New regulators for quantum field theories with compactified extra dimensions. I. Fundamentals

Sky Bauman* and Keith R. Dienes+

Department of Physics, University of Arizona, Tucson, Arizona 85721, USA

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In this paper, we propose two new regulators for quantum field theories in spacetimes with compactified extra dimensions. We refer to these regulators as the "extended hard cutoff" and "extended dimensional regularization." Although based on traditional four-dimensional regulators, the key new feature of these higher-dimensional regulators is that they are specifically designed to handle mixed spacetimes in which some dimensions are infinitely large and others are compactified. Moreover, unlike most other regulators which have been used in the extra-dimension literature, these regulators are designed to respect the original higher-dimensional Lorentz and gauge symmetries that exist prior to compactification, and not merely the four-dimensional symmetries which remain afterward. This distinction is particularly relevant for calculations of the physics of the excited Kaluza-Klein modes themselves, and not merely their radiative effects on zero modes. By respecting the full higher-dimensional symmetries, our regulators avoid the introduction of spurious terms which would not have been easy to disentangle from the physical effects of compactification. As part of our work, we also derive a number of ancillary results. For example, we demonstrate that in a gauge-invariant theory, analogues of the Ward-Takahashi identity hold not only for the usual zero-mode (four-dimensional) photons, but for all excited Kaluza-Klein photons as well.

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I. INTRODUCTION

Extra dimensions are among the leading candidates for physics beyond the standard model. However, despite the vast amount of work done in this area, phenomenological studies of higher-dimensional models still face limitations. A fundamental issue is that virtually all realistic theories in higher dimensions are nonrenormalizable. Because parameters in a nonrenormalizable theory are extremely sensitive to a ultraviolet (UV) cutoff, and because an infinite number of counterterms are needed to absorb divergences, our ability to make meaningful predictions at different energy scales appears to be compromised. Additionally, regulators of UV divergences can introduce unphysical artifacts. For example, a hard cutoff in QED artificially generates a large photon mass term. The problem of artifacts should be especially severe in higher-dimensional theories since the nonrenormalizability will magnify any such radiative effect.

Unfortunately, such artifacts will be introduced by many of the regulators which are typically used to perform calculations in spacetimes with compactified extra dimensions. This happens because these regulators artificially treat momentum components along compactified extra dimensions as if they were separate from the other components. To be more explicit, let us consider a typical one-loop diagram in a theory with a single universal compactified extra dimension. The amplitude corresponding to such a diagram can be expressed as a mode-number sum over a four-momentum integral, i.e.,

$$\mathcal{M} = \sum_r \int \frac{d^4k}{(2\pi)^4} I(k, r), \tag{1.1}$$

where $\mathcal{M}$ represents the one-loop amplitude, k is the four-momentum of a Kaluza-Klein (KK) state running in the loop, and r is its KK mode number. The function I depends on k and r, as well as the couplings in the theory and momenta and mode numbers of any external particles. Of course, both the four-momentum integral and the KK sum contribute to possible divergences, and both potential sources of infinities must be regularized.

The typical approach is to apply a standard four-dimensional regulator (such as a hard cutoff or dimensional regularization) to the integral, and to truncate the sum at large but finite limits. Thus, the sum and the integral are regulated independently. Unfortunately, independent regularizations artificially violate the higher-dimensional Lorentz invariance that originally existed in the theory, thereby leading to unphysical artifacts in $\mathcal{M}$. This is because the variables k and r/R from Eq. (1.1) are actually part of a single five-momentum running in the loop. Our regulator should therefore reflect this higher-dimensional Lorentz symmetry, just as a hard cutoff in four dimensions (4D) is always imposed on the total Euclidean four-momentum running in a loop, and not a particular subset of momentum components. This is why separate regularizations of four-momentum integrals and KK sums violate higher-dimensional Lorentz invariance. Without respecting the full five-dimensional Lorentz symmetry, any such regulator has the potential to introduce unphysical artifact terms into the results of any calculation.

*bauman@physics.arizona.edu
+dienes@physics.arizona.edu

Of course, it might seem that we can always subtract unphysical artifacts at the end of a calculation. However, this is not generally possible because the compactification itself, which breaks the higher-dimensional Lorentz invariance globally, can also induce local violations of the higher-dimensional Lorentz invariance in an effective field theory (EFT) at finite energy. We would therefore not know which terms to subtract, since it would be extremely difficult to distinguish these unphysical artifacts from the expected *bona-fide* violations of five-dimensional Lorentz invariance which arise due to the compactification.

Given this situation, our goal in this paper is to develop a set of regulators which are based firmly on two fundamental higher-dimensional symmetries:

(i) higher-dimensional Lorentz invariance; and

(ii) higher-dimensional gauge invariance, when appropriate.

Regulators which are based on these symmetries should therefore be broadly applicable and free of unphysical artifacts. Moreover, we shall also require that our regulators be *theory independent*. In other words, we shall require that our regulators be insensitive to the specific particle content and interactions characterizing the field theory in question.

In this paper, we shall develop two distinct regulator schemes which meet these criteria. Indeed, in each case, these regularization methods can be viewed as higher-dimensional generalizations of well-known four-dimensional regulators. However, as discussed above, their distinguishing property is that they control four-momentum integrals and KK sums *collectively*, as appropriate for higher-dimensional calculations. Under this scheme, the constraints on the integral and the sum in Eq. (1.1) become coupled.

Our first regulator will be a generalization of a four-dimensional hard-cutoff scheme to the case of theories with KK modes. We shall refer to this regulator as an "extended hard-cutoff" (EHC) regulator. To do this, we shall consider the case of a single extra dimension compactified on a circle. Instead of separately regulating four-momentum integrals and KK sums, we shall implement a cutoff on the *total five-momenta* of virtual KK states running through internal loops. This procedure is Lorentz invariant, and therefore does not introduce unphysical artifacts.

By contrast, our second regulator will be a generalization of dimensional regularization, to be referred to as "extended dimensional regularization" (EDR). Specifically, we shall use standard dimensional-regularization techniques to control four-momentum integrals. However, we shall also demand that KK sums be truncated at limits which depend on the dimensional-regularization parameter ϵ. The critical point, then, is to determine an appropriate balancing relation between this KK cutoff and the dimensional-regularization parameter ϵ which preserves not only higher-dimensional Lorentz invariance, but also higher-dimensional gauge invariance. To do this, we shall consider the case of five-dimensional QED compactified on a circle and show explicitly that preserving both higher-dimensional Lorentz invariance and gauge invariance in this theory leads to a unique relation between ϵ and the KK cutoff. Our criterion of theory independence will then guarantee that this relation between the KK cutoff and ϵ should hold for *all* higher-dimensional field theories, regardless of whether or not they contain gauge symmetries.

At this stage, one might be tempted to offer two possible objections to the approach we shall be following in this paper. First, since the compactification itself distinguishes extra dimensions from the ones we currently observe, one could argue that there is no need to respect higher-dimensional Lorentz invariance. Indeed, one might even argue that the very process of compactification forces us to employ regulators that do *not* respect higher-dimensional Lorentz invariance: since the momentum components along compactified dimensions are discrete variables and components along large dimensions are continuous, it might seem that no regularization scheme can truly put these variables on equal footing. However, it is important to realize that compactification is an effect at finite distance and therefore finite energies. In the UV limit, by contrast, this discreteness fades away and higher-dimensional Lorentz invariance is restored. Since regulators are designed to control UV divergences, it is therefore essential that they respect whatever symmetries exist at short distances.

Second, one might object that higher-dimensional theories are nonrenormalizable. Therefore, it would seem that we should obtain meaningless results regardless of which regulator we use, in which case there is no point in trying to extract exact predictions from such theories. However, despite the nonrenormalizability, it is possible to derive precise, *finite* relationships between the *renormalized* parameters in our effective field theories that characterize KK states. Indeed, as we shall explicitly demonstrate in Ref. [1], the use of proper regulators will allow us to relate the parameters describing excited KK modes to the corresponding parameters describing zero modes, after each has received radiative corrections. If the zero-mode parameters are taken to be experimental inputs, then the entire KK spectrum is determined. We emphasize that this only works when regulators are designed to respect higher-dimensional symmetries.

Although our extended hard-cutoff and extended dimensional-regularization procedures ultimately achieve the same goal, there are two significant conceptual differences between them. First, our extended hard cutoff is designed to treat all components of loop momenta in the same way, and hence this cutoff never breaks higher-dimensional Lorentz invariance. By contrast, our extended dimensional-regularization procedure controls divergences from four-momentum integrals and KK sums through very different means. Higher-dimensional symmetries thus do not appear to be preserved from the outset, but survive in the end only because of a special relation between their regularization parameters.

There also is a second important difference. Because a hard cutoff explicitly violates gauge invariance, our extended hard-cutoff regulator will not be suitable for higher-dimensional theories in which gauge symmetries are present. By contrast, our extended dimensional-regularization procedure is designed to respect higher-dimensional gauge invariance as well as higher-dimensional Lorentz invariance. As such, this is the regulator of choice when dealing with gauge-invariant theories. In this connection, we remark that while the process of compactification explicitly violates Lorentz invariance globally (and this can translate into local Lorentz violations below the UV limit), the process of compactification in and of itself does *not* violate any higher-dimensional gauge symmetry which exists in the UV limit. Specifically, as we shall demonstrate for the case of five-dimensional QED compactified on a circle, a full five-dimensional gauge invariance survives after compactification, even at low energy scales. Our extended dimensional-regularization procedure will reflect this explicitly through the preservation of Ward identities and Ward-Takahashi identities; indeed, such identities will continue to hold not only for the (zero-mode) photon, but for all of the excited (KK) photons as well.

This paper is organized as follows. In Sec. II, we introduce our EHC regulator, and explain how it regularizes divergences in a Lorentz-invariant fashion. In Sec. III, we then introduce our EDR in the context of higher-dimensional gauge theory. In Sec. IV, we turn to a discussion of other regulators which have been utilized in the literature, and compare our regulators with those. We also demonstrate, through explicit examples, the kinds of difficulties that can arise when one uses a regulator which does not respect higher-dimensional Lorentz invariance. We also discuss the relations between our EHC and EDR regulators and several other Lorentz-invariant methods which have already been developed in the literature. Finally, Sec. V contains our conclusions and ideas for possible extensions.

This paper is the first in a two-part series. In this paper, we shall focus on the development of two new regulators, as sketched above. By contrast, in the following article [1], we shall discuss how these new regulators may be employed in order to derive effective field theories at different energy scales. We shall also discuss how these regulator techniques can be used to extract finite results for physical observables that relate the physics of excited KK modes to the physics of KK zero modes. In this context, it should be noted that one of our primary motivations for developing these new EHC and EDR regulators has been to enable us to study the way in which the Kaluza-Klein mass and coupling parameters in any higher-dimensional effective field theory evolve as a function of energy scale. For example, we might wish to study how the well-known tree-level relations amongst the tower of KK masses and amongst their couplings are "deformed" when radiative effects are included. This will be the subject of a third paper [2]. However, each of these subsequent papers will rely on the regulators and calculational techniques that we shall be developing here.

II. THE EHC REGULATOR

In this section, we introduce our higher-dimensional EHC regulator. For simplicity, we consider the case of a single extra dimension compactified on a circle; generalizations to other compactifications will be straightforward. As discussed in the Introduction, our cutoff will be purely five dimensional in nature, and will respect five-dimensional Lorentz invariance from the outset. Of course, if our higher-dimensional theory is also gauge invariant, then a hard cutoff will not be applicable; in such cases, the EDR regulator in Sec. III should be used.

To illustrate our procedure, let us consider a generic one-loop diagram of the form shown in Fig. 1 in which an external particle with four-momentum p^μ and mode number n interacts with a tower of KK particles of bare mass M. Enforcing 5D momentum conservation at the vertices (as appropriate for compactification on a circle) and assuming that the solid lines correspond to scalar fields leads to a one-loop integral of the form

$$L_n(p) = \sum_r \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - r^2/R^2 - M^2} \times \frac{1}{(k-p)^2 - (r-n)^2/R^2 - M^2} \tag{2.1}$$

where k is the four-momentum of a particle in our loop and

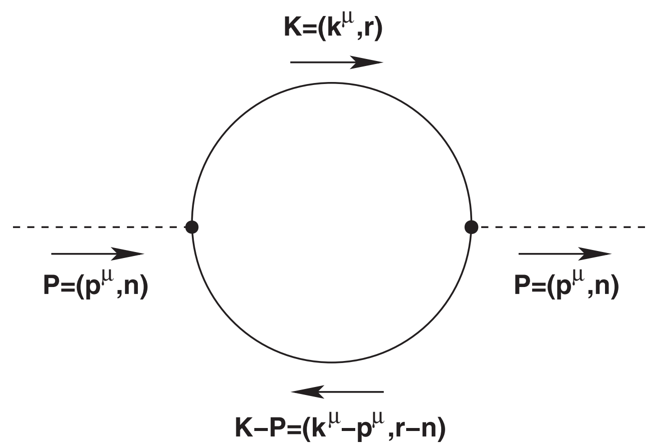

FIG. 1. A generic one-loop diagram: an external Kaluza-Klein particle (dotted line) with four-momentum p^μ and Kaluza-Klein index n interacts with a tower of Kaluza-Klein particles (solid lines) of bare mass M.

r is its mode number. Although we are considering a particular form for a loop integral, we will keep our discussion as general as possible.

Following standard techniques, we may immediately rewrite this loop integral as

$$L_n(p) = i\int_0^1 dx \sum_r \int \frac{d^4\ell_E}{(2\pi)^4}\left[\frac{1}{\ell_E^2 + \ell^{42} + \mathcal{M}^2(x)}\right]^2, \tag{2.2}$$

where x is a Feynman parameter, where ℓ represents the shifted momentum

$$\ell \equiv k - xp, \qquad \ell^4 \equiv (r - xn)/R, \tag{2.3}$$

where ℓ_E is the Euclidean (Wick-rotated) momentum

$$\ell_E^0 \equiv -i\ell^0, \qquad \vec{\ell}_E \equiv \vec{\ell}, \tag{2.4}$$

and where the effective mass in Eq. (2.2) is given by

$$\mathcal{M}^2(x) \equiv M^2 + x(x-1)\left[p^2 - \frac{n^2}{R^2}\right]. \tag{2.5}$$

Note that throughout this paper, vector and tensor components corresponding to the fifth dimension will be denoted with a superscript "4." We have chosen this somewhat unorthodox convention in order to emphasize the preservation of five-dimensional Lorentz invariance, so that our five-dimensional Lorentz indices are given as $M = 0, 1, 2, 3, 4$.

We now introduce our hard momentum cutoff Λ. We shall apply this directly to the Euclidean *five-momentum* running in the loop, as appropriate for an intrinsically five-dimensional calculation, so that

$$\ell_E^2 + \ell^{42} \leq \Lambda^2. \tag{2.6}$$

Of course, this constraint equation correlates the cutoff for the integration over the four-momentum ℓ_E with the cutoff for the summation over the KK index r. In particular, the constraint in Eq. (2.6) can be implemented by restricting the KK summation to integers in the range

$$-\Lambda R + xn \leq r \leq \Lambda R + xn \tag{2.7}$$

and then restricting our ℓ_E integration to the corresponding range

$$\ell_E^2 \leq \Lambda^2 - \ell^{42} \leq \Lambda^2 - (r - xn)^2/R^2. \tag{2.8}$$

Clearly, Eqs. (2.7) and (2.8) are nested constraints on the components of the momentum of the particle running in the loop. However, this "nesting" is unavoidable if our regulator is to preserve five-dimensional Lorentz invariance and avoid distinguishing a special direction in spacetime. Since ℓ_E is continuous and ℓ^4 is discrete, one might argue at first glance that these variables are fundamentally different, and that Eq. (2.6) does not truly respect a five-dimensional Lorentz symmetry. However, as discussed in the Introduction, the discreteness is an effect at *finite* energy scales, originating from the compactification. This discreteness is not apparent in the UV limit, where the gaps between KK masses are effectively negligible. Therefore, Eq. (2.6) will indeed allow us to regularize five-dimensional UV divergences in a Lorentz-invariant fashion.

Equations (2.6), (2.7), and (2.8) define our EHC regularization procedure. Indeed, unlike the case with dimensional regularization to be discussed in Sec. III, the maintenance of five-dimensional Lorentz invariance in this case has not been particularly difficult or profound. However, this is not enough, since we also need to know how to perform calculations which implement these constraints. Equation (2.7) is particularly unpleasant, since it puts the Feynman parameter and the mode number of the external particle in the summation limits. One might hope that we can neglect these terms when Λ is large. However this is ultimately not possible due to the hypersensitivity to the exact value of a cutoff in a nonrenormalizable theory. The rest of this section is therefore devoted to the calculational issue of converting such expressions for loop diagrams into useful forms.

For the special case of $n = 0$, our loop diagram can be written as

$$L_0(p) = i\int_0^1 dx \sum_{r=-\Lambda R}^{\Lambda R} f_0(p, r, x) \tag{2.9}$$

where f_0 is the integral over ℓ_E from Eq. (2.2), subject to the constraint in Eq. (2.8). In general, f_0 is a function of p, r, and x, but we will not need to evaluate f_0 for this discussion. Note that in writing Eq. (2.9), we have treated ΛR as an integer. In the limit of a large cutoff, this assumption will have no effect on our results.

For $n \neq 0$, however, the cutoff on the r summation depends on the Feynman parameter x. Fortunately, we can eliminate this dependence through a series of variable substitutions. Let us first assume that $n > 0$. In this case, our summation is over all integers r in the range $-\Lambda R + xn \leq r \leq \Lambda R + xn$. In the following, we shall adopt a notation whereby $\sum_{r=a}^{b}$ denotes a summation over integer values of r within the range $a \leq r \leq b$ even if a and b are not themselves integers. We can then write

$$L_n(p) = i\int_0^1 dx \sum_{r=-\Lambda R+xn}^{\Lambda R+xn} f_n(p, r, x), \tag{2.10}$$

where f_n is the analog of f_0 for nonzero n. For $n > 0$, we may express this summation as

$$\begin{aligned}\int_0^1 dx \sum_{r=-\Lambda R+xn}^{\Lambda R+xn} &= \frac{1}{n}\int_0^n du \sum_{r=-\Lambda R+u}^{\Lambda R+u} \quad \text{where } u \equiv xn \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1} du \sum_{r=-\Lambda R+u}^{\Lambda R+u} \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\int_0^1 d\hat{u} \sum_{\hat{r}=-\Lambda R+\hat{u}}^{\Lambda R+\hat{u}} \quad \text{where } \begin{cases}\hat{u}\equiv u-j \\ \hat{r}\equiv r-j\end{cases} \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\int_0^1 d\hat{u} \sum_{\hat{r}=-\Lambda R+1}^{\Lambda R}. \end{aligned} \tag{2.11}$$

In passing to the last line, we have continued to treat ΛR as an integer. We have also used the fact that the exact $\hat{u} = \{0, 1\}$ endpoints of the $\hat{u}$-integration region are sets of measure zero.

For general $n \neq 0$ of either sign, we can make an analogous set of substitutions, resulting in the general identity

$$\int_0^1 dx \sum_{r=-\Lambda R+xn}^{\Lambda R+xn} = \frac{1}{|n|}\sum_{j=0}^{|n|-1}\int_0^1 d\hat{u}\sum_{\hat{r}=-\Lambda R+1}^{\Lambda R}, \tag{2.12}$$

where

$$\hat{u} \equiv x|n| - j \tag{2.13}$$

and

$$\hat{r} \equiv \text{sign}(n)r - j. \tag{2.14}$$

Together, Eqs. (2.13) and (2.14) imply that $R\ell^4 = \text{sign}(n)[\hat{r} - \hat{u}]$.

Note that the mode number n has disappeared from the magnitude of ℓ^4. This is precisely as we expect, since ℓ^4 is merely a summation variable and should not depend on the magnitude of n. Likewise, the dependence on sign(n) arises by convention and can be absorbed into coefficients of ℓ^4. Ultimately, this removal of n from ℓ^4 was possible only because of the limits we chose for r at the beginning of our calculation.

However, it is important to realize that n has not vanished from our calculation. Because $\hat{u}$ is now viewed as an independent Feynman-like variable in Eq. (2.12), x must now be expressed in terms of $\hat{u}$, and this reintroduces a dependence on n into any expressions which previously depended on x. For example, the quantity x appears within $\mathcal{M}^2$, as defined in Eq. (2.5). However, the important point is that this dependence on n is now wholly within the four-dimensional integrand, and no longer appears within the KK summation limits.

Given these variable substitutions, our loop-diagram expression for nonzero n can now be rewritten as

$$L_n(p) = i\int_0^1 d\hat{u}\frac{1}{|n|}\sum_{j=0}^{|n|-1}\sum_{\hat{r}=-\Lambda R+1}^{\Lambda R} f_n(p, \hat{r}, \hat{u}, j). \tag{2.15}$$

We shall henceforth drop the hats from $\hat{r}$ and $\hat{u}$. Note that the functions $f_0(r, u)$ and $f_n(r, u, j)$ each depend on the cutoff Λ because they are integrals whose limits contain Λ. For example, if our original diagram is of the form (2.2), then these functions f_0 and f_n are given by

$$\begin{aligned} f_0(r, u) &= \int \frac{d^4\ell_E}{(2\pi)^4}\left[\frac{1}{\ell_E^2 + r^2/R^2 + M^2 + u(u-1)p^2}\right]^2, \\ f_n(r, u, j) &= \int \frac{d^4\ell_E}{(2\pi)^4}\left[\frac{1}{\ell_E^2 + (r-u)^2/R^2 + M^2 + (u+j)(u+j-|n|)(\frac{p^2}{n^2} - \frac{1}{R^2})}\right]^2 \end{aligned} \tag{2.16}$$

where these integrals are subject to the cutoffs

$$f_0\colon \ \ell_E^2 \leq \Lambda^2 - r^2/R^2, \qquad f_n\colon \ \ell_E^2 \leq \Lambda^2 - (r-u)^2/R^2, \tag{2.17}$$

respectively. Note that f_n is the same as f_0, but with the simultaneous algebraic substitutions $r \to \rho \equiv r - u$, $u \to y \equiv (u + j)/|n|$, and $p^2 \to p^2 - n^2/R^2$.

Equations (2.9) and (2.15) are the main results of this section. Once loop diagrams are in these forms, they can be evaluated directly using standard four-dimensional techniques. Similarly, although we restricted ourselves to the case of a single external particle, generalizations to more complicated situations are straightforward.

Finally, before concluding our discussion of our EHC regulator, we remark that the identity we have outlined in Eq. (2.12) relies rather fundamentally on the assumption that the one-loop diagrams we are regulating can be evaluated through the introduction of only a single Feynman parameter x (or u). However, this procedure readily generalizes to diagrams that would utilize arbitrary numbers of Feynman parameters.

As a concrete example, let us consider a diagram such as the one-loop vertex correction in Fig. 2 which would require two Feynman parameters. In general, such a diagram will take the algebraic form

$$L_{n_1,n_2}(p_1, p_2) = i\int_0^1 dx_1\int_0^1 dx_2 \sum_r f_{n_1,n_2}(p_1, p_2, r, x_1, x_2) \tag{2.18}$$

where x_1 and x_2 are our two Feynman parameters and f is our four-momentum integral. However, unlike the case of a single Feynman parameter, our shifted momentum within

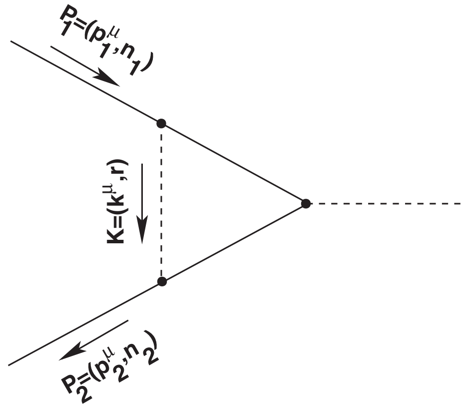

FIG. 2. A generic one-loop diagram with three external particles and three internal propagators. Such a one-loop diagram will require two Feynman parameters.

f will now be given by

$$\ell \equiv k - x_1 p_1 - x_2 p_2, \qquad \ell^4 = (r - x_1 n_1 - x_2 n_2)/R. \tag{2.19}$$

Despite this change in the definition of ℓ, our EHC regularization condition continues to take the same form as in Eq. (2.6). The cutoffs on our KK summation therefore now take the form

$$-\Lambda R + x_1 n_1 + x_2 n_2 \le r \le \Lambda R + x_1 n_1 + x_2 n_2, \tag{2.20}$$

while our corresponding four-momentum integral is subject to the cutoff

$$\ell_E^2 \le \Lambda^2 - (r - x_1 n_1 - x_2 n_2)^2 / R^2. \tag{2.21}$$

As before, the primary difficulty here is the presence of the Feynman parameters x_1 and x_2 in the upper and lower limits of the KK summation in Eq. (2.19). However, these can be eliminated in a manner completely analogous to the method outlined in Eq. (2.12). First, we observe that when $n_1 = n_2 = 0$, the Feynman parameters are eliminated trivially, and Eq. (2.20) reduces to $-\Lambda R \le r \le \Lambda R$. Moreover, when one $n_i = 0$ but the other is nonzero, only one Feynman parameter appears in Eq. (2.20). The variable transforms introduced in Eq. (2.12) may therefore be employed to disentangle the remaining Feynman parameter from the summation limits. As a result, the only new nontrivial case is the one in which both n_1 and n_2 are nonzero.

Let us first consider the case in which both n_1 and n_2 are positive. Repeating the steps in Eq. (2.11), we can then write

$$\begin{aligned}
\int_0^1 dx_1 \int_0^1 dx_2 \sum_{r=-\Lambda R + x_1 n_1 + x_2 n_2}^{\Lambda R + x_1 n_1 + x_2 n_2} &= \frac{1}{n_1 n_2} \int_0^{n_1} du_1 \int_0^{n_2} du_2 \sum_{r=-\Lambda R + u_1 + u_2}^{\Lambda R + u_1 + u_2} \quad \text{where } u_i \equiv x_i n_i \\
&= \frac{1}{n_1 n_2} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \int_{j_1}^{j_1+1} du_1 \int_{j_2}^{j_2+1} du_2 \sum_{r=-\Lambda R + u_1 + u_2}^{\Lambda R + u_1 + u_2} \\
&= \frac{1}{n_1 n_2} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \int_0^1 d\hat{u}_1 \int_0^1 d\hat{u}_2 \sum_{\hat{r}=-\Lambda R + \hat{u}_1 + \hat{u}_2}^{\Lambda R + \hat{u}_1 + \hat{u}_2} \quad \text{where } \begin{cases} \hat{u}_i \equiv u_i - j_i \\ \hat{r} \equiv r - j_1 - j_2. \end{cases} \\
&= \frac{1}{n_1 n_2} \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \int_0^1 d\hat{u}_1 \left[\int_0^{1-\hat{u}_1} d\hat{u}_2 \sum_{\hat{r}=-\Lambda R+1}^{\Lambda R} + \int_{1-\hat{u}_1}^1 d\hat{u}_2 \sum_{\hat{r}=-\Lambda R+2}^{\Lambda R+1} \right].
\end{aligned} \tag{2.22}$$

In passing to the final line, we have continued to treat ΛR as an integer and recognized that while the combination $\hat{u}_1 + \hat{u}_2$ ranges from 0 to 2, the summation index $\hat{r}$ ranges over the following values:

$$\begin{cases} -\Lambda R + 1 \le \hat{r} \le \Lambda R & \text{for } 0 < \hat{u}_1 + \hat{u}_2 < 1, \\ -\Lambda R + 2 \le \hat{r} \le \Lambda R + 1 & \text{for } 1 < \hat{u}_1 + \hat{u}_2 < 2. \end{cases} \tag{2.23}$$

Dropping the hats, it follows that under the EHC regulator, the diagram $L_{n_1,n_2}(p_1, p_2)$ in Eq. (2.18) with $n_1, n_2 > 0$ can be rewritten as

$$\begin{aligned} L_{n_1,n_2}(p_1, p_2) &= i\int_0^1 dx_1 \int_0^1 dx_2 \sum_r f_{n_1,n_2}(p_1, p_2, r, x_1, x_2) \\ &= \frac{i}{n_1 n_2}\sum_{j_1=0}^{n_1-1}\sum_{j_2=0}^{n_2-1}\int_0^1 du_1\bigg[\int_0^{1-u_1} du_2 \sum_{r=-\Lambda R+1}^{\Lambda R} + \int_{1-u_1}^1 du_2 \sum_{r=-\Lambda R+2}^{\Lambda R+1}\bigg] f_{n_i}(p_i, r, u_i, j_i) \\ &= \frac{i}{n_1 n_2}\sum_{j_1=0}^{n_1-1}\sum_{j_2=0}^{n_2-1}\int_0^1 du_1 \int_0^1 du_2 \sum_{r=-\Lambda R}^{\Lambda R} f_{n_i}(p_i, r, u_i, j_i) + E \end{aligned} \tag{2.24}$$

where $\ell^4 = (r - u_1 - u_2)/R$, where the one-loop integrals f_{n_1,n_2} are regulated according to Eq. (2.21), and where E denotes an "endpoint contribution" which depends on the particular values of f at or near the cutoff endpoints of the KK summation, as given below.

The above results are given for the case in which n_1 and n_2 are both positive. However, we can handle the general case in which both n_1 and n_2 are nonzero as follows. Let us define $s_i \equiv \text{sign}(n_i)$, and likewise let us define $\hat{u}_i \equiv x_i|n_i| - j_i$ and $\hat{r} \equiv r - s_1 j_1 - s_2 j_2$. Note that in terms of these variables, we have $R\ell^4 = \hat{r} - s_1\hat{u}_1 - s_2\hat{u}_2$. Dropping the hats, we then find the general identity

$$L_{n_1,n_2}(p_1, p_2) = \frac{i}{|n_1 n_2|}\sum_{j_1=0}^{|n_1|-1}\sum_{j_2=0}^{|n_2|-1}\int_0^1 du_1 \times \int_0^1 du_2 \sum_{r=-\Lambda R}^{\Lambda R} f_{n_i}(p_i, r, u_i, j_i) + E_{s_1,s_2} \tag{2.25}$$

where the endpoint contributions $E_{\pm,\pm}$ are given as

$$E_{s_1,s_2} = -\frac{i}{|n_1 n_2|}\sum_{j_1=0}^{|n_1|-1}\sum_{j_2=0}^{|n_2|-1}\int_0^1 du_1 \hat{E}_{s_1,s_2} \tag{2.26}$$

with

$$\begin{aligned} \hat{E}_{++} &\equiv \int_0^1 du_2 f(-\Lambda R) + \int_{1-u_1}^1 du_2[f(-\Lambda R + 1) - f(\Lambda R + 1)], \\ \hat{E}_{--} &\equiv \int_0^1 du_2 f(\Lambda R) + \int_{1-u_1}^1 du_2[f(\Lambda R - 1) - f(-\Lambda R - 1)], \\ \hat{E}_{+-} &\equiv \int_0^{u_1} du_2 f(-\Lambda R) + \int_{u_1}^1 du_2 f(\Lambda R), \\ \hat{E}_{-+} &\equiv \int_0^{u_1} du_2 f(\Lambda R) + \int_{u_1}^1 du_2 f(-\Lambda R). \end{aligned} \tag{2.27}$$

In writing Eq. (2.27), we have suppressed all indices and variables for the f-functions except their dependence on the KK mode number r. These results have obvious generalizations to one-loop diagrams with additional Feynman parameters.

We see, then, that our EHC regulator is quite general, and that the methods outlined here enable us to eliminate the resulting Feynman parameters from the upper and lower limits on our KK summations.

III. THE EDR PROCEDURE

In this section, we turn to our 5D EDR procedure. Unlike the case of the hard cutoff in Sec. II, our extended dimensional-regularization procedure is designed to respect not only five-dimensional Lorentz invariance, but also five-dimensional gauge invariance. As discussed in the Introduction, this will happen as the result of a careful balancing between the dimensional-regularization parameter ϵ, which regulates the four-dimensional momentum integral, and the KK cutoff Λ, which regulates the KK sum.

This section is organized as follows. We start with a preliminary exposition of our procedure in Sec. III A. Then, in Sec. III B, we discuss the method by which gauge invariance is maintained by demonstrating that the Ward(-Takahashi) identities must hold not only for the zero-mode photon, but also for all KK excitations of the photon. In Sec. III C, we then use this in order to generate a relation between the cutoff parameters used for the momentum integrals and the KK mode-number sums. Finally, in Sec. III D, we deal with a number of loose ends. For example, we show that this relation implies that five-dimensional Lorentz invariance will be preserved as well.

A. Preliminary steps

We begin by considering a generic one-loop amplitude in five dimensions, with one dimension compactified on a circle. As with the diagram in Fig. 1, we will assume that we have a certain fixed number of external particles with four-momenta p_i^μ and KK mode numbers n_i which enter the diagram as initial states or exit as final states. We shall also assume that only one Feynman parameter is needed; the generalization to multiple Feynman parameters is straightforward. Such an amplitude then generally takes the form

$$L_n^{MN\cdots}(p_1, p_2, \ldots) = i\int_0^1 dx \sum_r \int \frac{d^4\ell_E}{(2\pi)^4}\Omega_n^{MN\cdots}(\ell_E, r, x) \tag{3.1}$$

where $\Omega_n^{MN\cdots}(\ell_E, r, x)$ is an appropriate unspecified integrand and where the overall n subscript denotes the collection of external KK indices. Here $M, N, \ldots$ are five-dimensional Lorentz indices appropriate for the diagram in question; thus, unlike the situation in Sec. II, we are now explicitly indicating that these amplitudes need not be Lorentz scalars. We shall also assume that our theory contains a five-dimensional gauge invariance prior to compactification.

We now seek to develop a regularization procedure for such amplitudes, which is based on the traditional 't Hooft-Veltman dimensional-regularization procedure [3] for the four-momentum integral. However, we need to regulate not only the four-dimensional momentum integral, but also the KK sum, and our goal is to implement these two regulators in such a balanced way that both five-dimensional Lorentz invariance and five-dimensional gauge invariance are maintained. It is this "balancing" feature which extends the 't Hooft-Veltman dimensional-regularization procedure to spacetimes with compactified extra dimensions, and which results in our name "extended dimensional regularization."

As we shall see, the EDR procedure will consist of three separate components:

(i) First, we shift the 4D momentum integral into $d \equiv 4 - \epsilon$ spacetime dimensions.

(ii) At the same time, we deform the integrand $\Omega_n^{MN\cdots}(\ell_E, \ell^4, x)$ to reflect the fact that our integral is now in $d \equiv 4 - \epsilon$ dimensions. For example, one standard integrand substitution which is familiar from traditional dimensional regularization in four dimensions is to replace $\ell^\mu \ell^\nu \to \ell^2 g^{\mu\nu}/(4-\epsilon)$ where $\ell^2 \equiv g_{\mu\nu}\ell^\mu\ell^\nu$. However, we now expect there to be a similar deformation for the terms in the integrand which depend on the (discrete) fifth component ℓ^4 of the momentum. Deriving the precise form of this deformation is the first of our tasks. Note that since the introduction of a fifth dimension does not introduce any additional Dirac γ matrices, the usual deformation of the γ-matrix algebra that one must perform for 4D dimensional regularization is unchanged for 5D.

(iii) Finally, we apply lower and upper cutoffs $\{r_1(\epsilon), r_2(\epsilon)\}$ to our KK sum, so that this sum is over the range $r_1(\epsilon) \le r \le r_2(\epsilon)$. These cutoffs will be functions of ϵ, and deriving the precise relation between ϵ and these limits is our second task.

Indeed, the precise deformation of terms involving ℓ^4 in the integrand, as well as the precise forms of the cutoffs $\{r_1(\epsilon), r_2(\epsilon)\}$ as functions of ϵ, will be determined by the fact that *five-dimensional* Lorentz invariance and gauge invariance must be maintained.

Even before imposing five-dimensional gauge invariance, there are certain simplifications we can make. First, we know that we must have $r_1(\epsilon) \to -\infty$ and $r_2(\epsilon) \to +\infty$ as $\epsilon \to 0$. Second, however, just as in Eq. (2.7), we claim that $r_{1,2}(\epsilon)$ must actually take the form

$$r_1(\epsilon) = -\Lambda(\epsilon)R + xn, \qquad r_2(\epsilon) = \Lambda(\epsilon)R + xn \quad (3.2)$$

in terms of a single as-yet-undetermined function $\Lambda(\epsilon)$. In other words, although our summation cutoffs are not symmetric in the r variable, we claim that they must be symmetric in the ℓ^4 variable, where $R\ell^4 \equiv r - xn$. The reason for this is simple. At first glance, it might appear that since the four-momentum integrals in dimensional regularization are over infinite domains, there is no difference between integrating over the internal loop four-momentum k^μ or the shifted loop four-momentum $\ell^\mu \equiv k^\mu - xp^\mu$, and we might expect the same to hold for the KK sums. However, integrals which are odd with respect to ℓ vanish by convention in dimensional regularization. This means that it is the domain of integration for ℓ which is symmetric, even if it tends to an infinite size. Therefore, higher-dimensional Lorentz invariance requires that the limits on ℓ^4 also be the ones which are symmetric. Indeed, we have verified that any other choice will ultimately lead to inconsistencies—specifically, the sorts of checks that we will perform at the end of Sec. IV would not be successful with any other choice.

We can also further refine the form of the deformations within the integrand $\Omega_n^{MN\cdots}(\ell_E, r, x)$ itself. As mentioned above, we know that terms of the form $\ell^\mu\ell^\nu$ should be replaced by $\ell^2 g^{\mu\nu}/(4-\epsilon)$. In flat space (which is the only case we consider in this paper), this amounts to a deformation for terms $(\ell^i)^2$ for $i = 0, 1, 2, 3$. Five-dimensional Lorentz invariance therefore requires a corresponding deformation for the discrete *fifth* component $\ell^4\ell^4$ that arises within expressions of the form $\ell^M\ell^N$. In general, we can parametrize this deformation in the form

$$\ell^4\ell^4 \to [1 + \lambda\epsilon + \mathcal{O}(\epsilon^2)](\ell^4)^2 \quad (3.3)$$

where λ is an unknown parameter we seek to determine. As we shall see, determining the deformation to this order in ϵ will be sufficient for our purposes. We stress, however, that the deformation in Eq. (3.3) is only appropriate for terms that arise within a Lorentz-covariant expression of the form $\ell^M\ell^N$. By contrast, terms $(\ell^4)^2$ which might arise from other Lorentz-covariant forms such as $[\ell^2 - (\ell^4)^2]g^{MN}$ remain undeformed, in accordance with our expectations from ordinary dimensional regularization in four dimensions.

Given these observations, we can then proceed by implementing the variable substitutions described in Sec. II. We thus have

$$L_0^{MN\cdots} = i\int_0^1 dx \sum_{r=-\Lambda R}^{\Lambda R} \int \frac{d^d\ell_E}{(2\pi)^d}\Omega_0^{MN\cdots}(\ell_E, r, x) \quad (3.4)$$

where the zero KK subscript indicates that all external particles are zero modes, and

$$L_n^{MN\ldots} = i\int_0^1 d\hat{u}\frac{1}{|n|} \times \sum_{j=0}^{|n|-1}\sum_{\hat{r}=-\Lambda R+1}^{\Lambda R}\int\frac{d^d\ell_E}{(2\pi)^d}\Omega_n^{MN\ldots}(\ell_E,\hat{r},\hat{u},j) \quad (3.5)$$

where the transformed variables $\hat{u}$ and $\hat{r}$ are defined in Eqs. (2.13) and (2.14). As discussed above, the KK cutoffs Λ are to be viewed as functions of ϵ.

Again, we stress that it is remarkable that there will exist solutions for $\Lambda(\epsilon)$ and λ which can simultaneously preserve both higher-dimensional Lorentz invariance and higher-dimensional gauge invariance. After all, our four-momentum integrals are unrestricted, while our KK summations are truncated. Likewise, our four-momenta are continuous, while our KK momenta are discrete. Nevertheless, we shall find that the proper solutions for $\Lambda(\epsilon)$ and λ will conspire to simultaneously maintain both of these higher-dimensional symmetries at the end of any calculation.

Thus, the complete development of our EDR regulator now rests on determining two remaining unknowns. First, we seek to determine $\Lambda(\epsilon)$ as a function of ϵ. Second, we seek to determine the value of the parameter λ in Eq. (3.3).

B. Ward-Takahashi identities for KK photons

We now demand that our EDR regulator preserve whatever five-dimensional gauge invariance exists prior to compactification. However, before proceeding further, it is important to determine the extent to which the process of compactification, in and of itself, might break the full five-dimensional gauge invariance. In other words, we need to understand the extent to which five-dimensional gauge invariance can be expected to survive the process of spacetime compactification.

In this section, we shall address this issue within the framework of the specific case of five-dimensional QED compactified on a circle. Although the usual Ward identities (and indeed the more general Ward-Takahashi identities) are expected to hold for the usual four-dimensional zero-mode photon (as a result of the residual *four-dimensional* gauge invariance), we shall demonstrate that *analogues of these identities actually hold for all of the KK excitations of the photon as well*. In other words, five-dimensional gauge invariance is manifested in our compactified theory through the existence of a whole tower of Ward(-Takahashi) identities, one for each KK-photon excitation; compactification does not break gauge invariance at the level of these identities. As such, these identities can be taken as the signature of the original full five-dimensional gauge invariance, and demanding that these identities continue to hold in our compactified theory will ultimately enable us to determine the value for the parameter λ as well as the relation between Λ and ϵ.

Let us begin by quickly reviewing the usual four-dimensional Ward and Ward-Takahashi identities. Let $\mathcal{M}(p; q_1, \ldots, q_N; q'_1, \ldots, q'_N)$ represent the full Fourier-transformed correlation function for some QED process with N incoming fermions of four-momenta $\{q_1, \ldots, q_N\}$, N outgoing fermions with four-momenta $\{q'_1, \ldots, q'_N\}$, and an incoming photon γ with four-momentum p. In general, these fermion momenta need not be on shell, and we can write $\mathcal{M}$ in the form $\mathcal{M} = \epsilon_\mu \mathcal{M}^\mu$ where ϵ_μ represents the photon polarization four-vector. Likewise, let $\mathcal{M}_0$ represent the full Fourier-transformed correlation function for the same process except without the photon γ. Then the usual four-dimensional Ward-Takahashi identity states that

$$\begin{aligned} p_\mu\mathcal{M}^\mu(p; q_1, \ldots, q_N; q'_1, \ldots, q'_N) &= e\sum_i[\mathcal{M}_0(q_1, \ldots, q_n; q'_1, \ldots, (q'_i - p), \ldots) \\ &\quad - \mathcal{M}_0(q_1, \ldots, (q_i + p), \ldots; q'_1, \ldots, q'_N)], \end{aligned} \quad (3.6)$$

where e is the unit of electric charge carried by each fermion. Moreover, if we then use the LSZ reduction procedure to obtain the corresponding amplitude for the corresponding amputated diagrams, we find that the right side of Eq. (3.6) does not contribute. We thus obtain the simpler Ward identity

$$p_\mu\mathcal{M}^\mu(p; q_1, \ldots, q_N; q'_1, \ldots, q'_N) = 0 \quad (3.7)$$

which holds when each of the external momenta (including that of the external photon) is on shell. Of course, the quantity $\mathcal{M}$ in Eq. (3.7) now represents the amplitude of the corresponding amputated diagram, and the external momenta are now restricted to be on shell.

Before we consider whether and how these identities can be extended to the case of a compactified higher-dimensional spacetime, we first review their derivation. The usual diagrammatic proof of the Ward-Takahashi identity (see, e.g., any standard reference such as Ref. [4]) proceeds by realizing that by summing over each of the diagrams that contribute to $\mathcal{M}_0$, and then by summing over all possible ways of inserting an extra external photon into each of these diagrams, we produce all of the diagrams contributing to $\mathcal{M}$. Thus, we can focus on any individual diagram contributing to $\mathcal{M}_0$, and consider all possible ways in which an additional external photon line can be inserted into such a diagram. In QED, a photon line can only be inserted onto an already-existing fermion line, and there are only two possible types of fermion lines such a diagram may contain: a closed internal loop (as illustrated in Fig. 3), or a line which ultimately connects an incoming fermion to an outgoing fermion.

If the additional photon connects to a fermion line in the former class, the sum over insertion locations cancels identically upon integrating over the internal fermion loop momentum. Specifically, the sum over all insertion points for a photon of momentum p^μ into the diagram in

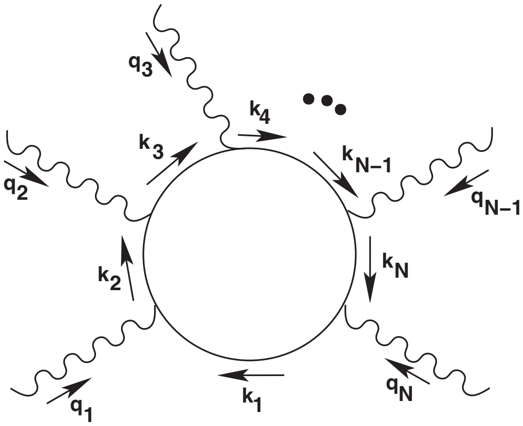

FIG. 3. A closed fermion loop with N photon lines, with momentum labeling conventions as indicated. Summing over all possible insertion locations of an additional photon with momentum p^μ into this diagram produces the amplitude in Eq. (3.8).

Fig. 3 is proportional to

$$e^{N+1}\int\frac{d^4k_1}{(2\pi)^4}\,\mathrm{tr}\bigg[\bigg(\frac{i}{\not k_N-m}\bigg)\gamma^{\lambda_N}\bigg(\frac{i}{\not k_{N-1}-m}\bigg)\gamma^{\lambda_{N-1}}\ldots\times\bigg(\frac{i}{\not k_1-m}\bigg)\gamma^{\lambda_1}-\bigg(\frac{i}{\not k_N+\not p-m}\bigg)\gamma^{\lambda_N}\times\bigg(\frac{i}{\not k_{N-1}+\not p-m}\bigg)\gamma^{\lambda_{N-1}}\ldots\bigg(\frac{i}{\not k_1+\not p-m}\bigg)\gamma^{\lambda_1}\bigg]. \quad (3.8)$$

The γ^{λ_j} factors are from the vertices of the photons already shown in Fig. 3, and m is the mass of the internal fermion running in the loop. [Note that these momenta k_i, q_i and the integer N have no relations to the similarly named quantities in Eqs. (3.6) and (3.7).] However, it is easy to see that Eq. (3.8) vanishes. Rewriting Eq. (3.8) as the difference of two integrals, we can shift the variable of integration in the second term from k_1 to k_1+p. These two integrals thus cancel against each other identically. We see, then, that the sum over all insertion points of a photon into a closed loop is zero; such diagrams do not contribute to the right side of the Ward-Takahashi identity.

By contrast, the right side of Eq. (3.6) arises from the subclass of diagrams in which the additional photon line attaches to a fermion line that connects an incoming fermion to an outgoing fermion. The treatment of such diagrams is standard, and the derivation can be found in Ref. [4]. The upshot is that the summation over diagrams contributing to $\mathcal{M}_0$ then yields Eq. (3.6). Although this is only a diagrammatic proof of the Ward-Takahashi identity, it is sufficient for our purposes and can be replaced by a more general path-integral derivation if needed.

We now wish to extend this derivation of the Ward-Takahashi identity to the case of five-dimensional QED compactified on a circle. Our first step will be to repeat this derivation in five *uncompactified* dimensions. However, it is immediately clear that there is no change to the basic result. Indeed, the entire diagrammatic proof sketched above survives intact, and we obtain a five-dimensional Ward-Takahashi identity which is identical to Eq. (3.6) except with the replacement of Lorentz indices $\mu\to M\equiv(\mu,4)$ and the understanding that all momenta are now *five-momenta*. Thus, each five-momentum now contains the usual four-momentum as well as an additional fifth component. The same is true, of course, for the external photon momentum p.

Given this, our second and final step is to determine the extent to which this five-dimensional Ward-Takahashi identity survives the process of compactification. Of course, compactification has the net effect of changing each of these fifth components from continuous to discrete. For cases in which the external photon attaches to a fermion line stretching between incoming and outgoing fermions, this discretization of the fifth component has no net effect on the analysis and our algebraic results survive as before.

However, we must also verify that there are no new features for the cases in which the external photon attaches to a fermion line which forms a closed internal loop. This case is special because our integral over the internal loop five-momentum now becomes an integration over the four-dimensional loop-momentum components as well as a discrete summation over the fifth component (i.e., a summation over the Kaluza-Klein index of the internal fermion). To be more specific, we now wish to consider the compactified five-dimensional analogue of Fig. 3 in which each of the momenta shown represents a discretized five-momentum, with $k_i\equiv(k_i^\mu,k_i^4)$ and $q_i\equiv(q_i^\mu,q_i^4)$ where $k_i^4\equiv r_i/R$ and $q_i^4\equiv s_i/R$ for some integers $r_i,s_i\in\mathbb{Z}$. If our external photon has five-momentum $p\equiv(p^\mu,n/R)$, the sum over insertion locations for this external photon now leads to the compactified five-dimensional amplitude

$$e^{N+1}\sum_{r\in\mathbb{Z}}\int\frac{d^4k_1}{(2\pi)^4}\,\mathrm{tr}\bigg[\bigg(\frac{i}{\not k_N-m}\bigg)\gamma^{\lambda_N}\bigg(\frac{i}{\not k_{N-1}-m}\bigg)\gamma^{\lambda_{N-1}}\ldots\times\bigg(\frac{i}{\not k_1-m}\bigg)\gamma^{\lambda_1}-\bigg(\frac{i}{\not k_N+\not p-m}\bigg)\gamma^{\lambda_N}\times\bigg(\frac{i}{\not k_{N-1}+\not p-m}\bigg)\gamma^{\lambda_{N-1}}\ldots\bigg(\frac{i}{\not k_1+\not p-m}\bigg)\gamma^{\lambda_1}\bigg] \quad (3.9)$$

where quantities such as $\not k$ are now understood to represent five-dimensional contractions, i.e., $\not k\equiv k_M\gamma^M\equiv k_\mu\gamma^\mu-(r/R)\tilde\gamma^4$ where $\tilde\gamma^4\equiv i\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$. Just as with Eq. (3.8), we can once again separate these terms into distinct integrations/summations and recognize that the second term is the same as the first term except for the algebraic replacements $k_i^\mu\to k_i^\mu+p^\mu$ and $r_i\to r_i+n$. The first of these replacements has no net effect because the four-momentum integration in Eq. (3.9) has infinite range; indeed, this range remains infinite even when the integrand is regulated through 4D dimensional regulation. However, the situation with the second replacement is slightly more subtle. Of course, the shift $r_i \to r_i + n$ does not disturb the form of our KK summation because each integer r_i in the summation range is merely being shifted by another integer n. However, in this case the summation range is not infinite, since there is an implicit cutoff. It is therefore only as this cutoff is removed at the end of the calculation that the replacement $r_i \to r_i + n$ has no net effect on the KK summation, and Eq. (3.9) holds. Of course, for the special $n = 0$ case (corresponding to a zero-mode external photon), this last issue does not arise, and the KK summation is unaltered regardless of the value of the cutoff.

Putting the pieces together, then, we obtain a Ward-Takahashi identity which is suitable for five-dimensional spacetimes with a single compactified dimension:

$$\begin{aligned} p_M \mathcal{M}^M(p; k_1, \ldots, k_N; q_1, \ldots, q_N) \\ = e\sum_i [\mathcal{M}_0(k_1, \ldots, k_n; q_1, \ldots, (q_i - p), \ldots) \\ - \mathcal{M}_0(k_1, \ldots, (k_i + p), \ldots; q_1, \ldots, q_N)]. \end{aligned} \quad (3.10)$$

Here M is the five-dimensional Lorentz index, and all momenta are understood to be *five-momenta*. As usual, this identity holds in the presence of a suitable regulator. In the special case of a zero-mode external photon, this identity holds exactly; by contrast, for all other cases, this identity holds *up to terms which vanish as the regulator is removed*. The identity in Eq. (3.10) is quite powerful, however: it implies not only that our ordinary (zero-mode) photon satisfies the Ward-Takahashi identity (as we might have always expected), but also that *each of our excited KK photons satisfies a Ward-Takahashi identity as well*. In this sense, our original five-dimensional gauge invariance has survived the process of compactification—even though our original five-dimensional Lorentz invariance is broken.

Given this result, we can then generate a corresponding five-dimensional Ward identity in the usual way. In general, Ward identities follow from the Ward-Takahashi identities through LSZ reductions, but we do not really require the full LSZ machinery. The critical observation is that the two sides of Eq. (3.10), just like the two sides of its four-dimensional version Eq. (3.6), have differing pole structures in momentum space: the left sides of these equations have $2N + 1$ poles, while the right sides of these equations have $2N$ poles. Nothing pertaining to the dimensionality of the spacetime or the process of compactification reconciles this mismatch in the pole structure. Consequently, passing to the amplitudes of the corresponding *amputated* diagrams and placing our external particles on shell, we find that the right sides of these equations cannot contribute, and thus we obtain a five-dimensional Ward identity which holds for each KK photon:

$$p_M \mathcal{M}^M(p; k_1, \ldots, k_N; q_1, \ldots, q_N) = 0. \quad (3.11)$$

As with Eq. (3.10), it is understood that this is an exact relation which holds for zero-mode external photons in the presence of a regulator; for excited KK photons, by contrast, this relation holds up to terms which vanish as the regulator is removed. However, this will be sufficient for our purposes.

Finally, note that unlike the Ward-Takahashi identities in Eq. (3.10), the Ward identities in Eq. (3.11) hold only when the external photon is on shell. However, in the special case of amplitudes with no external fermions, the right side of Eq. (3.10) vanishes identically. In such cases, we expect the Ward identity in Eq. (3.11) to hold regardless of whether the external photon momentum is on shell or off shell.

One important special case that we will shortly consider is the case of diagrams with two external photons and no external fermions—i.e., a five-dimensional vacuum-polarization diagram. By momentum conservation, the five-momentum $p^M = (p^\mu, n/R)$ of the incoming photon will be equal to the five-momentum of the outgoing photon. In this case, our amplitude $\mathcal{M}^{MN}$ will have two five-dimensional Lorentz indices, and our Ward identities take the form

$$p_M \mathcal{M}^{MN} = p_N \mathcal{M}^{MN} = 0. \quad (3.12)$$

Expanded out, these identities imply

$$p_\mu \mathcal{M}^{\mu\nu} = \frac{n}{R}\mathcal{M}^{4\nu} \quad \text{and} \quad p_\nu \mathcal{M}^{\mu\nu} = \frac{n}{R}\mathcal{M}^{\mu 4} \quad (3.13)$$

as well as

$$p_\mu \mathcal{M}^{\mu 4} = \frac{n}{R}\mathcal{M}^{44} \quad \text{and} \quad p_\nu \mathcal{M}^{4\nu} = \frac{n}{R}\mathcal{M}^{44}. \quad (3.14)$$

Combining these two results, we thus obtain the relation

$$p_\mu p_\nu \mathcal{M}^{\mu\nu} = \left(\frac{n}{R}\right)^2 \mathcal{M}^{44}. \quad (3.15)$$

Of course, our derivation of these identities has been purely diagrammatic and restricted to the special case of five-dimensional QED compactified on a circle. Despite these limitations, the arguments of this section should easily generalize to the case of multiple extra dimensions compactified on square tori. Moreover, we expect identities like these to hold for even more general spacetimes and compactifications. After all, Ward(-Takahashi) identities are merely expressions of Noether's theorem (and resulting Schwinger-Dyson equations) applied to gauge symmetries. As such, they can generally be proven using path-integral techniques which should survive compactification as long as no spacetime boundary is introduced (to produce new surface terms). Thus, we expect a Ward identity of this type to emerge whenever our higher-dimensional Lagrangian exhibits a gauge symmetry and the spacetime is compactified on a smooth manifold.

Needless to say, the situation can be significantly different for compactifications on orbifolds. The presence of fixed points (or fixed lines/planes, etc.) can give rise to surface terms (such as brane-kinetic terms) which render the would-be Ward identities invalid for all but the usual four-dimensional Ward identity on the brane. Moreover, even for compactifications on manifolds, we stress that the corresponding Ward identities may not always take a recognizable form. Implicit in our derivation above was the assumption that the Kaluza-Klein eigenfunctions coincide with momentum eigenfunctions. While this is true for compactifications on square tori, this will not be true in general: for example, compactification on a sphere produces Legendre polynomials which have no interpretations in terms of individual plane waves. Since our Ward identities are usually written in terms of momentum-space wave functions, such compactifications can lead to Ward identities involving many nontrivial interactions between individual plane-wave modes.

Finally, we remind the reader that not every regulator will respect these identities: certain UV divergences can spoil the argument we made about insertions into a KK fermion loop. For example, some regulators (e.g., the hard cutoff) are known to violate these identities in four dimensions. Thus, only certain regulators will respect these five-dimensional Ward(-Takahashi) identities, and it is the goal of this section to determine for which regulators this is the case.

C. Imposing the Ward-Takahashi identities for KK photons

We now impose our higher-dimensional Ward-Takahashi identities in order to derive a relationship between the dimensional-regularization parameter ϵ and the summation cutoff Λ introduced in Sec. III A. We shall also determine the precise value for λ introduced in Eq. (3.3).

To do this, we consider the special case of Fig. 1 in which the external particles are on-shell KK photons and the particles running in the loop are a tower of KK fermions with bare mass M (so that the tree-level squared mass of the rth excitation is given by $r^2/R^2 + M^2$). Such a diagram is indeed nothing but a five-dimensional vacuum-polarization diagram with two Lorentz indices (M, N) corresponding to the external photons, and this is precisely the sort of diagram for which we expect the higher-dimensional Ward identities given in Eqs. (3.12), (3.13), (3.14), and (3.15) to hold.

Prior to regularization, the different components of the vacuum-polarization amplitude take the form

$$L_n^{MN} = -4e^2 \int_0^1 dx \sum_r \int \frac{d^4\ell}{(2\pi)^4} \times \left[\frac{1}{\ell^2 - \ell^{42} - \mathcal{M}^2(x)}\right]^2 \Omega_n^{MN} \tag{3.16}$$

where

$$\begin{aligned}
\Omega_n^{\mu\nu} &= 2\ell^\mu\ell^\nu + 2x(x-1)p^\mu p^\nu + g^{\mu\nu}[-\ell^2 + \ell^{42} \\
&\quad + (2x-1)(n/R)\ell^4 - \mathcal{M}^2(x) + 2M^2], \\
\Omega_n^{\mu 4} &= p^\mu[(2x-1)(n/R)\ell^4 + 2x(x-1)n/R], \\
\Omega_n^{44} &= \ell^2 + \ell^{42} + (2x-1)(n/R)\ell^4 \\
&\quad + 2x(x-1)n^2/R^2 + \mathcal{M}^2(x) - 2M^2.
\end{aligned} \tag{3.17}$$

Here n is the mode number of the external photon, and in writing these expressions, we have continued to use the notation and conventions listed at the beginning of Sec. II. The procedure outlined in Sec. III A then demands that we regularize four-momentum integrals by taking their dimensionality to be $d = 4 - \epsilon$, truncate KK sums according to Eq. (3.2), and also deform the integrands according to Eq. (3.3). After performing the momentum loop integrations, we then find that these components take the form

$$L_n^{MN} = -\frac{ie^2}{4\pi^2} R^\epsilon \int_0^1 dx \sum_{r=-\Lambda(\epsilon)R+xn}^{\Lambda(\epsilon)R+xn} f_n^{MN} \tag{3.18}$$

where

$$\begin{aligned}
f_n^{\mu\nu} &= \{[(2x-1)(n/R)\ell^4 + 2M^2 - 2\mathcal{M}^2(x)]g^{\mu\nu} \\
&\quad + 2x(x-1)p^\mu p^\nu\}W, \\
f_n^{\mu 4} &= p^\mu[(2x-1)\ell^4 + 2x(x-1)(n/R)]W, \\
f_n^{44} &= [3\ell^{42} + (2x-1)(n/R)\ell^4 + 2x(x-1)(n/R)^2 \\
&\quad + 3\mathcal{M}^2(x) - 2M^2]W + (1-2\lambda)\ell^{42} + \mathcal{M}^2(x)
\end{aligned} \tag{3.19}$$

with

$$W \equiv \frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[(\ell^4 R)^2 + (\mathcal{M}(x)R)^2] + \mathcal{O}(\epsilon). \tag{3.20}$$

Here γ is the Euler-Mascheroni constant. Note that the KK summation in Eq. (3.18) does not necessarily force the terms which are linear with respect to ℓ^4 to vanish. This is an important distinction from the case in which ℓ^4 is a continuous variable.

Given these expressions for the vacuum-polarization diagrams, we now demand that they respect the Ward identities (3.13) and (3.14) for the KK-photon modes. First, we immediately observe from the above results that

$$p_\mu f_n^{\mu\nu} = \left(\frac{n}{R}\right) f_n^{4\nu}. \tag{3.21}$$

Thus, we find that the full Ward identity in Eq. (3.13) for the *amplitudes* L_n^{MN} is satisfied identically as a result of a Ward identity for the *integrands* f_n^{MN} for all external KK-photon mode numbers n. This implies that the Ward identity in Eq. (3.13) holds regardless of whether the external KK photon is on shell or off shell, and regardless of how ϵ and Λ are related in the internal KK sum in Eq. (3.18).

Moreover, because this amplitude contains no external fermions, the fact that the Ward identities hold when the external photon momenta are off shell implies that the full Ward-Takahashi identities hold as well. Thus, while Eq. (3.21) is an important self-consistency check on our approach, it does not yield any new information that helps us determine $\Lambda(\epsilon)$ or λ.

The situation, however, is different for the Ward identity in Eq. (3.14). Examining the integrands, we find that

$$p_\mu f^{\mu 4} - \left(\frac{n}{R}\right) f^{44} = \left[-\left(\frac{n}{R}\right)(3\ell^{42} + \mathcal{M}^2(x)) + (2x-1)\ell^4\left(p^2 - \frac{n^2}{R^2}\right)\right]W - \left(\frac{n}{R}\right)[(1-2\lambda)\ell^{42} + \mathcal{M}^2(x)]. \quad (3.22)$$

Note that this vanishes identically when our external photon is the zero-mode photon (i.e., $n = 0$) and when it is on shell. Thus, we find that the Ward identity in Eq. (3.14) also holds automatically for zero-mode photons, as we expect. Moreover, even when the external zero-mode photon is not on shell, the Ward(-Takahashi) identity continues to hold because the nonzero factor $(p^2 - n^2/R^2)$ in Eq. (3.22) comes multiplied by a single power of ℓ^4, which vanishes over the symmetric r summation in Eq. (3.18). Together, this is nothing but the preservation of four-dimensional gauge invariance, which once again occurs regardless of the precise relations between Λ, λ, or ϵ.

By contrast, in order to preserve *five-dimensional* gauge invariance, we require that the Ward(-Takahashi) identities in Eq. (3.14) hold for *all* KK photons—i.e., for *all* values of n. We must therefore concentrate on the cases when $n \neq 0$, and determine a value for λ and a relation between the KK summation cutoff Λ and ϵ such that Eq. (3.14) holds. At first glance, our main complication is that our cutoffs Λ appear only in the KK summation limits. However, since $n \neq 0$, we can utilize the variable-transformation methods we developed in Sec. II. Specifically, following the steps outlined in Sec. II, we change variables from x to $\hat{u}$ defined in Eq. (2.13) and from r to $\hat{r}$ defined in Eq. (2.14), and then drop the hats from $\hat{u}$ and $\hat{r}$. This amounts to the algebraic substitution $x \to (u+j)/|n|$, and we shall define $y \equiv (u + j)/|n|$. Following Eq. (3.5), we can then write

$$p_\mu L^{\mu 4} - \frac{n}{R} L^{44} = \frac{ie^2}{4\pi^2} \frac{\text{sign}(n) R^\epsilon}{R} \sum_{r=-\Lambda R+1}^{\Lambda R} \frac{1}{|n|} \times \sum_{j=0}^{|n|-1} \int_0^1 du f_n \quad (3.23)$$

where the integrand f_n is the variable-shifted version of Eq. (3.22), i.e.,

$$f_n = \left[|n|\left(\frac{3(r-u)^2}{R^2} + \mathcal{M}^2(y)\right) + (1-2y)(r-u) \times \left(p^2 - \frac{n^2}{R^2}\right)\right]W + |n|\left[(1-2\lambda)\frac{(r-u)^2}{R^2} + \mathcal{M}^2(y)\right]. \quad (3.24)$$

Here W continues to represent the quantity in Eq. (3.20), now written with the algebraic substitutions $(R\ell^4)^2 \to (r-u)^2$ and $\mathcal{M}^2(x) \to \mathcal{M}^2(y)$.

It is not immediately clear which relationships between Λ, λ, and ϵ would force the expression in Eq. (3.23) to vanish as $\Lambda \to \infty$ (or as $\epsilon \to 0$), or whether such a relation even exists. However, we may consider the special case in which the external KK photons are on shell. In other words, we can restrict our attention to the Ward identities rather than the full Ward-Takahashi identities. Once we determine the appropriate relationships between Λ, λ, and ϵ for the purposes of maintaining the Ward identities, we can then verify that the full Ward-Takahashi identities hold as well.

When the external KK photons are on shell, $p^2 - n^2/R^2 = 0$ and $\mathcal{M}^2(y) = M^2$. Our integrand is also independent of j, which enables us to explicitly perform the j summation in Eq. (3.23) and soak up the overall factor of $|n|$. We then see that Eq. (3.23) is given by

$$\begin{aligned}
p_\mu L^{\mu 4} - \left(\frac{n}{R}\right) L^{44} &= \frac{ie^2 n R^\epsilon}{4\pi^2 R} \sum_{r=-\Lambda R+1}^{\Lambda R} \int_0^1 du \left[\left(\frac{3(r-u)^2}{R^2} + M^2\right)W + (1-2\lambda)\frac{(r-u)^2}{R^2} + M^2\right] \\
&= \frac{ie^2 n R^\epsilon}{4\pi^2 R} \sum_{r'=-\Lambda R}^{\Lambda R-1} \int_{r'}^{r'+1} dw \left[\left(\frac{3w^2}{R^2} + M^2\right)W + (1-2\lambda)\frac{w^2}{R^2} + M^2\right] \\
&= \frac{ie^2 n R^\epsilon}{4\pi^2 R} \int_{-\Lambda R}^{\Lambda R} dw \left[\left(\frac{3w^2}{R^2} + M^2\right)W + (1-2\lambda)\frac{w^2}{R^2} + M^2\right] \\
&= \frac{ie^2 n R^\epsilon}{4\pi^2 R} \left\{\frac{2\tilde{\Lambda}^3}{R^2}\left[1 + c - \frac{2\lambda}{3} - \log(\tilde{\Lambda}^2 + M^2R^2)\right] + 2\tilde{\Lambda} M^2[1 + c - \log(\tilde{\Lambda}^2 + M^2R^2)]\right\}. \quad (3.25)
\end{aligned}$$

Note that the second equality above follows from defining $w \equiv u - r$ and $r' = -r$, and the third follows from explicitly performing the truncated KK sum. The fourth equality is obtained by substituting $W = 2/\epsilon - \gamma + \log(4\pi) - \log[w^2 + (MR)^2] + \mathcal{O}(\epsilon)$ and explicitly evaluating the w integral. Finally, in writing the final line, we have defined $\tilde{\Lambda} \equiv \Lambda R$ and $c \equiv 2/\epsilon - \gamma + \log(4\pi)$.

Given these results, we see that there are many different ways in which this final expression can be made to vanish as $\tilde{\Lambda} \to \infty$, as required by our Ward identity for excited KK photons. One possibility, for example, is to demand that Λ and ϵ be related to each other such that $1 + c = \log(\tilde{\Lambda}^2 + M^2R^2)$ up to terms which vanish more strongly than $1/\tilde{\Lambda}^3$ as $\tilde{\Lambda} \to \infty$. If we additionally take $\lambda = 0$, then both of the terms in the final expression in Eq. (3.25) will vanish as $\tilde{\Lambda} \to \infty$ (or as $\epsilon \to 0$). However, such relations are not suitable for a bona-fide regulator because they depend on M. They thus depend on the particular fermions in the theory, and are not theory independent.

It turns out that there is only one possible M-independent regulator which does the job. For large Λ, we can write $\log(\tilde{\Lambda}^2 + M^2R^2) \approx 2\log(\tilde{\Lambda}) + (MR/\tilde{\Lambda})^2$, whereupon Eq. (3.25) takes the form

$$p_\mu L^{\mu 4} - \left(\frac{n}{R}\right)L^{44} = \frac{ie^2nR^\epsilon}{4\pi^2R}\left[\frac{2\tilde{\Lambda}^3}{R^2}\left(1 + c - \frac{2\lambda}{3} - 2\log\tilde{\Lambda}\right) + 2\tilde{\Lambda}M^2(c - 2\log\tilde{\Lambda}) + \mathcal{O}(MR/\tilde{\Lambda})\right]. \tag{3.26}$$

We therefore demand that $c = 2\log\tilde{\Lambda}$ up to terms which vanish faster than $1/\tilde{\Lambda}^3$ as $\tilde{\Lambda} \to 0$, and we likewise choose $\lambda = 3/2$. These choices guarantee that $p_\mu L^{\mu 4} - (n/R)L^{44} \to 0$ as $\epsilon \to 0$, i.e., as $\Lambda \to \infty$.

Thus, to summarize, we conclude that the proper relationship between Λ and ϵ is given by

$$\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) = 2\log(\Lambda R) + \delta \tag{3.27}$$

where $\delta \to 0$ as $\Lambda \to \infty$. [For example, for the expression in Eq. (3.26), we know that $\delta\Lambda^3 \to 0$ as $\Lambda \to \infty$.] We shall discuss the role played by δ below. We also conclude that

$$\lambda = 3/2. \tag{3.28}$$

Equations (3.27) and (3.28) are the relations between Λ, λ, and ϵ which preserve higher-dimensional gauge invariance as well as higher-dimensional Lorentz invariance. As such, these relations therefore define our EDR procedure. Moreover, as we shall see, these relations are *universal* (as demanded by our criterion of theory independence): as we shall soon discuss, they apply for any loop diagram in any theory with a circular extra dimension, even though we derived them via a study of five-dimensional QED.

Finally, although we have shown above that these relations are sufficient to satisfy the Ward identities for all KK photons, we have also verified through an explicit calculation that they actually satisfy the full Ward-Takahashi identities for KK photons as well. In other words, the Ward identities are satisfied regardless of whether the external photon momenta are on shell or off shell.

We should also emphasize an important point. Clearly, our EDR regulator should be applicable for all values of the compactification radius R. As such, the EDR regulator should be applicable even in the $R \to \infty$ limit in which flat five-dimensional Minkowski space is restored and our KK sum becomes an integral. However, even in this limit, our EDR regulator does *not* reduce to ordinary 't Hooft-Veltman 5D dimensional regularization. This is because we are continuing to treat the resulting five-dimensional momentum integral in an asymmetric way, even in the $R \to \infty$ limit, using 4D dimensional regularization for the large spacetime dimensions and a hard cutoff for the extra spacetime dimension. Thus, while we continue to have a self-consistent regulator even in the $R \to \infty$ limit, this is not the flat five-dimensional version of the ordinary 't Hooft-Veltman regulator. Note that this situation was entirely different for our extended hard-cutoff regulator in Sec. II. In that case, the $R \to \infty$ limit does reproduce an ordinary five-dimensional hard cutoff.

Another example of this difference between the $R \to \infty$ limit of the EDR procedure and the ordinary 5D 't Hooft-Veltman dimensional-regularization procedure is the fact that EDR involves a deformation of the four-momentum components of the form $\ell^\mu\ell^\nu \to \ell^2 g^{\mu\nu}/(4-\epsilon)$, but a deformation of the extra fifth component of the form in Eq. (3.3) with $\lambda = 3/2$. These deformations are intrinsically different, and remain so even in the $R \to \infty$ limit; indeed, neither of these deformations is what would be encountered in 5D 't Hooft-Veltman dimensional regularization. These inequivalent deformations in some sense compensate for the inequivalent regularizations applied to the four-momenta and the KK momenta, and are precisely what are required in order to maintain the Ward-Takahashi identities. Moreover, as we shall discuss below, this is also necessary for the maintenance of five-dimensional Lorentz invariance for all values of R.

Despite these differences, the overall form of the relation (3.27) is expected at a certain intuitive level. We know, for example, that the $1/\epsilon$ pole in ordinary 4D dimensional regulation corresponds to a logarithmic divergence, and a logarithmic divergence manifests itself as the logarithm of a cutoff Λ. Thus, a relation of the form in Eq. (3.27), which relates $1/\epsilon$ to $\log(\Lambda)$, is to be expected. What is nontrivial, by contrast, is that this relation also preserves five-dimensional *gauge* invariance, as expressed through the preservation of the Ward identities. This, of course, was the objective of our entire analysis.

D. Loose ends

Thus far, our development of the EDR regulator has led us to the conditions in Eqs. (3.27) and (3.28). However, there are a number of issues which we have not yet addressed:

(i) We have not yet demonstrated that these conditions preserve higher-dimensional Lorentz invariance.

(ii) We have not yet demonstrated that these conditions are *universal*—i.e., that they suitably regulate the divergences that might appear in any potential diagram in a gauge-invariant five-dimensional theory compactified on a circle, so that *all* possible amplitudes satisfy appropriate Ward-Takahashi identities.

(iii) And finally, we have not yet discussed the significance of the quantity δ which appears in Eq. (3.27).

All of these issues must be addressed before we can claim to have a bona-fide regulator for five-dimensional theories compactified on a circle. The purpose of this section is to address each of these issues, one at a time.

1. Higher-dimensional Lorentz invariance

We begin by considering the issue of higher-dimensional Lorentz invariance.

It is, of course, unavoidable that reducing the dimensionality of our uncompactified spacetime from four dimensions to $D \equiv 4 - \epsilon$ dimensions breaks higher-dimensional Lorentz invariance, since this dimensional-alteration process cannot regularize discrete KK sums. Therefore, the best one can do in a dimensional-regularization setup is to restore the higher-dimensional Lorentz symmetry at the end of a calculation, just as we restore the Ward identities (and more generally, the Ward-Takahashi identities) in the $\Lambda \to \infty$ limit. However, we already know that our extended hard-cutoff (EHC) regularization procedure in Sec. III preserves five-dimensional Lorentz invariance, by construction. Therefore, within the context of a five-dimensional theory without gauge invariance, if we can demonstrate that our EHC and EDR procedures lead to identical results after the cutoffs are removed, we will have demonstrated that our extended dimensional-regularization procedure preserves higher-dimensional Lorentz invariance. Fortunately, we have done this calculation within the context of the effective field theories of KK modes discussed in Ref. [2], and the results are positive.

Moreover, even within the calculation we have done in Sec. III C, it is straightforward to verify that five-dimensional Lorentz invariance is preserved. Recall that we began with a vacuum-polarization amplitude in Eq. (3.16) which *a priori* transforms as a five-dimensional Lorentz tensor. However, after we imposed our regulator, this expression took the form in Eq. (3.18) where the integrands for the different Lorentz components are given in Eq. (3.19). Clearly, the forms of these different Lorentz components are quite different, and it seems that higher-dimensional Lorentz invariance is broken. However, if we take the $R \to \infty$ limit, the KK sum in Eq. (3.18) becomes an integral. Imposing the relations in Eqs. (3.27) and (3.28) and assuming that $\mathcal{M}^2(x) \geq 0$, we then find that these different components all collapse into the single form

$$L^{MN} = -\frac{ie_5^2}{8\pi^3}\int_0^1 dx 2x(1-x)\{[p^2 - (p^4)^2]g^{MN} - p^M p^N\}W' \tag{3.29}$$

where $e_5 \equiv \sqrt{2\pi R}e$ is the 5D gauge coupling and where

$$W' = 4\Lambda - 2\pi\sqrt{\mathcal{M}^2(x)} + \mathcal{O}(m^2/\Lambda). \tag{3.30}$$

Likewise, similar expressions can be derived for the case with $\mathcal{M}^2(x) < 0$. Clearly, the expression in Eq. (3.29) transforms as a higher-dimensional Lorentz tensor. We note that this happens only if we impose the relations in Eqs. (3.27) and (3.28).

We shall present further explicit evidence of the preservation of five-dimensional Lorentz invariance in Sec. IV.

2. Universality

In this section, we discuss the question of *universality*—i.e., whether our EDR regulator can suitably regulate the divergences that might appear in any potential one-loop diagram in a gauge-invariant five-dimensional theory compactified on a circle.

Thus far, we have only demonstrated that EDR preserves the higher-dimensional Ward-Takahashi identities for vacuum-polarization diagrams with two external KK photons. However, our regulator should respect higher-dimensional gauge symmetry in general. This can only happen if our extended dimensional-regularization procedure preserves KK Ward identities and Ward-Takahashi identities for *arbitrary* QED processes in higher dimensions.

Even though there are an infinite number of possible amplitudes in QED, it is sufficient for our regulator to preserve KK Ward-Takahashi identities for loop diagrams of the type shown in Fig. 3, with no external fermions. This is because a divergence from this type of diagram is the only effect which has the potential to spoil the proof of the Ward-Takahashi identity that we outlined in Sec. III B. Furthermore, power counting in 5D implies that diagrams with six or more external KK photons should be finite. Hence, we only need to check that the Ward-Takahashi identities hold for amplitudes with at most five external photons and no external fermions. Note that for such amplitudes, the Ward-Takahashi identity reduces to the same form as the Ward identity, except that the external photons need not be on shell.

We can therefore consider the cases with $0 \leq N \leq 5$ external photons individually. Just as elsewhere in this paper, we restrict our attention to one-loop diagrams.

(i) $N = 0$.—Diagrams of this form with no external photons are mere vacuum bubbles which never contribute to physical amplitudes.

(ii) $N = 1, 3, 5$.—In these cases, our amplitudes have odd numbers of external photons and vanish as a consequence of Furry's theorem. Note that Furry's theorem is itself a direct consequence of charge-conjugation symmetry, and does not rely on gauge invariance *per se*. Since our regulator respects charge-conjugation invariance, the KK Ward-

Takahashi identities are thus trivially satisfied in each of these cases.

(iii) $N = 2$.—This is the case we already examined, and we have already shown that our dimensional-regularization procedure respects KK Ward-Takahashi identities for such vacuum-polarization diagrams.

Given these conclusions, it only remains to check that our regulator preserves the Ward-Takahashi identities in the $N = 4$ case, i.e., for "box" diagrams of the type shown in Fig. 3 with four external KK photons.

Of course, if gauge invariance is truly maintained, then power counting actually overestimates the degree of divergence in each diagram. This is because gauge invariance generally removes several powers of divergence from each diagram. For example, we have already seen that gauge invariance forces the vacuum-polarization diagrams to diverge linearly in the summation cutoff Λ rather than cubically. In general, inserting extra external photons will also decrease the degree of divergence. Therefore, if we can show that the $N = 4$ box diagram is actually finite, then our demonstration of universality is complete.

Evaluating the box diagram is a rather complicated undertaking, even in four dimensions [5]. Therefore, rather than providing a direct evaluation in five dimensions, we shall instead provide an indirect argument that this diagram is indeed finite. Our argument proceeds as follows. Let us first consider the $R \to \infty$ limit in which our extra dimension is completely uncompactified. In this case, we know that the ordinary 't Hooft-Veltman 5D dimensional-regularization procedure [3] provides a valid regulator which preserves the Ward-Takahashi identities. Given that the Ward-Takahashi identities are satisfied for this regulator, it can be shown that our 5D box amplitude is finite; this will be demonstrated explicitly below. Thus, we conclude that the box amplitude is finite in the $R \to \infty$ limit. However, the process of compactifying the extra spacetime dimension cannot change the leading-order divergence structure of an amplitude; an amplitude which is finite as $R \to \infty$ must be finite for all values of R. This radius independence of the leading divergence structure follows from the fact that the UV behavior of an amplitude should be independent of the large-scale geometry of our smooth spacetime manifold. (Indeed, one of the primary alternative regularization methods to be discussed in Sec. IV will depend on this fact.)

The only missing step, then, is to demonstrate that our five-dimensional box amplitude is finite in the $R \to \infty$ limit if the Ward-Takahashi identities hold. However, this result is well known in the four-dimensional case (see, e.g., Ref. [4]), and every step of the proof carries directly over to the case of the one-loop box amplitude in five dimensions. The only difference is that rather than having a degree of divergence of -4 (as in four dimensions), this amplitude now has a degree of divergence of -3.

One might worry that this proof has a potential loophole. Since the individual diagrams contributing to the box amplitude are separately superficially divergent, a bad choice of regulator could disturb the cancellation between diagrams triggered by gauge invariance, thereby yielding an incorrect, divergent result. However, it is always possible to use a gauge-invariant regulator such as the Pauli-Villars (PV) regulator in order to render each diagram individually superficially convergent. There is then no danger of destroying the cancellations between diagrams, and the Pauli-Villars regulator can be lifted at the end of the calculation. Indeed, this "pretreating" of each diagram with a Pauli-Villars regulator can also be used to justify the Furry-theorem cancellations inherent in the $N = 1, 3, 5$ diagrams.

Within box diagrams, such cancellations are actually rather robust. For example, in the four-dimensional case, the required cancellations are known to occur in a special case (so-called "Delbrück scattering" [5]) even when a simple hard cutoff is used.

We thus conclude that the EDR procedure preserves the Ward-Takahashi identities for all possible one-loop diagrams in five-dimensional QED compactified on a circle.

3. The fate of δ

Thus far, we have shown that our momentum integrations and KK sums must have cutoff parameters ϵ and Λ which are related through Eq. (3.27). This expression is sufficient to describe the manner in which ϵ and Λ are correlated as $\epsilon \to 0$ (or as $\Lambda \to \infty$).

However, each side of this relation contains additional terms [$\mathcal{O}(\epsilon)$ and δ, respectively] which vanish in these limits. Even though these terms individually vanish, it may seem that determining these terms can be critical for performing radiative calculations. For example, in a given calculation, δ may eventually be multiplied by terms which grow as $\Lambda \to \infty$; this structure is already apparent in expressions such as Eq. (3.26). Thus, it may appear that δ can give rise to nonzero terms which contribute to the final results of radiative calculations, even after the cutoff is removed.

Clearly, the precise form of the $\mathcal{O}(\epsilon)$ terms will depend on the specific diagram in question, much as we expect in ordinary 4D dimensional regularization. Consequently, we expect that δ will also be a diagram-dependent quantity. We stress, however, that the *relation* (3.27) is itself general. Indeed, the only diagram dependence is in how certain terms (which vanish as the cutoffs are removed) are reallocated between $\mathcal{O}(\epsilon)$ and δ in Eq. (3.27).

We shall now discuss the fate of δ as a contributing factor in any field-theory calculation. As we shall explain, no physical observable can possibly depend on δ. Therefore, it is never necessary to calculate δ for any given diagram, and the universal relation in Eq. (3.27) is sufficient for the calculation of any physical observable.

This claim ultimately rests on the observation that any physical observable must be finite and regulator independent. For example, a diagram such as that in Fig. 1 represents a one-loop mass shift for the external particle. If L_n represents the value of this diagram when the external particle carries KK mode number n, we know that each L_n might individually be divergent; it is only after *renormalization* that such a one-loop corrected mass becomes finite. However, *differences* such as $L_n - L_0$ represent one-loop radiative contributions to the mass differences between different KK modes. Since such mass differences are physical observables, quantities such as $L_n - L_0$ should be both finite and regulator independent. In an upcoming paper [2], we shall demonstrate that such differences are indeed regulator independent: even though the raw expressions for the loop-diagram differences appear to contain the regulator cutoffs, these cutoffs can all be eliminated through resummations and cancellations. However, imposing the requirement of finiteness on these differences will lead us to our observation about the irrelevance of δ.

We begin by considering the result of any single diagram. Our interest is in the behavior of such a diagram as our cutoff is removed (i.e., as $\Lambda \to \infty$), so we shall concentrate on only those contributions which potentially survive as $\Lambda \to \infty$. In general, following steps such as those which led to Eq. (3.26), we may express the value of any particular diagram $L^{(i)}$ in the form

$$L^{(i)} \sim \alpha_0^{(i)} + \alpha^{(i)}(\Lambda) + \delta^{(i)}(\Lambda)\beta^{(i)}(\Lambda) \qquad (3.31)$$

where the symbol "$\sim$" indicates that we are only retaining terms which survive as $\Lambda \to \infty$. In Eq. (3.31), $\alpha_0^{(i)}$ is a diagram-dependent constant term, while $\alpha^{(i)}$ and $\beta^{(i)}$ are diagram-dependent diverging functions of Λ. Likewise, $\delta^{(i)}$ is our diagram-dependent δ parameter. Even though $\delta^{(i)}$ is assumed to vanish as $\Lambda \to \infty$, it multiplies a potentially divergent function $\beta^{(i)}(\Lambda)$ and thus can still give rise to a contribution which survives as $\Lambda \to \infty$. In general, this contribution will take the form

$$\delta^{(i)}(\Lambda)\beta^{(i)}(\Lambda) \sim b_0^{(i)} + b^{(i)}(\Lambda) \qquad (3.32)$$

where once again $b_0^{(i)}$ is a potential constant (Λ-independent) term and $b^{(i)}(\Lambda)$ is a divergent function of Λ.

Given these individual diagrams $L^{(i)}$, the correction to a physical observable at one-loop order will always take the form of a linear combination $\sum c_i L^{(i)}$. Such a physical observable will therefore have the divergence behavior

$$\sum c_i L^{(i)} \sim \sum_i c_i \alpha_0^{(i)} + \sum_i c_i \alpha^{(i)}(\Lambda) + \sum_i c_i b_0^{(i)} + \sum_i c_i b^{(i)}(\Lambda). \qquad (3.33)$$

However, because this corresponds to a physical observable, we know that this expression must be finite as $\Lambda \to \infty$. We therefore have that

$$\sum_i c_i \alpha^{(i)}(\Lambda) = -\sum_i c_i b^{(i)}(\Lambda). \qquad (3.34)$$

Moreover, as we shall explain below, we further claim that

$$\sum_i c_i b_0^{(i)} = 0. \qquad (3.35)$$

Thus, regardless of the precise value of the $\delta^{(i)}(\Lambda)$ functions, we see that their entire purpose is simply to soak up all other potential divergences from physically observable quantities. In the end, the final result for any physical observable in the $\Lambda \to \infty$ limit is given by $\sum_i c_i \alpha_0^{(i)}$, and this quantity is completely $\delta^{(i)}$ independent.

Of course, a critical step here was the assumption in Eq. (3.35) that $\sum_i c_i b_0^{(i)} = 0$. However, this quantity must cancel because it is regulator dependent (depending ultimately on the individual $\delta^{(i)}$'s). Indeed, as we have discussed above, this quantity is related to the regulator-dependent $\mathcal{O}(\epsilon)$ terms through Eq. (3.27), and as such these $b_0^{(i)}$ terms are analogous to the factors of $\log(4\pi)$ or the Euler-Mascheroni constant γ which appear in dimensional-regularization calculations but have no observable effects. The cancellation in Eq. (3.35) is merely the expression of the fact that such terms will always cancel in the calculation of any physical observable.

Thus, we conclude that the δ terms in Eq. (3.27)—although potentially important for the value of any individual diagram $L^{(i)}$—will ultimately be irrelevant for the calculation of any physical observable. Therefore, as indicated above, it is never necessary to calculate δ for any given diagram, and the universal relation in Eq. (3.27) is sufficient for the calculation of any physical observable.

IV. COMPARISONS WITH OTHER REGULATORS

In this section, we shall compare our techniques with other regulators that exist in the literature for dealing with higher-dimensional quantum field theories with compactified extra dimensions. We shall pay particular attention to existing methods which respect to higher-dimensional symmetries, with the purpose of demonstrating that our regulator successfully reproduces results that can be obtained by these methods. However, we also shall explain why our particular regulators are useful, despite the existence of alternatives. We shall also illustrate the unwanted complications that can emerge when one employs a regulator which does not respect higher-dimensional symmetries.

A. Review of existing techniques

We begin by reviewing various regularization techniques which have already appeared in the literature.

The most straightforward way to analyze radiative corrections on extra dimensions is to decompose our higher-dimensional fields in terms of KK modes, and to treat these modes as heavy 4D particles. One defines the theory up to some large but finite cutoff Λ, and the Euclidean four-momenta of particles and their KK masses are assumed to lie below this cutoff, i.e.,

$$p_E^2 \leq \Lambda^2, \tag{4.1}$$

and

$$m_n^2 \leq \Lambda^2, \tag{4.2}$$

where m_n is the mass of the nth KK mode. For compactifications on a circle, these masses are given by the usual dispersion relation:

$$m_n^2 = m^2 + \frac{n^2}{R^2}. \tag{4.3}$$

In the usual treatments, Eqs. (4.1) and (4.2) are taken to be *independent* constraints, since such a regulator is insensitive to the original higher-dimensional nature of the KK theory. By contrast, the dispersion relation in Eq. (4.3) is nothing but the expression of 5D Lorentz invariance which exists at tree level.

This sort of regulator has been applied in a number of calculations going all the way back to the original work in Ref. [6], in which it was shown that gauge coupling unification can occur with a significantly reduced unification scale in a higher-dimensional context, and that large fermion mass hierarchies can also be generated. Since then, regulators such as these have been applied in a variety of contexts having to do with precision studies of extra dimensions and their diverse effects on ordinary four-dimensional (zero-mode) physics.

These studies all have one feature in common: they are concerned with the properties of the zero modes and the radiative corrections to these properties which are induced by the existence of the excited KK states. Because the properties of the zero modes are sensitive to only four-dimensional symmetries, regulators which break five-dimensional symmetries but preserve four-dimensional symmetries are sufficient for such calculations. For example, it is straightforward to demonstrate that for calculations involving only zero modes, the sort of 4D regulator defined in Eqs. (4.1) and (4.2) and the 5D regulator we introduced in Sec. II will yield results whose divergences differ by at most an overall multiplicative constant. However, such a constant can be absorbed into the definition of the cutoff itself (which is particularly ambiguous in a nonrenormalizable theory), and these effects necessarily vanish as the regulator is removed. Thus, both types of regulators will produce identical results for all zero-mode calculations.

Unfortunately, such four-dimensional regulators are insufficient for calculations of the properties of the excited KK modes themselves. Such regulators are therefore also insufficient for calculations that aim to compare the properties of the excited KK modes (such as their masses or couplings) with those of the zero modes, as might be extracted in a collider experiment. Indeed, as we shall show explicitly in Sec. IV C, such four-dimensional regulators lead to unphysical artifacts which are difficult to disentangle from true, physical effects.

To date, there are very few calculational methods in the literature which preserve the original higher-dimensional symmetries that existed prior to compactification. However, there are three notable exceptions which we shall now discuss.

First, it can sometimes happen that no regulator is needed, even in higher dimensions. For example, in Ref. [7], a practical example of a regulator-independent calculation in higher dimensions was given. Specifically, the authors of Ref. [7] calculated $g-2$ for the muon in a higher-dimensional standard model compactified on universal extra dimensions. For the case of a single extra dimension, they found that $g-2$ received only finite corrections from KK modes at one-loop order. Of course, no regulator was needed in this case. However, they found that such corrections diverged logarithmically in six dimensions.

Second, it can sometimes happen that a four-dimensional regulator might itself be sufficient in higher dimensions. An example of this phenomenon appears in Ref. [8]. Applying ordinary 4D dimensional regularization, the author of Ref. [8] showed that it was possible to obtain regulator-independent results for QED on a universal extra dimension. *A priori*, one would have expected an infinite number of counterterms for this theory, due to its nonrenormalizability. However, it was shown in Ref. [8] that only a counterterm for the electric charge was needed for describing corrections to the zero-mode coupling at one-loop order. Specifically, the author of Ref. [8] calculated the vacuum-polarization diagram $L^{\mu\nu}(p) = \Pi(p^2)\times(p^\mu p^\nu - g^{\mu\nu}p^2)$ for a photon zero mode with four-momentum p, and found that the regulator ϵ canceled in the difference $\Pi(p^2) - \Pi(0)$. Any divergence in a correction to a higher-order coupling operator (e.g., the electron magnetic moment) is therefore solely a consequence of the charge renormalization. Quantities such as $g-2$ receive finite (hence, regulator-independent) corrections. However, the author of Ref. [8] showed that this sort of cancellation occurs only at one-loop order in 5D, and explicitly demonstrated that additional counterterms are needed when there are two extra dimensions. Moreover, there was no discussion of vertex corrections, which are needed for calculating corrections to higher-order operators.

To the best of our knowledge, there is only one other regulator that has appeared in the literature which is intrinsically higher-dimensional and which preserves higher-dimensional Lorentz and gauge symmetries. This is the regularization method developed in Ref. [9]. This method rests upon the observation that the effects of compactification should evaporate in the UV limit, and consequently the

UV divergence of a given diagram evaluated on a four-dimensional space with a single compactified extra dimension should be the same as the UV divergence of the same diagram evaluated on a five-dimensional flat (uncompactified) space. One can thus extract a finite result from any given loop diagram in the compactified theory by subtracting the value of the corresponding diagram in a theory where all of the dimensions are infinite. In this way, one therefore obtains [9] a recipe for extracting finite values from loop diagrams, which respects the full higher-dimensional Lorentz invariance as well as whatever higher-dimensional gauge invariance might exist.

Operationally, the technique in Ref. [9] employs a Poisson resummation in order to recast the sum over Kaluza-Klein momentum mode numbers n within a loop diagram on a compactified extra dimension as a convergent sum over a "dual" set of *winding* numbers w. It turns out that the $w = 0$ contribution is nothing but the contribution from the corresponding diagram evaluated on the uncompactified space. This "regularization" procedure therefore amounts to transforming to the dual winding-number basis and then disregarding the contribution from the $w = 0$ winding mode.

As an example, using this method, the authors of Ref. [9] examined five-dimensional QED with massless fermions, compactified on a circle. Although the zero-mode photon does not gain a mass as a result of four-dimensional gauge invariance, it was found that the masses of the excited KK-photon modes are each shifted by a uniform amount,

$$\Delta m_n^2 = -\frac{e^2}{2\pi R^2}\sum_{w\neq 0}\frac{2}{|2\pi w|^3} = -\frac{e^2\zeta(3)}{4\pi^4 R^2}, \qquad (4.4)$$

where e is the unit of electric charge and where the ζ function represents the winding-number sum:

$$\zeta(n) \equiv \sum_{w=1}^{\infty}\frac{1}{w^n}. \qquad (4.5)$$

Indeed, most of the results obtained using this method involve the ζ function as a sum over winding numbers.

We note that it was strictly for gauge fields that the authors of Ref. [9] found such a splitting pattern. In an upcoming paper [2], we shall show that such splittings also occur for other types of particles, even when there is no gauge symmetry. However, we find that these types of splittings occur only when the four-dimensional masses of our particles are nonzero (a case which was not considered in Ref. [9]).

It is important to note that the procedure introduced in Ref. [9] is *not*, strictly speaking, a regulator. Indeed, a regulator is a way of temporarily deforming a divergent expression to render it finite; such deformed expressions are then parametrized by a continuous deformation parameter (such as Λ or ϵ) which is removed at the end of the calculation. For example, let us assume that two expressions A and B are each separately divergent, but their difference is a physical quantity and therefore finite. Rather than separately evaluating A and B, we might instead evaluate A' and B', where A' and B' are regulated, finite expressions. We would then find that $A' - B'$ is either regulator independent, or tends to a finite value as the regulator is removed.

By contrast, the procedure introduced in Ref. [9] is simply a method of extracting a finite expression from a single, infinite diagram. In general, we have no assurance that this finite expression corresponds to any physical quantity unless the particular calculation we are doing happens to lead to this expectation for other reasons. For example, let $L_n|_R$ denote the value of a one-loop vacuum-polarization diagram with an external KK photon with mode number n, evaluated when our extra spacetime dimension has radius R, and let $L_n|_\infty$ denote the value of the corresponding vacuum-polarization diagram on an infinite extra dimension. (The subscript n in the uncompactified case indicates that the fifth component of our external photon momentum is still given by n/R, just as in the compactified case.) Let us also define $\tilde{L}_n|_R$ as that portion of $L_n|_R$ which renormalizes the mass (i.e., $\tilde{L}_n^{\mu\nu}|_R$ would represent the piece within $L_n^{\mu\nu}|_R$ which is proportional to the metric $g^{\mu\nu}$). Within such a setup, we can then write expressions such as $\tilde{L}_n|_R - \tilde{L}_0|_R$ in the form

$$\tilde{L}_n|_R - \tilde{L}_0|_R = (\tilde{L}_n|_R - \tilde{L}_n|_\infty) - (\tilde{L}_0|_R - \tilde{L}_0|_\infty) \qquad (4.6)$$

where we have taken $\tilde{L}_n|_\infty = \tilde{L}_0|_\infty$ (as occurs when appropriate renormalization conditions are applied, such as placing the external photons on shell in each case). Now, the residual four-dimensional gauge symmetry requires that $\tilde{L}_0|_R$ should vanish for all R (including $R \to \infty$), whereupon we conclude that the physical difference $\tilde{L}_n|_R - \tilde{L}_0|_R$ is actually finite and given by $\tilde{L}_n|_R - \tilde{L}_n|_\infty$. Indeed, it is for this reason that this technique is capable of evaluating radiative shifts to individual KK masses, even though it was designed only to yield differences between corrections to quantities in a compactified theory and an uncompactified one.[1]

[1]Note that in this specific example of KK-photon mass renormalization, the above results also imply that $\tilde{L}_n|_\infty = 0$ for all n. Of course, this can be easily understood as a result of *five-dimensional* gauge invariance. Thus, in this particular case, our original diagram $\tilde{L}_n|_R$ was already finite by itself, and indeed the subtracted term $\tilde{L}_n|_\infty$ vanishes. We have nevertheless chosen to present this somewhat "null" example because this is the original example given in Sec. II of Ref. [9]. In this context, we remark that although the result [9] quoted in our Eq. (4.4) is correct, it would be incorrect to make the further assumption that the $w = 0$ contribution follows the same functional form as the $w \neq 0$ contributions, diverging as $1/w$ with $w \to 0$. Indeed, as we have explained above, the $w = 0$ contribution actually vanishes by five-dimensional gauge invariance, and a direct calculation of the $w = 0$ contribution will yield an expression which is either identically zero, or occasionally indeterminate in the absence of a suitable regulator.

Even though the method of Ref. [9] is not, strictly speaking, a regulator, it is nevertheless possible to generalize this method slightly in order to make it a full-fledged regulator. For example, we could always write any (divergent) expression $L_n|_R$ in the form

$$L_n|_R = (L_n|_R - L_n|_\infty) + L_n|_\infty. \tag{4.7}$$

The first term would then clearly be finite, and the second term could be regularized using any of the standard higher-dimensional regulators that apply in an uncompactified space. Together, we would then have a bona-fide regulator prescription which could be universally applied for any expression $L_n|_R$. However, such a regulator would involve two separate methods, one for each of the terms in Eq. (4.7), and would thus be relatively awkward to employ in practical settings.

B. Comparisons with previous results

If our EHC and EDR regulators are valid, they must reproduce the results derived via the winding-number technique discussed above. In this section, we shall show that this is indeed the case.

We first consider the squared-mass shift described by Eq. (4.4). This shift is derived from the part of the vacuum-polarization diagram in Eqs. (3.18) and (3.19) which is proportional to $g^{\mu\nu}$. As above, we define $\tilde{L}^{\mu\nu}$ to be this part of the diagram. Let us first evaluate this expression following our EDR procedure. Utilizing our $\Lambda(\epsilon)$ relation in Eq. (3.27) and explicitly performing the sum over KK modes, we obtain

$$\tilde{L}^{\mu\nu} = -\frac{ie^2 g^{\mu\nu}}{4\pi^2 R^2}\lim_{\Lambda R\to\infty}\left\{\frac{4}{9}(\Lambda R)^3 - \frac{\Lambda R}{3} - \left[\frac{4}{3}(\Lambda R)^3 + 2(\Lambda R)^2 + \frac{2\Lambda R}{3}\right]\log(\Lambda R) + 2\sum_{r=1}^{\Lambda R} r^2\log(r^2)\right\}. \tag{4.8}$$

Therefore, our regulator will not reproduce the result in Eq. (4.4) unless

$$\lim_{\Lambda R\to\infty}\left\{\frac{4}{9}(\Lambda R)^3 - \frac{\Lambda R}{3} - \left[\frac{4}{3}(\Lambda R)^3 + 2(\Lambda R)^2 + \frac{2\Lambda R}{3}\right]\times\log(\Lambda R) + 2\sum_{r=1}^{\Lambda R} r^2\log(r^2)\right\} \stackrel{?}{=} \frac{\zeta(3)}{\pi^2}. \tag{4.9}$$

On the surface, such an identity would seem somewhat improbable, since the left side involves individual terms which are each manifestly divergent, while the right side is finite. Indeed, some of the terms on the left side of Eq. (4.9) are simple polynomials in ΛR, while the expression on the second line is a discrete sum in which ΛR appears as an upper limit.

Surprisingly, however, it is easy to verify numerically that Eq. (4.9) holds to any precision desired. Indeed, the expression on the left side of this identity experiences a remarkably fast convergence to $\zeta(3)/\pi^2$, already coming within 10% of this value for $\Lambda R = 1$, and coming within 1% for $\Lambda R = 9$. In fact, Eq. (4.9) is an entirely novel mathematical representation for the ζ function as the limit of an infinite summation. Equivalently, inverting this relation provides an analytical form for the infinite sum $\sum_r r^2\log(r^2)$, which can be useful in many contexts dealing with KK summations.

This, then, provides a highly nontrivial check of our EDR procedure. By demonstrating that EDR is consistent with the technique in Ref. [9], we once again verify that EDR indeed preserves both higher-dimensional Lorentz invariance and higher-dimensional gauge invariance, as promised. Although we have only shown a comparison for one particular diagram, it is straightforward to verify that similar cross-checks hold for other diagrams as well.

We can also verify that our EHC regulator is consistent with the method of Ref. [9]. However, in order to make such a comparison, we should examine a theory which exhibits higher-dimensional Lorentz invariance but not higher-dimensional gauge invariance.

For this purpose, let us examine a toy five-dimensional model consisting of a single scalar ϕ and a single fermion ψ compactified on a circle and experiencing a Yukawa interaction of the form $G\phi(\bar{\psi}\psi)$ where G is the five-dimensional Yukawa coupling. Indeed, this theory will be analyzed more extensively in Ref. [2]. Within this theory, let us examine the one-loop diagram which renormalizes the squared mass of a KK excitation of the scalar field with mode number n. This diagram is shown in Fig. 1, where we now take the external lines to represent KK modes of the scalar ϕ and the internal lines to represent KK modes of the fermion ψ. As before, we shall write $L_n|_R$ to denote the value of this diagram when our extra spacetime dimension has radius R, and we shall write $L_n|_\infty$ to denote the corresponding diagram on an infinite extra dimension. Note that in the latter case, despite the disappearance of discrete KK modes, the subscript n continues to be specified as a reminder that the fifth component of the external momentum in such a diagram should continue to carry the value n/R.

Because gauge invariance is not a symmetry of this theory, it will be sufficient to employ our EHC regulator in evaluating this diagram. Following the procedure outlined in Sec. II, we then obtain the expression

$$L_n|_R - L_n|_\infty = \frac{ig^2}{4\pi^2 R^2}\lim_{\Lambda R\to\infty}\left\{\frac{4}{9}(\Lambda R)^3 - \frac{\Lambda R}{3} - \left[\frac{4}{3}(\Lambda R)^3 + 2(\Lambda R)^2 + \frac{2\Lambda R}{3}\right]\log(\Lambda R) + 2\sum_{r=1}^{\Lambda R} r^2\log(r^2)\right\}. \tag{4.10}$$

The quantity Λ now represents our hard cutoff, which is the same for both diagrams, and $g \equiv G/\sqrt{2\pi R}$ represents the Yukawa coupling of each individual KK mode. Note that the above result holds for any value of n, including $n = 0$, and holds independently of whether the 5D scalar is real or complex (since the scalar does not run in the loop). By contrast, the regularization technique of Ref. [9] leads to the expression

$$L_n|_R - L_n|_\infty = \frac{ig^2}{4\pi^4 R^2}\zeta(3) = \frac{iG^2}{8\pi^5 R^3}\zeta(3). \quad (4.11)$$

However, once again, the identity in Eq. (4.9) ensures that these results are equivalent. Indeed, we see that Eq. (4.9) essentially serves as a mapping between the results derived using the methods of this paper and those derived using the methods of Ref. [9].

Although these UV regulators yield the same results for mass corrections, they nevertheless treat *infrared* (IR) divergences differently. Because there is no direct relationship between the IR divergence that results in a given diagram when an extra dimension is compactified and the IR divergence that results when the extra dimension is infinite, the regularization method of Ref. [9] does not eliminate IR divergences. Indeed, the discrete KK sum that results for a compactified extra dimension and the KK integral that would result in the case of an infinite dimension only become more dissimilar in the IR limit. Of course, the regulators in this paper also leave IR divergences intact. However, because the method of Ref. [9] requires that we pass from a KK momentum basis to a KK winding basis in order to eliminate the UV divergence, any IR divergence which remains is redistributed across all winding modes, particularly those with large winding numbers, and can no longer easily be isolated. By contrast, because our methods do not require any such reorganization, the IR divergences that remain in our method continue to be easily identified and treated.

As a concrete example of these ideas, let us consider the vacuum-polarization diagram $L_n^{\mu\nu}$ in the case in which the external KK photon of mode number n is on shell and the bare (five-dimensional) mass M of the fermion running in the loop is zero. Using our EDR procedure, we obtain the results in Eqs. (3.18) and (3.19). Although the integrands in Eq. (3.19) are finite for each nonzero r, the quantity W in Eq. (3.20) diverges for $n = r = 0$, i.e., for a zero-mode external photon with a zero-mode fermion running in the loop. This is the IR divergence, encapsulated entirely within the zero-mode contribution to the KK sum in Eq. (3.18). By contrast, if we were to use the methods of Ref. [9] to analyze the same vacuum-polarization diagram, we would obtain the result

$$L_0^{\mu\nu}|_R - L_0^{\mu\nu}|_\infty = \frac{ie^2}{4\pi^2}\frac{p^\mu p^\nu}{3}\sum_{w\neq 0}\frac{1}{|w|}. \quad (4.12)$$

In this case, the IR divergence is reflected in the divergence of the winding-number sum, and cannot be isolated to a particular term within Eq. (4.12).

Note that IR divergences can generally be regularized through the introduction of small masses. For example, the IR divergence discussed above is eliminated when the fermion is given a small, nonzero, four-dimensional mass or the external photon is slightly off shell. The introduction of such a mass is relatively straightforward to implement within the framework of the regulators in this paper. However, the introduction of such a mass within the framework of Ref. [9] might be significantly more complicated. Such an IR regulator would inevitably be redistributed across every contribution to the winding-number sum (rendering it finite), but such a sum is not likely to have a simple mathematical form. Alternatively, one could imagine regulating a sum such as that in Eq. (4.12) directly (e.g., by inserting a small Boltzmann-like suppression factor), but such an insertion is likely to break higher-dimensional Lorentz invariance or gauge invariance. Moreover, it is not clear that transforming such a factor back to the KK momentum basis would provide it with any clear physical interpretation.

We have seen, then, that the regulators we have proposed in this paper are able to reproduce the corresponding results of Ref. [9] when appropriate. However, to be truly useful, our techniques also must apply in situations where other methods do not. Since the technique in Ref. [9] operates strictly in the winding-number basis, it loses information about contributions to radiative corrections from different physical momentum scales. This poses no problem in calculations of radiative corrections to physical parameters (e.g., masses and couplings) which would be observed in experiments. However, it is not possible to calculate Wilsonian renormalization-group evolutions of such parameters in this scheme. If extra dimensions are discovered at a future collider, it may be desirable to define EFT's for KK modes below the center-of-mass (CM) energy. Calculating the parameters in such a theory would require the use of the Wilsonian renormalization group, with the corresponding evolution of parameters running from the UV to the CM energy. As we shall see in Refs. [1,2], our regulators can handle such calculations explicitly. Indeed, this was one of our original motivations for developing the new regulators in this paper.

C. The necessity of preserving higher-dimensional Lorentz invariance

In this section, we illustrate the pathologies which appear when using regulators that break higher-dimensional Lorentz invariance. As a concrete example, we shall again consider our toy five-dimensional model consisting of a single scalar ϕ and a single fermion ψ compactified on a circle and experiencing a Yukawa interaction of the form $G\phi(\bar{\psi}\psi)$ where G is our five-dimensional Yukawa coupling. Within this theory, we shall attempt to calculate the radiative corrections to the KK masses of the scalar using a regulator which preserves four-dimensional Lorentz invariance but breaks five-dimensional Lorentz invariance.

Once again, we shall do this by calculating the difference between a loop diagram which renormalizes the squared mass of a scalar mode in Yukawa theory and the corresponding diagram for the zero mode. We define $L_n(p)$ to be the squared-mass renormalization diagram for a scalar with mode number n and four-momentum p (shown in Fig. 1). For simplicity, we take the zero-mode masses m_ψ and m_ϕ of these two fields to vanish. We then find

$$L_n = 4ig^2 \int_0^1 dx \sum_r \int \frac{d^4\ell_E}{(2\pi)^4} \times \left[\frac{\ell_E^2 + r(r-n)/R^2 + x(1-x)n^2/R^2}{(\ell_E^2 + (r-xn)^2/R^2)^2}\right] \quad (4.13)$$

where $g \equiv G/\sqrt{2\pi R}$. Note that we now write $(r-xn)/R$ rather than ℓ^4 because we are no longer treating this quantity as the fifth component of a five-vector.

The expression in Eq. (4.13) is badly divergent, and must be regularized. Let us therefore place a 4D cutoff Λ on ℓ_E and truncate the KK sum at this cutoff. In other words, we shall take our integration limits to be given by $\ell_E^2 \leq \Lambda^2$ and our summation limits to be given by $-\Lambda R \leq r \leq \Lambda R$. Note that these constraints break higher-dimensional Lorentz invariance, since they separately regularize four-momentum integrals and KK sums. Nevertheless, imposing these constraints, we find that

$$\begin{aligned} L_n - L_0 = -\frac{ig^2}{4\pi^2 R^2} \sum_{r=-\Lambda R}^{\Lambda R} \int_0^1 dx \bigg\{ &(-2x^2 + x)n^2 \\ &+ \frac{(r-xn)^4 + (r-xn)^2[x(x-1)n^2 - r(r-n)]}{\Lambda^2 R^2 + (r-xn)^2} \\ &+ [2(r-xn)^2 + x(x-1)n^2 - r(r-n)] \\ &\times [\log(\Lambda^2 R^2 + (r-xn)^2) - 2\log(r-xn)] \\ &- r^2[\log(\Lambda^2 R^2 + r^2) - 2\log r]\bigg\}. \end{aligned} \quad (4.14)$$

Clearly, this expression diverges linearly with Λ. This is a problem, since this quantity corresponds to the difference between squared masses, which should be finite.

The reason this divergence appears is that the loop diagrams in this equation do not determine renormalized masses by themselves. Rather, each KK mode should have a counterterm for its squared mass, and a calculation of a squared-mass difference must include these counterterms. Such terms would indeed cancel artificial violations of Lorentz invariance. However, they also would break the KK dispersion relation for the underlying theory, since they are part of the bare Lagrangian.

This situation has an analogue in four-dimensional QED. If we use a hard cutoff to regularize divergences in that theory, we then generate a photon mass which is proportional to the cutoff. As well as being divergent, such a mass term violates gauge symmetry. However, as is well known (see, e.g., Ref. [4]), this problem can be remedied by introducing counterterms which break gauge invariance and cancel the unphysical effects from loop diagrams. However, our bare Lagrangian is then no longer gauge invariant.

In 5D Yukawa theory, the relevant symmetry is higher-dimensional Lorentz invariance. In the spirit of QED, it may therefore appear straightforward to introduce counterterms to cancel regulator-induced violations of 5D Lorentz invariance. However, the compactification of an extra dimension also breaks higher-dimensional Lorentz invariance at finite scales. This violation can manifest itself in an EFT as a violation of the usual 5D dispersion relation, as in the case of Eq. (4.4). Therefore, counterterms would not only have to cancel unphysical violations induced by our regularization scheme, but nevertheless preserve the bona-fide effects induced by the compactification itself. Without *a priori* knowledge of what the results should be, it would be quite difficult to determine which effects would be physical and which would not. Indeed, it would be difficult to deduce the form of appropriate counterterms if we limit ourselves to this sort of regulator. Such a regulator, therefore, does not lend itself to a straightforward calculation involving the relative renormalizations of the parameters describing a KK spectrum.

As required, the regulators developed in this paper yield finite loop-diagram differences and thus avoid this problem. We therefore did not need to introduce counterterms, since the squared masses of KK states—which are renormalized by our loop diagrams—all carry the same divergence at tree level. Indeed, the dimensionless squared masses are given by the relation $m_n^2 R^2 = m_0^2 R^2 + n^2$ at tree level, and only the $m_0^2 R^2$-term diverges in the UV. Hence, only one counterterm is needed for the entire mass spectrum of KK states, and the effects of such a counterterm cancel when calculating *differences* between squared masses. Similar results hold for other types of loop diagrams. It is for this reason that our techniques can produce regulator-independent EFT's. These issues will be discussed in more detail in Ref. [1].

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we proposed two new regulators (EHC and EDR) for quantum field theories in spacetimes with compactified extra dimensions. Although they are based on traditional four-dimensional regulators, the key new feature of these higher-dimensional regulators is that they are specifically designed to handle mixed spacetimes in which some dimensions are infinitely large and others are compactified. Moreover, unlike most other regulators which have been used in the extra-dimension literature, these regulators are designed to respect the original higher-dimensional Lorentz and gauge symmetries that exist prior to compactification, and not merely the four-dimensional symmetries which remain afterward.

As we have discussed, these regulators should be particularly useful for calculations of the physics of the excited Kaluza-Klein modes in any higher-dimensional theory, and not merely the radiative effects that these excited KK modes induce on zero modes. Indeed, by respecting the full higher-dimensional symmetries, our regulators avoid the introduction of spurious terms which would not have been easy to disentangle from the physical effects of compactification.

Moreover, as part of our work, we also derived a number of ancillary results. For example, in gauge-invariant theories, we demonstrated that analogues of the Ward-Takahashi identity hold not only for the usual zero-mode (four-dimensional) photons, but for all excited Kaluza-Klein photons as well.

Clearly, the analysis we have done in this paper only begins to scratch the surface of what is possible. For example, this analysis has been restricted to five dimensions and, in many places, to one-loop amplitudes. While this clearly covers the most pressing situation that might emerge if extra dimensions are ultimately discovered, it would be interesting to extend our discussion to multiloop amplitudes (where appropriate) and to even higher dimensions. In particular, both of these extensions would involve additional KK sums which would require their own cutoffs, and thus there will be additional balancing constraints that must be imposed between these cutoffs and the regulator of the four-dimensional momentum integral in order to preserve higher-dimensional Lorentz and gauge symmetries.

Other sorts of extensions are also possible. For example, in more than five dimensions, we can consider compactifications not just on flat spaces (such as we have considered here), but also spaces with their own intrinsic curvatures or warpings. Moreover, even for flat compactification manifolds, there remains the possibility of having nontrivial *shape* moduli [10]. All of these possibilities represent different types of mixed spacetimes which would have unusual KK spectra and which would, in principle, require their own analysis.

There are also other important geometric extensions to consider, even in five dimensions. For example, although the analysis of this paper has been restricted to compactification on a smooth manifold, it is important to extend these results to orbifolded spacetimes which contain boundaries (i.e., branes, or orbifold fixed points). Indeed, compactification on such orbifolded geometries is ultimately required in order to obtain a chiral theory in four dimensions. In such theories, some processes are purely four dimensional (occurring on the branes) while others are five dimensional and still others are mixed. Although the existence of brane-kinetic terms [11] can have a profound effect on the physics on the brane, we nevertheless expect our higher-dimensional Ward identities to be preserved in the bulk. Regulators such as those we have developed here should therefore continue to have application for the bulk physics in such situations. This will be discussed in more detail in Ref. [2].

Even within the framework of compactification of a single extra dimension on a circle, there remain important extensions of our work which we have not considered. For example, we have primarily focused on Abelian gauge theories and their associated Ward identities, but we have not considered their non-Abelian extensions. This will be important for ultimately calculating radiative corrections within, say, a higher-dimensional standard model. Likewise, in this paper we often considered five-dimensional QED. Although this theory is nonrenormalizable, we restricted our attention to the usual electron/photon coupling and did not allow additional nonrenormalizable interactions. Even though such interactions should continue to respect our higher-dimensional Lorentz and gauge symmetries (therefore requiring the use of a regulator such as we have developed here), the existence of such interactions can be expected to lead to complications beyond those considered in this paper.

Finally, it should be stressed that this work focused on only one rather narrow type of regulator, namely, one in which our KK sums were regulated through a hard cutoff Λ. However, other types of regulators are possible. For example, an infinite KK sum might alternatively be regulated through the introduction of Boltzmann-like suppression factors, e.g.,

$$\sum_r \frac{1}{r} \longrightarrow \sum_r \frac{1}{r} e^{-y|r|} \tag{5.1}$$

where $y > 0$ is a regulator parameter. One would then take $y \to 0$ at the end of the calculation, while simultaneously maintaining a certain relation between y and ϵ (analogous to our EDR relation between Λ and ϵ) so that five-dimensional Lorentz invariance and gauge invariance are maintained. However, it is not clear what physical interpretation might be ascribed to such a regulator parameter y. Similarly, we again mention the possibility of preserving gauge invariance even with a hard cutoff, but with suitable counterterms as well. However, such counterterms will necessarily break the original higher-dimensional symmetries of our bare Lagrangian.

Another approach, first advanced in Ref. [12], is to rewrite the KK sum as a contour integral in which the different terms of the sum emerge from the poles of the integrand. One can then apply a regularization procedure akin to 't Hooft-Veltman dimensional regularization to the integral [12,13]. However, this still results in two independent regulators, one for the KK integral and another for the four-momentum integral, and five-dimensional symmetries will generally not be protected unless these two regulators are balanced in a manner similar to what we have outlined in this paper.

There are, of course, other potential methods of deforming our KK summations. For example, we might Poisson-resum our KK summation, and attempt to apply one of the above regulators to the Poisson-resummed version instead.

Note that Poisson resummation of the KK sum was originally introduced into the large extra-dimension context in Ref. [14]. There are also other techniques which might be employed, such as proper-time regulators, zeta-function regularization, etc. Indeed, these methods ultimately play various roles in the different approaches sketched here. Other approaches towards treating the KK summation based on dimensional regularization have also been utilized in various calculations [15].

Another possibility might be to employ a so-called "mixed propagator" formalism [16]. In such a formalism, the four large dimensions are treated in momentum space, as usual, while the compactified extra dimension is treated in position space. This avoids the introduction of a KK sum altogether. However, in such situations the higher-dimensional divergences are not eliminated—they are the same as would appear in the corresponding higher-dimensional *uncompactified* theory, as this formalism makes abundantly clear. This formalism thus lends itself naturally to the treatment given in Ref. [9].

Of course, it is possible that the true UV limit of a given higher-dimensional theory is not higher dimensional at all [17]. Such "deconstructed" extra dimensions would change the UV divergence structure of the theory in a profound way that would eliminate the need for many of these different regularization techniques. Indeed, deconstruction can also be used as an alternative technique for performing many of the sorts of radiative calculations for excited KK modes that have been our focus in this paper [18]. Similarly, radiative corrections may be finite in cases in which there exist additional symmetries (either unbroken or softly broken) to protect against divergences. A well-known example of this would include radiative corrections in theories with supersymmetry broken through the Scherk-Schwarz mechanism [19] (leading to so-called "KK regularization," in which the full KK summations lead to finite results), or in theories in which the Higgs is identified as a component of a higher-dimensional gauge field and consequently has a mass for which radiative corrections are protected by gauge symmetries [20].

Likewise, such higher-dimensional theories may ultimately be embedded into string theory. String theory provides entirely new methods of eliminating divergences which transcend what is possible in quantum field theories based on point particles. Indeed, there even exist several string-inspired methods of regularizing field theories directly [21–23].

Another possibility is to retain the full higher-dimensional space but take a nonperturbative approach towards extracting exact solutions for the excited KK masses and couplings. Ideas in this direction have been advanced, e.g., in Ref. [24].

In this connection, it might seem strange that we have not discussed the PV regulator. Indeed, such a regulator preserves both Lorentz invariance and gauge invariance, even in higher dimensions, and may be more than sufficient for certain calculations (see, e.g., Refs. [25,26]). However, there are several reasons why such a regulator may not ultimately be suitable for general calculations in mixed spacetimes, especially those focusing on the radiative corrections to the properties of the excited KK modes. First, the PV regulator does not preserve non-Abelian gauge symmetries, even in four dimensions. Second, even for the Abelian theories which have been our main focus in this paper, compactification introduces a major algebraic problem: the PV regulator parameter Λ becomes inextricably entangled in our KK mode-number sum except in particular situations (see, e.g., Ref. [26]) in which the radiative corrections are already known to be finite. Thus, this regulator is particularly unsuited for the mixed spacetimes which have been our main focus in this paper. Of course, it might seem that such a PV regulator might nevertheless be suitable for numerical studies which do not require closed-form analytical expressions. However, even this is not possible, because there is a third complication: *unitarity* is not preserved using a PV regulator unless the regulator parameter Λ is sent to infinity. Thus, it is likely to be difficult to treat such a system numerically with any confidence when our PV regulator is in force.

By contrast, the regulators we have developed in this paper are designed to be relatively straightforward, intuitive, and easy to use for practical calculations. Indeed, as mentioned at the end of the Introduction, this paper is only the first in a two-part series. In the following article [1], we shall discuss how these new regulators may be employed in order to derive regulator-independent effective field theories at different energy scales. We shall also discuss how these regulator techniques can be used to extract finite results for physical observables that relate the physics of excited KK modes to the physics of KK zero modes. Moreover, in a third paper [2], we shall study the manner in which the KK masses and couplings in various higher-dimensional effective field theories evolve as functions of energy scale, and as extra spacetime dimensions are slowly integrated out in passing from the UV to the IR. In particular, in Ref. [2], we shall study how the well-known tree-level relations amongst the tower of KK masses and amongst their couplings are "deformed" when radiative effects are included. In each case, we shall see that it is the regulators we have developed here which will enable these calculations to be performed.

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[1] S. Bauman and K. R. Dienes, Phys. Rev. D **77**, 125006 (2008).
[2] S. Bauman and K. R. Dienes (to be published).
[3] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. **B44**, 189 (1972).
[4] See, e.g., M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).
[5] See, e.g., H. Cheng and T. T. Wu, Phys. Rev. **182**, 1873 (1969); H. Cheng, E. C. Tsai, and X. q. Zhu, Phys. Rev. D **26**, 908 (1982); **26**, 922 (1982).
[6] K. R. Dienes, E. Dudas, and T. Gherghetta, Phys. Lett. B **436**, 55 (1998); Nucl. Phys. **B537**, 47 (1999); arXiv:hep-ph/9807522.
[7] T. Appelquist and B. A. Dobrescu, Phys. Lett. B **516**, 85 (2001).
[8] D. M. Ghilencea, Phys. Rev. D **70**, 045011 (2004); **70**, 045018 (2004).
[9] H. C. Cheng, K. T. Matchev, and M. Schmaltz, Phys. Rev. D **66**, 036005 (2002).
[10] K. R. Dienes, Phys. Rev. Lett. **88**, 011601 (2001); K. R. Dienes and A. Mafi, Phys. Rev. Lett. **88**, 111602 (2002); **89**, 171602 (2002).
[11] H. Georgi, A. K. Grant, and G. Hailu, Phys. Lett. B **506**, 207 (2001).
[12] S. Groot Nibbelink, Nucl. Phys. **B619**, 373 (2001).
[13] R. Contino and A. Gambassi, J. Math. Phys. (N.Y.) **44**, 570 (2003).
[14] I. Antoniadis, Phys. Lett. B **246**, 377 (1990).
[15] D. M. Ghilencea, J. High Energy Phys. 03 (2005) 009; D. M. Ghilencea, D. Hoover, C. P. Burgess, and F. Quevedo, J. High Energy Phys. 09 (2005) 050; D. M. Ghilencea, H. M. Lee, and K. Schmidt-Hoberg, J. High Energy Phys. 08 (2006) 009.
[16] M. Puchwein and Z. Kunszt, Ann. Phys. (N.Y.) **311**, 288 (2004).
[17] N. Arkani-Hamed, A. G. Cohen, and H. Georgi, Phys. Rev. Lett. **86**, 4757 (2001).
[18] See, e.g., A. Falkowski, C. Grojean, and S. Pokorski, Phys. Lett. B **581**, 236 (2004).
[19] J. Scherk and J. H. Schwarz, Phys. Lett. **82B**, 60 (1979); Nucl. Phys. **B153**, 61 (1979).
[20] Y. Hosotani, Phys. Lett. **126B**, 309 (1983).
[21] K. R. Dienes, Nucl. Phys. **B429**, 533 (1994); K. R. Dienes, M. Moshe, and R. C. Myers, Phys. Rev. Lett. **74**, 4767 (1995); K. R. Dienes, Nucl. Phys. **B611**, 146 (2001).
[22] D. M. Ghilencea, H. P. Nilles, and S. Stieberger, New J. Phys. **4**, 15 (2002).
[23] D. M. Ghilencea and S. Groot Nibbelink, Nucl. Phys. **B641**, 35 (2002).
[24] Y. Ohtake and Y. Wakabayashi, Prog. Theor. Phys. **115**, 611 (2006).
[25] K. Agashe, A. Delgado, and R. Sundrum, Nucl. Phys. **B643**, 172 (2002).
[26] R. Contino and L. Pilo, Phys. Lett. B **523**, 347 (2001).