $SU(N_\mathrm{F})$ [the overall $U(1)$ is gauged in this model]. The vacua constitute the Higgs branch, which is isomorphic to a hyper-Kähler manifold $T^*G_{N_\mathrm{F},N_\mathrm{C}}$, the cotangent bundle over the Grassmann manifold [42]

$$G_{N_\mathrm{F},N_\mathrm{C}} \simeq \frac{SU(N_\mathrm{F})}{SU(N_\mathrm{C}) \times SU(N_\mathrm{F} - N_\mathrm{C}) \times U(1)}. \tag{2.4}$$

The coset structure reflects the fact that the global symmetry $SU(N_\mathrm{F})$ is spontaneously broken and that the Nambu-Goldstone bosons for the broken symmetry appear.

When masses are fully nondegenerate the flavor symmetry is explicitly broken down to $U(1)^{N_\mathrm{F}-1}$ and the vacua reduce to a finite number of discrete points [43]. The number of vacua is given by $\frac{N_\mathrm{F}!}{N_\mathrm{C}!(N_\mathrm{F}-N_\mathrm{C})!}$. All the Nambu-Goldstone bosons become pseudo-Nambu-Goldstone bosons with masses. Domain wall solutions interpolating between these vacua have already been discussed [21–25].

On the other hand, when the Higgs masses are partially degenerate the flavor symmetry is enhanced as

$$U(1)^{N_\mathrm{F}-1} \to SU(N_1) \times SU(N_2) \times \cdots \tag{2.5}$$

while N_i masses are degenerate ($i = 1, 2, \cdots$). There appear Nambu-Goldstone modes continuously parametrizing degenerate vacua, which constitute a submanifold of the massless Higgs branch $T^*G_{N_\mathrm{F},N_\mathrm{C}}$. This is the situation which we consider in this paper.

B. BPS equations and the moduli matrix

The 1/2 BPS equations for domain walls interpolating the discrete vacua can be obtained by usual Bogomol'nyi completion of the energy

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dy \mathrm{Tr}\bigg[(\mathcal{D}_y H - HM + \Sigma H)^2 \\ &\quad + \frac{1}{g^2}\left(\mathcal{D}_y\Sigma - \frac{g^2}{2}(c\mathbf{1} - HH^\dagger)\right)^2 + c\mathcal{D}_y\Sigma\bigg] \\ &\geq c[\mathrm{Tr}\Sigma(\infty) - \mathrm{Tr}\Sigma(-\infty)]. \end{aligned} \tag{2.6}$$

The first-order differential equations for the configurations saturating this energy bound are of the form [21]

$$\mathcal{D}_y H = HM - \Sigma H, \qquad \mathcal{D}_y\Sigma = \frac{g^2}{2}(c\mathbf{1} - HH^\dagger). \tag{2.7}$$

Here we consider static configurations depending only on the y direction.

Let us solve these 1/2 BPS equations. First, the first equation can be solved by [21]

$$H = S^{-1}(y)H_0 e^{My}, \qquad \Sigma + iW_y = S^{-1}(y)\partial_y S(y). \tag{2.8}$$

Here H_0, called the moduli matrix, is an $N_\mathrm{C} \times N_\mathrm{F}$ constant complex matrix of rank N_C and contains all the moduli parameters of solutions. The matrix-valued quantity $S(y) \in GL(N_\mathrm{C}, \mathbf{C})$ is determined by the second equation in (2.7) which can be converted to the following equation for $\Omega \equiv SS^\dagger$:

$$\frac{1}{cg^2}[\partial_y(\Omega^{-1}\partial_y\Omega)] = \mathbf{1}_{N_\mathrm{C}} - \Omega^{-1}\Omega_0, \qquad \Omega_0 \equiv \frac{1}{c}H_0 e^{2My}H_0^\dagger. \tag{2.9}$$

This equation is called the master equation for domain walls. From the vacuum conditions at spatial infinities $y \to \pm\infty$, we can see that the solution Ω of the master equation should satisfy the boundary condition $\Omega \to \Omega_0$ as $y \to \pm\infty$. It determines S for a given moduli matrix H_0 up to the gauge transformations $S^{-1} \to US^{-1}$, $U \in U(N_\mathrm{C})$ and then the physical fields can be obtained through (2.8). Note that the master equation is symmetric under the following V-transformations:

$$H_0 \to VH_0 \quad \text{and} \quad S(y) \to VS(y) \quad \text{with} \quad V \in GL(N_\mathrm{C}, \mathbf{C}), \tag{2.10}$$

and if the moduli matrices are related by the V-transformations $H_0' = VH_0$, they give physically equivalent configurations. We call this equivalence relation the V-equivalence relation and denote it as $H_0 \sim VH_0$. The master equation was shown to be nonintegrable [44], and the existence and uniqueness of its solution for any given H_0 was rigorously proved at least for the $U(1)$ gauge theory [20].

In the effective action on the domain walls, the moduli parameters ϕ^i appearing in the moduli matrix H_0 are promoted to fields $\phi^i(x^\mu)$ which depend on the coordinates of the world volume. Then the effective theory is described as a nonlinear sigma model whose target space is the moduli space endowed with a Kähler metric. The Kähler metric of the effective action can be obtained through the Kähler potential which is written down as the following integral form [11,34]:

$$K(\phi, \phi^*) = \int_{-\infty}^{\infty} dy[\mathcal{K}(y, \phi, \phi^*) - \mathcal{K}_{ct}(y, \phi) - \bar{\mathcal{K}}_{ct}(y, \phi^*)], \tag{2.11}$$

$$\mathcal{K}(y, \phi, \phi^*) = \mathrm{Tr}\left[c \log\Omega + c\Omega^{-1}\Omega_0 + \frac{1}{2g^2}(\Omega^{-1}\partial_y\Omega)^2\right], \tag{2.12}$$

where $\mathcal{K}_{ct}(y, \phi)$ and $\bar{\mathcal{K}}_{ct}(y, \phi^*)$ are counterterms, which are added to subtract the divergent part contained in $\mathcal{K}(y, \phi, \phi^*)$. Note that this addition of the counterterms can be interpreted as the Kähler transformation, and the Kähler metric $K_{ij} = \frac{\partial K}{\partial \phi^i \partial \phi^{j*}}$ does not change by the addition of the counterterms $\mathcal{K}_{ct}(y, \phi)$, $\bar{\mathcal{K}}_{ct}(y, \phi^*)$ which are purely holomorphic and antiholomorphic with respect to the moduli parameters, respectively.

According to this formula, as a matter of course we can confirm that the total inertial mass of the walls, T_{inertial}, agrees with the total static energy (tension) of the BPS walls, T_{BPS}, by the following discussion. Assume a field $w(x^\mu)$ consists of the center of masses, $\mathrm{Re}w$, and a Nambu-Goldstone mode for the overall phase, $\mathrm{Im}w$. The total inertial mass T_{inertial} is given by the coefficient of the kinetic term of the center of mass, $2K_{ww^*}$. Because of the translational invariance, w and the coordinate y appear in the Kähler potential density $\mathcal{K}$ (2.12) through a form, $y - \mathrm{Re}w$. This fact leads to the statement above as

$$T_{\mathrm{inertial}} = 2\frac{\partial^2 K}{\partial w \partial w^*} = \frac{1}{2}\int_{-\infty}^{\infty} dy \frac{\partial^2 \mathcal{K}}{\partial y^2} = \frac{c}{2}\left[\frac{\partial}{\partial y}\mathrm{Tr}\log\Omega\right]_{y=-\infty}^{y=\infty} = c[\mathrm{Tr}\Sigma]_{y=-\infty}^{y=\infty} = T_{\mathrm{BPS}}. \tag{2.13}$$

For well-separated walls, this statement is also applicable to each wall and determines an asymptotic metric for their position moduli. Combining this and the flavor symmetry of the system, we can often determine the asymptotic metric for full moduli space. This is a main strategy in Secs. III and IV.

The technique introduced here to solve BPS equations, the moduli matrix formalism, was generalized to non-Abelian vortices in various cases [45]: changing the manifold from $\mathbf{R}^2$ to a cylinder or a torus T^2, non-Abelian string reconnection, an extension to the semilocal case, and the finite temperature. See Refs. [11,46] for a review including other composite BPS solitons.

C. Domain walls and local structure of the moduli space

We now discuss the domain wall solutions interpolating between different vacua. Domain walls in the case of the fully nondegenerate Higgs masses were constructed and discussed previously [17–24]. In the $U(1)$ gauge theory, the model admits the N ordered vacua and the $N - 1$ domain walls connecting them. Each wall carries a zero mode of broken $U(1)$ symmetry and a broken translational symmetry. Rigorously speaking, only one massless field is the exact Nambu-Goldstone mode for the broken translational symmetry. The others are approximate Nambu-Goldstone modes when all walls are far away from each other. Then each wall carries a zero mode locally in moduli space

$$\mathbf{C}^*(= \mathbf{C} - \{0\}) \simeq \mathbf{R} \times U(1). \tag{2.14}$$

However, one has to note that the moduli space of the full solution is *not* a direct product of them. For instance let us consider $U(1)$ gauge theory with three flavors. This model contains three isolated vacua and admits two walls. The moduli space of two domain walls is not a direct product of two $\mathbf{C}^*$'s but $\mathbf{C}^* \times \mathbf{C}$. This is because two walls cannot pass through, and one of the $U(1)$ modulus shrinks when they are compressed to a single wall.

Continuously degenerate vacua occur when a global symmetry G is spontaneously broken. If it breaks to its subgroup H, the Nambu-Goldstone bosons parametrizing a coset space G/H appear. Let us consider the situation such that a path of a wall configuration, connecting two isolated vacua, passes near the continuously degenerate vacua. Once a wall solution is found, another solution can be obtained by acting the global symmetry G on it. Then we obtain a continuous series of solutions parametrized by G/H as shown in Fig. 1. In other words, non-Abelian Nambu-Goldstone modes of G/H are localized on the wall solution since G fixes the two isolated vacua.

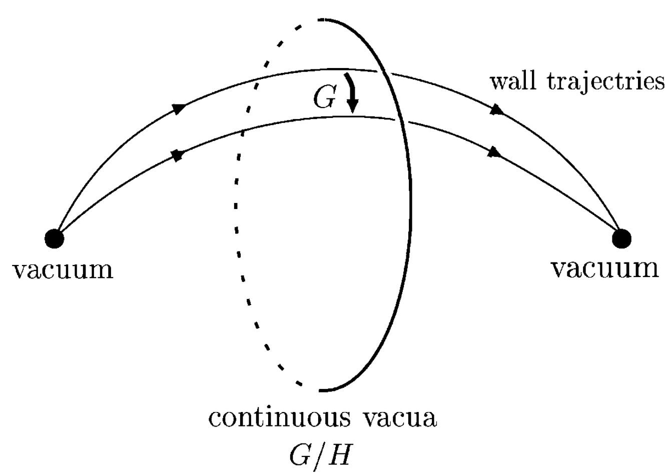

FIG. 1. Continuous series of wall solutions parametrized by G/H are obtained when trajectories pass near the continuously degenerate vacua.

Actually the condition that both vacua on both sides of the wall are isolated is not necessary. Rather these localized non-Abelian zero modes usually occur when a wall configuration passes near continuously degenerate vacua as seen in the next subsection.

D. D-brane configurations

The wall configurations are realized as a kinky D-brane configuration [24]. (See [47] for the case of $U(1)$ gauge group.) In this subsection we generalize the discussion of [24] to the case of partially degenerate Higgs masses. We will see the D-brane configuration is very useful to understanding a local structure of the moduli space of domain wall solutions.

First of all the model in $d = p + 1$ dimensions ($p = 1$, 2, 3, 4) can be realized on a $p + 1$-dimensional world volume of Dp-branes in a Dp − D$(p + 4)$ system. The four co-dimensional direction $\mathbf{C}^2$ of the Dp-branes along D$(p + 4)$-branes are divided by $\mathbf{Z}_2$ in order to remove unwanted adjoint Higgs fields describing the positions of the Dp-branes inside the D$(p + 4)$-branes. Then we can regard Dp-branes as fractional D$(p + 2)$-branes stacked at orbifold singularity of $\mathbf{C}^2/\mathbf{Z}_2$. [Taking T-duality we can map the brane configuration to a D$(p + 1)$ − D$(p + 3)$-NS5 system of the Hanany-Witten setup, but we do not do that in this paper.] Hypermultiplets containing Higgs fields are obtained from strings connecting the Dp- and D$(p + 4)$-branes whereas vector multiplets containing gauge fields appear from strings connecting the Dp-branes. When the positions of D$(p + 4)$-branes split along their co-dimensions in ten dimensions, the Higgs fields (the hypermultiplets) get masses. In order to discuss domain walls we consider here real masses which are allowed for any dimensions.[6] Previously we considered the fully nongenerate masses and therefore completely separated D$(p + 4)$-branes [24]. Now we consider the case that $N_i (i = 1, 2, \cdots)$ coincident D$(p + 4)$-branes realizing the flavor symmetry (2.5). In a vacuum where each Dp-brane sits in one of the D$(p + 4)$-branes, at most n Dp-branes can coexist in the n coincident D$(p + 4)$-branes due to the requirement of the so-called s-rule [52]. The vacuum configuration can be illustrated as Fig. 2.

We can find the vacuum structure from this configuration. When r Dp-branes sit in $n(>r)$ coincident D$(p + 4)$-branes, we obtain degenerate vacua, the cotangent bundle $T^*G_{n,r}$ over the Grassmann manifold (see Fig. 2),

$$G_{n,r} \simeq \frac{SU(n)}{SU(r) \times SU(n - r) \times U(1)}. \tag{2.15}$$

This manifold is a submanifold of the massless Higgs branch (2.4). We thus find that the moduli space of vacua is the direct product of the Grassmann manifolds (2.15): $\prod_i G_{n_i,r_i}$ $(0 \le r_i \le n_i)$ with $\sum n_i = N_F$ and $\sum r_i = N_C$.

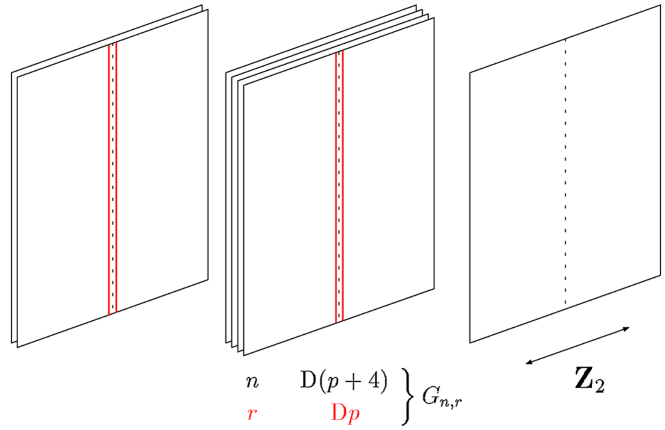

FIG. 2 (color online). D-brane configurations for a degenerate vacuum $G_{n,r}$.

We now consider domain wall configurations. Eigenvalues of the adjoint Higgs field Σ correspond to the positions of Dp-branes. When there exists a domain wall, some (not necessarily one) Dp-branes exhibit a kink, namely, they travel from one D$(p + 4)$-brane to another D$(p + 4)$-brane. The BPS condition dictates that these kinks have to move in one direction. An example of domain walls in a $U(1)$ gauge theory is drawn in Fig. 3. From this configuration we can find zero modes associated with symmetry breaking. For instance on the middle D$(p + 4)$-branes in Fig. 3 a Dp-brane breaks $SU(3)$ flavor symmetry to $SU(2) \times U(1)$. Therefore associated with this symmetry breaking, there appear zero modes $\mathbf{C}P^2 \simeq SU(3)/[SU(2) \times U(1)]$. These modes are normalizable

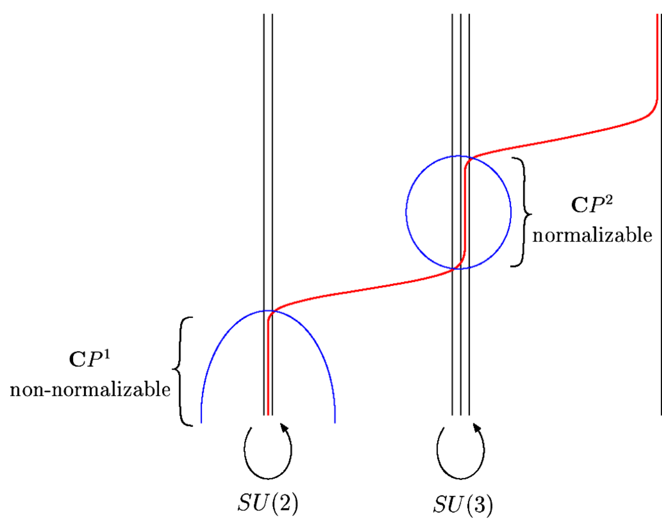

FIG. 3 (color online). The D-brane configuration for two degenerate walls at the left, three degenerate walls in the middle, and a single wall at the right.

[6] We need complex Higgs masses when we construct domain wall junctions (network or webs) [48,49] or dyonic domain walls [50,51]. The complex masses are possible up to four dimensions ($p = 1$, 2, 3).

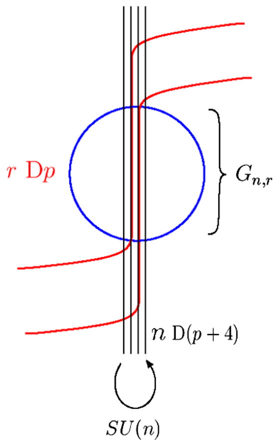

FIG. 4 (color online). The $r(<n)$ Dp-branes residing in a finite region of n D$(p+4)$-branes give the zero modes forming the Grassmann manifold $G_{n,r}$.

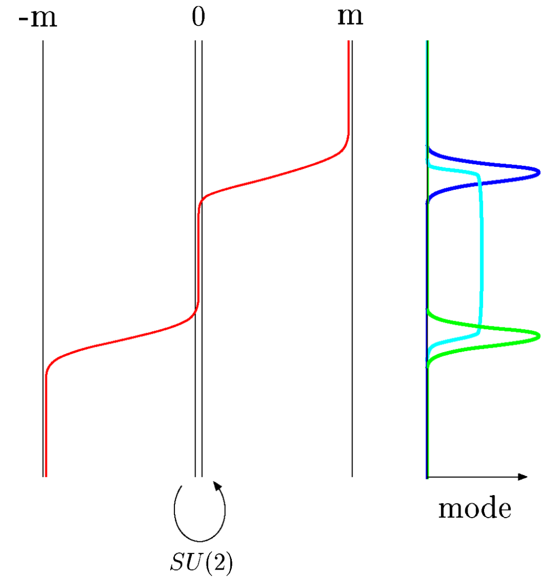

FIG. 5 (color online). D-brane picture for a domain wall with non-Abelian clouds.

because this symmetry breaking occurs in a finite region between the upper kink and the lower kink. This means that these modes have a support between the two domain walls. We call these modes "non-Abelian clouds" as in the case of non-Abelian monopoles [30]. In general when $r(<n)$ Dp-branes exist at finite region of n D$(p+4)$-branes, there appear zero modes of the Grassmann manifold $G_{n,r}$ given in Eq. (2.15). See Fig. 4. On the other hand, there also exist usual modes (2.14) localized on a wall which we call "wall-localized modes."

When a symmetry breaking occurs in an infinite or semi-infinite region as in the leftmost part of Fig. 3, the modes for this symmetry breaking have an infinite or semi-infinite support, and therefore they are nonnormalizable. These bulk modes do not appear in the effective theory on walls and do not contribute to the moduli space of walls.

In summary there in general appear normalized modes, classified into wall-localized modes and non-Abelian clouds, as well as nonnormalizable modes. We can find a local structure of the moduli space but unfortunately at this stage we cannot find a global structure of the moduli space from the brane configuration. In general each part is not a direct product in the whole moduli space because of a nontrivial bundle structure. We have to integrate the modes over the co-dimension in order to obtain the whole moduli space. We perform the integration explicitly in two examples in the succeeding sections.

III. NON-ABELIAN CLOUDS IN ABELIAN GAUGE THEORIES

A. A simple example of non-Abelian clouds

Let us see the non-Abelian clouds in a simple example of the Abelian gauge theory coupled with the $N_F = 4$ Higgs fields. The corresponding brane configuration is shown in Fig. 5. The massless vacuum manifold is $T^*\mathbf{C}P^3$ where the base manifold is parametrized by

$$\mathbf{C}P^3 = \{HH^\dagger = c\}/U(1), \qquad H = \sqrt{c}(h_1, h_2, h_3, h_4), \tag{3.1}$$

where the quotient is the overall $U(1)$. The vacuum manifold is expressed as (the inside and the surface of) a triangular pyramid in the three-dimensional space $(|h_1|^2, |h_2|^2, |h_3|^3)$, as shown in Fig. 6(a). When the mass matrix containing a small parameter ϵ $(0 \leq \epsilon \in \mathbf{R})$

$$M = \mathrm{diag}\left(m, \frac{m\epsilon}{2}, -\frac{m\epsilon}{2}, -m\right) \tag{3.2}$$

is turned on, the vacuum manifold is lifted except for four points and the flavor symmetry breaks from $SU(4)$ to $U(1)^3$. These discrete vacua are the four vertices of the pyramid shown in Fig. 6(b). We label those vacua as $\langle A\rangle$ $(A = 1, 2, 3, 4)$. The vacuum expectation value of the vacuum $\langle A\rangle$ is $h_B = \delta_{AB}$. Taking a limit of $\epsilon \to 0$, the second and the third Higgs fields become degenerate so that the flavor symmetry enhances from $U(1)^3$ to $U(1)^2 \times SU(2) \in SU(4)$. There are two isolated vacua and one degenerate vacuum $\mathbf{C}P^1 \simeq SU(2)/U(1)$ represented by a line connecting $\langle 2\rangle$ and $\langle 3\rangle$ as shown by a thick line in Fig. 6(c). We denote this degenerate vacuum as $\langle 2-3\rangle$.

There exist domain wall solutions interpolating vacua in the model with fully or partially nondegenerate Higgs masses. In the case of $N_C = 1$, the moduli matrix and the V-equivalence (2.11) take the form of

$$H_0 = (\phi_1, \phi_2, \phi_3, \phi_4) \sim \lambda(\phi_1, \phi_2, \phi_3, \phi_4), \qquad \lambda \in \mathbf{C}^*. \tag{3.3}$$

In terms of the moduli matrix the vacua $\langle A\rangle$ is described by $\phi_B = \delta_{BA}$ for $B = 1, 2, 3, 4$. Since we want to consider the domain wall interpolating the vacua $\langle 1\rangle$ and $\langle 4\rangle$ (passing by

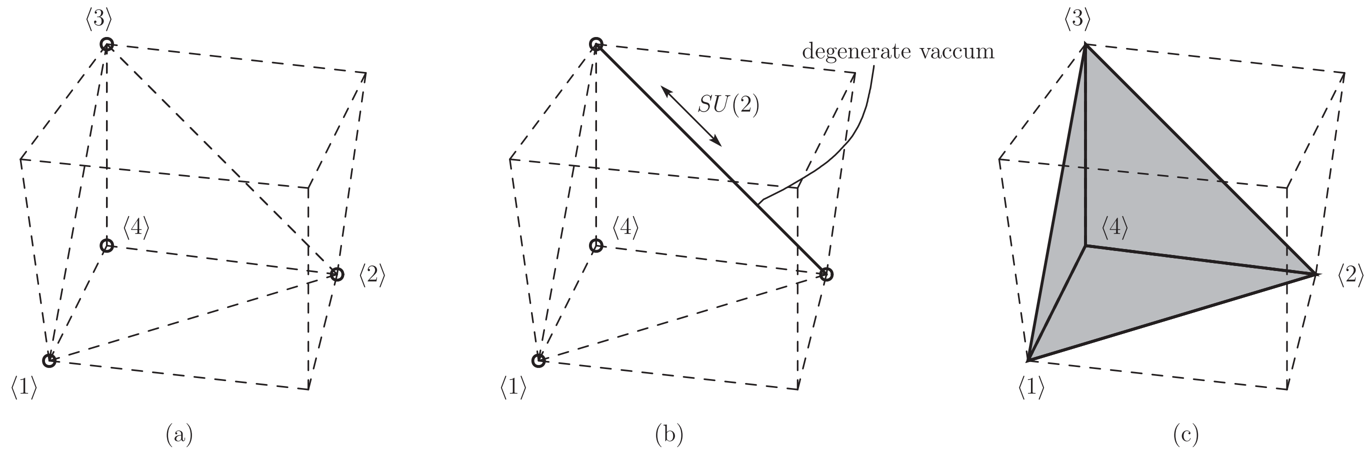

FIG. 6. Vacua for various cases of mass configurations plotted in the three-dimensional space of Higgs fields h_i^2, $i = 1, 2, 3$ with $\sum_{i=1}^{4} h_i^2 = 1$. (a) nondegenerate massive vacua, (b) massive degenerate and nondegenerate vacua, (c) massless vacuum.

$\langle 2\rangle, \langle 3\rangle$ on the way), the parameter ϕ_1 and ϕ_4 should not be zero while ϕ_2, ϕ_3 can become zero. So the moduli space corresponding to the multiple domain walls which connect $\langle 1\rangle$ and $\langle 4\rangle$ is

$$\mathcal{M} \simeq (\mathbf{C}^2 \times (\mathbf{C}^*)^2)//\mathbf{C}^* \simeq \mathbf{C}^* \times \mathbf{C}^2, \qquad (3.4)$$

where the double slash denotes identification by the V-transformation. Here the part $\mathbf{C}^* \simeq \mathbf{R} \times U(1)$ represents the translational modulus and the associated phase modulus.

When we take the gauge coupling g to infinity, the model reduces to a nonlinear sigma model whose target space is the Higgs branch of the vacua in the original theory. To make the discussion simple, we take this limit for a while. One benefit to consider the nonlinear sigma model is that the BPS equations are analytically solved. In fact, the solutions are expressed as [22]

$$H = \frac{1}{\sqrt{\Omega_0}} H_0 e^{My} \quad \text{with} \quad \Omega_0 \equiv H_0 e^{2My} H_0^\dagger. \qquad (3.5)$$

A domain wall solution corresponds to a trajectory connecting the vertex $\langle 1\rangle$ and $\langle 4\rangle$. Flows from $\langle 1\rangle$ to $\langle 4\rangle$ inside the pyramid are shown in Fig. 7.

Physical meaning of the moduli parameters becomes much clearer by using the V-equivalence relation (3.3) to fix the form of the moduli matrix as

$$H_0 = (1, e^{\varphi_1}, e^{\varphi_1+\varphi_2}, e^{\varphi_1+\varphi_2+\varphi_3}). \qquad (3.6)$$

Furthermore, one may be visually able to see the "kink" configuration in the profile of the field $\Sigma = (1/2)\partial_y \log\Omega_0$. In the vacuum region $\langle A\rangle$ the function $\Sigma(y)$ takes the value $\Sigma = m_A$. Several solutions are shown in Fig. 8. The domain wall positions can be roughly read from the moduli matrix in Eq. (3.6) as

$$\frac{y_+}{L_+} = \varphi_1 + \varphi_1^*, \qquad \frac{y_0}{L_0} = \varphi_2 + \varphi_2^*, \qquad \frac{y_-}{L_-} = \varphi_3 + \varphi_3^*, \qquad (3.7)$$

where y_+ is the position of the right wall and y_0, y_- are the positions for the middle and the left walls, respectively. Here $L_{\pm,0}$ stands for the width of each wall

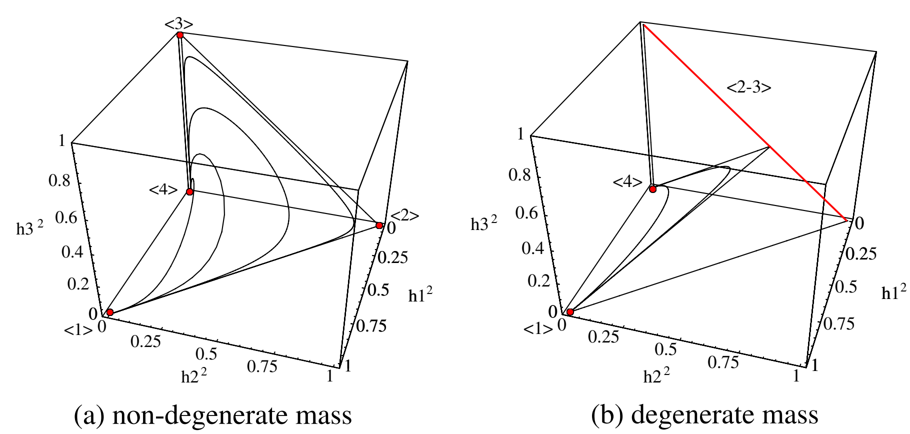

FIG. 7 (color online). Domain wall trajectories in the target space $\mathbf{C}P^3$ for nondegenerate mass (a) and for degenerate mass (b).

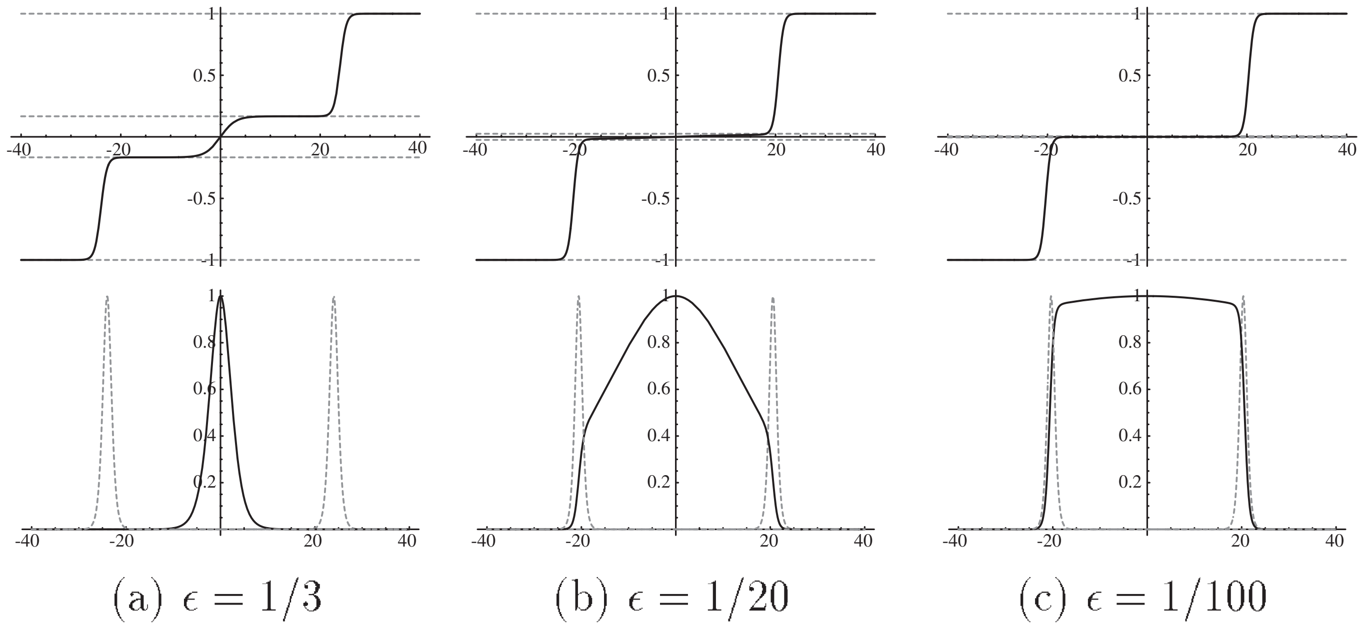

FIG. 8. Configuration of Σ (first row) and density of the Kähler metric of φ_1, φ_2, and φ_3 (second row). Moduli parameters are $(\varphi_1, \varphi_2, \varphi_3) = (20, 0, -20)$ and $m = 1$.

$$L_+ \equiv \frac{2}{m(2-\epsilon)}, \qquad L_0 \equiv \frac{1}{m\epsilon}, \qquad L_- \equiv \frac{2}{m(2-\epsilon)}. \tag{3.8}$$

This rough estimation is, of course, valid only for well-separated walls whose positions are aligned as $y_- \ll y_0 \ll y_+$, see Fig. 8(a). Each domain wall is accompanied by a complex moduli parameter φ_i whose real part is related to the wall position and imaginary part is the $U(1)$ internal symmetry [the Nambu-Goldstone mode associated with the broken $U(1)$ flavor symmetry].

To argue symmetry aspects of the moduli parameters, first let us consider a model which has completely non-degenerate masses and domain walls interpolating between those vacua. The global symmetry explicitly breaks from $SU(4)$ to $U(1)^3 \subset SU(4)$. We take, as the unbroken global symmetries, $U_1(1), U_2(1)$ and $U_3(1)$ with generators diag$(1, -1, -1, 1)$, diag$(1, 0, 0, -1)$, and diag$(0, 1, -1, 0)$, respectively. Each vacua $\langle A \rangle$ preserves all of these symmetries. However, once domain walls connecting those vacua appear, they break all or part of these symmetries. For example, the moduli matrix $H_0 = (1, 0, 0, \phi_4)$ corresponding to a domain wall connecting two vacua $\langle 1 \rangle$ and $\langle 4 \rangle$ breaks $U_2(1)$ but still preserves $U_1(1)$ and $U_3(1)$. Here note that the overall phase can be absorbed by the V-transformation (3.3). Therefore, the phase of the moduli parameter ϕ_4 corresponds to nothing but the broken global symmetry $U_2(1)$. This implies that the Nambu-Goldstone mode localizes around the domain wall as we saw above. For the moduli matrix $H_0 = (1, \phi_2, 0, \phi_4)$, which corresponds to two domain walls connecting three vacua $\langle 1 \rangle \rightarrow \langle 2 \rangle \rightarrow \langle 4 \rangle$, the symmetry $U_3(1)$ in addition to $U_2(1)$ breaks while a combination of $U_1(1)$ and $U_3(1)$ is still preserved.

Moreover, when we turn on the third element in the moduli matrix as $H_0 = (1, \phi_2, \phi_3, \phi_4)$, the third vacuum region appears and then the configuration has three domain walls connecting four vacua $\langle 1 \rangle \rightarrow \langle 2 \rangle \rightarrow \langle 3 \rangle \rightarrow \langle 4 \rangle$. In this case all of $U(1)^3$ are broken by the domain walls, so that the corresponding three Nambu-Goldstone modes appear. These three Nambu-Goldstone modes are described by imaginary parts of $\log\phi$, which are combined with the three positions (3.7), to form three complex coordinates of the moduli space $\mathbf{C}^2 \times \mathbf{C}^*$.

Next we consider a limit where the second and the third masses are degenerate [$\epsilon \rightarrow 0$ in the mass matrix (3.2)]. In this limit the global symmetry $U_1(1) \times U_2(1) \times U_3(1)$ is enhanced to $U_1(1) \times U_2(1) \times SU(2)$. At the same time, the degenerate vacuum $\langle 2-3 \rangle$ appears instead of the two isolated vacua $\langle 2 \rangle$ and $\langle 3 \rangle$ as shown in Fig. 6(c). At the degenerate vacuum, $U_1(1)$, $U_2(1)$ are preserved but $SU(2)$ is broken to $U_3(1)$. Therefore, the degenerate vacuum $\langle 2-3 \rangle$ is $SU(2)/U_3(1) = \mathbf{C}P^1$. Nonvanishing $\phi_4 \neq 0$ causes the wall interpolating two vacua $\langle 1 \rangle \rightarrow \langle 4 \rangle$ and breaks only $U_2(1)$ again. Once the degenerate vacuum appears in the configuration such as two domain walls connecting vacua like $\langle 1 \rangle \rightarrow \langle 2-3 \rangle \rightarrow \langle 4 \rangle$, the breaking pattern of the global symmetry becomes different from that in the case of fully nondegenerate masses. The moduli matrix $H_0 = (1, \phi_2, \phi_3, \phi_4)$ describes such domain walls. Note that the second and the third elements breaks $SU(2)$ completely. The global symmetry $U_1(1) \times U_2(1) \times SU(2)$ are broken to $U(1)$ which is a mixture of $U_1(1)$ and $H \in SU(2)$. Emergence of the second wall and further $U(1)$-symmetry breaking are related to the facts that $|\phi_2|^2 + |\phi_3|^2 \neq 0$ and $\phi_4 \neq 0$. These facts imply that the modes corresponding to the two broken $U(1)$'s localize around the walls accompanied by the two position moduli and the mode corresponding to $SU(2)/H$ have support in a region around the degenerate vacuum $\langle 2-3\rangle$. This is consistent with the observation from the view point of the D-brane picture in Fig. 5. We can count the number of the moduli parameters as follows. Two real parameters $\{|\phi_2|^2+|\phi_3|^2, |\phi_4|^2\}$ correspond to the positions of the two walls whereas the remaining four parameters correspond to the broken global symmetry $U_1(1)\times U_2(1)\times SU(2)/U(1)$. This is again consistent with $\dim_{\mathbf{R}}(\mathbf{C}^2\times\mathbf{C}^*)$.

In Fig. 8 we showed domain wall configurations of the three domain walls connecting the four vacua. As the parameter ϵ decreases, the width of the middle domain wall connecting the vacua $\langle 2\rangle$ and $\langle 3\rangle$ becomes broad and the tension of the wall becomes small since they are proportional to $1/\epsilon$ and ϵ, respectively. When the width of the middle wall becomes lager than the separation of the two outside walls, $L_0 \gtrsim y_+ - y_-$, we can no longer see the middle wall. The density of the Kähler metric for the moduli parameters φ_1, φ_2, and φ_3 in the strong gauge coupling limit are shown in the second row of Fig. 8. The Kähler potential in the strong coupling limit is given by $K = c\int dy \log\Omega_0$ [34]. When three walls are well isolated as in Fig. 8(a), three modes corresponding to the moduli parameters φ_1, φ_2, and φ_3 are localized on the respective domain walls. As ϵ decreases, the density of the Kähler metric of φ_2 is no longer localized but is stretched between two outside domain walls. In the limit where $\epsilon\to 0$ the physical meaning of φ_2 as the position and the internal phase associated with the middle domain wall should be completely discarded. Instead, φ_2 gives the non-Abelian cloud which comes from the flat direction $\mathbf{C}P^1$ of the degenerate vacua $\langle 2-3\rangle$. For each fixed moduli parameters φ_1, φ_2, φ_3, the domain wall solution as a function of y sweeps out a trajectory in the target space $\mathbf{C}P^3$. These domain wall trajectories are shown for various values of moduli parameters in Fig. 7: nondegenerate mass case (a) and degenerate mass case (b). For degenerate mass case, the trajectories do not go out from the triangular plane whose vertices are $\langle 1\rangle, \langle 4\rangle$ and one point on the edge between $\langle 2\rangle$ and $\langle 3\rangle$.

B. Effective action of non-Abelian clouds and their dynamics

Next we construct the effective action for the non-Abelian clouds while leaving the gauge coupling to be finite. In this subsection we consider a more general model with N_F flavors with masses

$$M = (m_1, 0, 0, \cdots, 0, -m_2), \qquad m_1, m_2 > 0. \tag{3.9}$$

There exist two isolated points of vacua and one continuously degenerate vacua $\mathbf{C}P^{N_F-3}$.

This model admits two domain walls interpolating between two isolated vacua at $y=-\infty$ to $y=+\infty$ with the degenerate vacua $\mathbf{C}P^{N_F-3}$ between the two domain walls. The full moduli space is

$$\mathcal{M} \simeq \mathbf{C}^*\times\mathbf{C}^{N_F-2}. \tag{3.10}$$

In the following we do not consider the $\mathbf{C}^*$ corresponding to the center of the mass and the overall phase. Then let us take the moduli matrix

$$H_0 = (1, \phi_2, \phi_3, \cdots, \phi_{N_F-1}, 1). \tag{3.11}$$

The positions of the two walls can be estimated as

$$y_1 = \frac{1}{2m_1}\log|\vec{\phi}|^2, \qquad y_2 = -\frac{1}{2m_2}\log|\vec{\phi}|^2, \tag{3.12}$$

with a vector $\vec{\phi}\equiv(\phi_2, \phi_3, \cdots, \phi_{N_F-1})$. Notice that we have fixed the center of mass of the two walls as $m_1y_1+m_2y_2=0$. The distance of the two walls is defined as

$$R = y_1 - y_2 = \frac{1}{\mu}\log|\vec{\phi}|^2, \qquad \mu\equiv\frac{2m_1m_2}{m_1+m_2}. \tag{3.13}$$

The function $\Omega_0(y)$ in the master Eq. (2.9) in this case is given by

$$\Omega_0 = c^{-1}(e^{2m_1y}+|\vec{\phi}|^2+e^{-2m_2y}). \tag{3.14}$$

Although we have to solve the master Eq. (2.9) to obtain the explicit expression of the quantity Ω, we do not need it for the later analysis: it is easy to see that the Kähler potential (2.11) depends only on[7] $\mu R = \log|\vec{\phi}|^2$

$$K(\phi, \phi^*) = f(\mu R). \tag{3.15}$$

We give the asymptotic form of the function f below. The effective action is obtained from the Kähler potential via the Kähler metric as $L_{\rm eff} = K_{ij^*}\partial_\mu\phi^i\partial^\mu\phi^{j*}$. After changing the variables as

$$\vec{\phi} = e^{(\mu R+i\xi)/2}\vec{n}, \qquad |\vec{n}|^2 = 1, \tag{3.16}$$

we can obtain the following expression:

$$\begin{aligned}\mathcal{L}_{\rm eff} &= \tfrac{1}{4}f''(\mu R)[\mu^2(\partial_\mu R)^2 + (\partial_\mu\xi - 2i\vec{n}^\dagger\partial_\mu\vec{n})^2] \\ &\quad + f'(\mu R)[|\partial_\mu\vec{n}|^2 - |\vec{n}^\dagger\partial_\mu\vec{n}|^2].\end{aligned} \tag{3.17}$$

Here the complex vector $\vec{n}$ consists of the coordinates of the vacua $\mathbf{C}P^{N_F-3}$ between the two walls, that is, the non-Abelian clouds.[8]

[7]The Kähler potential of this type was studied in [53] where the Ricci-flat metric on a line bundle over the projective space was obtained by enforcing the Ricci-flat condition. Here, the metric does not have to be Ricci-flat of course.

[8]The target space metric of the effective Lagrangian (3.17) locally looks like a complex line bundle over $\mathbf{C}P^{N_F-3}$, namely, $\mathcal{O}(-1)\to\mathbf{C}P^{N_F-3}$. However it does not hold for $R\to-\infty$ (coincident walls) where the metric tends to a single point as found in Eq. (3.18), below. Therefore the base space $\mathbf{C}P^{N_F-3}$ of the bundle is blown down to a point to obtain $\mathbf{C}^{N_F-2}$ in the moduli space (3.10).

Since two walls become independent as they are separated by a large distance, the kinetic term of the relative distance R should be a free action $\mathcal{L}_{\rm free} = \frac{\mu c}{4}(\partial_\mu R)^2$ for sufficiently large R. Note that the coefficient $\mu c/4$ is calculated by using Eq. (2.13). Furthermore, the Kähler metric written in the moduli fields ϕ^i which are original entries in the moduli matrix should be smooth everywhere, especially at $|\vec{\phi}| = 0$ ($R \to -\infty$). From these two facts we can find the asymptotic behavior of the function $f(\mu R)$ as

$$f(\mu R) = \begin{cases} \frac{c\mu R^2}{2} - (d_1 + d_2)c\mu R + \mathcal{O}(1) & \text{for } R \to \infty \\ Ae^{\mu R} + \mathcal{O}(e^{2\mu R}) & \text{for } R \to -\infty, \end{cases} \tag{3.18}$$

where $d_1 \equiv m_1/g^2c$ and $d_2 \equiv m_2/g^2c$ are half of the widths of walls and A is a constant determined by solving the BPS equations. The derivation of the subleading term for well-separated walls, which is proportional to R, is given in Appendix A. Note that in the region of $R < d_1 + d_2$, the parameter R no longer has the meaning of the distance between the walls and the two walls are nearly compressed into one wall. Especially at $|\vec{\phi}| = 0 (R \to -\infty)$, the two walls are completely compressed and the degrees of freedom of the non-Abelian clouds between the two walls disappear with the shrinking of $\mathbf{C}P^{N_{\rm F}-3}$ to a point. Figure 9 (i) and (ii) show the typical profiles of the functions $f'(\mu R)$ and $f''(\mu R)$ for various values of the gauge coupling constant.

In order to consider the dynamics in detail, let us concentrate on the minimal case of $N_{\rm F} = 4$ and consider the kinks in the $d = 1 + 1$ gauge theory with the $(1 + 0)$ dimensional world volume (the world volume is time only $\mu = 0$) in the rest of this section. It is convenient to redefine the parameters as

$$(\phi_2, \phi_3) = \vec{\phi} = e^{(\mu R + i\xi)/2}\vec{n} = e^{(\mu R + i\xi)/2}\left(e^{i\varphi/2}\cos\frac{\theta}{2}, e^{-i\varphi/2}\sin\frac{\theta}{2}\right). \tag{3.19}$$

The Lagrangian in these coordinates takes the form

$$L_{\rm eff} = \frac{f''(\mu R)}{4}[\mu^2\dot{R}^2 + (\dot{\xi} + \cos\theta\dot{\varphi})^2] + \frac{f'(\mu R)}{4}(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) \tag{3.20}$$

$$\to \frac{c}{4\mu}[\mu^2\dot{R}^2 + (\dot{\xi} + \cos\theta\dot{\varphi})^2] + \frac{c}{4}(R - d_1 - d_2)(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2). \tag{3.21}$$

The asymptotic behavior of the coefficients of kinetic terms of θ and φ reflects the fact that the wave functions of the non-Abelian clouds are extending in the interval between the walls which is effectively reduced by the widths of the walls. As mentioned above, the Kähler potential depends only on $\mu R = \log|\vec{\phi}|^2$, so that there exist four conserved quantities defined by

$$Q = \frac{i}{2}K_{ij^*}(\dot{\phi}^{j*}\phi^i - \dot{\phi}^i\phi^{j*}), \tag{3.22}$$

$$q_a = \frac{i}{2}K_{ij^*}(\dot{\phi}^{j*}(\sigma_a)^i{}_k\phi^k - \dot{\phi}^i\phi^{k*}(\sigma_a)^j{}_k), \tag{3.23}$$

where $\sigma_a (a = 1, 2, 3)$ are the Pauli matrices. These conserved charges originate from $U(2)$ symmetry which rotates the complex vector $\vec{\phi}$. Note that they are not independent, but related as

$$Q = \frac{1}{|\vec{\phi}|^2}(\phi^*\sigma_a\phi)q_a. \tag{3.24}$$

By using these conserved charges, we can rewrite the

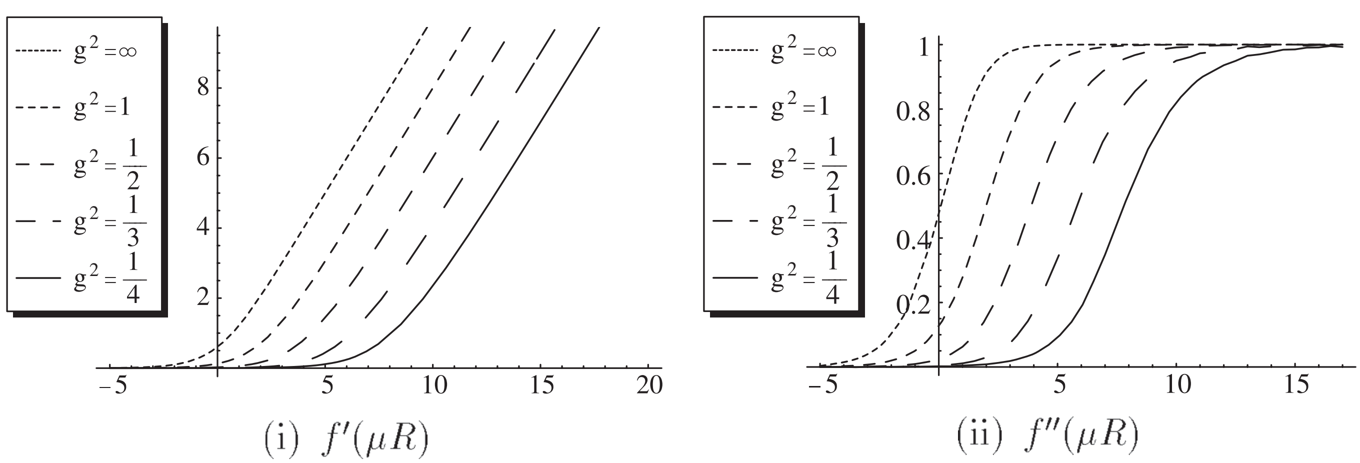

FIG. 9. Typical profiles of the functions $f'(\mu R)$ and $f''(\mu R)$ with $c = 1, m_1 = m_2 = 1$. The function $f(\mu R)$ is numerically calculated for the gauge coupling $g^2 = \infty, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

Lagrangian as

$$\tilde{L}_{\rm eff} = \frac{\mu^2}{4} f''(\mu R)\dot{R}^2 - \frac{Q^2}{f''(\mu R)} - \frac{q_a q_a - Q^2}{f'(\mu R)}. \quad (3.25)$$

Let us consider the dynamics of kinks when these conserved charges take nonzero values. Then we have two types of potential between two walls: one is given by $V_1(R) = 4Q^2/f''(\mu R)$, which exponentially approaches to a constant as the distance R becomes larger, and the other is $V_2(R) = 2(q_a q_a - Q^2)/f'(\mu R)$, which behaves as $V_2(R) \approx 1/R$ for large R. The former type of the potential also exists in the case of fully nondegenerate masses. The novel feature here is the existence of the potential $V_2(R)$ which leads to a long-range repulsive force between two walls. Physical interpretation of these potentials is quite interesting. In the case of fully nondegenerate masses, there are only massive modes which propagate between two walls, so that the potential falls off rapidly for large $R \gg 1/\mu$. In the case of degenerate masses, we have some massless Nambu-Goldstone modes propagating between two walls. They are nothing but the non-Abelian clouds and mediate the long-range repulsive force: the motion in the internal space induces a repulsive force between the two kinks.

C. Kink bound state stabilized by non-Abelian clouds

We can let the degenerate mass split by giving imaginary masses for the Higgs scalar fields. Then an attractive force between the two kinks is induced, and a bound state of the two kinks can be formed. When we add the additional masses of the scalar fields in the original theory as

$$M \to M + i\tilde{M} = {\rm diag}(m_1, i\tilde{m}/2, -i\tilde{m}/2, -m_2), \qquad \tilde{m} > 0, \quad (3.26)$$

the vacuum manifold is lifted and the continuous degeneracy of the vacua disappears. These imaginary masses make domain walls Abelian. We can, however, easily keep a part of continuous degeneracy of vacua, by extending the system to a model with real and imaginary masses in the case of $N_{\rm F} > 4$. In that case, domain walls remain non-Abelian. Here we consider (3.26) for simplicity. For a small $\tilde{m} \ll \mu$, the additional masses induce a potential which is given by the squared norm of the Killing vector $k = \tilde{m}\partial_\varphi$ on the moduli space as

$$V_{\rm eff} = \frac{\tilde{m}^2}{4}[f''(\mu R)\cos^2\theta + f'(\mu R)\sin^2\theta]. \quad (3.27)$$

This is an attractive potential with minimum at $|\vec{\phi}|^2 = e^{\mu R} = 0$, namely, two walls tend to be compressed into one wall. Once the additional masses are turned on, not all charges Q and q_a are conserved, but Q and q_3 are left to be conserved. The charges Q and q_3 are conjugate momenta of ξ and φ, respectively,

$$\frac{\partial L_{\rm eff}}{\partial \dot{\xi}} = Q = \frac{f''(\mu R)}{2}(\dot{\xi} + \cos\theta\dot{\varphi}), \quad (3.28)$$

$$\frac{\partial L_{\rm eff}}{\partial \dot{\varphi}} = q_3 = \frac{f''(\mu R)}{2}\cos\theta(\dot{\xi} + \cos\theta\dot{\varphi}) + \frac{f'(\mu R)}{2}\sin^2\theta\dot{\varphi}. \quad (3.29)$$

Therefore, we effectively obtain the following potential

$$\tilde{V}_{\rm eff} = \frac{\tilde{m}^2}{4}[f''(\mu R)\cos^2\theta + f'(\mu R)\sin^2\theta] + \frac{Q^2}{f''(\mu R)} + \frac{(q_3 - \cos\theta Q)^2}{f'(\mu R)\sin^2\theta}. \quad (3.30)$$

The potential is composed of four terms with two different types of asymptotic behaviors, namely, long-range and short-range forces: the first and third terms exponentially approach to constants for large R, while the second and fourth terms are proportional to R and $1/R$, respectively.

The effective potential is bounded from below

$$\tilde{V}_{\rm eff} \geq \tilde{m}|q_3|. \quad (3.31)$$

This lower bound of the effective potential is saturated if R and θ satisfy

$$\frac{\tilde{m}}{2} f''(\mu R)\cos\theta = \eta Q, \qquad \frac{\tilde{m}}{2} f'(\mu R)\sin^2\theta = \eta(q_3 - \cos\theta Q), \qquad \eta \equiv {\rm sign}(q_3). \quad (3.32)$$

The solution of these equations shows various properties for given values of the conserved charges Q and q_3. In the following we consider two cases, (1) $|Q| = |q_3|$ and (2) $Q = 0$.

(1) As an example, let us consider the case where the absolute values of two charges are the same $|Q| = |q_3|$. In this case, the relative distance R and the phase θ are stabilized at

$$\theta = \begin{cases} 0 & \text{for } Q = q_3 \\ \pi & \text{for } Q = -q_3 \end{cases}, \qquad R = R_0 \quad \text{with} \quad f''(\mu R_0) = \frac{2|q_3|}{\tilde{m}}. \quad (3.33)$$

In this case, the positions of two walls are stabilized at the points where the two short-range forces balance. Because of this short-range force the two walls stabilize with either small separation $R \lesssim 1/\mu$ or large separation $R \gtrsim 1/\mu$ with exponentially weak binding force. The squared mass of the fluctuation of the relative distance is given by $m_{\delta R}^2 = (\tilde{m} f^{(3)}(\mu R_0)/f''(\mu R_0))^2$, which becomes exponentially small for large R_0. Especially, if the two wall system has too much conserved charge $|q_3| \geq \max(f''(\mu R)) = \tilde{m}c/\mu$, an instability appears: the minimum of the potential disappears to infinity $R \to \infty$ (runaway potential). This type of the stabilized wall also exists as Q-walls (dyonic walls) in models with fully nondegenerate masses [50,51]. Actually, the corresponding configuration to the solution (3.33) can be obtained by embedding the Q-wall solution in a model with nondegenerate masses into the model we are now considering.

(2) Another example is the case with $Q = 0$. In this case, the relative distance R and the phase θ are stabilized at

$$\theta = \frac{\pi}{2}, \qquad R = \tilde{R}_0, \quad \text{with} \quad f'(\mu \tilde{R}_0) = \frac{2|q_3|}{\tilde{m}}. \tag{3.34}$$

The two walls are stabilized at the point where the two long-range forces balance. Because of these long-range forces the positions of the two walls can be stabilized with a large separation. The squared masses of the fluctuations around the minimum of the potential are given by $m_{\delta R}^2 = m_{\delta\theta}^2 = \tilde{m}^2 f''(\mu\tilde{R}_0)/f'(\mu\tilde{R}_0)$, which behave as $1/\tilde{R}_0$ for large relative distance. There is no instability even if $|q_3| \gg \tilde{m}c/\mu$, since $f'(\mu R)$ grows linearly for large R. These properties are in contrast to the case of fully nondegenerate masses. All these differences between fully nondegenerate and degenerate masses originate from the existence of the non-Abelian clouds in the degenerate case which gives the long-range interactions.

Finally, let us make a comment on supersymmetry. The masses (3.26) for Higgs fields (hypermultiplets) are possible in dimensions 3 + 1 or less. The stable configurations of the Q-walls (dyonic walls) considered in this subsection are 1/4 BPS states [50,51].

IV. THE GENERALIZED SHIFMAN-YUNG MODEL

A. The model and its vacua

In this section we consider non-Abelian gauge theory with degenerate masses of the Higgs fields. The simplest such situation may be provided by two sets of two degenerate mass parameters of the Higgs fields. A previously considered model is the $U(2)$ gauge theory with four Higgs fields in the fundamental representation with the mass matrix $M = \mathrm{diag}(m, m, -m, -m)$ [26,27], which we call the Shifman-Yung model. The model enjoys a flavor symmetry $SU(2)_\mathrm{L} \times SU(2)_\mathrm{R} \times U(1)_\mathrm{A}$. This model admits two domain walls which can pass through each other, in contrast to the Abelian gauge theory where walls do not pass through each other. It has been demonstrated that the coincident domain wall configurations break the flavor symmetry to $SU(2)_\mathrm{V}$ and the Nambu-Goldstone bosons corresponding to $[SU(2)_\mathrm{L} \times SU(2)_\mathrm{R} \times U(1)_\mathrm{A}]/SU(2)_\mathrm{V} \simeq U(2)$ appear in the effective action on the walls. The symmetry breaking is the same as that of the chiral symmetry in hadron physics. The kinky D-brane configuration for this wall configuration is shown in Fig. 10(a). Up to two Dp-branes are allowed to lie inside D$(p+4)$-branes by the s-rule [52].

To generalize the non-Abelian flavor symmetry $SU(2)_\mathrm{L} \times SU(2)_\mathrm{R}$ to $SU(N)_\mathrm{L} \times SU(N)_\mathrm{R}$, we consider the $U(N)$ gauge theory with $N_\mathrm{F} = 2N$ Higgs fields in the fundamental representation whose mass matrix is given by

$$M = m\frac{\sigma_3}{2} \otimes \mathbf{1}_N = \frac{1}{2}\mathrm{diag}(\overbrace{m, \cdots, m}^{N}, \overbrace{-m, \cdots, -m}^{N}). \tag{4.1}$$

This system has a non-Abelian flavor symmetry $SU(N)_\mathrm{L} \times SU(N)_\mathrm{R} \times U(1)_\mathrm{A}$. Since we have only two mass parame-

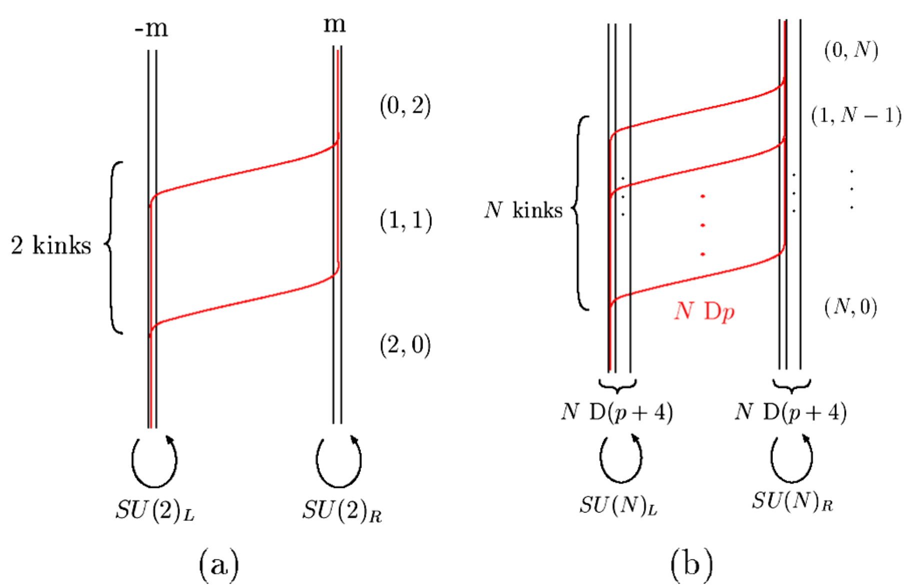

FIG. 10 (color online). The kinky D-brane configurations for the SY model (a) and for the GSY model (b).

ters m and $-m$, possible vacua are classified by an integer $0 \le k \le N$: in the kth vacua, there is a configuration in which k flavors of the first half and $N-k$ flavors of the latter half take nonvanishing values and then Σ and H are

$$\Sigma|_{\rm vacuum} = \frac{1}{2}{\rm diag}(\overbrace{m,\cdots,m}^{k},\overbrace{-m,\cdots,-m}^{N-k}),\qquad H|_{\rm vacuum} = \sqrt{c}\begin{pmatrix}\mathbf{1}_k & \mathbf{0} & \mathbf{0}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_{N-k} & \mathbf{0} & \mathbf{1}_{N-k}\end{pmatrix}. \tag{4.2}$$

This vacuum is labeled as $(k, N-k)$. The flavor symmetry $SU(N)_{\rm L}$ is broken down to $SU(k)_{\rm C+L}\times SU(N-k)_{\rm L}\times U(1)_{\rm C+L}$, and $SU(N)_{\rm R}$ is broken down to $SU(k)_{\rm R}\times SU(N-k)_{\rm C+R}\times U(1)_{\rm C+R}$. Therefore in this vacua there emerge $4k(N-k)$ Nambu-Goldstone modes, which parametrize the direct product of two Grassmann manifolds,

$$G^{\rm L}_{N,k}\times G^{\rm R}_{N,k}. \tag{4.3}$$

Consequently the number of the discrete components of the vacua is $N+1$ in this system.

The unbroken symmetries of the vacua $(N, 0)$ and $(0, N)$ which we consider in the next subsections as the boundary condition of domain walls, are $SU(N)_{\rm C+L}\times U(1)_{\rm C+L}\times SU(N)_{\rm R}$ and $SU(N)_{\rm L}\times SU(N)_{\rm C+R}\times U(1)_{\rm C+R}$, respectively.

B. General solution of domain walls

Walls are obtained by interpolating between a vacuum at $y=-\infty$ and another vacuum at $y=\infty$. The boundary conditions at both infinities define topological sectors. For a given topological sector, we may find several walls. The maximal number of walls in this system is N, which are obtained for the following maximal topological sector:

$$H = \begin{cases}\sqrt{c}(\mathbf{1}_N,\mathbf{0}_N) & \text{at } y=+\infty\\ \sqrt{c}(\mathbf{0}_N,\mathbf{1}_N) & \text{at } y=-\infty.\end{cases} \tag{4.4}$$

In this case, the moduli matrix H_0 can be set into the following form without loss of generality:

$$H_0 = \sqrt{c}(\mathbf{1}_N, e^{\phi}) \sim \sqrt{c}(e^{-\phi},\mathbf{1}_N), \tag{4.5}$$

where e^{ϕ} is an element of $GL(N,\mathbf{C})$ and ϕ describes the moduli space of walls of this system, and the two forms are related by the V-transformation (2.10). The $GL(N,\mathbf{C})$ matrix e^{ϕ} can always be rewritten as a product of a unitary matrix U and a Hermitian matrix $e^{\hat{x}}$ as

$$e^{\phi} = e^{\hat{x}}U^{\dagger},\qquad (\hat{x} = \tfrac{1}{2}\log(e^{\phi}e^{\phi^{\dagger}})). \tag{4.6}$$

With these two matrices, S is an $N\times N$ matrix and is given by the following form:

$$S^{-1} = \mathcal{U}(y,U)\exp\{-\psi(y\mathbf{1}_N - \hat{x}/m) - \hat{x}/2\}, \tag{4.7}$$

where $\mathcal{U}(y,U)$ is an element of the $U(N)$ gauge group satisfying

$$\mathcal{U}(y,U)\to\begin{cases}\mathbf{1}_N & \text{for } y\to+\infty\\ U & \text{for } y\to-\infty\end{cases} \tag{4.8}$$

so that the boundary conditions (4.4) are satisfied, and $\psi(y)$ is a certain real smooth function of y and satisfies the boundary conditions,

$$\psi(y)\to\begin{cases}\frac{1}{2}my & \text{for } y\to+\infty\\ -\frac{1}{2}my & \text{for } y\to-\infty.\end{cases} \tag{4.9}$$

The BPS equations determine the function $\psi(y)$ uniquely, which has been investigated numerically.

The moduli space of domain walls in the GSY model is parameterized by e^{ϕ} and therefore turns out to be

$$\mathcal{M} \simeq GL(N,\mathbf{C})[=U(N)^{\mathbf{C}}]\simeq \mathbf{C}^*\times SL(N,\mathbf{C}). \tag{4.10}$$

This moduli space admits the isometry

$$e^{\phi}\to e^{i\alpha}g_{\rm L}e^{\phi}g_{\rm R}^{\dagger} \tag{4.11}$$

with $(g_{\rm L}, g_{\rm R})\in SU(N)_{\rm L}\times SU(N)_{\rm R}$ and $e^{i\alpha}\in U(1)_{\rm A}$. This is because the domain wall solutions break the symmetry of the two vacua $(N,0)$ and $(0,N)$, $G = SU(N)_{\rm C+L}\times SU(N)_{\rm C+R}\times U(1)_{\rm C+L-R}$, down to its subgroup. The unbroken subgroup is not unique as explained in the next subsection. Here the y-dependence of the gauge transformations varies for different factors of G. For instance the gauge transformation $g(y)\in U(N)_{\rm C}$ in $SU(N)_{\rm C+L}$ has the y-dependence such as

$$g(y)\to\begin{cases}g_{\rm L}^{-1} & \text{for } y\to\infty\\ 0 & \text{for } y\to-\infty,\end{cases} \tag{4.12}$$

with $g_{\rm L}\in SU(N)_{\rm L}$. The opposite dependence is for $SU(N)_{\rm C+R}$. In the following we do not explicitly write "C" as the indices of the groups.

C. Symmetry structure of the moduli space

The $GL(N,\mathbf{C})$ matrix e^{ϕ} can be always diagonalized with two unitary matrices $U_{\rm L}, U_{\rm R}$ as

$$e^{\phi} = U_{\rm L}e^{\phi_0}U_{\rm R}^{\dagger},\qquad \phi_0 = m\,{\rm diag}(y_1,y_2,\cdots,y_N). \tag{4.13}$$

These matrices $U_{\rm L}$, $U_{\rm R}$, and ϕ_0 give another parametrization of the moduli space and are related to $\hat{x}$ and U as

$$U = U_{\rm R}U_{\rm L}^{\dagger},\qquad \hat{x} = U_{\rm L}\phi_0U_{\rm L}^{\dagger}. \tag{4.14}$$

Here the flavor symmetries $g_{\rm L}\in SU(N)_{\rm L}$, $g_{\rm R}\in SU(N)_{\rm R}$ and $e^{-i\alpha}\in U(1)_{\rm A}$ act on e^{ϕ} as

$$H_0\begin{pmatrix}g_{\rm L}e^{\frac{i}{2}\alpha} & 0\\ 0 & g_{\rm R}e^{-\frac{i}{2}\alpha}\end{pmatrix} = \sqrt{c}(g_{\rm L}e^{\frac{i}{2}\alpha}, e^{\phi}g_{\rm R}e^{-\frac{i}{2}\alpha}) \sim \sqrt{c}(\mathbf{1}_N, e^{-i\alpha}g_{\rm L}^{\dagger}e^{\phi}g_{\rm R}), \tag{4.15}$$

where the last equivalence is due to the V-transformation (2.10). Therefore, by using the flavor symmetries, the matrix ϕ always reduces to the real diagonal matrix ϕ_0

as in Eq. (4.13) and each real parameter y_r indicates the position of the rth wall. In this parametrization of the moduli space, there is a redundancy such that

$$U_{\rm L,R} \to U'_{\rm L,R} = U_{\rm L,R} e^{i\lambda}, \qquad e^{i\lambda} \in U(1)^N, \tag{4.16}$$

with a real diagonal matrix λ. Furthermore, when some walls are coincident, the redundancy is enhanced to a larger group. For instance when first and second walls are coincident $y_1 = y_2$, the redundancy is enhanced to $U(1)^{N-1} \times SU(2)$. This means that U_L, U_R, ϕ_0 do not parametrize the moduli space correctly when some of the walls are coincident. Therefore, this parametrization is applicable only for separated walls, although physical meanings of the moduli parameters are manifest.[9]

In the rest of this subsection we discuss the Nambu-Goldstone modes and the quasi-NG modes in our model. For a while let us consider the case that the "chiral" symmetry $SU(N)_{\rm L} \times SU(N)_{\rm R} \times U(1)_{\rm A}$ acts on e^{ϕ} as (4.11). When ϕ is eventually proportional to the unit matrix, the chiral symmetry (4.11) is spontaneously broken down to its diagonal subgroup $SU(N)_{\rm V}$ defined by $g_{\rm L} = g_{\rm R}$ in (4.11):

$$e^{\phi} \to g e^{\phi} g^{\dagger}, \qquad g \in SU(N). \tag{4.17}$$

This breaking results in the appearance of the massless Nambu-Goldstone bosons (pions), parametrizing the coset space $[SU(N)_{\rm L} \times SU(N)_{\rm R} \times U(1)_{\rm A}]/SU(N)_{\rm V} \simeq U(N)_{\rm A}$. It is known in the supersymmetric case that there must appear more massless bosons called the quasi-Nambu-Goldstone bosons [35] in order to have Kähler target spaces. In this case we can have the quasi-NG modes as many as NG modes. It was found in [36] that the numbers of NG modes and quasi-NG modes can change from point to point in the moduli space in noncompact nonlinear sigma models, although the total number of massless bosons is unchanged. This is because the vacuum expectation values along directions corresponding to quasi-NG bosons can further break the symmetry. The most general effective Kähler potential compatible with the symmetry is given in Appendix B to describe the low energy dynamics of massless fields. The exchange of NG and quasi-NG modes occurs also in the moduli space of multiple non-Abelian vortices [54].

Note the fact that the global symmetry $G = SU(N)_{\rm L} \times SU(N)_{\rm R} \times U(1)_{\rm A}$ in (4.11) acts on the moduli space metric as an isometry whereas the complexified group $G^{\bf C} = SL(N, {\bf C})_{\rm L} \times SL(N, {\bf C})_{\rm R} \times {\bf C}^*$ acts on it transitively but not as an isometry. Therefore, the $G^{\bf C}$ action may change the point in moduli space to another with a different symmetry structure. By using the $G^{\bf C}$ action, an arbitrary moduli parameter ϕ can be brought to zero:

$$e^{\phi} = {\bf 1}_N. \tag{4.18}$$

At this point in moduli space, the global symmetry G is broken down to $H_{\rm max} = SU(N)_{\rm V}$ defined in (4.17). Then the number of the NG modes is $\dim G/H_{\rm max} = N^2$. Since the total number of massless bosons is $\dim G^{\bf C}/H^{\bf C} = 2N^2$, the number of quasi-NG modes[10] is $2N^2 - N^2 = N^2$ at this point of moduli space.

Since the symmetry of Lagrangian is G but not $G^{\bf C}$ we can use only G when we discuss the symmetry structure at each point in moduli space. General ϕ can be transformed by G to

$$e^{\phi} = {\rm diag.}(v_1, v_2, \cdots, v_N) \tag{4.19}$$

with v_i real parameters. When all v_i's are different from each other, $H_{\rm max} = SU(N)_{\rm V}$ is further broken down to $H_{\rm min} = U(1)_{\rm V}^{N-1}$. Here the numbers of NG bosons and quasi-NG bosons are $2N^2 - (N-1)$ and $N-1$, respectively. These $N-1$ quasi-NG bosons correspond to the $N-1$ parameters v_i without the overall factor. Therefore, some quasi-NG bosons at the symmetric point (4.18) in the moduli space change to the NG bosons parametrizing $H_{\rm max}/H_{\rm min} = SU(N)_{\rm V}/U(1)_{\rm V}^{N-1}$ reflecting this further symmetry breaking. When some v_i's coincide, some non-Abelian groups are recovered: $H = U(1)_{\rm V}^r \times \prod U(n_i)_{\rm V}$. Then the NG modes $H_{\rm max}/H$ are supplied from quasi-NG modes.[11] All these points in the moduli space with different unbroken symmetries are of course degenerate. This "vacuum alignment" was first pointed out by Shore [36] in the context of supersymmetric nonlinear sigma models.

An interesting point is that the diagonal moduli parameters v_i (quasi-NG bosons) in Eq. (4.19) correspond to the positions of N domain walls, see Eq. (4.13). When all domain walls are separated, the unbroken symmetry is $U(1)_{\rm V}^{N-1}$. When positions of n domain walls coincide, $U(n)_{\rm V}$ symmetry is recovered. This phenomenon has a resemblance to the case of D-branes. However, there is a crucial difference: the symmetry in our case of domain walls is a global symmetry, whereas that of D-branes is a local gauge symmetry. However, in the case of the $d = 2 + 1$ wall world volume, massless scalars can be dualized to gauge fields. Shifman and Yung [26] expected that the off-diagonal gauge bosons of $U(N)$ [which are originally the off-diagonal NG bosons of $U(N)$ before taking a dual-

[9]Similar pathology exists in a parametrization of the moduli space of the non-Abelian vortices by using their position moduli and orientational moduli in the internal space. In such a parametrization, separated vortices are well described but coincident vortices cannot be described [9]. The smooth coordinates parametrizing the moduli space are linear parameters in the moduli matrix (see [45(c)]).

[10]This situation that the number of the NG bosons and quasi-NG bosons coincide is called maximal realizations [35] or fully doubled realizations [36].

[11]The space $H_{\rm max}/H_{\rm min}$ or $H_{\rm max}/H$ is fibered over $G/H_{\rm max}$ and the total space of NG bosons is of course $G/H_{\rm min}$ or G/H. These spaces are G-orbits in the full moduli space $\mathcal{M} \simeq GL(N, {\bf C})$, and the latter is stratified by these spaces as leaves.

ity] will become massive when domain walls are separated, in order to interpret domain walls as D-branes. However, our analysis shows that the off-diagonal NG bosons of $U(N)$ remain massless, and instead some of the quasi-NG bosons become NG bosons for further symmetry breaking with the total number of massless bosons unchanged as explained. We will take a duality explicitly in Sec. V in the case that the dimension of the wall world volume is 3 + 1.

D. The Effective action of domain walls

Elements of the matrix ϕ are holomorphic coordinates of the moduli space. In the effective action, the matrix ϕ is promoted to a matrix-valued field. Note that matrix-valued fields U and $\hat{x}$ (U_R, U_L, and ϕ_0) depend on both ϕ and ϕ^* (neither holomorphic nor antiholomorphic with respect to ϕ). With this knowledge, the Kähler potential for the effective action is calculated by the formulas (2.11) and (2.12), where Ω and Ω_0 are given by

$$\begin{aligned}\Omega &= \exp\{2\psi(y\mathbf{1}_N - \hat{x}/m) + \hat{x}\},\\ \Omega_0 &= e^{\hat{x}}(e^{m(y\mathbf{1}_N - \hat{x}/m)} + e^{-m(y\mathbf{1}_N - \hat{x}/m)}).\end{aligned} \tag{4.20}$$

If we are interested only in the Kähler metric $K_{ij^*}(\phi, \phi^*)$ rather than the density $\mathcal{K}_{ij^*}(y, \phi, \phi^*)$ of the Kähler metric, we can calculate the Kähler metric directly without any approximations. The formulas (2.11) and (2.12) tell us that the quantity $\hat{x}$ in the Kähler potential is the only matrix which is not proportional to the unit matrix. Moreover the matrix-valued fields ϕ and $\phi^\dagger$ appear only through the matrix $\hat{x}$. Therefore, we can write the Kähler potential in terms of a function F of the matrix $\hat{x}$ as

$$K(\phi, \phi^\dagger) = \mathrm{Tr}[F(\hat{x})]. \tag{4.21}$$

This result reflects the fact that if the matrix ϕ (thus $\hat{x}$) is diagonal, the solution reduces to a direct sum of the solutions for independent walls. Because of the Kähler invariance, the Kähler metric receives no contribution from purely holomorphic or purely antiholomorphic additive terms in the Kähler potential. This fact implies that the function $F(x)$ is equivalent under arbitrary linear transformations,

$$F(x) \simeq F(x) + ax + b, \tag{4.22}$$

since $\mathrm{Tr}(\hat{x})$ can be written as a sum of holomorphic and antiholomorphic functions

$$\begin{aligned}2\mathrm{Tr}(\hat{x}) &= \mathrm{logdet}(e^{\phi}e^{\phi^\dagger}) = \mathrm{logdet}(e^{\phi}) + \mathrm{logdet}(e^{\phi^\dagger})\\ &= \mathrm{Tr}(\phi + \phi^\dagger).\end{aligned} \tag{4.23}$$

Actually, the Kähler potential in Eq. (2.11) is well-defined only after using this Kähler transformation, since it contains divergent parts due to constant terms and linear terms with respect to $\hat{x}$. Since the function F is independent of the size of the matrix, N, the function F can be determined by considering the Abelian case ($N = 1$). In the Abelian case, the complex field ϕ consists of two real fields corresponding to two Nambu-Goldstone modes: $\mathrm{Re}\phi/m$ and $\mathrm{Im}\phi$, which are the Nambu-Goldstone modes for broken translation and $U(1)$ phase, respectively. The low energy theorem (2.13) for these Nambu-Goldstone modes tells us that the Kähler potential for $N = 1$ is given by $K = c\hat{x}^2/m$. Thus we obtain the Kähler potential for general N in a compact form

$$K(\phi, \phi^\dagger) = \frac{c}{m}\mathrm{Tr}[\hat{x}^2] = \frac{c}{4m}\mathrm{Tr}[(\mathrm{log}e^{\phi}e^{\phi^\dagger})^2]. \tag{4.24}$$

This is a Kähler potential on $\mathcal{M} \simeq GL(N, \mathbf{C})$.

Next let us derive the Kähler metric from this Kähler potential. To this end, it is convenient to define derivative operators δ_μ and $\delta_\mu^\dagger$ such that $\delta_\mu \equiv \partial_\mu\phi\frac{\partial}{\partial\phi}$, $\delta_\mu^\dagger \equiv \partial_\mu\phi^\dagger\frac{\partial}{\partial\phi^\dagger}$. For instance, δ_μ acts on $\hat{x}$ as,[12]

$$\begin{aligned}2\delta_\mu\hat{x} &= \frac{2L_{\hat{x}}}{e^{2L_{\hat{x}}} - 1} \times \pi_\mu\\ &= \pi_\mu - [\hat{x}, \pi_\mu] + \frac{1}{3}[\hat{x}, [\hat{x}, \pi_\mu]] + \cdots,\end{aligned} \tag{4.26}$$

where π_μ is defined by $\pi_\mu \equiv (\partial_\mu e^{\phi})e^{-\phi}$, and $L_{\hat{x}}$ is an anti-Hermitian operator acting as $L_V \times A = [V, A]$. The effective Lagrangian is, thus, calculated as

$$\mathcal{L} = \delta^\mu\delta_\mu^\dagger K(\phi, \phi^\dagger) = \frac{c}{2m}\mathrm{Tr}\left[\pi_\mu^\dagger\frac{2L_{\hat{x}}}{e^{2L_{\hat{x}}} - 1} \times \pi^\mu\right]. \tag{4.27}$$

Here we have used the identity $2\,\mathrm{Tr}[\hat{x}\delta_\mu^\dagger\hat{x}] = \mathrm{Tr}[\hat{x}\pi_\mu^\dagger]$.

E. Localization properties in the strong coupling limit

Here, we examine the localization properties of various massless modes. We will use the density of the Kähler metric or Kähler potential as physical quantities to examine the localization properties of massless modes.

To this goal, it is convenient to consider the strong coupling limit $g^2 \to \infty$ where we know the exact solution for the matrix-valued function Ω which is given in terms of the moduli matrix H_0 as

$$\begin{aligned}\Omega = \Omega_0 &= c^{-1}H_0e^{2My}H_0^\dagger\\ &= e^{\hat{x}}(e^{m(y\mathbf{1}_N - \hat{x}/m)} + e^{-m(y\mathbf{1}_N - \hat{x}/m)}).\end{aligned} \tag{4.28}$$

Equation (2.13) gives the density of the Kähler metric in

[12]The relation between infinitesimal deformations $\delta e^{2\hat{x}}$ and $\delta\hat{x}$ is generally given by

$$\begin{aligned}\delta e^{2\hat{x}}e^{-2\hat{x}} &= 2\int_0^1 dt(e^{2t\hat{x}}\delta\hat{x}e^{2(1-t)\hat{x}})e^{-2\hat{x}}\\ &= 2\int_0^1 dte^{2tL_{\hat{x}}} \times \delta\hat{x}\\ &= \frac{e^{2L_{\hat{x}}} - 1}{L_{\hat{x}}} \times \delta\hat{x}.\end{aligned} \tag{4.25}$$

the strong coupling limit as

$$\delta^\mu \delta^\dagger_\mu \mathcal{K}(y, \phi, \phi^\dagger) = \delta^\mu \delta^\dagger_\mu \mathrm{Tr}[c \log \Omega] = c\mathrm{Tr}[\pi^\dagger_\mu \Omega^{-1} \pi^\mu \Omega^{-1} e^{2\hat{x}}]. \quad (4.29)$$

By integrating over y one can easily check that this density of the Kähler metric leads to the effective Lagrangian (4.27). Let us introduce an $N \times N$ matrix τ_μ as

$$\begin{aligned}\tau_\mu &\equiv U^\dagger_{\mathrm{L}}(2(e^{L_{\hat{x}}} + 1)^{-1} \times \pi_\mu) U_{\mathrm{L}} \\ &= \partial_\mu \phi_0 + 2(e^{L_{\phi_0}} + 1)^{-1} \times (U^\dagger_{\mathrm{L}} \partial_\mu U_{\mathrm{L}}) \\ &\quad - 2(e^{-L_{\phi_0}} + 1)^{-1} \times (U^\dagger_{\mathrm{R}} \partial_\mu U_{\mathrm{R}}).\end{aligned} \quad (4.30)$$

If rth and sth walls are well separated $y_r \gg y_s$, the (r, s) component of the matrix τ_μ is given by

$$(\tau_\mu)_{rs} \approx \begin{cases} -(U^\dagger_{\mathrm{R}} \partial_\mu U_{\mathrm{R}})_{rs} & \text{for } r > s, \\ m\partial_\mu y_r + (U^\dagger_{\mathrm{L}} \partial_\mu U_{\mathrm{L}})_{rr} - (U^\dagger_{\mathrm{R}} \partial_\mu U_{\mathrm{R}})_{rr} & \text{for } r = s, \\ (U^\dagger_{\mathrm{L}} \partial_\mu U_{\mathrm{L}})_{rs} & \text{for } r < s. \end{cases} \quad (4.31)$$

In terms of this τ_μ, the density of the Kähler metric is given by

$$\mathcal{K}_{ij^*} \partial_\mu \phi^i \partial^\mu \phi^{j*} = \frac{c}{4} \sum_r^N \frac{|(\tau_\mu)_{rr}|^2}{\cosh^2(m(y - y_r))} + c \sum_{r \neq s}^N \frac{\cosh^2(\frac{m}{2}(y_r - y_s)) |(\tau_\mu)_{rs}|^2}{\cosh(m(y - y_r)) \cosh(m(y - y_s))}. \quad (4.32)$$

This formula contains full information of the localization properties of the massless modes.

Equation (4.32) shows that the fields y_r indicate that the fluctuation field of the position of the rth wall and the wave function corresponding to the rth diagonal element $(\tau_\mu)_{rr}$ is localized on the rth wall. On the other hand, the fluctuation modes of the off-diagonal elements $(\tau_\mu)_{rs}$, $(r \neq s)$ are not localized on the individual wall. To see where these off-diagonal modes have nonvanishing wave functions, we take the limit of well-separated walls $y_r \gg y_s$. Then we obtain

$$\frac{\cosh^2(\frac{m}{2}(y_r - y_s))}{\cosh(m(y - y_r)) \cosh(m(y - y_s))} \approx \begin{cases} 0, & \text{for } y \gg y_r \\ 1, & \text{for } y_s \ll y \ll y_r \\ 0, & \text{for } y \ll y_s. \end{cases} \quad (4.33)$$

Therefore we find that the off-diagonal elements $(\tau_\mu)_{rs}$ correspond to the non-Abelian clouds which have support between the rth wall and the sth wall. As we showed in the previous section, nonvanishing fluctuation of these modes causes a repulsive force between the two walls. In contrast, the bulk modes have support over the entire space including infinity, and the localized modes have support between (possibly coincident) walls. Note that $4k(N - k)$ modes of the non-Abelian clouds, which correspond to $(\tau_\mu)_{rs}$ and $(\tau_\mu)_{sr}$ with $1 \leq r \leq k$ and $k + 1 \leq s \leq N$, have support in the kth vacua in Eq. (4.2) and just constitutes the NG modes of that vacua.

F. Dynamics of non-Abelian cloud fluctuations: chiral dynamics

If we restrict our attention to coincident walls $y_1 = y_2 = \cdots = y_N = 0$, the matrix e^ϕ reduces to a unitary matrix $U^\dagger = U_{\mathrm{L}} U^\dagger_{\mathrm{R}}$ leading to $\hat{x} = 0$, then the Lagrangian reduces to the chiral Lagrangian plus a kinetic term for fluctuations of $\hat{x}$ as

$$\mathcal{L} = -\frac{c}{2m} \mathrm{Tr}[U^\dagger \partial_\mu U U^\dagger \partial^\mu U] + \frac{c}{2m} \mathrm{Tr}[\partial_\mu \hat{x} \partial^\mu \hat{x}] + \mathcal{O}(\hat{x}^4). \quad (4.34)$$

This is nothing but the chiral Lagrangian for the chiral symmetry breaking if we set all the quasi-NG bosons to zero; $\hat{x} = 0$. There NG bosons are interpreted as "pions." In Ref. [27] we placed two domain walls at the same position in order to realize the chiral Lagrangian.

Conversely, in the case of well-separated walls, $y_1 \gg y_2 \gg \cdots \gg y_N$, the Lagrangian asymptotically reduces to

$$\mathcal{L}|_{\mathrm{well-separated}} \approx \sum_r \frac{cm}{2} |\partial_\mu y_r|^2 + \frac{c}{2m} |(A^-_\mu)_{rr}|^2 + \frac{c}{4} \sum_{r \neq s} |y_r - y_s| (|(A^+_\mu)_{rs}|^2 + |(A^-_\mu)_{rs}|^2), \quad (4.35)$$

where the vector fields $A^\pm_\mu$ give the fluctuations of the unitary matrices U_{L} and U_{R} as

$$\begin{aligned} A^-_\mu &= iU^\dagger_{\mathrm{L}} \partial_\mu U_{\mathrm{L}} - iU^\dagger_{\mathrm{R}} \partial_\mu U_{\mathrm{R}} = -U^\dagger_{\mathrm{L}}(U^\dagger i \partial_\mu U) U_{\mathrm{L}}, \\ A^+_\mu &= iU^\dagger_{\mathrm{L}} \partial_\mu U_{\mathrm{L}} + iU^\dagger_{\mathrm{R}} \partial_\mu U_{\mathrm{R}}. \end{aligned} \quad (4.36)$$

Because of the redundancy (4.16), the diagonal elements of A^+_μ turn out to be unphysical modes as we observe in (4.35). Note that kinetic terms for the off-diagonal elements of $A^\pm_\mu$ are proportional to the distance of walls $|y_r - y_s|$. This fact tells us that these are non-Abelian clouds as we expected.

Let us consider well-separated domain walls (kinks) in the case of $N = 2$ for simplicity. We again ignore the center of mass position and the Nambu-Goldstone mode for broken overall $U(1)$ phase. We parametrize $U_{\mathrm{L}}, U_{\mathrm{R}} \in SU(2)$ as

$$U_{\mathrm{L}} = \begin{pmatrix} \cos(\frac{\theta_L}{2}) \exp(i\frac{\varphi_L + \xi_L}{2}) & \sin(\frac{\theta_L}{2}) \exp(i\frac{\varphi_L - \xi_L}{2}) \\ -\sin(\frac{\theta_L}{2}) \exp(-i\frac{\varphi_L - \xi_L}{2}) & \cos(\frac{\theta_L}{2}) \exp(-i\frac{\varphi_L + \xi_L}{2}) \end{pmatrix}, \quad (4.37)$$

and similarly for $U_{\mathrm{R}} \in SU(2)$. Then the Lagrangian for well-separated walls $R = y_1 - y_2 \gg 1/m$ reduces to

$$\mathcal{L}|^{N=2}_{\text{well-separated}} \approx \frac{c}{4m^2}[m^2(\partial_\mu R)^2 + (\partial_\mu \xi + \cos\theta_L \partial_\mu \varphi_L - \cos\theta_R \partial_\mu \varphi_L)^2] + \frac{cR}{4}[(\partial_\mu \theta_L)^2 + \sin^2\theta_L(\partial_\mu \varphi_L)^2 + (\partial_\mu \theta_R)^2 + \sin^2\theta_R(\partial_\mu \varphi_R)^2], \tag{4.38}$$

where $\xi \equiv \xi_L - \xi_R$. The mode $\xi_L + \xi_R$ is unphysical and does not appear in this effective Lagrangian. As in the case of the walls discussed in Sec. III, the fields $\theta_{L,R}$ and $\varphi_{L,R}$ have kinetic terms whose coefficients are proportional to the distance of the walls R for large R. They correspond to the non-Abelian clouds which parametrize the vacuum between the walls $\mathbf{C}P^1 \times \mathbf{C}P^1 \simeq S^2 \times S^2$. Therefore, the conserved charges for the non-Abelian clouds lead to a long-range repulsive force as in Sec. III.

On the other hand, the addition of small imaginary masses for the Higgs fields leads a long-range attractive force. For instance, let us consider a deformation of the mass matrix

$$M = \tfrac{1}{2}\text{diag}(m + i\tilde{m}_L, m - i\tilde{m}_L, -m + i\tilde{m}_R, -m - i\tilde{m}_R) \tag{4.40}$$

with small mass parameters $\tilde{m}_{L,R} \ll m$. These mass parameters $\tilde{m}_{L,R}$ break the chiral symmetry $SU(2)_L \times SU(2)_R$ to $U(1)_L \times U(1)_R$ and induce a long-range attractive potential through Killing vectors $k_{L,R} = \tilde{m}_{L,R}\frac{\partial}{\partial \varphi_{L,R}}$. With very small charges $Q \ll c\frac{\tilde{m}_{L,R}}{m}(\ll c)$, an expectation value of R is guessed to be small, $R \approx Q/(c\tilde{m}_{L,R}) \ll 1/m$. In the low energy limit, $E \gtrsim Q\tilde{m} \to 0$, we obtain the dressed chiral Lagrangian (4.34) with a potential made of the sum of the squares of the Killing vectors $k_L = \frac{\tilde{m}_L}{2} \times (\sigma_3 U)_{ij}\frac{\partial}{\partial U_{ij}} + i\frac{\tilde{m}_L}{2}[\sigma_3, \hat{x}]_{ij}\frac{\partial}{\partial \hat{x}_{ij}}$ and $k_R = -\frac{\tilde{m}_R}{2}(U\sigma_3)_{ij}\frac{\partial}{\partial U_{ij}}$:

$$V = \frac{c}{4m}\left(\tilde{m}_L^2 + \tilde{m}_R^2 - \tilde{m}_L\tilde{m}_R \text{Tr}[U^\dagger \sigma_3 U \sigma_3] + \frac{\tilde{m}_L^2}{2}\text{Tr}[i\sigma_3, \hat{x}]^2\right). \tag{4.41}$$

Most quasi-NG bosons become massive by the third term while the quasi-NG boson corresponding to $\hat{x}$ commuting with σ_3 remains massless.

We have obtained a mass term for pions in the second term. However, it does not agree with the usual form induced by the quark mass terms in the chiral perturbation theory. The same situation occurs in the context of the holographic QCD [55].

V. DUALITY AND NON-ABELIAN 2-FORM FIELDS

Up to here we did not restrict the dimensionality of the space-time; BPS domain walls can be constructed in dimensions ranging from $d = 1 + 1$ to $d = 4 + 1$. In this section we restrict the dimension to be the maximal one $d = 4 + 1$ to discuss the duality on the $3 + 1$-dimensional world volume of walls, which is realistic for brane-world applications. In $3 + 1$ dimensions, scalar fields are dual to 2-form fields. In the framework of supersymmetry with four supercharges, the chiral superfields $\Phi(x, \theta, \bar{\theta})$ ($\bar{D}_{\dot{\alpha}}\Phi = 0$) are dual to the chiral spinor superfields $B_\alpha(x, \theta, \bar{\theta})$ ($\bar{D}_{\dot{\alpha}}B_\beta = 0$) [56].

In the simplest model considered in Sec. III, the moduli space of domain walls is toric Kähler, namely, it admits $U(1)^n$ holomorphic isometry with n its complex dimension. In this case the dual theory can be obtained by using the n Abelian dualities along n $U(1)$ isometries [42]. The dual theory is an interacting theory of Abelian 2-form fields.

In this section we discuss the dual theory of the GSY model considered in Sec. IV. In the paper of Shifman and Yung [26], they considered the $d = 3 + 1$ bulk dimension and so the $2 + 1$ dimensional wall world volume. They claimed that the dual theory of the $U(2)$ NG bosons in $2 + 1$ dimensions is $U(2)$ gauge theory, although they were not able to obtain a nontrivial interaction term of non-Abelian gauge fields. Here we construct the full dual theory by restricting the bulk dimension to $d = 4 + 1$ so that the wall world volume has $3 + 1$ dimensions. We thoroughly perform the duality transformation of the $GL(N, \mathbf{C})$ sigma model and find the action of non-Abelian 2-form fields. Its bosonic counterpart is known as the Freedman-Townsend model [39]. In fact the Kähler potential (4.24) precisely coincides with the one proposed for supersymmetric extension of the Freedman-Townsend model [38].

We start from the Lagrangian of the 2-form fields. Here we use the superfield formalism basically following the notation in [57]. The 2-form field $B_{\mu\nu}(x)$ in $3 + 1$ dimensions belong to the (anti-)chiral spinor superfields $B_\alpha(x, \theta, \bar{\theta})$ [$\bar{B}_{\dot{\alpha}}(x, \theta, \bar{\theta})$], satisfying the constraints [56,58]

$$\bar{D}_{\dot{\alpha}}B_\beta(x, \theta, \bar{\theta}) = 0, \qquad D_\alpha \bar{B}_{\dot{\beta}}(x, \theta, \bar{\theta}) = 0. \tag{5.1}$$

These superfields can be expanded in terms of component fields as

$$\begin{aligned} B^\alpha(y, \theta) &= \psi^\alpha(y) + \tfrac{1}{2}\theta^\alpha(C(y) + iD(y)) + \tfrac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}\theta_\beta B_{\mu\nu}(y) + \theta\theta\eta^\alpha(y), \\ \bar{B}_{\dot{\alpha}}(y^\dagger, \bar{\theta}) &= \bar{\psi}_{\dot{\alpha}}(y^\dagger) + \tfrac{1}{2}\bar{\theta}_{\dot{\alpha}}(C(y^\dagger) - iD(y^\dagger)) + \tfrac{1}{2}(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}B_{\mu\nu}(y^\dagger) + \bar{\theta}\bar{\theta}\bar{\eta}_{\dot{\alpha}}(y^\dagger), \end{aligned} \tag{5.2}$$

where $(\sigma^{\mu\nu})^\alpha{}_\beta = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)^\alpha{}_\beta$, $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$, and $y^{\mu\dagger} = x^\mu - i\theta\sigma^\mu\bar{\theta}$. If one fixes y^μ ($y^{\mu\dagger}$), one finds $\bar{D}_{\dot{\alpha}} = \partial/\partial\bar{\theta}^{\dot{\alpha}}$ ($D_\alpha = -\partial/\partial\theta^\alpha$). See Ref. [57] for details. We consider the non-Abelian 2-form field with the group $G = U(N)$: $B_\alpha(x, \theta, \bar{\theta}) = B^A_\alpha(x, \theta, \bar{\theta})T_A$. Let us introduce a $\mathcal{U}(N)$-valued auxiliary vector superfield $A(x, \theta, \bar{\theta}) = A^A(x, \theta, \bar{\theta})T_A$, satisfying the constraint $A^{A\dagger} = A^A$. Its field strengths are (anti-)chiral spinor superfields,

$$W_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}(e^{-A}D_\alpha e^A), \qquad \bar{W}_{\dot\alpha} = \tfrac{1}{4}DD(e^A\bar{D}_{\dot\alpha}e^{-A}). \tag{5.3}$$

The first-order Lagrangian is given as [38]

$$\mathcal{L} = -\frac{1}{2f}\left[\int d^2\theta \mathrm{Tr}(W^\alpha B_\alpha) + \int d^2\bar\theta \mathrm{Tr}(\bar{W}_{\dot\alpha}\bar{B}^{\dot\alpha})\right] + \frac{1}{4f}\int d^4\theta \mathrm{Tr}A^2. \tag{5.4}$$

See also [42] for the Abelian case. This Lagrangian is invariant under the antisymmetric tensor gauge transformation,[13] parameterized by a $\mathcal{U}(N)$-valued vector superfield $\Omega(x,\theta,\bar\theta) = \Omega^A(x,\theta,\bar\theta)T_A$ (with $\Omega^{A\dagger} = \Omega^A$):

$$\begin{aligned}\delta B_\alpha &= -\frac{i}{4}\bar{D}\bar{D}\mathcal{D}_\alpha(e^{-A}\Omega),\\ \delta\bar{B}_{\dot\alpha} &= -\frac{i}{4}DD\bar{\mathcal{D}}_{\dot\alpha}(\Omega e^{-A}), \qquad \delta A = 0\end{aligned} \tag{5.5}$$

with the covariant spinor derivative $\mathcal{D}_\alpha = D_\alpha + [e^{-A}D_\alpha e^A, \cdot]$. With this invariance we can take the Wess-Zumino gauge: $D = \psi_\alpha = 0$. The Lagrangian (5.4) is invariant under the *global* $U(N)$-transformation

$$\begin{aligned}B_\alpha \to B'_\alpha = g^{-1}B_\alpha g, &\qquad \bar{B}_{\dot\alpha} \to \bar{B}'_{\dot\alpha} = g^{-1}\bar{B}_{\dot\alpha}g,\\ A \to A' = g^{-1}Ag, &\qquad W_\alpha \to W'_\alpha = g^{-1}W_\alpha g,\end{aligned} \tag{5.6}$$

with $g \in U(N)$.

In principle the second order Lagrangian of the 2-form fields B_α can be obtained by eliminating the auxiliary field A with solving its equations of motion. On the other hand, if we eliminate $B_\alpha(x,\theta,\bar\theta)$, we can obtain the $GL(N,\mathbf{C})$ sigma model. The equation of motion for B_α

$$-4W_\alpha(x,\theta,\bar\theta) = \bar{D}\bar{D}(e^{-A}D_\alpha e^A) = 0 \tag{5.7}$$

implies that A is in a pure gauge:

$$e^{A(x,\theta,\bar\theta)} = e^{\phi(x,\theta,\bar\theta)}e^{\phi^\dagger(x,\theta,\bar\theta)}, \qquad \bar{D}_{\dot\alpha}\phi(x,\theta,\bar\theta) = 0. \tag{5.8}$$

Here we have introduced the $\mathcal{U}(N)$-valued chiral superfield $\phi = \phi^A T_A$. By substituting (5.8) back into the Lagrangian (5.4), we obtain the Lagrangian for ϕ [38]:

$$\mathcal{L} = \int d^4\theta \frac{1}{4f}\mathrm{Tr}[(\log e^\phi e^{\phi^\dagger})^2]. \tag{5.9}$$

This coincides with the Kähler potential (4.24) with identifying $c/m = 1/f$. In the Wess-Zumino gauge $D = \psi_\alpha = 0$ physical bosonic fields are the 2-form fields $B^A_{\mu\nu}$ and the associated scalar fields C^A. When all domain walls are coincident we can identify the $B^A_{\mu\nu}$ as the NG bosons of $U(N)$ and C^A as the quasi-NG bosons. When some walls are separated, identification is rather complicated.

[13]This transformation is Abelian, though Ω is $\mathcal{G}$-valued.

VI. A COMMENT ON NON-ABELIAN MONOPOLES AND A MONOPOLE BOUND STATE

Our work is straightforwardly applicable to a system of confined monopoles in the Higgs phase. Those monopoles can be identified with kinks inside a non-Abelian vortex [28]. It is well known that a single BPS vortex in $U(1) \times SU(N)$ gauge theory coupled to N Higgs fields in the fundamental representation with the FI term has the orientational moduli $\vec\phi \in \mathbf{C}P^{N-1}$ ($\vec\phi \simeq \lambda\vec\phi$ with $\lambda \in \mathbf{C}^*$) [6,7]. Its Kähler potential is given by

$$K = \frac{4\pi}{g^2}\log|\vec\phi|^2 \tag{6.1}$$

with g the coupling constant of $SU(N)$. If we add real masses described by a diagonal mass matrix M in the original theory, a contribution to the effective theory on the vortex is calculated by a Killing vector $\delta\vec\phi = iM\vec\phi$; the potential is written as the square of the Killing vector [13,14,28,29]. This system is the same as the one we considered in the strong gauge coupling limit in Sec. III, if we replaced $4\pi/g^2$ by c. Here, kinks (domain walls) of the effective action correspond to monopoles confined by vortices attached from both sides. Actually, the coefficient in the potential (6.1) can be determined so that the tension of the kink coincides with the mass of the monopole [13,14]. So far only nondegenerate masses M were considered for the Higgs scalar fields [13,14,28,29]. In this case a confined monopole is Abelian (of the 't Hooft-Polyakov type) and attached vortices are also Abelian (of the ANO type). A new aspect in this paper is that if we choose the degenerate masses M as discussed in Sec. III, non-Abelian monopoles are confined by non-Abelian vortices. This precisely gives a correspondence between non-Abelian domain walls and non-Abelian monopoles as discussed in the introduction. In particular, we expect correspondence of non-Abelian clouds in both solitons. We expect that in the original theory we can take a limit of usual non-Abelian monopoles without vortices in an unbroken phase by turning off the FI parameter. In this limit the $U(1)$ magnetic flux spreads out and the vortex vanishes. This is because the Kähler potential is independent of the FI parameter and the $U(1)$ gauge coupling. More precise correspondence to non-Abelian monopoles deserves to be studied further, in particular, for the application to non-Abelian duality.

An interesting application of this correspondence is a monopole-monopole bound state. A mass splitting in the imaginary part between masses, which are degenerate in the real part, can be considered by taking another Killing vector; if we take masses like Eq. (3.26), we see the existence of the long-range repulsive force by charges Q and the confining force by imaginary masses $\tilde{m}$ between the monopoles. The distance of these monopoles are stabilized as $g^2Q/2\pi\tilde{m}$. This bound state is made of Abelian monopoles, but we can construct a bound state of non-

Abelian monopoles by considering a set of masses degenerate in both real and imaginary parts, instead of Eq. (3.26).

VII. CONCLUSION AND DISCUSSION

In this paper, we have studied domain walls in Abelian and non-Abelian gauge theories with degenerate masses for Higgs fields. In the model with degenerate masses, discrete components of the vacua are not necessarily isolated points but have continuous flat directions. Then the domain walls interpolating between these vacua have normalizable as well as nonnormalizable zero-modes corresponding to the Nambu-Goldstone modes of the broken non-Abelian flavor symmetry. When spatial infinities have such a degeneracy the wall solutions possess a nonnormalizable mode whose wave functions extend to infinity. On the other hand, when such a degeneracy appears between two domain walls, the wall solutions possess normalizable wave functions spreading between those domain walls. The latter are called non-Abelian clouds and appear in the effective theory on the domain walls. In the effective theory, these non-Abelian clouds give the long-range forces between two walls. We have constructed domain walls with stabilized relative position, which were supported by the long-range forces. They have different properties from those of Q-walls in models with fully nondegenerate masses. The properties of the domain walls in the model with degenerate masses have been discussed by using the D-brane configurations. We have determined the Kähler potential of the effective theory of the walls in the generalized Shifman-Yung model and have found that the effective dynamics of coincident walls are described by the chiral Lagrangian. In addition, we also have found that they are described by the chiral Lagrangian with mass terms if we introduce complex mass parameters which break the non-Abelian flavor symmetry in the original theory. We have performed the electromagnetic duality transformations to the massless scalars on the $3 + 1$-dimensional world volume of the walls. We have obtained the antisymmetric tensor field with non-Abelian symmetry by applying the dual transformation of Freedman and Townsend. We have given a brief discussion on the application to the non-Abelian monopoles confined by non-Abelian vortices. The possibility of a monopole-monopole bound state has been pointed out.

We give several discussions here.

We have obtained supersymmetric extension of the $U(N)$ chiral Lagrangian. It is obviously interesting to include higher derivative corrections to it. It was partially done [27] to obtain a four derivative term, which turned out to be the Skyrme term. Duality between Nambu-Goto type action and tensor gauge theory with higher derivative terms was discussed in [59]. So the dual tensor theory should be obtainable in the case with higher derivative terms.

Inclusion of a SUSY breaking term deserves to be studied. Since masslessness of quasi-NG bosons is ensured only by supersymmetry they will acquire mass of the scale of the SUSY breaking term. There was large degeneracy of vacua so the question is which vacuum is chosen by the SUSY breaking. This problem was studied in SUSY nonlinear sigma models [36]; the answer is that there remains the vacua with the maximal unbroken symmetry. This implies that an attractive force exists between (non-BPS) domain walls and then all the domain walls are compressed in the end.

Our work can be generalized to the case of domain wall networks which are $1/4$ BPS states [48]. Non-Abelian clouds appear inside a domain wall loop there [49]. In that case, a repulsive force caused by the charge given to non-Abelian clouds is proportional to the inverse of the area of the loop.

Another interesting $1/4$ BPS composite system is a system of vortices stretched between domain walls (called D-brane soliton) [23,60]. So far this system was studied in theories with nondegenerate Higgs masses, where domain walls are Abelian and possess only $U(1)$ internal moduli. Although the full exact solutions were already obtained [23] (in the strong gauge coupling limit), it is interesting to understand this configuration from the viewpoint of the domain wall world volume. Vortex strings attached to domain walls can be regarded as sigma model lumps in the viewpoint of the *total* domain wall moduli $\mathcal{M}_{\rm total} \simeq G_{N_{\rm F},N_{\rm C}} \simeq SU(N_{\rm F})/[SU(N_{\rm C}) \times SU(N_{\rm F} - N_{\rm C}) \times U(1)]$, which can be constructed by patching all topological sectors together [22].[14] The topological stability is ensured by $\pi_2(\mathcal{M}_{\rm total}) \simeq \mathbf{Z}$. The other interpretation is that vortex strings can be regarded as *global* vortices of the $U(1)$ moduli of domain walls in the wall effective action. In this case, the topological stability is ensured by $\pi_1[U(1)] \simeq \mathbf{Z}$. Domain walls with a non-Abelian cloud possess non-Abelian moduli $U(N)$ as discussed in this paper. The total moduli space in the GSY model is $\mathcal{M}_{\rm total} \simeq G_{2N,N} \simeq SU(2N)/[SU(N) \times SU(N) \times U(1)]$, so $\pi_2(\mathcal{M}_{\rm total}) \simeq \mathbf{Z}$ as in the nondegenerate case. Sigma model lumps in this case, however, are more interesting. This is because the moduli space (with a fixed topological sector) is $\mathcal{M} \simeq T^* U(N)$ as we have seen in (4.10). This gives non-Abelian global vortices supported by $\pi_1[U(N)] \simeq \mathbf{Z}$ which are expected to form in the chiral phase transition [61].

ACKNOWLEDGMENTS

We would like to thank Youichi Isozumi for a collaboration at the early stage of this work. M. E., M. N., and K. O. would like to thank David Tong for a collaboration in [27], Eric Weinberg and Ki-Myeong Lee for a fruitful discussion on non-Abelian clouds, and Korea Institute of Advanced Study (KIAS) for their hospitality. This work is

[14]The reason why we have to consider different topological sectors together is that the number of domain walls is reduced at the center of vortices [22].

supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science, and Technology, Japan No. 17540237 and No. 18204024 (N. S.). The work of T. F. is supported by the Japan Society for the Promotion of Science for Young Scientists. The work of M. E. and K. O. is supported by the Japan Society for the Promotion of Science for Research Abroad.

APPENDIX A: ASYMPTOTIC BEHAVIOR OF THE KÄHLER POTENTIAL

First, we consider the Kähler potential for one domain wall in the $N_F = 2$ case as a simplest example. Let us take the moduli matrix and mass parameter as

$$H_0 = (1, \phi), \qquad M = \mathrm{diag}(m, 0). \tag{A1}$$

The position of the wall is given by

$$y_0 \equiv \frac{1}{m}\log|\phi|. \tag{A2}$$

If we define $\psi \equiv \log\Omega$, the master Eq. (2.9) for one domain wall is written as

$$\partial_y^2\psi = g^2c(1 - (e^{2my} + e^{2my_0})e^{-\psi}). \tag{A3}$$

The asymptotic behavior of the solution ψ far away from the wall position y_0 is given by

$$\psi \simeq \log(e^{2my} + e^{2my_0}) \simeq \begin{cases} 2my & \text{for } y \gg y_0 \\ 2my_0 & \text{for } y \ll y_0, \end{cases} \tag{A4}$$

with exponentially suppressed correction terms of order $\mathcal{O}(e^{-my})$ or $\mathcal{O}(e^{-yg\sqrt{c}})$ [19,20]. The density of the Kähler potential (2.12) can be written in terms of ψ as

$$\mathcal{K} = c\psi + c(e^{2my} + e^{2my_0})e^{-\psi} + \frac{1}{2g^2}(\partial_y\psi)^2. \tag{A5}$$

The counterterms $\mathcal{K}_{ct}(\phi)$ and $\bar{\mathcal{K}}_{ct}(\phi^*)$, which are holomorphic and antiholomorphic with respect to the moduli parameter ϕ, are determined from the asymptotic behavior (A4) as

$$\mathcal{K}_{ct}(\phi) + \bar{\mathcal{K}}_{ct}(\phi^*) = c[2my\theta(y) + (\log\phi + \log\phi^*)\theta(-y)] + c + \frac{2m^2}{g^2}\theta(y), \tag{A6}$$

where $\theta(y)$ is the step function. The Kähler potential can be calculated by using the transformation property under the translation such that $\psi(y + y_0, y_0) = \psi(y, 0) + 2my_0$. Then we obtain the asymptotic behavior of the Kähler potential of one wall for large values of y_0 as

$$K = \int_{-\infty}^{\infty} dy(\mathcal{K} - \mathcal{K}_{ct}(\phi) - \bar{\mathcal{K}}_{ct}(\phi^*)) = mcy_0^2 - \frac{2m^2}{g^2}y_0 + \text{const.} \tag{A7}$$

Next, let us calculate the Kähler potential for the walls with degenerate masses discussed in Sec. III. The function $f(\mu R)$, which has been defined in (3.15), is independent of the number of the flavors with degenerate masses, so we can calculate the function $f(\mu R)$ in the $N_F = 3$ case. The moduli matrix, mass parameters and the master equation are given by

$$H_0 = (1, \phi, 1), \qquad M = \mathrm{diag}(m_1, 0, m_2), \tag{A8}$$

$$\partial_y^2\psi = g^2c(1 - (e^{2m_1y} + |\phi|^2 + e^{-2m_2y})e^{-\psi}). \tag{A9}$$

The positions of the walls are related to the parameter ϕ as

$$y_1 = \frac{1}{m_1}\log|\phi|, \qquad y_2 = -\frac{1}{m_2}\log|\phi|, \tag{A10}$$

and the relative distance of the walls is given by $R = y_1 - y_2 = 2/\mu\log|\phi|$ with $\mu \equiv 2m_1m_2/(m_1 + m_2)$. First, let us consider the asymptotic behavior of the Kähler potential for sufficiently large R. The solution of this master equation for sufficiently large R is given by

$$\psi \approx \psi_1 + \psi_2 - \mu R, \tag{A11}$$

where ψ_1 and ψ_2 is the solution of the master equation for one wall (A3) with replacements $(y, y_0, m) \to (y, y_1, m_1)$ and $(y, y_0, m) \to (-y, -y_2, m_2)$, respectively. The correction to the solution (A11) is exponentially small for sufficiently large R. For the solution of the master equation ψ, the density of the Kähler potential is written as

$$\mathcal{K} = c\psi + c(e^{2m_1y} + e^{\mu R} + e^{-2m_2y})e^{-\psi} + \frac{1}{2g^2}(\partial_y\psi)^2 \approx \mathcal{K}_1 + \mathcal{K}_2 - c\mu R - c, \tag{A12}$$

where we have used the fact that $\partial_y\psi_1\partial_y\psi_2$ is exponentially small for large R. The counterterms are chosen to be

$$\begin{aligned}\mathcal{K}_{ct} &= 2cy[m_1\theta(y) - m_2\theta(-y)] + c \\ &\quad + \frac{2}{g^2}[(m_1)^2\theta(y) + (m_2)^2\theta(-y)], \\ &= (\mathcal{K}_{ct})_1 + (\bar{\mathcal{K}}_{ct})_1 + (\mathcal{K}_{ct})_2 + (\bar{\mathcal{K}}_{ct})_2 - c\mu R - c.\end{aligned} \tag{A13}$$

Here the quantities with subscript 1, 2 are given by the corresponding quantities (A5) and (A6) with the replacements $(y, y_0, m) \to (y, y_1, m_1)$ and $(y, y_0, m) \to (-y, -y_2, m_2)$, respectively. Then we find the asymptotic Kähler potential for large R as

$$K = \int_{-\infty}^{\infty} dy(\mathcal{K} - \mathcal{K}_{ct}) \approx K_1 + K_2 = \frac{c\mu}{2}R^2 - \frac{m_1 + m_2}{g^2}\mu R + \text{const.} \tag{A14}$$

The correction to this Kähler potential is exponentially small for large R. Next, let us consider asymptotic behavior for small $|\phi|^2 = e^{\mu R}$. The Kähler potential for sufficiently small $|\phi|$ can be easily obtained by assuming that the moduli space is smooth and ϕ is a good coordinate of the moduli space at $|\phi| = 0$. Then the metric of the moduli space in terms of the coordinate ϕ can be expanded as

$$g(|\phi|^2) \equiv \frac{\partial^2 K}{\partial \phi \partial \phi^*} = A + \mathcal{O}(|\phi|^2). \tag{A15}$$

Here the constant term A cannot be zero since ϕ is a good coordinate at $|\phi| = 0$. Therefore, the Kähler potential for small $|\phi|$ is given by

$$\begin{aligned} f(\mu R) = K(|\phi|^2) &= A|\phi|^2 + \mathcal{O}(|\phi|^4) \\ &= Ae^{\mu R} + \mathcal{O}(e^{2\mu R}). \end{aligned} \tag{A16}$$

Here we have ignored constant terms which do not contribute to the Kähler metric.

APPENDIX B: GENERAL KÄHLER POTENTIAL DETERMINED BY SYMMETRY

In this subsection we construct the most general Kähler potential compatible with the symmetry (4.11) in the spirit of the method of nonlinear realization. If we define

$$X \equiv e^{\phi} e^{\phi^\dagger}, \tag{B1}$$

it transforms as

$$X \to g_{\rm L} X g_{\rm L}^\dagger. \tag{B2}$$

Then the most general Kähler potential invariant under the symmetry (4.11) is given using an arbitrary function F of $N-1$ variables:

$$K = F(\mathrm{Tr}X, \mathrm{Tr}X^2, \cdots, \mathrm{Tr}X^{N-1}). \tag{B3}$$

Traces of higher order of X's are not independent because of the Cayley-Hamilton theorem of N by N matrices A: $A^N - \mathrm{Tr}(A)A^{N-1} \cdots \pm (\det A)\mathbf{1}_N = 0$. The Kähler potential (B3) was obtained by Shore a long time ago [36]. The target space of this nonlinear sigma model is the complexification of $U(N)$: $GL(N, \mathbf{C}) = U(N)^{\mathbf{C}} \simeq T^* U(N)$. By construction the isometry is not the transitive group $\mathbf{C}^* \times SL(N, \mathbf{C})_{\rm L} \times SL(N, \mathbf{C})_{\rm R}$. The metric is invariant under its real subgroup (4.11) generated by a real form of the complex Lie algebra. This always occurs if one constructs an effective Lagrangian of massless particles when a global symmetry is spontaneously broken with preserving supersymmetry [35]. It was shown in [36] that by setting quasi-NG modes to zero the Lagrangian reduces at the most symmetric points (where ϕ is proportional to the unit matrix) to the chiral Lagrangian of $U(N)$ $\mathcal{L} = \frac{1}{2} f_\pi^2 \mathrm{Tr}[(U^\dagger \partial_\mu U)^2]$, with f_π^2 a constant determined by the derivative of F. However, at generic points symmetry is further broken and more Nambu-Goldstone bosons appear. It is known that G-invariants which are not invariant under $G^{\mathbf{C}}$, namely, the variables $\mathrm{Tr}X, \mathrm{Tr}X^2, \cdots, \mathrm{Tr}X^{N-1}$ in (B3), parametrize quasi-NG bosons at generic points in the moduli space [36].

Returning to our case of domain walls we have additional symmetry other than (4.11) so that we can further restrict the form of the Kähler potential (B3). It is the translational symmetry of space-time broken by the presence of the domain walls:

$$\phi \to \phi + \lambda \mathbf{1}_N, \qquad X \to X e^{\lambda + \lambda^*}. \tag{B4}$$

Interestingly this can be understood as the imaginary part of $e^{i\alpha} \in U(1)_{\rm A}$ in (4.11). The Kähler potential (B3) is reduced to

$$K = c_1 \mathrm{Tr}[(\log X)^2] + \tilde{F}\left[\frac{(\mathrm{Tr}X)^2}{\mathrm{Tr}(X^2)}, \frac{(\mathrm{Tr}X)^3}{\mathrm{Tr}(X^3)}, \frac{(\mathrm{Tr}X)(\mathrm{Tr}X^2)}{\mathrm{Tr}(X^3)}, \cdots\right]. \tag{B5}$$

Here $\tilde{F}$ is an arbitrary function of variables with zero weight of X. The first term is invariant up to the Kähler transformation under the translational symmetry (B4), and the second term is strictly invariant under it. The first term is the Kähler potential (4.24) with the identification of the overall constant $c_1 = c/4m$, and the second term describes the deformation of the metric along the noncompact directions with preserving the isometry [36].

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