New regulators for quantum field theories with compactified extra dimensions. II. Ultraviolet finiteness and effective field theory implementation

Sky Bauman* and Keith R. Dienes+

Department of Physics, University of Arizona, Tucson, Arizona 85721, USA

(Received 29 January 2008; published 3 June 2008)

In the previous article, we proposed two new regulators for quantum field theories in spacetimes with compactified extra dimensions. Unlike most other regulators that have been used in the extra-dimension literature, these regulators are specifically designed to respect the original higher-dimensional Lorentz and gauge symmetries that exist prior to compactification, and not merely the four-dimensional symmetries which remain afterward. In this paper, we use these regulators in order to develop a method for extracting ultraviolet-finite results from one-loop calculations. This method also allows us to derive Wilsonian effective field theories for Kaluza-Klein modes at different energy scales. Our method operates by ensuring that divergent corrections to parameters describing the physics of the excited Kaluza-Klein modes are absorbed into the corresponding parameters for zero modes, thereby eliminating the need to introduce independent counterterms for parameters characterizing different Kaluza-Klein modes. Our effective field theories can therefore simplify calculations involving Kaluza-Klein modes, and be compared directly to potential experimental results emerging from collider data.

DOI: 10.1103/PhysRevD.77.125006 PACS numbers: 11.10.Kk, 11.10.Hi, 11.25.Mj

I. INTRODUCTION

If all goes according to plan, the CERN Large Hadron Collider (LHC) will uncover exciting new phenomena at the TeV scale. These phenomena are likely to hold clues pertaining to some of the most pressing current mysteries of particle physics, including the nature of electroweak symmetry breaking and the origin of the stability of the energy scale at which this occurs. Indeed, through such discoveries, data from the LHC are likely to change the paradigm of high-energy physics, eventually leading to a new “standard model” for the next generation of particle physic(ist)s.

Of course, if we subscribe to the belief that the truly fundamental energy scales of physics are unreachably high (e.g., at or near $M_{\rm Planck} \approx 10^{19}$ GeV, or at least significantly above the electroweak scale), then this new standard model will be at best yet another effective field theory (EFT), valid only within a well-prescribed energy range. Interpreting this data-produced effective Lagrangian will then require comparisons to the EFT's which can be derived from various potential theoretical models of possible new physics. For example, weak-scale supersymmetry (SUSY) is widely considered to be a compelling candidate for new physics, and most phenomenological studies of weak-scale SUSY focus on specific EFT's (e.g., the minimal supersymmetric standard model) in which the supersymmetry is broken but in which the origin of this breaking is not included.

Extra spacetime dimensions are also leading candidates for new physics beyond the current standard model. However, while there has been considerable work analyzing the cumulative effects that the corresponding towers of Kaluza-Klein (KK) states might have on ordinary four-dimensional physics, there have been almost no studies concerned with the EFT's of the towers of excited KK modes themselves. Analyses which do exist are qualitative, focus on special interactions (e.g., brane-kinetic terms), or contain special implicit assumptions.

Yet there are general EFT questions which might be asked in this context. For example, if there exists a single extra flat dimension compactified on a circle of radius R, then the masses of the corresponding KK modes can be expected to follow the well-known relation $m_n^2 = m_0^2 + n^2/R^2$. Likewise, the couplings of these modes will all be equal: $\lambda_{n,n',\ldots} = \lambda_{0,0,\ldots}\delta_{n+n'+\ldots}$. These relations are nothing but the reflection of the higher-dimensional Lorentz invariance which holds in the ultraviolet (UV) limit, and such patterns will be taken as strong evidence in judging whether newly discovered particles are indeed KK states. However, as one passes to lower energies (e.g., through a Wilsonian renormalization-group analysis), these masses and couplings are subject to radiative corrections. As a result, we expect that these simple mass and coupling relations will be deformed as the heavy KK states are integrated out of the spectrum. Indeed, at relatively low energies, the spectrum of low-lying KK modes may be significantly distorted relative to our naive tree-level expectations, and this can potentially be important for experimental searches for (and the identification/interpretation of) such states.

The goal of this paper is to develop methods of deriving and analyzing the EFT's of such KK towers as functions of energy scale. Indeed, if extra dimensions are ultimately observed at the LHC through the discovery of KK reso-

*bauman@physics.arizona.edu
+dienes@physics.arizona.edu

nances, it will be important to understand the radiative corrections to the masses and couplings of such states since this information will ultimately feed into precision calculations of their cross sections and decay rates. However, aside from potential experimental consequences, analyzing the EFT's of towers of KK resonances as functions of the energy scale is also interesting from a purely theoretical perspective, since this provides the only systematic way of understanding what happens as extra dimensions are ''integrated'' out in passing from a higher-dimensional UV limit to a four-dimensional infrared limit.

One fundamental obstacle to performing such a renormalization-group analysis of the KK spectrum has been that general techniques for regularizing loop effects in KK theories were not known. While quantum-mechanical regulators exist which preserve the four-dimensional symmetries (such as Lorentz invariance and gauge invariance) which remain after compactification, such regulators are sufficient only for radiative calculations of the physics of the zero modes. By contrast, calculations of the *excited* KK modes will require techniques which preserve the full set of *higher-dimensional* symmetries. While there has been a small amount of literature concentrating on radiative corrections in KK theories (see, e.g., Refs. [1–17]), relatively few approaches actually satisfy this latter requirement.

In Ref. [18], we developed two new regulators for quantum field theories in spacetimes with compactified extra dimensions. We refer to these regulators as the ''extended hard cutoff'' (EHC) and ''extended dimensional regularization'' (EDR). Although based on traditional four-dimensional regulators, the key new feature of these higher-dimensional regulators is that they are specifically designed to handle mixed spacetimes in which some dimensions are infinitely large and others are compactified. Moreover, unlike most other regulators which have been used in the extra-dimension literature, these regulators are designed to respect the original higher-dimensional Lorentz and gauge symmetries that exist prior to compactification, and not merely the four-dimensional symmetries which remain afterward.

By respecting the full higher-dimensional symmetries, the regulators of Ref. [18] avoid the introduction of spurious terms which would not have been easy to disentangle from the physical effects of compactification. Moreover, by preserving the physics associated with higher-dimensional symmetries, they maintain the associated Ward identities. For example, in a gauge-invariant theory, analogues of the Ward-Takahashi identity should hold not only for the usual zero-mode (four-dimensional) photons, but for all excited Kaluza-Klein photons as well. It is the regulators in Ref. [18] which preserve such identities for the excited KK modes as well as the zero modes.

In this paper, we will extend the techniques in Ref. [18] in two directions.

(i) First, we shall show how the regulators of Ref. [18] can be used in order to extract ultraviolet-finite results from one-loop calculations. Our method operates by ensuring that divergent corrections to parameters describing the physics of the excited Kaluza-Klein modes are absorbed into the corresponding parameters for zero modes, thereby eliminating the need to introduce independent counterterms for parameters characterizing different Kaluza-Klein modes.

(ii) Second, we shall show how these finite results can be used in order to construct EFT's for towers of KK modes. Our EFT approach will therefore provide a framework for comparing an effective Lagrangian extracted from LHC data to higher-dimensional theoretical models. Additionally, as we shall discuss, our EFT's will carry special advantages for calculations of loop effects in experiments involving excited KK modes.

In this paper, we shall follow a Wilsonian approach towards deriving our EFT's. Specifically, we shall employ the regulators of Ref. [18] to calculate the masses and couplings of the KK states as functions of a Wilsonian renormalization-group scale. In other words, we shall explicitly integrate out heavy KK states above a given scale μ, and observe how the parameters describing the remaining light (but nevertheless excited) KK states are affected as a function of the scale μ. One key observation will be essential to this analysis: Although the masses and couplings of individual KK states can be expected to experience strong divergences, the *relative differences* of these parameters between excited KK modes and the zero mode are physical observables and thus can be expected to remain finite and regulator independent. Thus, if the parameters describing the zero modes at a given energy scale are assumed to be determined from experiment, then these finite differences can be used to obtain the parameters describing all of the other excited KK states at this scale. We thus obtain all the parameters needed to define EFT's describing the tower of KK states as functions of the energy scale.

Although the techniques presented here are more general than most previously existing methods, our analysis in this paper will be restricted in certain significant ways. First, our procedures will apply to calculations in theories for which the compactification space is a smooth manifold rather than an orbifold. As such, we will not be considering the effects of extra terms which might arise at singularities of the compactification space, such as brane-kinetic terms. Moreover, as discussed above, although the differences between KK parameters should be finite regardless of any perturbative expansion, this paper will focus exclusively on one-loop calculations. Finally, although our techniques can easily be generalized, for concreteness we shall primarily consider the case of a single extra dimension compactified on a circle.

This paper is organized as follows. In Sec. II, we shall show how to use the regulators from Ref. [18] in order to extract finite, regulator-independent predictions in KK theories at one-loop order. Specifically, we shall provide a general procedure for deriving finite, regulator-independent expressions for differences between renormalized KK parameters. Then, in Sec. III, we shall provide two explicit examples illustrating how this procedure is implemented. In Sec. IV, we shall then demonstrate how to obtain regulator-independent Wilsonian EFT's from these differences. Specifically, we shall show how to calculate Wilsonian evolutions of EFT parameters with respect to the energy scale. Our conclusions can be found in Sec. V.

This paper is the second in a two-part series, and relies on the results from the preceding paper [18]. As such, we shall assume complete familiarity with the methods of Ref. [18], and shall not review results which can be found there.

II. ULTRAVIOLET FINITENESS

In this section, we shall provide a general procedure for calculating finite, regulator-independent corrections to differences between parameters characterizing excited modes and zero modes in KK towers. As indicated in the Introduction, we shall rely on the methods developed in Ref. [18], and we shall assume that the reader is familiar with these techniques. Section III will then provide two explicit examples illustrating how this procedure is implemented.

We begin by considering a generic one-loop diagram of the form shown in Fig. 1 in which an external particle with four-momentum p^μ and mode number n interacts with a tower of KK particles of bare (five-dimensional) mass M. For example, the mass of the rth KK mode in this tower is given by $m_r^2 = M^2 + r^2/R^2$ if the extra dimension is compactified on a circle. Enforcing 5D momentum conservation at the vertices (as appropriate for compactification on a circle) then leads to a one-loop expression of the general form

$$L_n(p) = i\int_0^1 dx \sum_r f_n(p, r, x), \tag{2.1}$$

where x is a Feynman parameter and where f_n represents an appropriate four-dimensional loop momentum integral.

In general, such an expression will diverge badly. Meaningful algebraic manipulations are therefore only possible in the presence of a regulator. In this case, there are two possible sources of divergence: the four-momentum integral f_n, and the internal KK summation $\sum_r$. Both must therefore be regulated, and, as discussed in the Introduction, we need to utilize regulators which preserve the full *five-dimensional* symmetries which exist prior to compactification. These include not only five-dimensional Lorentz symmetry, but also five-dimensional gauge symmetry when appropriate. The fact that five-dimensional symmetries must be preserved implies that we must somehow *correlate* the regulator for the four-momentum integral with the regulator for the KK summation so that they are both imposed and lifted together.

In Ref. [18], two such regulator procedures were introduced. In our EHC procedure, the four-momentum integral f_n is regulated through the introduction of a hard cutoff, while our EDR procedure utilizes a generalization of ordinary 't Hooft-Veltman dimensional regularization for f_n. In either case, the KK summation is regulated through the introduction of a hard cutoff Λ, and all appropriate five-dimensional symmetries are protected through the introduction of strict relations between this cutoff Λ and the regulator parameters involved in regulating f_n. These relations are given in Ref. [18]. Note that while our EDR procedure is completely general, preserving both five-dimensional Lorentz and gauge symmetries, our EHC regulator preserves only Lorentz symmetries and thus is suitable for theories without gauge symmetries. In either case, the net result is that the general expression in Eq. (2.1) then takes the regulated form

$$L_n(p) = i\int_0^1 dx \sum_{r=-\Lambda R+xn}^{\Lambda R+xn} f_n(p, r, x), \tag{2.2}$$

where we now understand f_n to denote an appropriate *regulated* four-momentum integral, and where the particular form of the KK limits is explained in Ref. [18], with the notation $\sum_{r=a}^{b}$ denoting a summation over integer values of r within the range $a \le r \le b$ (even if a and b are not themselves integers). We shall also assume that ΛR can be treated as an integer; in the $\Lambda \to \infty$ limit as our cutoff is removed, this assumption will not affect our final results. Of course, it is understood when writing expressions such as Eq. (2.2) that we are to take the limit $\Lambda \to \infty$ at the end of the calculation (along with a simultaneous, correlated

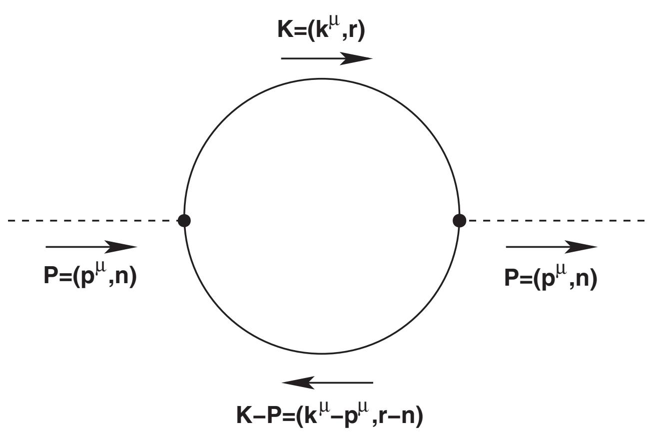

FIG. 1. A generic one-loop diagram, as in Ref. [18]: an external Kaluza-Klein particle (dotted line) with four-momentum p^μ and Kaluza-Klein index n interacts with a tower of Kaluza-Klein particles (solid lines) of bare mass M.

removal of the regulator within the four-momentum integral).

Our goal is to obtain finite, regulator-independent expressions for differences such as $L_n - L_0$ for $n \neq 0$. In order to do this, we begin by utilizing an identity discussed in Ref. [18]. Specifically, for any $n \neq 0$, we can perform a series of variable substitutions in order to write

$$\int_0^1 dx \sum_{r=-\Lambda R+xn}^{\Lambda R+xn} = \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 d\hat{u} \sum_{\hat{r}=-\Lambda R+1}^{\Lambda R}, \tag{2.3}$$

where

$$\hat{u} \equiv x|n| - j \tag{2.4}$$

and

$$\hat{r} \equiv \mathrm{sign}(n) r - j. \tag{2.5}$$

This identity serves to render the KK summation cutoffs independent of the Feynman parameter. Thus, rewriting L_n for $n \neq 0$ in this way and dropping the hats from $\hat{u}$ and $\hat{r}$, we have

$$-i(L_n - L_0) = \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du \sum_{r=-\Lambda R+1}^{\Lambda R} f_n(r, u, j) - \int_0^1 dx \sum_{r=-\Lambda R}^{\Lambda R} f_0(r, x). \tag{2.6}$$

Relabeling $x \to u$ in the second term and using the fact that f_0 is j independent to join the integrands, we thus have

$$-i(L_n - L_0) = \sum_{r=-\Lambda R}^{\Lambda R} \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du[f_n(r, u, j) - f_0(r, u)] - \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du f_n(-\Lambda R, u, j). \tag{2.7}$$

Note that we have dropped the four-momentum p from the expressions for L_0 and L_n, since these expressions are presumed to be evaluated after appropriate renormalization conditions have been applied.

As discussed in the Introduction, physical observables such as the *relative* masses and couplings between different KK states must remain finite even though the masses and couplings for individual KK states might accrue divergent radiative corrections. Indeed, relative differences such as these are originally finite at tree level (modulo potential effects due to classical rescalings), and are also finite to all orders in the UV limit (or equivalently the $R \to \infty$ limit), where five-dimensional Lorentz invariance is restored and the effects of compactification become irrelevant.[1] Since the divergence structure of the theory should not be altered by changing R, we expect such relative differences to remain finite regardless of the radius or effective energy scale. As a result, we expect that expressions such as those in Eq. (2.7) should be finite either exactly as written, or with L_0 and L_n replaced with those subexpressions within L_0 and L_n which are responsible for renormalizing observables. (For example, if the external KK particle in Fig. 1 is a KK photon carrying the Lorentz index μ, then the relevant subexpression would consist of those terms $\tilde{L}_n^{\mu\nu}$ within the full $L_n^{\mu\nu}$ which are proportional to the metric $g^{\mu\nu}$ and which therefore renormalize the masses of the excited KK photons.) We shall assume that our generic expressions L_n consist of only such terms in what follows. Note that since we are restricting our attention to one-loop diagrams, radiative corrections to relative KK parameters will indeed correspond to linear differences of the forms $L_n - L_0$.

Even though Eq. (2.7) is finite, our goal is to write $L_n - L_0$ in a manifestly finite, regulator-independent fashion. In other words, we seek to be able to write differences such as $L_n - L_0$ in the analogous form

$$-i(L_n - L_0) = \sum_{r=-\infty}^{\infty} \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du[\alpha_n(r, u, j) - \alpha_0(r, u)] + \Delta_n, \tag{2.8}$$

where the functions α_0, α_n, and Δ_n are each manifestly finite and regulator independent. However, comparing Eqs. (2.7) and (2.8), we see that we are nearly there. Indeed, looking at Eq. (2.7), we see that there are only two cases we need to consider.

These cases can be distinguished by two properties. If
(i) $f_n(-\Lambda R, u, j)$ remains finite as $\Lambda R \to \infty$, and
(ii) $f_n(r, u, j) - f_0(r, u)$ remains finite as $\Lambda R \to \infty$ for each value of r,

then our first case will apply. Our second case will arise in all other situations, when either one or both of these conditions fail.

In the first case, $f_n(-\Lambda R, u, j)$ remains finite as $\Lambda R \to \infty$. This situation arises when the UV divergence from the four-momentum integration within f_n is canceled by the diverging Kaluza-Klein number $r \equiv -\Lambda R \to -\infty$ in the denominator of the integrand. In such cases, the second line of Eq. (2.7) is finite by itself and may be identified as Δ_n:

$$\Delta_n = -\lim_{\Lambda R \to \infty} \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du f_n(-\Lambda R, u, j). \tag{2.9}$$

The first line of Eq. (2.7) must then also be individually finite, which implies that the KK summation over the difference $f_n - f_0$ should also be finite as $\Lambda R \to \infty$. By itself, this need not imply that each $f_n - f_0$ should be finite for each individual term in the KK sum, for it is possible that divergences of individual $f_n(r) - f_0(r)$ as $\Lambda R \to \infty$ are canceled across the increasingly many terms in the sum as $\Lambda R \to \infty$. (We shall see an explicit example of this

[1] This is not true in orbifold theories, due to the possible presence of brane-kinetic terms.

phenomenon in Sec. III.) However, if we additionally know that each $f_n(r, u, j) - f_0(r, u)$ remains finite as $\Lambda R \to \infty$ for each value of r (our second defining criterion above), it then follows that all regulator dependence must cancel within the difference $f_n - f_0$ in Eq. (2.7). In such cases, we can therefore proceed to identify α_0 and α_n as the cutoff-independent parts of f_0 and f_n, respectively.

Alternatively, it may happen that one or both of the two conditions itemized above are not satisfied. This is therefore the second possible case we need to face. For example, if $f_n(-\Lambda R, u, j)$ diverges as $\Lambda R \to \infty$, then neither expression within Eq. (2.7) is finite by itself, and a further rearrangement of terms within Eq. (2.7) is needed. However, we can generally handle this situation as follows. In general, we can identify α_n as the cutoff-independent part of the difference $f_n - \tilde{f}_n$, where $\tilde{f}_n$ is the value of f_n when all of the bare masses in our theory vanish and renormalization conditions for massless particles have been applied. Subtracting $\tilde{f}_n$ from f_n then cancels the cutoff-dependent terms that do not contain a bare mass, and subtracting L_0 from L_n then cancels whatever cutoff dependence remains. However, the price we pay is that these extra $\tilde{f}_n$ and $\tilde{f}_0$ terms are now shifted into Δ_n, so that Δ_n is now given by

$$\Delta_n = \lim_{\Lambda R \to \infty} \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du \left\{ \sum_{r=-\Lambda R}^{\Lambda R} [\tilde{f}_n(r, u, j) - \tilde{f}_0(r, u)] - f_n(-\Lambda R, u, j) \right\}. \tag{2.10}$$

Of course, these extra terms are precisely what are needed in order to cancel the divergence of $f_n(-\Lambda R)$ as $\Lambda \to \infty$ in cases in which it diverges, and render Δ_n finite. Even when $f_n(-\Lambda R)$ remains finite as $\Lambda \to \infty$, these extra terms will preserve the finiteness of Δ and compensate for the shifted definition of α functions relative to our first case above.

Even though Δ_n is finite in each case, it is still important to be able to write Δ_n in an explicitly regulator-independent form. We shall show how to do this in Sec. III.

Thus, we conclude that as our regulators are removed, the difference between loop diagrams will always take the form of Eq. (2.8) regardless of whether $f_n(-\Lambda R, u, j)$ or $f_n(r, u, j) - f_0(r, u)$ have finite limits as $\Lambda R \to \infty$. In Eq. (2.8), all dependence on a cutoff has been absorbed into observed parameters, and it is understood that the Kaluza-Klein r summation in Eq. (2.8) is to be evaluated symmetrically, with equal and opposite diverging limits. Expressions of this form will then enable us to obtain regulator-independent equations for such renormalized KK parameters as masses and couplings.

It may seem suspicious that we have defined the α- and Δ functions differently for the cases in which $f_n(-\Lambda R)$ and $f_n - f_0$ either remain finite or diverge as $\Lambda \to \infty$. However, this was done simply as a matter of convenience. In all cases, the most general procedure is the second one that we have outlined above, and this procedure is also valid even when both of our defining criteria are met. In such cases, this procedure merely introduces extraneous terms to the α- and Δ functions, but these new additions will always cancel in the loop diagram difference $L_n - L_0$.

Finally, before concluding, we remark that the procedure we have outlined here has relied rather fundamentally on the assumption that the one-loop diagrams we are regulating can be evaluated through the introduction of a single Feynman parameter x (or u). However, this procedure readily generalizes to one-loop diagrams that would utilize arbitrary numbers of Feynman parameters. For example, in the case of two Feynman parameters, we have already shown in Ref. [18] that the identity in Eq. (2.3) generalizes to take the form

$$L_{n_1,n_2}(p_1, p_2) = \frac{i}{|n_1 n_2|} \sum_{j_1=0}^{|n_1|-1} \sum_{j_2=0}^{|n_2|-1} \int_0^1 du_1 \times \int_0^1 du_2 \sum_{r=-\Lambda R}^{\Lambda R} f_{n_i}(p_i, r, u_i, j_i) + E_{s_1,s_2} \tag{2.11}$$

where $s_i \equiv \text{sign}(n_i)$ and $u_i \equiv x_i|n_i| - j_i$. We have also defined $\hat{r} \equiv r - s_1 j_1 - s_2 j_2$, and then dropped the hat on $\hat{r}$. The quantity E_{s_1,s_2} in Eq. (2.11) represents a so-called "endpoint" contribution [analogous to the final term in Eq. (2.7)] which depends on f evaluated at or near the limits of the KK summation [18]. In such cases, the α functions are defined analogously to the case of a single Feynman parameter, with the endpoint contributions E leading to corresponding Δ functions. Indeed, the only changes to the basic formalism we have sketched are that there are now two variables of integration, two mode-number indices for f, α, and Δ, and slightly less trivial endpoint contributions.

III. TWO EXPLICIT EXAMPLES

In this section, we shall provide two explicit examples of the general procedure outlined in Sec. II. These two examples are designed to illustrate the two different cases sketched at the end of Sec. II.

A. First example: Pure scalar theory

Our first example will assume that our external particles in Fig. 1 are Lorentz scalars, and that the solid lines in Fig. 1 represent scalars as well. In this case, the generic diagram $L_n(p)$ is given by

$$L_n(p) = \sum_r \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - r^2/R^2 - M^2} \times \frac{1}{(k-p)^2 - (r-n)^2/R^2 - M^2} \tag{3.1}$$

where k is the four-momentum of a particle in our loop and r is its mode number. Combining the denominators using standard Feynman-parameter methods, we can then cast this expression into the form in Eq. (2.1), where

$$f_n(p, r, x) \equiv \int \frac{d^4\ell_E}{(2\pi)^4}\left[\frac{1}{\ell_E^2 + \ell^{42} + \mathcal{M}^2(x)}\right]^2, \qquad n \in \mathbb{Z}, \tag{3.2}$$

where $\ell \equiv k - xp$ is the shifted *five*-momentum [i.e., $\ell^\mu \equiv k^\mu - xp^\mu$ and $\ell^4 \equiv (r - xn)/R$)], where ℓ_E is the standard Euclidean (Wick-rotated) version of ℓ, and where the effective mass in Eq. (3.2) is given by

$$\mathcal{M}^2(x) \equiv M^2 + x(x-1)\left[p^2 - \frac{n^2}{R^2}\right]. \tag{3.3}$$

As it stands, these expressions are divergent. We can regulate them, while preserving the full *five*-dimensional Lorentz invariance, using either of the two regulators introduced in Ref. [18]. Either regulator leads to an expression of the form in Eq. (2.2), and after the variable substitutions in Eq. (2.3), these f functions take the forms

$$\begin{aligned} f_0(r, u) &= \int \frac{d^4\ell_E}{(2\pi)^4}\left[\frac{1}{\ell_E^2 + r^2/R^2 + M^2 + u(u-1)p^2}\right]^2, \\ f_n(r, u, j) &= \int \frac{d^4\ell_E}{(2\pi)^4}\left[\frac{1}{\ell_E^2 + (r-u)^2/R^2 + M^2 + (u+j)(u+j-|n|)(\frac{p^2}{n^2} - \frac{1}{R^2})}\right]^2. \end{aligned} \tag{3.4}$$

In the case of the EHC regulator, the domain of integration for f_0 and f_n within Eq. (3.2) is regulated according to the prescriptions [18]

$$f_0\colon\ \ell_E^2 \le \Lambda^2 - r^2/R^2, \qquad f_n\colon\ \ell_E^2 \le \Lambda^2 - (r-u)^2/R^2, \tag{3.5}$$

while in the case of the EDR regulator, we merely shift the measure $d^4\ell_E/(2\pi)^4 \to d^{4-\epsilon}\ell_E/(2\pi)^{4-\epsilon}$, where ϵ and Λ are related according to [18],

$$\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) = 2\log(\Lambda R) + \delta. \tag{3.6}$$

Here γ is the Euler-Mascheroni constant, and δ is an inconsequential parameter which vanishes as $\Lambda \to \infty$ (or as $\epsilon \to 0$). In either case, the prescriptions in Eq. (3.5) or (3.6) are precisely designed to preserve five-dimensional Lorentz invariance [18].

Using either regulator, these functions f_0 and f_n can be evaluated explicitly. Applying appropriate renormalization conditions in each case (for example, $p^2 = M_e^2$ and $p^2 = M_e^2 + n^2/R^2$ respectively, where M_e is the bare four-dimensional mass of the external particles in Fig. 1) and explicitly performing the integrals in Eq. (3.4) with the EHC regulator, we obtain

$$f_0(r, u) = \frac{1}{16\pi^2}\left\{\frac{r^2 - \Lambda^2R^2}{[\Lambda^2 + \mathcal{M}^2(u)]R^2} + \log[(\Lambda^2 + \mathcal{M}^2(u))R^2] - \log[r^2 + \mathcal{M}^2(u)R^2]\right\} \tag{3.7}$$

where $\mathcal{M}^2(u) \equiv M^2 + u(u-1)M_e^2$. The function $f_n(r, u, j)$ is given by an identical expression with the replacements $r \to \rho \equiv r - u$ and $u \to y \equiv (u+j)/|n|$. As promised, we see that the Λ dependence is completely canceled in the difference $f_n - f_0$ as $\Lambda \to \infty$.

Given the forms in Eq. (3.7), it may easily be verified that $f_n(-\Lambda R, u, j)$ remains finite (and in fact vanishes) as $\Lambda R \to \infty$. Likewise, it is easy to check that $f_n(r, u, j) - f_0(r, u)$ remains finite as $\Lambda R \to \infty$ for each value of r. This is therefore an example of the first case discussed at the end of Sec. II, whereupon we see that we can identify α_0 and α_n as the cutoff-independent parts of f_0 and f_n, respectively. Looking at Eq. (3.7), we see that the only cutoff-independent term that survives in the difference $f_n - f_0$ as $\Lambda \to \infty$ is the final term in Eq. (3.7). We can therefore identify

$$\begin{aligned} \alpha_0(r, u) &= -\frac{1}{16\pi^2}\log[r^2 + \mathcal{M}^2(u)R^2], \\ \alpha_n(r, u, j) &= -\frac{1}{16\pi^2}\log[\rho^2 + \mathcal{M}^2(y)R^2] \end{aligned} \tag{3.8}$$

where $\rho \equiv r - u$ and $y \equiv (u+j)/|n|$.

We can also explicitly evaluate Δ_n for this example. Since $f_n(-\Lambda R, u, j)$ remains finite as $\Lambda R \to \infty$, we know that Δ_n is given by Eq. (2.9). However, since $f_n(-\Lambda R, u, j)$ actually vanishes as $\Lambda R \to \infty$, Eq. (2.9) implies that Δ_n vanishes as well. We therefore conclude that $\Delta_n = 0$ for this example. Indeed, we have found this to be a common result for theories without gauge invariance (for which our EHC regulator is appropriate). However, these results are of course independent of the specific regulator employed as long as the regulator respects all of the five-dimensional symmetries that exist in the theory prior to compactification.

B. Second example: Five-dimensional QED

As a somewhat more complicated example, let us now consider the case of a vacuum polarization diagram in five-dimensional QED. We can therefore take the external lines in Fig. 1 to correspond to an incoming/outgoing KK photon, while our internal lines correspond to a tower of KK fermions. As in the previous example, we shall assume that this tower of KK fermions has bare five-dimensional mass M. However, unlike the situation in the previous example, this theory has both five-dimensional Lorentz invariance and five-dimensional gauge invariance prior to compactification.

Because the incoming and outgoing photons carry five-dimensional Lorentz vector indices $M, N = 0, 1, 2, 3, 4$, this diagram L_n^{MN} will have a Lorentz two-tensor structure. Thus, in general, after integration over the internal momentum running in the loop, L_n^{MN} will contain one part which is proportional to g^{MN} and another which is proportional to $p^M p^N$. We shall restrict our attention to the part of $L_n^{\mu\nu}$ which is proportional to $g^{\mu\nu}$, since this is the component which gives rise to renormalizations of the squared masses of the KK photons. We shall henceforth denote these terms as $\tilde{L}_n^{\mu\nu}$.

Clearly, $\tilde{L}_0^{\mu\nu}$ vanishes for the photon zero mode; this is because four-dimensional gauge invariance protects the zero-mode photon from gaining a mass. However, using the techniques we have sketched above, it is relatively straightforward to calculate $\tilde{L}_n^{\mu\nu}$ for nonzero n. Employing the EDR regularization procedure from Ref. [18] (as appropriate for theories with gauge invariance) and applying the renormalization condition $p^2 - n^2/R^2 = 0$ for on-shell external KK photons (as appropriate for calculating mass corrections), we then find that $\tilde{L}_n^{\mu\nu}$ takes the form in Eq. (2.2) with

$$f_n^{\mu\nu}(r, x) \equiv -\frac{e^2 R^\epsilon}{4\pi^2 R^2}(2x - 1)n(r - xn)W g^{\mu\nu}, \qquad n \neq 0, \tag{3.9}$$

where

$$W \equiv \frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[(r - xn)^2 + (MR)^2] + \mathcal{O}(\epsilon). \tag{3.10}$$

Equivalently, after the variable substitutions in Eq. (2.3) and dropping the hats on $\hat{r}$ and $\hat{u}$, we find that $f_n^{\mu\nu}(r, u, j)$ is given by Eqs. (3.9) and (3.10) where we simply replace $x \to y \equiv (u + j)/|n|$ and $r - xn \to (r - u) \cdot \mathrm{sign}(n)$.

Given these results, we can now examine the behavior of $f_n^{\mu\nu}(-\Lambda R, u, j)$ as $\Lambda R \to \infty$ and $\epsilon \to 0$. At first glance, it may appear that this expression diverges. However, we must recall that ΛR and ϵ are related [18] according to Eq. (3.6). Substituting Eq. (3.6) into Eq. (3.10), we find that $W \approx -2u/(\Lambda R) + \delta$ as $\Lambda R \to \infty$. Since δ also vanishes as $\Lambda R \to \infty$ [and generally does so more quickly than $1/(\Lambda R)$], we see that $f_n^{\mu\nu}(-\Lambda R, u, j)$ actually remains finite in this limit. On the other hand, it is immediately apparent that $f_n(r, u, j)$ diverges as $\Lambda R \to \infty$, while $f_0(r, u) = 0$. Thus, our second defining condition at the end of Sec. II is not satisfied, whereupon we see that five-dimensional QED is an example of the second case discussed at the end of Sec. II.

Before going further, it is important to stress that these observations do not contradict the overall finiteness of Eq. (2.7). Indeed, the finiteness of $f_n^{\mu\nu}(-\Lambda R, u, j)$ in this limit implies that the second line of Eq. (2.7) is finite by itself, and this in turn implies that the first line of Eq. (2.7) must also be finite as $\Lambda R \to \infty$. In order to see how these divergences cancel, let us consider the simple case where $n = 1$. In this case, $j = 0$ and $y = x$, whereupon we find that the first line of Eq. (2.7) takes the form

$$\sum_{r=-\Lambda R}^{\Lambda R} \int_0^1 du(2u - 1)(r - u)\{\log[(\Lambda R)^2] - \log[(r - u)^2 + (MR)^2]\}. \tag{3.11}$$

Note that both terms in Eq. (3.11) diverge like $(\Lambda R) \times \log(\Lambda R)$ as $\Lambda R \to \infty$; this is a subtle interplay between terms in which the cutoff ΛR appears explicitly in the integrand/summand (as in the first term above) and in which the cutoff appears only as the upper limit on the r summation (as in the second term above). It is nevertheless straightforward to verify that these two divergences cancel directly in Eq. (3.11), leading to a finite expression as $\Lambda R \to \infty$. A similar cancellation happens for each value of n.

According to our general prescription in Sec. II, we must therefore identify $\alpha_n(r, u, j)$ as the cutoff-independent part of the difference $f_n - \tilde{f}_n$, where $\tilde{f}_n$ is the value of f_n when all of the bare masses in our theory vanish. We thus find that

$$\alpha_n^{\mu\nu}(r, u, j) = \frac{e^2 g^{\mu\nu}}{4\pi^2 R^2}(2y - 1)|n|(r - u)\{\log[(r - u)^2 + (MR)^2] - \log[(r - u)^2]\} \tag{3.12}$$

for all $n \neq 0$, while $\alpha_0^{\mu\nu} = 0$. We can also calculate $\Delta_n^{\mu\nu}$. Using the definition in Eq. (2.10) and incorporating the relation in Eq. (3.6), we find

$$\Delta_n^{\mu\nu} = -\frac{e^2 g^{\mu\nu}}{4\pi^2 R^2} \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du \times \lim_{\Lambda R \to \infty} \sum_{r=-\Lambda R+1}^{\Lambda R} (2y - 1) \times |n|(r - u)\{\log[(\Lambda R)^2] - \log[(r - u)^2]\}. \tag{3.13}$$

While these are indeed the correct results, we note that the summand/integrand in Eq. (3.13) is still regulator dependent, depending explicitly on Λ. Thus, in contrast to the regulator-independent $\alpha^{\mu\nu}$-terms from Eq. (3.12), we see that we have not yet succeeded in writing $\Delta_n^{\mu\nu}$ in a manifestly finite, *regulator-independent* manner. We emphasize that it is not the cutoffs on the upper and lower limits of the KK sum in Eq. (3.13) which cause difficulty; after all, at an algebraic level, these KK summation cutoffs may be smoothly removed without difficulty. By contrast, it is the explicit factor of ΛR within the summand itself which causes algebraic difficulty and which prevents this expression from being truly regulator independent.

Our goal, of course, is to show that the expression for $\Delta_n^{\mu\nu}$ in Eq. (3.13)—just like our expressions for $\alpha_n^{\mu\nu}$ and $\alpha_0^{\mu\nu}$—can be rewritten in a manifestly finite, regulator-independent manner. In other words, without affecting the value of $\Delta_n^{\mu\nu}$, we wish to replace the second and third lines of Eq. (3.13) with an expression of the general form

$$\sum_{r=-\infty}^{\infty} h(r, u, j) \tag{3.14}$$

where $h(r, u, j)$ is a regulator-independent function.

Clearly, in order to derive the appropriate function $h(r, u, j)$, we need to find a way of algebraically redistributing the explicit Λ dependence in the summand of Eq. (3.13) across all of the terms in the KK sum. This can be accomplished as follows. First, we observe that explicitly performing the KK summation and Feynman integration for the Λ-dependent term in Eq. (3.13) yields

$$\begin{aligned}\int_0^1 du \sum_{r=-\Lambda R+1}^{\Lambda R} (2y-1)|n|(r-u)\log[(\Lambda R)^2] \\ = -\frac{1}{3}(\Lambda R)\log[(\Lambda R)^2].\end{aligned} \tag{3.15}$$

Note that this result is an odd function of the cutoff ΛR. However, we can now "invert" this and rewrite any odd function of the cutoff $F(\Lambda R)$ in the desired form using the identity

$$\begin{aligned}F(\Lambda R) &= \frac{1}{2}[F(\Lambda R) - F(-\Lambda R)] \\ &= \frac{1}{2}\int_{-\Lambda R}^{\Lambda R} dz f(z) \quad \text{where } f(z) \equiv dF(z)/dz \\ &= \frac{1}{2}\sum_{r=-\Lambda R+1}^{\Lambda R}\int_{r-1}^{r} dz f(z) \\ &= -\frac{1}{2}\sum_{r=-\Lambda R+1}^{\Lambda R}\int_1^0 du f(r-u) \quad \text{where } u \equiv r - z \\ &= \frac{1}{2}\sum_{r=-\Lambda R+1}^{\Lambda R}\int_0^1 du f(r-u).\end{aligned} \tag{3.16}$$

As we see, this identity has the net effect of throwing an arbitrary, explicit (odd) dependence on ΛR into the upper and lower limits of a discrete sum, just as desired. In the case at hand, we have $F(z) \equiv -\frac{1}{3} z \log(z^2)$, whereupon we see that $f(z) \equiv -\frac{1}{3}(\log z^2 + 2)$. We thus find that $h(r, u, j) \equiv -\frac{1}{3}(\log[(r-u)^2] + 2)$ in Eq. (3.14), whereupon we conclude that $\Delta_n^{\mu\nu}$ can be written in the regulator-independent form

$$\begin{aligned}\Delta_n^{\mu\nu} = \frac{e^2 g^{\mu\nu}}{4\pi^2 R^2}\frac{1}{|n|}\sum_{j=0}^{|n|-1}\int_0^1 du \sum_{r=-\infty}^{\infty}\bigg\{\bigg[(2y-1)|n|(r-u)+\frac{1}{6}\bigg] \\ \times \log[(r-u)^2] + \frac{1}{3}\bigg\}.\end{aligned} \tag{3.17}$$

Similar algebraic manipulations can also be performed for other diagrams of interest.

It is straightforward to verify that the expression in Eq. (3.17) converges to a finite result. As an example, let us consider the case with $n = 1$. For $n = 1$, we have $j = 0$ and $y = u$. Explicitly performing the u integration and defining $s \equiv r - 1/2$, we then find

$$\begin{aligned}\Delta_1^{\mu\nu} = \frac{e^2 g^{\mu\nu}}{144\pi^2 R^2}\sum_{s\in\mathbb{Z}+1/2}\bigg\{(4-24s^2) + 3s(4s^2-1) \\ \times \bigg[\log\bigg[\bigg(s+\frac{1}{2}\bigg)^2\bigg] - \log\bigg[\bigg(s-\frac{1}{2}\bigg)^2\bigg]\bigg]\bigg\}.\end{aligned} \tag{3.18}$$

We immediately observe that the summand has a symmetry under $s \to -s$, which implies that contributions from positive values of s are identical to contributions from negative values of s. Thus, the finiteness of Eq. (3.18) does *not* rely on a cancellation between contributions from positive and negative KK mode numbers; sums over positive or negative values of s are each *separately* convergent. Furthermore, we see that the contributions from $|s| = 1/2$ are also finite, since the divergent logarithm in Eq. (3.18) for $|s| = 1/2$ is multiplied by the factor $(4s^2 - 1)$, which vanishes even more strongly. This cancellation of the logarithmic divergence can also be verified by evaluating the original integral in Eq. (3.18) directly with $r = 0$, 1. Finally, for large $|s|$, it is straightforward to verify that the summand in Eq. (3.18) scales as $\sim 1/s^2$. Thus the KK sum in Eq. (3.18) is absolutely convergent, as required. In fact, it can easily be shown that the s summation in Eq. (3.18) converges to $-36\zeta(3)/\pi^2$, where $\zeta(n)$ is the Riemann zeta function. We therefore find $\Delta_1^{\mu\nu} = -[e^2 g^{\mu\nu}/(4\pi^4 R^2)]\zeta(3)$, in agreement with results quoted in Refs. [4,18]. It turns out that $\Delta_n^{\mu\nu}$ takes this value for all $n \neq 0$.

IV. EFFECTIVE FIELD THEORIES FOR KALUZA-KLEIN MODES

Thus far, we have described how to calculate radiative shifts to physical KK parameters such as KK masses and couplings. In doing this, we have included effects from all energy scales from the deep infrared to the ultraviolet, as required.

However, as a question of both practical importance and mathematical curiosity, it is useful to have an EFT description of our KK system which is appropriate for any arbitrary finite cutoff μ. In the context of KK theories, EFT's are particularly useful tools for doing calculations and making predictions because they only contain finite numbers of KK states, and the presence of a cutoff eliminates problems of nonrenormalizability. Indeed, it is ultimately only an EFT (with finitely many relevant parameters) which can be fit to experiment.

Towards this end, we now seek to obtain EFT's which can be used to describe our KK systems at lower energy scales. In order to do so, we need to determine how these radiative corrections accrue as we move from the ultraviolet limit (where the full 5D Lorentz invariance of the theory is restored and the corrections vanish) to the infrared. In other words, we seek to express these radiative corrections as evolution functions of the energy scale μ at which a collider might operate.

In this section, we will present a procedure for doing this which is based on a Wilsonian approach. We begin, in Sec. IV A, with some general comments concerning novel features of the Wilsonian approach which arise for KK theories. In Sec. IV B, we then present our general results describing the flow of KK parameters using a Wilsonian treatment.

A. Wilsonian flow in Kaluza-Klein theories

Given a Lagrangian in the UV limit, a Wilsonian approach to analyzing the behavior of the corresponding effective field theories at different lower energy scales μ consists of integrating out all physics with (Euclidean) momentum or energy scales exceeding μ. In general, this has two effects: it produces new effective interactions which were not present in the original UV Lagrangian, and it changes the values of the bare parameters which were already present in the original Lagrangian, rendering them μ dependent. The resulting Lagrangian then describes a Wilsonian EFT at energy scale μ. Within the framework of such a theory, we would then perform calculations in the usual way based on this effective Lagrangian, except that μ now serves as a hard UV cutoff for such calculations. This is appropriate because the contributions from the physics at scales above μ has already been absorbed directly into the EFT Lagrangian.

In an ordinary four-dimensional setup, this process of integrating out physics above the scale μ is performed uniformly across all sectors of the theory, for all particle species that may appear in the Lagrangian. Following the strict approach outlined above, we do not eliminate heavy particles from our theory; it is simply that their contributions become small because of the kinematic constraints that operate within a restrictive cutoff μ. For example, let us imagine that our 4D theory contains different particle species with masses m_i for $i = 1, \ldots, n$. Evaluating a one-loop radiative correction within such a theory, we would sum over the contributions from each particle that is allowed to propagate in the loop. Likewise, each of these contributions is evaluated by integrating the possible loop momentum over all possible values up to infinity. Of course, for each individual particle running in the loop, the contributions to the momentum integral will be greatest from those values of the loop momentum for which the internal particle is closest to being on shell. As a result, the contributions from particles whose masses exceed the cutoff will be exceedingly small because they will be significantly off shell for all allowed values of the loop momentum below the cutoff. However, within the framework of a Wilsonian-derived EFT, we are only instructed to truncate each of these momentum integrals at a scale μ. We are *not* instructed to eliminate any of the particle species themselves, even if their masses m_i significantly exceed μ.

This situation changes significantly for KK theories. At first glance, one might think that a KK theory is simply another four-dimensional theory with an infinite set of increasingly heavy particles. However, we must remember that in a KK theory, the masses of these particles receive contributions from (and are therefore the reflections of) the fifth components of a higher-dimensional momentum. It would be acceptable to disregard this fact if we were not aiming to develop EFT's that respect our original higher-dimensional symmetries as far as possible. This includes higher-dimensional Lorentz invariance. However, in the present case, we seek to follow an intrinsically higher-dimensional approach so as to avoid the introduction of spurious Lorentz-violating contributions.

As a result, we must follow an intrinsically *higher-dimensional* Wilsonian approach to deriving our EFT's. In the case of five dimensions, this means that one-loop integrals such as that in Eq. (3.2) must be truncated with an intrinsically *five-dimensional* cutoff μ. In other words, we must impose the five-dimensional Lorentz-invariant constraint

$$\ell_E^2 + (\ell^4)^2 \leq \mu^2. \tag{4.1}$$

Because $R\ell^4 \equiv r - xn$, such a constraint equation correlates the cutoff for the integration over the four-momentum ℓ_E with the cutoff for the summation over the KK index r. In particular, the constraint in Eq. (4.1) can be implemented by restricting the KK summation to integers in the range

$$-\mu R + xn \leq r \leq \mu R + xn \tag{4.2}$$

and then restricting our ℓ_E integration to the corresponding range

$$\ell_E^2 \leq \mu^2 - (\ell^4)^2 \leq \mu^2 - (r - xn)^2/R^2. \tag{4.3}$$

Thus, we see that in a truly five-dimensional setup, a Wilsonian treatment not only implies a truncation for four-dimensional loop integrations, but *it also implies a truncation in the KK tower.* In other words, we not only eliminate certain momentum scales from consideration, but we also eliminate heavy KK states entirely.

It is, of course, no accident that the KK constraint in Eq. (4.2) resembles that in Eq. (2.2), and that the four-momentum cutoff in Eq. (4.3) resembles that in Eq. (3.5), except with the replacement $\Lambda \to \mu$. Equations (2.2) and (3.5) together stem from our EHC regulator, whose defining characteristic is also the imposition of a five-dimensional Lorentz-invariant cutoff. There is, however, a major conceptual difference between the two cutoffs Λ and μ: while Λ is a *regulator* cutoff which is always removed at the end of a calculation, μ is a finite *physical* (Wilsonian) cutoff which defines an associated momentum scale for our EFT and which serves as a finite physical cutoff for calculations performed within the context of that EFT. Such a cutoff is not taken to infinity at the end of such an EFT calculation. It is therefore only at the algebraic level that these two cutoffs appear to be similar.

B. Deriving Wilsonian EFT's

Given these observations, it is now straightforward to derive our Wilsonian EFT's as a function of the momentum scale μ. We shall do this to one-loop order, and shall provide a general method of deriving the μ dependence that our KK parameters (masses and couplings) accrue. There are, of course, new effective operators that will also be generated in these EFT's. The procedure we shall be outlining below also applies to their coefficients as well.[2]

We shall let λ_n collectively denote these KK parameters. Of course, these parameters λ_n will receive classical rescalings in cases where they are dimensionful. In order to eliminate these classical rescalings, we shall henceforth assume that each λ_n has been multiplied by a sufficient power of the radius R so as to be dimensionless. Likewise, we know that each λ_n will receive quantum corrections which are divergent when our cutoffs are removed; indeed, it is only differences such as $\lambda_n - \lambda_0$ which can be expected to remain finite. Our strategy will therefore be to derive the μ dependence of differences such as $\lambda_n - \lambda_0$. The parameters $\lambda_{n\neq 0}$ describing the excited KK states in our EFT associated with any scale μ can then be obtained in terms of the parameters λ_0 for the zero modes.

Let us first consider the full, one-loop radiative correction to the difference $\lambda_n - \lambda_0$. According to Eq. (2.8), this one-loop radiative correction $\Delta(\lambda_n - \lambda_0)$ is given by

$$\Delta(\lambda_n - \lambda_0) = \sum_{r=-\infty}^{\infty} \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du[\alpha_n(r, u, j) - \alpha_0(r, u)] + \Delta_n \tag{4.4}$$

where the α_n- and Δ_n functions are determined according to the procedures described in Sec. II. Note that since the λ_n are dimensionless, the α_n functions corresponding to these λ_n are dimensionless as well. Clearly Eq. (4.4) represents the complete radiative shift in $\lambda_n - \lambda_0$ that is accrued from momentum scales all the way from the ultraviolet limit to the infrared.

Our goal, however, is to evaluate the corrections to the bare parameters in the UV Lagrangian that accrue due to integrating out only that portion of the physics associated with momenta exceeding μ. We therefore wish to calculate the *partial* radiative correction from the ultraviolet limit down to an arbitrary nonzero momentum scale μ. To do this, we calculate the same one-loop radiative correction, only now imposing the constraint

$$\ell_E^2 + (\ell^4)^2 \geq \mu^2, \tag{4.5}$$

where μ is treated as an *infrared* cutoff. In other words, we must integrate out the contributions from KK modes satisfying

$$r \leq -\mu R + xn \quad \text{or} \quad r \geq \mu R + xn, \tag{4.6}$$

where the corresponding ℓ_E integrations are restricted to the region

$$\ell_E^2 \geq \mu^2 - (r - xn)^2/R^2. \tag{4.7}$$

In analogy with Eqs. (2.6) and (2.7), we see that this then leads to the partial radiative correction

$$\begin{aligned}\Delta(\lambda_n - \lambda_0)|_\mu &= \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du \lim_{\Lambda R\to\infty} \bigg[\bigg(\sum_{r=-\Lambda R+1}^{-\mu R} + \sum_{r=\mu R+1}^{\Lambda R}\bigg) f_n(r; \mu, \Lambda) - \bigg(\sum_{r=-\Lambda R}^{-\mu R} + \sum_{r=\mu R}^{\Lambda R}\bigg) f_0(r; \mu, \Lambda) \\ &\quad + \delta_{\mu R,0} f_0(0; \mu, \Lambda)\bigg] \\ &= \frac{1}{|n|} \sum_{j=0}^{|n|-1} \int_0^1 du \lim_{\Lambda R\to\infty} \bigg\{\bigg(\sum_{r=-\Lambda R}^{-\mu R} + \sum_{r=\mu R}^{\Lambda R}\bigg)[f_n(r; \mu, \Lambda) - f_0(r; \mu, \Lambda)] - f_n(-\Lambda R; \mu, \Lambda) - f_n(\mu R; \mu, \Lambda) \\ &\quad + \delta_{\mu R,0} f_0(0; \mu, \Lambda)\bigg\}.\end{aligned} \tag{4.8}$$

[2]Of course, even our original higher-dimensional theory can be viewed as an EFT of its own, and hence would contain all possible operators consistent with our UV symmetries. From this perspective, the only "new" operators that would be generated are those which break the higher-dimensional symmetries but not the four-dimensional symmetries. In any case, the procedure we are outlining here is applicable to the coefficients of any operators, whether they are higher order or not. However, for convenience, we shall refer to these coefficients as KK masses and couplings in what follows.

Here the f_n functions are the dimensionless functions appropriate for calculations of the λ_n parameters. Note that in writing these functions, we have suppressed their (u, j) arguments relative to the notation in previous sections. However, we have also added two explicit cutoff arguments, writing $f(r; \mu_1, \mu_2)$ where μ_1 and μ_2 respectively represent the infrared and ultraviolet cutoffs which truncate the four-momentum integrals contained within these functions. In Eq. (4.8), we have also defined $\delta_{\mu R,0} \equiv 1$ only if $\mu R = 0$, and zero otherwise. Terms proportional to $\delta_{\mu R,0}$ in Eq. (4.8) compensate for the notational overcounting which arises in the $\mu R = 0$ special case. Finally, note that the $\mu R \to 0$ limit of Eq. (4.8) indeed reproduces the full radiative correction in Eq. (2.7), which may then be replaced by the explicitly finite form in Eq. (2.8).

In the expressions in Eq. (4.8), the scale μ always appears as an *infrared* cutoff for the f functions. However, there also exists an alternative but equivalent set of expressions in which μ appears as an *ultraviolet* cutoff within the f functions. This alternative formulation exists because $\Delta(\lambda_n - \lambda_0)|_\mu$ can equivalently be calculated by integrating all the way down to zero energy, yielding the full contribution in Eq. (2.8), but then subtracting the contributions that emerge if we restore the partial contributions from zero energy back up to μ. These latter contributions are given in Eq. (2.7), where the ultraviolet cutoff Λ is replaced by μ and kept finite. Combining the results in Eqs. (2.7) and (2.8), we therefore have

$$\begin{aligned}\Delta(\lambda_n - \lambda_0)|_\mu = {}& \frac{1}{|n|}\sum_{j=0}^{|n|-1}\int_0^1 du \sum_{r=-\infty}^{\infty}[\alpha_n(r,u,j) \\ & - \alpha_0(r,u)] + \Delta_n \\ & - \frac{1}{|n|}\sum_{j=0}^{|n|-1}\int_0^1 du\bigg\{\sum_{r=-\mu R}^{\mu R}[f_n(r;0,\mu) \\ & - f_0(r;0,\mu)] - f_n(-\mu R;0,\mu)\bigg\}. \qquad (4.9)\end{aligned}$$

Note that μ now appears as an ultraviolet cutoff within the f functions. For example, in the case of the first example discussed in Sec. III, we see that $f_0(r; 0, \mu)$ is given by the expression in Eq. (3.7) with the replacement $\Lambda \to \mu$. Note that in the final line of Eq. (4.9), we do *not* take the additional step of replacing our f functions with α functions in which the μ dependence is eliminated. While such a replacement would have been appropriate in the $\mu \to \infty$ limit, we are regarding μ as an arbitrary finite parameter in Eq. (4.9). Consequently, although any divergent behavior as a function of μ indeed continues to cancel in the difference $f_n - f_0$, there can be subleading terms which scale with inverse powers of μ. Such terms must be retained for finite μ. Also note that since $f(r; 0, 0) = 0$, the $\mu \to 0$ limit of Eq. (4.9) again restores the full radiative contribution in Eq. (2.8).

In either case, with $\Delta(\lambda_n - \lambda_0)|_\mu$ written as in Eq. (4.8) or as in Eq. (4.9), we can then write the value of λ_n at scale μ in terms of λ_0 at the same scale:

$$\lambda_n|_\mu = \lambda_0|_\mu + [\kappa_n + \Delta(\lambda_n - \lambda_0)|_\mu] \qquad (4.10)$$

where κ_n denotes the difference $\lambda_n - \lambda_0$ evaluated in the UV limit:

$$\kappa_n \equiv (\lambda_n - \lambda_0)|_{\mu=\infty}. \qquad (4.11)$$

For example, if λ corresponds to the squared (dimensionless) masses $(mR)^2$ of the KK modes arising from a one-dimensional compactification on a circle, then $\kappa_n = n^2$.

We conclude this section with two related comments. First, it is clear that the above analysis has been based on our EHC regulator, as introduced in Ref. [18]. This does not imply, however, that such a formalism cannot be used in theories with gauge invariance. If the higher-dimensional theory in question has gauge invariance, this formalism may continue to be utilized; the only difference is that gauge-dependent counterterms must be introduced in order to compensate for the (apparent) breaking of gauge invariance induced by the presence of a hard cutoff. Moreover, an analogous formalism can be developed which avoids the hard cutoff altogether, and which utilizes our EDR regulator from Ref. [18]. This formalism would preserve our higher-dimensional symmetries, both Lorentz invariance and gauge invariance, in a completely manifest way. Note that regardless of which specific regulator from Ref. [18] is employed, the crucial ingredient that enables this formalism to operate is the fact that such regulators preserve the original symmetries of our *higher-dimensional* Lagrangian, and not merely those four-dimensional symmetries which remain after compactification.

Second, we also emphasize that the Wilsonian approach we have followed here is different from the more canonical approach in which the renormalization scale μ is introduced through renormalization conditions involving the momenta of external particles. By contrast, the above approach does not impose any particular relationship between the external momenta and the scale μ; for example, we are free to choose to place our external particles on shell as in Sec. III, if desired.

Given the above effective field theories, we would then proceed to compare with experiment in the usual way. We would begin with our effective Lagrangian formulated in terms of our KK parameters λ evaluated for an arbitrary scale μ:

$$\mathcal{L} = \mathcal{L}[\lambda_i(\mu)]. \qquad (4.12)$$

Since experimentalists will measure physical observables such as cross sections and decay rates at a particular energy E, we would then calculate such quantities from our effective Lagrangian. Denoting such physical observables collectively as σ, we would thereby obtain predicted values for these quantities in terms of μ, E, and our unknown parameters $\lambda_i(\mu)$:

$$\sigma_{\text{pred}} = \sigma_{\text{pred}}[\lambda_i(\mu), \mu, E]. \tag{4.13}$$

Setting $\sigma_{\text{pred}} = \sigma_{\text{expt}}$ then allows us to constrain the $\lambda_i(\mu)$ parameters, and thereby deduce the effective Lagrangian experimentally. Note, in this context, that while the values of the $\lambda_i(\mu)$ parameters in the effective Lagrangian will depend on the specific chosen reference value of μ, the final results for the physically observable cross sections and decay rates are independent of μ, as always.

V. CONCLUSIONS AND FUTURE DIRECTIONS

This paper concludes the two-part series initiated in Ref. [18]. In Ref. [18], we proposed two new regulators for quantum field theories in spacetimes with compactified extra dimensions. Unlike most other regulators which have been used in the extra-dimension literature, these regulators are specifically designed to respect the original higher-dimensional Lorentz and gauge symmetries that exist prior to compactification, and not merely the four-dimensional symmetries which remain afterward.

In this paper, we continued this work by showing how these regulators may be used in order to extract ultraviolet-finite results from one-loop calculations. We provided a general procedure which accomplishes this, and demonstrated its use through two explicit examples. We also showed how this formalism allows us to derive Wilsonian effective field theories for Kaluza-Klein modes at different energy scales.

The key property underpinning our methods is that the divergent corrections to parameters describing the physics of the excited Kaluza-Klein modes are absorbed into the corresponding parameters for zero modes. This eliminates the need to introduce independent counterterms for parameters characterizing different Kaluza-Klein modes. Our effective field theories can therefore simplify radiative calculations involving towers of Kaluza-Klein modes, and should be especially relevant if data from the LHC should happen to suggest the existence of TeV-sized extra dimensions. Indeed, when the parameters describing the zero modes are taken as experimental inputs, the relative corrections $\Delta(\lambda_n - \lambda_0)$ that we have determined will enable us to predict the properties of the entire corresponding KK spectrum. Knowledge of these differences thus allows us to obtain regulator-independent (i.e., Λ-independent) EFT's for KK modes at various energy scales.

Despite the fact that our final results are regulator independent, as required, we stress that the approaches we have taken here rely rather crucially on the existence of such regulators in order to perform the explicit calculations. If we had not employed regulators which explicitly preserve the higher-dimensional Lorentz and gauge symmetries that exist prior to compactification, we would have obtained additional spurious terms which would not have been easy to disentangle from the physical effects of compactification. Note that, in some sense, all of the radiative corrections we have been calculating represent the breaking of five-dimensional Lorentz invariance due to compactification: this breaking, which originates *nonlocally* through the compactification of the spacetime geometry in the UV limit, becomes a *local* effect in the EFT at lower energy scales. It would thus have been difficult to separate these 5D Lorentz-violating terms from those that would have also emerged from a poor choice of UV regulator. However, by employing the regulators we developed in Ref. [18], we have avoided the introduction of such spurious terms altogether. The 5D Lorentz-violating effects of the radiative corrections we have calculated can thus be interpreted directly as the low-energy consequences of spacetime compactification, as expected.

As already noted at the end of Ref. [18], our analysis has been limited in a number of significant ways. For example, this analysis has been restricted to five dimensions and to one-loop amplitudes. We have also focused on flat compactification spaces without orbifold fixed points. However, compactification on orbifolded geometries is ultimately required in order to obtain a chiral theory in four dimensions. Therefore, although our results can be taken to apply to the bulk physics in such setups, they would require generalization before they could accommodate orbifold fixed-point contributions such as those which might emerge from, e.g., brane-kinetic terms.

Even within the framework of compactification of a single extra dimension on a circle, there remain important extensions which can also be considered. For example, when deriving our effective field theories in Sec. IV, we focused on the scale dependence of those parameters λ_i (e.g., KK masses and couplings) which already appeared in the original UV Lagrangian. However, as we pass to lower energies, new effective operators, i.e., new higher-order interactions, will be generated. Despite the intrinsic nonrenormalizability of higher-dimensional theories, such operators have generally been ignored in the higher-dimensional literature. Fortunately, our techniques have the advantage of being able to handle these new interactions just as easily as they handle the leading interactions we have already considered; all that changes is the form of the f functions. Although the extra contributions from such operators are generally suppressed compared with the leading contributions which have been our focus, they can give rise to important phenomenological effects and thus must be taken into account in any complete study of the low-energy phenomenologies of KK theories.

The methods developed in this paper open intriguing possibilities for future phenomenological studies. For example, in an upcoming paper [19], we shall calculate KK radiative corrections and derive EFT's for two specific higher-dimensional models. We shall also generally study how KK mass relations such as $m_n^2 = m_0^2 + n^2/R^2$ ''evolve'' as we pass from the UV limit to the lower energy scales at which such KK states might eventually be discovered. Such analyses can thus be of direct importance for the discovery and interpretation of such states in future collider experiments.

ACKNOWLEDGMENTS

We wish to thank Z. Chacko and U. van Kolck for discussions pertaining to effective field theories in the context of extra dimensions. This work is supported in part by the National Science Foundation under Grant No. PHY/0301998, by the Department of Energy under Grant No. DE-FG02-04ER-41298, and by the Research Corporation.

[1] K. R. Dienes, E. Dudas, and T. Gherghetta, Phys. Lett. B **436**, 55 (1998); Nucl. Phys. **B537**, 47 (1999); arXiv:hep-ph/9807522.
[2] T. Appelquist and B. A. Dobrescu, Phys. Lett. B **516**, 85 (2001).
[3] D. M. Ghilencea, Phys. Rev. D **70**, 045011 (2004); **70**, 045018 (2004).
[4] H. C. Cheng, K. T. Matchev, and M. Schmaltz, Phys. Rev. D **66**, 036005 (2002).
[5] S. Groot Nibbelink, Nucl. Phys. **B619**, 373 (2001).
[6] R. Contino and A. Gambassi, J. Math. Phys. (N.Y.) **44**, 570 (2003).
[7] I. Antoniadis, Phys. Lett. B **246**, 377 (1990).
[8] D. M. Ghilencea, J. High Energy Phys. 03 (2005) 009; D. M. Ghilencea, D. Hoover, C. P. Burgess, and F. Quevedo, J. High Energy Phys. 09 (2005) 050; D. M. Ghilencea, H. M. Lee, and K. Schmidt-Hoberg, J. High Energy Phys. 08 (2006) 009.
[9] M. Puchwein and Z. Kunszt, Ann. Phys. (N.Y.) **311**, 288 (2004).
[10] N. Arkani-Hamed, A. G. Cohen, and H. Georgi, Phys. Rev. Lett. **86**, 4757 (2001).
[11] See, e.g., A. Falkowski, C. Grojean, and S. Pokorski, Phys. Lett. B **581**, 236 (2004).
[12] K. R. Dienes, Nucl. Phys. **B429**, 533 (1994); K. R. Dienes, M. Moshe, and R. C. Myers, Phys. Rev. Lett. **74**, 4767 (1995); K. R. Dienes, Nucl. Phys. **B611**, 146 (2001).
[13] D. M. Ghilencea, H. P. Nilles, and S. Stieberger, New J. Phys. **4**, 15 (2002).
[14] D. M. Ghilencea and S. Groot Nibbelink, Nucl. Phys. B **641**, 35 (2002).
[15] Y. Ohtake and Y. Wakabayashi, Prog. Theor. Phys. **115**, 611 (2006).
[16] K. Agashe, A. Delgado, and R. Sundrum, Nucl. Phys. **B643**, 172 (2002).
[17] R. Contino and L. Pilo, Phys. Lett. B **523**, 347 (2001).
[18] S. Bauman and K. R. Dienes, preceding article, Phys. Rev. D **77**, 125005 (2008).
[19] S. Bauman and K. R. Dienes (to be published).