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Generating generalized G_{D-2} solutions

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We show how one can systematically construct vacuum solutions to Einstein field equations with $D-2$ commuting Killing vectors in $D > 4$ dimensions. The construction uses Einstein-scalar field seed solutions in four dimensions and is performed both for the case when all the Killing directions are spacelike, as well as when one of the Killing vectors is timelike. The later case corresponds to generalizations of stationary axially symmetric solutions to higher dimensions. Some examples representing generalizations of known higher dimensional stationary solutions are discussed in terms of their rod structure and horizon locations and deformations.

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I. INTRODUCTION

There has been a renewed interest in higher dimensional solutions to Einstein field equations. Several interesting vacuum static/stationary solutions [1], the methods of their generations [2] and study [3], as well as some general results on uniqueness of higher dimensional static vacuum black holes [4–6] in five dimensions have appeared recently. In the case of time-dependent geometries, the main interest is to study higher dimensional spacetimes as backgrounds for string propagation [7]. In cosmology [8], the new trends impose "lifting" the cosmological models to higher dimensions. Another, and probably the most relevant reason to study the higher dimensional generalizations of the Einstein equations, stems from the fact that we live in a four-dimensional world. It would be important then, if we were able to convince ourselves, by studying the higher dimensional solutions, that there is something special, unique, and deep about four dimensions. This could only be done if we study the alternatives.

Much work has been done previously [9], yet, since the interests move with time, the motivation, and with it the boundary/initial conditions imposed on the solutions change as well. In solving Einstein equations, one usually imposes some sort of symmetry. What we are good at is the situation where the spacetime does not depend on more than two, preferably non-null variables. In four dimensions, these are known as the so-called G_2 solutions. Almost all known interesting four-dimensional solutions of Einstein equations in vacuum, electrovacuum, or with some fundamental matter fields belong to this class and contain at least two commuting Killing directions. Once we are confronted with the situation where the line element *does* depends on at most two coordinates, we are in "business": whether these are stationary solutions with axial symmetry, boost-symmetric spacetimes, anisotropic (isotropic) and inhomogeneous (homogeneous) cosmologies, cylindrical, or plane gravity and matter waves—there are dozens of generating techniques, algorithms etc. to construct new solutions of ever increasing complexity [10] starting from more simple seeds. Because of their generality, on one hand, and applicability, on the other, the known G_2 solutions provide a perfect "seed" or a building block to construct further new solutions. The construction of families of higher dimensional solutions is not an exception: one can efficiently use the known G_2 solutions as the seeds in order to construct their higher dimensional generalizations and analogues. As usual, the physics enters via the boundary and initial conditions (asymptotic flatness, clean horizons, and singularities in the case of compact objects; singularity structure, inflation, late/early-time acceleration-for cosmological models etc.). Obtaining sufficiently general solutions, of course, does not necessarily mean alleviating the search for specialty and physical meaningfulness, nevertheless, the understanding of the sufficiently general class of solutions may shed some light on a problem.

The above considerations lead us to the main purpose of this paper: the construction, in a possibly most simple and controllable way of higher dimensional vacuum solutions to Einstein equations which depend, at most, on two variables. Our starting point would be the four-dimensional vacuum seed metrics with the G_2 symmetry. The vacuum seed spacetime will be generalized to include massless dilatons, which will serve to lift the solutions to higher dimensions. To include the scalar field, the vacuum line element would be written in the coordinates especially adopted for this matter. It happens so, that in these coordinates the scalar dilaton coupled to gravity in the G_2 case satisfies a linear differential equation. This equation can be easily solved with the general solution representing a linear combination of some elementary solutions. Any term, as well as, any linear combination of these solutions, may serve to lift the dilaton spacetime to higher dimensions, obtaining in such a way a new vacuum solution in any dimension greater than four. To get interesting solutions, one must choose the "right" G_2 seed as well as the right combination of elementary scalar solutions (dilatons). The scheme works both, in the case when the two commuting Killing vectors of the seed are spacelike (cosmologies, cylindrical waves, colliding waves etc.) as well as, when one of the Killing vectors is timelike. It is important to mention that there is no need in imposing the hypersurface orthogonality of the two Killing directions (diagonality of the metric) and therefore, the solutions obtained this way generalize, in the case when one of the Killing directions is timelike, for example, the static higher dimensional solutions due to [1], enabling one to obtain the generalized stationary solutions, in fact, an infinite dimensional family of such solutions.

In the following section we briefly review the generating technique for the case where the two Killing directions of the seed solution are both spacelike. In Sec. III the case with one timelike Killing vector is addressed. In Sec. IV the method is specialized to the 5D case; aspects such as asymptotic flatness and the interpretation of the method as the insertion of rod sources are as well included in this section. In Sec. V, we analyze the trapping of surfaces, horizons, and singularities of the generated solutions. In Secs. VI and VII some 5D static and stationary solutions are generated. Finally, some conclusions are drawn in the last section.

II. ALL-SPACELIKE KILLING VECTORS

The case and the algorithm where the two Killing directions of the seed solution are both spacelike was first presented and discussed in the context of string/M-theory cosmology [11]. It is worthwhile, however, to briefly review it here since actually passing to the situation when one of the Killing vectors is timelike can be formally made by complex coordinate changes which will be given subsequently.

The starting point is the *vacuum solution* of the Einstein field equations in the form

$$ds^2 = e^{f^{\mathrm{vac}}}(-dt^2 + dz^2) + \gamma_{ab}dx^a dx^b. \qquad (1)$$

Here f^{vac} and γ_{ab} are functions of the t and z coordinates alone, $(x, y) \equiv (x^2, x^3)$ and we denote $\sqrt{\det\gamma} \equiv K(t, z)$.

We also assume that the scalar field is normalized as in [11] and $\Phi \equiv \sum_{i=1}^{N} \varphi_i$ expressed as a sum of elementary scalar fields solves the following linear differential equation

$$\frac{\partial}{\partial t}[K(t, z)\dot{\Phi}(t, z)] - \frac{\partial}{\partial z}[K(t, z)\Phi'(t, z)] = 0. \qquad (2)$$

Then, the solution to the coupled Einstein-scalar field equations is obtained [12] by keeping the transverse part characterized by the metric functions $K(t, z)$ and γ_{ab} without being changed, but replacing the longitudinal function $f(t, z)^{\mathrm{vac}}$ by

$$f(t, z)^{\mathrm{vac}} \longrightarrow f(t, z)^{\mathrm{vac}} + f(t, z)^{\mathrm{sc}}. \qquad (3)$$

The function $f(t, z)^{\mathrm{sc}}$, then, is solved by quadratures from

$$\dot{f}(t, z)^{\mathrm{sc}} = \frac{K}{K'^2 - \dot{K}^2}\Big[2K'\sum_{i=1}^{N}\dot{\varphi}_i\varphi_i' - \dot{K}\Big(\sum_{i=1}^{N}\dot{\varphi}_i^2 + \sum_{i=1}^{N}\varphi_i'^2\Big)\Big], \qquad (4)$$

$$f'(t, z)^{\mathrm{sc}} = \frac{K}{K'^2 - \dot{K}^2}\Big[K'\Big(\sum_{i=1}^{N}\dot{\varphi}_i^2 + \sum_{i=1}^{N}\varphi_i'^2\Big) - 2\dot{K}\sum_{i=1}^{N}\dot{\varphi}_i\varphi_i'\Big]. \qquad (5)$$

To lift the solution to higher dimensions and to obtain the vacuum spacetime, we first construct the new scalars [11] $\psi_i = \mathcal{D}_{ij}\varphi_j$, where $\mathcal{D}_{ij} \in GL(N, \mathbf{R})$ is given by

$$\mathcal{D} = \begin{pmatrix} \mu_1^{-(1/2)} & \mu_2^{-(1/2)} & \mu_3^{-(1/2)} & \dots & \mu_{N-1}^{-(1/2)} & \mu_N^{-(1/2)} \\ -\mu_1^{-(1/2)} & \mu_2^{-(1/2)} & \mu_3^{-(1/2)} & \dots & \mu_{N-1}^{-(1/2)} & \mu_N^{-(1/2)} \\ 0 & -2\mu_2^{-(1/2)} & \mu_3^{-(1/2)} & \dots & \mu_{N-1}^{-(1/2)} & \mu_N^{-(1/2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu_{N-1}^{-(1/2)} & \mu_N^{-(1/2)} \\ 0 & 0 & 0 & \dots & -(N-1)\mu_{N-1}^{-(1/2)} & \mu_N^{-(1/2)} \end{pmatrix}, \qquad (6)$$

along with

$$\mu_n = \frac{2}{3}n(n + 1), \qquad (7)$$

$$\mu_N = \frac{1}{3}N(N + 2), \qquad (8)$$

where $n = 1, \dots, N - 1$.

Finally, the N-dimensional vacuum solution is given by

$$ds_{4+N}^2 = e^{-(2/\sqrt{3})\sum_{i=1}^{N}\psi_i}ds_4^2 + \sum_{i=1}^{N} e^{(4/\sqrt{3})\psi_i}(dw^i)^2, \qquad (9)$$

where ds_4^2 is the four-dimensional scalar field solution constructed previously. The new scalars ψ need to be constructed only when one is seeking a solutions in more than five dimensions. In the five-dimensional case, the lifting from four dimensions involves directly the field Φ and is straightforward.

The case of all-spacelike Killing vectors is phenomenologically rich. Depending on the behavior of the function $K(t, z)$, the so-called transitivity surface area, one encounters distinct physical situations depending on the character of the gradient of K. This function, in a vacuum, electrovacuum or massless scalar case, due to the vanishing of the two-trace in t, z of the stress tensor, is necessarily a solution of the wave equation of the form

$$\ddot{K}(t, z) - K''(t, z) = 0. \tag{10}$$

The solutions $K = t$, $K = \sinh t \sinh z$, or $K = \sin t \sin z$ are often used in the studies of anisotropic and inhomogeneous cosmologies and in colliding wave solutions. The case $K = z \equiv \rho$ corresponds to Einstein-Rosen cylindrical waves etc. For all these cases, the general solutions of the Klein-Gordon Eq. (2) are known and well understood. For example, the different modes of the solution for the case $K = t$ can be written as

$$\begin{aligned}\Phi = {}& \beta \log t + \mathcal{L}\{A_\omega \cos[\omega(z + z_0)]J_0(\omega t)\} \\ & + \mathcal{L}\{B_\omega \cos[\omega(z + z_0)]N_0(\omega t)\} \\ & + \sum_i d_i \mathrm{arc}\cosh\left(\frac{z + z_i}{t}\right),\end{aligned}$$

where $\mathcal{L}$ indicates linear combinations of the terms in curly brackets, ω can have a discrete or continuous spectrum, and β, A_ω, B_ω, d_i are constants. The arc cosh terms are somewhat special in the sense that these can not be written as Fourier-Bessel integrals [13] and are often referred to as gravitational solitons [14] due to the relation to the inverse scattering technique where these terms usually pop up. In a more general case, when the gradient of K may vary from point to point and we are interested in either cosmologies with S^3 topology of spatial sections, as it happens in the case, for example, of Bianchi IX models, Gowdy models, or in a colliding wave problem, the function K may be taken as $K \sim \sin t \sin z$, and the general solution of Eq. (2) can be expanded in Legendre polynomials of the first and second kind,

$$\begin{aligned}\Phi = {}& \alpha_1 \log\left|\tan\frac{t}{2}\right| + \alpha_2 \log\left|\tan\frac{z}{2}\right| + \alpha_3 \log|\sin t \sin z| \\ & + \sum_{\ell=0}^{\infty}[A_\ell P_\ell(\cos t) + B_\ell Q_\ell(\cos t)] \\ & \times [C_\ell P_\ell(\cos t) + D_\ell Q_\ell(\cos t)],\end{aligned}$$

where α_i, A_ℓ, B_ℓ, C_ℓ, D_ℓ are constants.

In cosmology one is often interested in the solutions for which the gradient of the transitivity surface K is globally timelike because most of the homogeneous models have this property. In this case one may choose $K = t$ as a solution of Eq. (10). The contribution of the scalar field to the function f becomes then

$$\dot{f}(t, z)^{\mathrm{sc}} = t\left(\sum_{i=1}^{N} \dot{\varphi}_i^2 + \sum_{i=1}^{N} \varphi_i'^2\right), \tag{11}$$

$$f'(t, z)^{\mathrm{sc}} = 2t \sum_{i=1}^{N} \dot{\varphi}_i \varphi_i'. \tag{12}$$

As a seed, let us take, for example, the vacuum Kasner solution for which one has

$$ds^2 = e^{f(t,z)}(-dt^2 + dz^2) + t(e^{p(t,z)}dx^2 + e^{-p(t,z)}dy^2), \tag{13}$$

with

$$p = k \log t, \qquad f = \frac{k^2 - 1}{2} \log t. \tag{14}$$

The scalar field equation becomes now $\ddot{\Phi} + \dot{\Phi}/t - \Phi'' = 0$, and one may take the simplest homogeneous solution as

$$\Phi = a \log t. \tag{15}$$

Choosing the inhomogeneous scalar field solution would have lead to an inhomogeneous scalar field generalization of the Kasner (Bianchi I) model, and when lifted to higher dimensions would have produced inhomogeneous vacuum solutions in higher dimensions.

The corresponding function $f = f^{\mathrm{vac}} + f^{\mathrm{sc}}$ becomes

$$f = \frac{k^2 + a^2 - 1}{2} \log t. \tag{16}$$

The vacuum 5D solution is then easily obtained using the expressions above, and has the following synchronous form:

$$ds^2 = -dt^2 + t^A dz^2 + t^B dx^2 + t^C dy^2 + t^D dw^2, \tag{17}$$

where

$$\begin{aligned}A &= \frac{6(k^2 + a^2 - 1) - 8\sqrt{3}a}{3k^2 + 3a^2 + 9 - 4\sqrt{3}a}, \\ B &= \frac{12(k + 1 - 2\sqrt{3}a/3)}{3k^2 + 3a^2 + 9 - 4\sqrt{3}a}, \\ C &= \frac{12(1 - k - 2\sqrt{3}a/3)}{3k^2 + 3a^2 + 9 - 4\sqrt{3}a}, \\ D &= \frac{16\sqrt{3}a}{3k^2 + 3a^2 + 9 - 4\sqrt{3}a}.\end{aligned} \tag{18}$$

III. ONE TIMELIKE KILLING DIRECTION

We now turn to the case where one of the Killing vectors is timelike. This situation corresponds to the stationary axially symmetric spacetimes. The procedure to build higher dimensional solutions is similar to the previously discussed one but with some minor sign changes, once an adapted coordinate system is used.

Surprisingly, little is known on scalar field generalizations of axially symmetric spacetimes. Basically, this is due to the fact that scalar field stationary axisymmetric solutions are probably not that exciting. Unlike in cosmology, where the scalar fields play central role in inflation, dark matter, and dark energy models, the interest in scalar field generalizations of axially symmetric solutions of the Einstein field equations is rather scarce. One of the reasons as to why the scalar fields in axisymmetric spacetimes are of little interest is because these do not admit, apart from some very special cases, a perfect fluid description as in cosmological case, where the scalar field serves as velocity potential for the fluid. Moreover, the various no-hair theorems [15] exclude scalar field black holes in four dimensions. Several specific solutions, however, in the spherical case, Kerr-type generalizations [16], and the case with conformally coupled scalar fields [17] are known. A specific algorithm which converts a gravitational degree of freedom into a scalar field is also known [18], but is less adapted for the purposes of this paper.

Our starting point, this time, is the following line element [19]:

$$ds^2 = e^{\sigma^{\rm vac}}(dr^2 + dz^2) + \gamma_{ab}dx^a dx^b, \tag{19}$$

which we take to be a *solution* of the vacuum Einstein equations in four dimensions. The functions $\sigma^{\rm vac}$ and γ_{ab} are now functions of z and r alone and $(\phi, t) \equiv (x^3, x^0)$. In this case, as long as the determinant of γ is not a constant, we may take without any lost of generality [19],

$$\det\gamma = -r^2. \tag{20}$$

We now assume, as in the previous section, that the scalar field $\Phi \equiv \sum_{i=1}^N \varphi_i$ is a solution of the following equation:

$$\Phi_{rr} + \frac{1}{r}\Phi_r + \Phi_{zz} = 0. \tag{21}$$

This equation is easily obtained from the Klein-Gordon Eq. (2) of the previous section with the following change of coordinates $t \rightarrow r$, $z \rightarrow iz$ and is a consequence of the formal relationship between the G_2—eneralized Einstein-Rosen class, and the G_2—stationary axially symmetric class metrics. If the vacuum solution is globally diagonalizable (the static Weyl case) then the "Weyl potential" U which appears as a metric function $e^U dt^2$ (see Appendix A), solves exactly the same Eq. (21) as Φ. The general solution to the linear Eq. (21) is obtained by considering the following integral:

$$\Phi = \int_{\alpha=-\infty}^{\infty}\int_{\beta=0}^{2\pi}\frac{F(\alpha)d\alpha d\beta}{\sqrt{r^2 + (z-\alpha)^2 + G^2(\alpha) - 2G^2(\alpha)\cos\beta}}. \tag{22}$$

It is convenient, nevertheless, in a way analogous to the solutions of Eq. (2), to express the solution of the Eq. (21) as a superposition of the following terms:

$$\Phi = \beta\log r + \mathcal{L}\{A_\omega\cosh[\omega(z+z_0)]J_0(\omega r)\} + \mathcal{L}\{B_\omega\cosh[\omega(z+z_0)]N_0(\omega r)\} + \sum_i d_i \text{arc sinh}\left(\frac{z+z_i}{r}\right),$$

where the arc sinh terms are the Weyl-analogs of the arc cosh terms, and can be written as

$$\text{arc sinh}\frac{z+m}{r} = \log[(z+m) + \sqrt{r^2 + (z+m)^2}] - \log r. \tag{23}$$

The arbitrary constant m is often called a soliton "pole" and may be either real or complex in which case one must take $\mathcal{Re}[\text{arc sinh}\frac{z+m}{r}]$ or $\mathcal{Im}[\text{arc sinh}\frac{z+m}{r}]$ as the solution. If the soliton has a real pole, then the pole is directly related to the "rod" structure of the Weyl solutions. In fact it is an interesting way to exactly perturb the solitonic solutions by allowing the poles m to "catch" some imaginary part $m + i\epsilon$, see for example [20].

It is simple to show that the solution to the coupled Einstein-scalar field equations (see Appendix A), independently whether static or stationary, can be obtained by keeping the transverse part characterized by the metric function γ_{ab} without a change, but with the longitudinal function $\sigma(r,z)^{\rm vac}$ replaced by

$$\sigma(r,z)^{\rm vac} \longrightarrow \sigma(r,z)^{\rm vac} + \sigma(r,z)^{\rm sc}. \tag{24}$$

The function $\sigma(r,z)^{\rm sc}$ is then solved by quadratures from

$$\sigma_r^{\rm sc} = r\left[\sum_{i=1}^N \varphi_{ir}^2 - \sum_{i=1}^N \varphi_{iz}^2\right], \tag{25}$$

$$\sigma_z^{\rm sc} = 2r\left[\sum_{i=1}^N \varphi_{ir}\varphi_{iz}\right]. \tag{26}$$

To find the above expressions one can either, directly work with the Einstein equations as in [12], or put in the previous all-spacelike Killing vector case $K = t$ and perform formally the following coordinate transformation:

$$t \mapsto r, \qquad z \mapsto iz, \qquad x \mapsto i\theta, \qquad y \mapsto t. \tag{27}$$

One should also perform a global signature change after Wick rotating the solution if one desires to maintain the same signature. Equationwise, but not solutionwise, there is a one-to-one correspondence between the case of all-spacelike Killing vectors and the case when one of the Killing directions is timelike. Some solutions do not have a stationary analogue and vice versa, especially, when the two pertinent Killing vectors are not hypersurface orthogonal. In the diagonal case, however, all the solutions can be formally "copied" from one case to another.

From here one may now use the previous lifting expressions for $\psi_i = \mathcal{D}_{ij}\varphi_j$ and the Eq. (9) to construct the higher dimensional solutions. Before proceeding any further, some remarks are in order. In the static axially symmetric case, the four-dimensional solutions of the vacuum Einstein equations depend just on one function U, the Weyl potential, which as mentioned above, solves Eq. (21) with $\Phi \to U$ in vacuum or in the scalar case. In the higher dimensional case there may be an extra scalar degree of freedom for each extra dimension. In the stationary case, one should allow for an additional rotational degree of freedom, but, at any rate, we assume that the vacuum four-dimensional solutions are already given and do not enter into their generation—there is little to add on what is already known in this field [21].

A separate remark is about what kind of solutions are "interesting" for the scalar field in the higher dimensional generalization of axially symmetric solutions. We believe that those of interest happen to be the same solutions as the ones producing interesting Weyl potentials (the solitonic terms): $\log[(z+a) + \sqrt{(z+a)^2 + r^2}] \equiv \log r + \mathrm{arc\,sinh}[(z+a)/r]$, $\log r$, and their linear combinations. Note that the linear Eq. (21) allows solutions obtained by reflection ($z \to -z$), by shift ($z \to z+a$), as well as by multiplying the solution by an arbitrary constant A. Moreover, if a complex function Φ is a solution of Eq. (21), then both the real and the imaginary parts of Φ are also solutions.

IV. THE 5D CASE

Let us now specialize to a five-dimensional case. The above described procedure of constructing a 5D generalization of the axially symmetric solutions may be put in a more compact statement. Consider we have a vacuum solution to the Einstein field equations in four dimensions of the form

$$ds^2 = -e^U(dt + Ad\phi)^2 + e^{-U}r^2 d\phi^2 + e^{\sigma^{\mathrm{vac}}}(dr^2 + dz^2). \tag{28}$$

The following line element

$$ds^2 = -e^{U-(2/\sqrt{3})\Phi}(dt + Ad\phi)^2 + e^{-U-(2/\sqrt{3})\Phi}r^2 d\phi^2 + e^{\sigma^{\mathrm{vac}}+\sigma^{\mathrm{sc}}-(2/\sqrt{3})\Phi}(dr^2 + dz^2) + e^{(4/\sqrt{3})\Phi}dw^2, \tag{29}$$

is a vacuum solution of the 5D Einstein equations, provided Φ is a solution of the Eq. (21) and σ^{sc} is given by

$$\sigma^{\mathrm{sc}}_r = r(\Phi_r^2 - \Phi_z^2), \tag{30}$$

$$\sigma^{\mathrm{sc}}_z = 2r\Phi_r\Phi_z. \tag{31}$$

Thus we see that the construction of the vacuum generalizations of the 4D axially symmetric solutions is reduced to simple algebra.

A. Coping the Weyl analogues

Before applying the algorithm to generate the stationary solutions we may start by copying the already known solutions obtained with the two spacelike Killing vectors into stationary solutions. Obviously most of these "copies" will not have interesting physical properties.

To exemplify such a direct copying of solutions from cosmology to their Weyl analogue, we consider the open scalar Friedmann-Robertson-Walker (FRW) universe. To construct a scalar field FRW cosmology with open spatial section one starts with the following solution to the vacuum Einstein equations:

$$ds^2_{\mathrm{vac}} = (\sinh 2t)^{-(1/2)}(\cosh 4t - \cosh 4z)^{(3/4)}(-dt^2 + dz^2) + \tfrac{1}{2}\sinh 2t \sinh 2z(\tanh z dx^2 + \mathrm{cotanh} z dy^2), \tag{32}$$

and "dresses" it with the scalar field

$$\Phi = \frac{\sqrt{3}}{2}\log(\tanh t). \tag{33}$$

Immediately one gets a solution which describes an isotropic homogeneous universe with spatial sections of negative curvature [11],

$$ds^2 = \sinh 2t(-dt^2 + dz^2) + \tfrac{1}{2}\sinh 2t \sinh 2z(\tanh z dx^2 + \mathrm{cotanh} z dy^2). \tag{34}$$

To pass to Weyl coordinates, we first choose

$$T = \sinh 2t \sinh 2z, \qquad Z = \cosh 2t \cosh 2z, \tag{35}$$

and express

$$\log(\tanh t) = \frac{1}{2}\mathrm{arccosh}\left(\frac{1-Z}{T}\right) + \frac{1}{2}\mathrm{arccosh}\left(\frac{1+Z}{T}\right), \tag{36}$$

and

$$\log(\tanh z) = \frac{1}{2}\mathrm{arccosh}\left(\frac{1-Z}{T}\right) - \frac{1}{2}\mathrm{arccosh}\left(\frac{1+Z}{T}\right). \tag{37}$$

After some algebra we find that the Weyl potential and the scalar field for the analogue of the open FRW universe are

$$e^U = \log r + \frac{1}{2}\log\left[\frac{\sqrt{r^2 + (z-1)^2} + (z-1)}{\sqrt{r^2 + (z+1)^2} + (z+1)}\right], \tag{38}$$

$$\Phi = \tfrac{1}{2}\log\{[\sqrt{r^2 + (z-1)^2} + (1-z)] \times [\sqrt{r^2 + (z+1)^2} + (z+1)]\}, \tag{39}$$

for the metric given in [11]. Physically, the Weyl analogues of the open FRW universe have nothing to do with the original solution and probably have little relevance as static solutions. We have presented this here only to exemplify the procedure of "copying."

B. Asymptotic flatness

Dealing with the stationary axisymmetric solutions, one often imposes the asymptotically flat behavior of the line element away from the axis ($r \gg z$). In five dimensions, this behavior translates into $g_{ww} \propto \sqrt{z^2 + r^2} + z$, $g_{tt} \propto -1$, and $g_{\phi\phi}/r^2 \propto g_{ww}^{-1} = \sqrt{z^2 + r^2} - z$. Therefore, to build asymptotically flat solutions we have that if the scalar field Φ is given by

$$\Phi = \sum_{i=1}^{N} a_i \varphi_i, \tag{40}$$

where

$$\varphi_i = \log[(m_i - z) + \sqrt{(m_i - z)^2 + r^2}], \tag{41}$$

one must have ($g_{ww} \propto \sqrt{z^2 + r^2} + z$)

$$\frac{4}{\sqrt{3}} \sum_{i=1}^{N} a_i = 1. \tag{42}$$

In the static case if we take the solution for U of the form

$$U = \sum_{i=1}^{N} b_i V_i, \tag{43}$$

where

$$V_i = \log[(n_i - z) + \sqrt{(n_i - z)^2 + r^2}], \tag{44}$$

to get the asymptotically flat solutions we must impose ($g_{tt} \propto -1$)

$$-\frac{2}{\sqrt{3}} \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} b_i = 0, \qquad \longrightarrow \sum_{i=1}^{N} b_i = \frac{1}{2}, \tag{45}$$

the last condition then ($g_{\phi\phi}/r^2 \propto g_{ww}^{-1}$) is trivially satisfied.

C. The method in terms of the rod structure

As mentioned before, the interesting Weyl potentials are those associated with the solitonic terms or "rods" in the z axis [22]. Considering these terms, the generating method can be described as adding up a source to the fifth dimension. Given the form of the new metric by (29), in fact, we are also "subtracting" the same source from the other two Killing directions, thus "compensating" the extra sources that we have introduced to the system.

We can add either a finite rod of length $(a_0 - a)$, or the interval (a, a_0), by choosing

$$\Phi = \log\left[\frac{\sqrt{r^2 + (z + a)^2} - (z + a)}{\sqrt{r^2 + (z - a_0)^2} - (z - a_0)}\right], \tag{46}$$

or a semi-infinite rod (a, ∞) ($(-\infty, -a)$ taking the lower sign) with

$$\Phi = \log\{\sqrt{r^2 + (z \mp a)^2} \mp (z \mp a)\}. \tag{47}$$

V. TRAPPED SURFACES AND HORIZONS OF THE GENERATED SOLUTIONS

In $N > 4$ is not easy to figure out topological features of spacetime; to extract interesting information one needs invariant objects. One of the most interesting properties to study in these spacetimes is the trapness of two-dimensional surfaces. These are imbedded spatial surfaces such that any portion of them has a decreasing area along any future evolution direction. A practical way to study the trapped surfaces and locate horizons was introduced in [23] through evaluating a certain scalar κ. The sign of this scalar defines the trapping of a surface S. We shall analyze the effect of introducing rods in the 5D generated spaces.

For completeness we include some steps in the construction of such scalar κ introduced in [23]. Let us consider the line element

$$ds^2 = g_{ab}dx^a dx^b + 2g_{aA}dx^a dx^A + g_{AB}dx^A dx^B, \tag{48}$$

and a family of $(D-2)$-dimensional spacelike surfaces S_{X^a} with intrinsic coordinates $\{\lambda_A\}, A, B, .. = 2, \ldots, D-1$, imbedded into the spacetime. There are fixed coordinates, $\{x^a = X^a\}$, $a, b = 0, 1, \ldots, X^a$, while x^A denotes the local coordinates on the surface. $G = \sqrt{\det g_{AB}} = e^U$ gives the canonical $(D-2)$ volume element of the surfaces S_{X^a}. Introducing $H_\mu = \delta^a_\mu (U_{,a} - \nabla \cdot g_a)$, $g_a =: g_{aA}dx^A$, where the divergence operator acts on vectors at S_{X^a}, the invariant κ is defined by

$$\kappa_{\{X^a\}} = -g^{bc}H_b H_c|_{X^a}. \tag{49}$$

The hypersurfaces $\mathcal{H}$, defined locally by the vanishing of κ, are the so-called S_{X^a} horizons [23], and coincide in many instances with the classical horizons.

In what follows we have found it convenient to work in prolate spheroidal coordinates (x, y) which make the algebra much easier. These are related to the Weyl coordinates by

$$r = \alpha\sqrt{(x^2 - 1)(1 - y^2)}, \qquad z = \alpha xy, \tag{50}$$

with ranges $x \geq 1$ and $-1 \leq y \leq 1$.

In prolate spheroidal coordinates the generated spacetimes (29) take the following form:

$$ds^2 = e^{-(2\Phi/\sqrt{3})}\left\{e^{\sigma^{\rm vac} + \sigma^{\rm sc}}\alpha^2(x^2 - y^2)\left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2}\right) + \gamma_{ab}dx^a dx^b\right\} + e^{4\Phi/\sqrt{3}}d\omega^2, \tag{51}$$

where $a, b = t, \phi$. We shall consider spacelike surfaces with $t = \text{const}$ and $x = \text{const}$. Taking $\{x^a\} = \{t, x\}$ and

$\{x^A\} = \{y, \phi, \omega\}$, the scalar $\kappa_{\{t,x\}}$ is given by

$$\kappa_{\{t,x\}} = -e^{2\Phi/\sqrt{3}-\sigma^{sc}-\sigma^{vac}} \frac{(x^2-1)}{4\alpha^2(x^2-y^2)^3\gamma_{\phi\phi}^2} U_x^2, \qquad (52)$$

with

$$U_x = \gamma_{\phi\phi}[(x^2-y^2)(\sigma_x^{vac}+\sigma_x^{sc})+2x] + (x^2-y^2)\gamma_{\phi\phi,x}. \qquad (53)$$

The invariant κ, as compared with that one of the seed, is modified by the factor $e^{2\Phi/\sqrt{3}-\sigma^{sc}}$. The inclusion of a semi-infinite rod $(-\infty, -a_0)$ produces therefore

$$e^{(2\Phi/\sqrt{3})-\sigma^{sc}} = (x+y)^{4A^2} a_0^{2A/\sqrt{3}} [(x+1)(1+y)]^{(2A/\sqrt{3})-4A^2}, \qquad (54)$$

while with the finite rod (a, a_0), the modification corresponds to

$$e^{(2\Phi/\sqrt{3})-\sigma^{sc}} = \left(\frac{a}{a_0}\right)^{2A/\sqrt{3}} (x^2-y^2)^{4A^2} \frac{(x-1)^{(2A/\sqrt{3})-4A^2}}{(x+1)^{(2A/\sqrt{3})+4A^2}}. \qquad (55)$$

Hence, using this method new horizons or singularities arise, depending on the value of the exponent $2A/\sqrt{3} - 4A^2$. For instance, in the case of a finite rod, if $2A/\sqrt{3} - 4A^2 > 0$ in Eq. (55), κ vanishes at $x = 1$ and there is a horizon. While if $2A/\sqrt{3} - 4A^2 < 0$, κ diverges at the same point.

VI. GENERATING STATIC 5D SOLUTIONS

We now illustrate how the method works by generating some static 5D solutions by taking the Minkowski 4D and the Schwarzschild 4D black hole as seeds.

A. Generating 5D solutions from Minkowski seed

We first consider Minkowski 4D spacetime in the Rindler coordinates of uniformly accelerated observers [1,21]

$$\begin{aligned} ds_{M4}^2 &= -e^{2U_1}dt^2 + e^{2U_2}d\phi^2 + e^{\sigma^{vac}}(dr^2+dz^2), \\ U_1 &= \tfrac{1}{2}\log[-a+z+\sqrt{(-a+z)^2+r^2}] + \text{const}, \\ U_2 &= \tfrac{1}{2}\log[a-z+\sqrt{(a-z)^2+r^2}] + \text{const}, \\ \sigma^{vac} &= -\log[r^2+(a-z)^2]. \end{aligned} \qquad (56)$$

The corresponding rod structure consists of a semi-infinite rod $(-\infty, a)$ in the ∂_t direction and a semi-infinite rod (a, ∞) in the ∂_ϕ direction, as is shown in Fig. 1.

Now, consider adding up a semi-infinite rod $(-\infty, -a_0)$ in a fifth dimension using $\Phi = A\log[a_0 + z + \sqrt{(a_0+z)^2+r^2}]$, where A is a constant characterizing the scalar charge and a_0 defines a new interval on the z

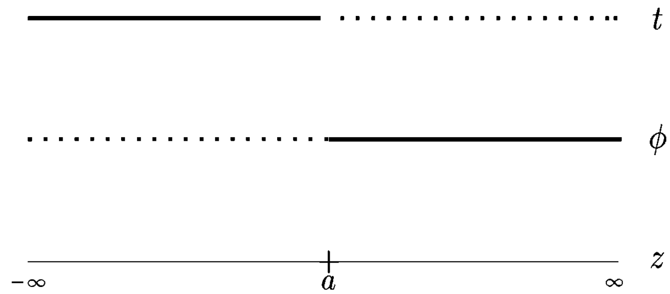

FIG. 1. Rod structure of the Minkowski 4D spacetime.

axis. The method produces a 5D solution given by

$$\begin{aligned} ds_{M5}^2 &= e^{-(2/\sqrt{3})\Phi+\sigma^{vac}+\sigma^{sc}}(dr^2+dz^2) \\ &\quad + e^{-(2/\sqrt{3})\Phi}[-e^{2U_1}dt^2 + e^{2U_2}d\phi^2] \\ &\quad + e^{(4/\sqrt{3})\Phi}d\omega^2, \end{aligned} \qquad (57)$$

with $\sigma^{sc} = 4A^2\log[\frac{a_0+z+\sqrt{r^2+(a_0+z)^2}}{\sqrt{r^2+(a_0+z)^2}}]$.

In prolate spheroidal coordinates it becomes

$$\begin{aligned} ds_{M5}^2 = a_0^{-(2A/\sqrt{3})}\Big\{&(x+y)^{1-4A^2}[(1+y)(1+x)]^{4A^2-(2A/\sqrt{3})} \\ &\times\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right) + (x+1)^{1-(2A/\sqrt{3})} \\ &\times(1+y)^{-(2A/\sqrt{3})}(1-y)d\phi^2 - (1+y)^{1-(2A/\sqrt{3})} \\ &\times(1+x)^{-(2A/\sqrt{3})}(x-1)dt^2 \\ &+ a_0^{6A/\sqrt{3}}[(x+1)(1+y)]^{4A/\sqrt{3}}d\omega^2\Big\}. \end{aligned} \qquad (58)$$

The value of the scalar charge A defines several important features of the generated spacetime. We illustrate it by analyzing the above generated solutions for two different values of scalar charge: $A = \frac{\sqrt{3}}{2}$ and $A = \frac{\sqrt{3}}{4}$.

1. Case with scalar charge $A = \frac{\sqrt{3}}{2}$

Choosing scalar charge as $A = \frac{\sqrt{3}}{2}$, (58) gives

$$\begin{aligned} ds_{M5}^2 = a_0^{-1}\Big\{&(x+y)^{-2}[(1+y)(1+x)]^2\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right) \\ &+\frac{(1-y)}{(1+y)}d\phi^2 - \frac{(x-1)}{(x+1)}dt^2\Big\} \\ &+ [a_0(x+1)(1+y)]^2 d\omega^2. \end{aligned} \qquad (59)$$

For $x = 1$ $(r = 2m)$, $g_{tt} = 0$ and a horizon is present, while $g_{\phi\phi}$ diverges at $y = -1$. The solution is not asymptotically flat as seen from the behavior of $g_{\omega\omega}$. The corresponding rod structure is as follows (Fig. 2).

A finite rod $(-a_0, a)$ in the ∂_t direction (event horizon), a semi-infinite rod (a, ∞) in the ∂_ϕ direction (axis of rotation of ϕ), a semi-infinite rod $(-\infty, -a_0)$ with negative

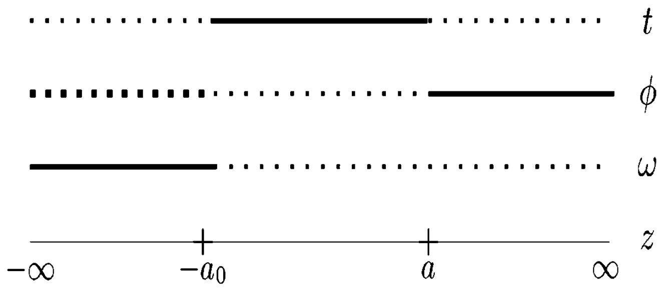

FIG. 2. Rod structure of the solution generated using the Minkowski 4D and inserting a semi-infinite rod $(-\infty, -a_0)$ in the fifth dimension with scalar charge $A = \frac{\sqrt{3}}{2}$. The bold dotted line along ∂_ϕ corresponds to a rod with negative mass density.

mass density in the ∂_ϕ direction, and a semi-infinite rod $(-\infty, -a_0)$ in the ∂_w direction.

The previous solution (59) can be compared with the static Myers-Perry (MPs) black hole [24] [Eq. (5.16) without rotation, $a_1 = a_2 = 0$ in [3]], whose line element and rod structure (Fig. 3) we include here for completeness

$$ds^2_{MPs} = \frac{r_0^2}{4}\left\{2(x+1)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x+1)(1-y)d\phi^2 - \frac{4}{r_0^2}\frac{(x-1)}{(x+1)}dt^2 + (x+1)(1+y)d\omega^2\right\}. \quad (60)$$

Comparing Fig. 2 with Fig. 3, note the similarity in rod structure, it is the same except for the presence of the negative mass density along the Killing direction ∂_ϕ.

We now consider the spacelike surfaces with $t = \text{const}$ and $x = \text{const}$ ($r = \text{const}$) for both solutions. The corresponding scalars κ that define the trapping of such surfaces are

$$\kappa_{MPs} = -\frac{9}{2r_0^2}\frac{(x-1)}{(x+1)^2} = -\frac{9m}{2r_0^2}\frac{(r-2m)}{r^2}, \quad (61)$$

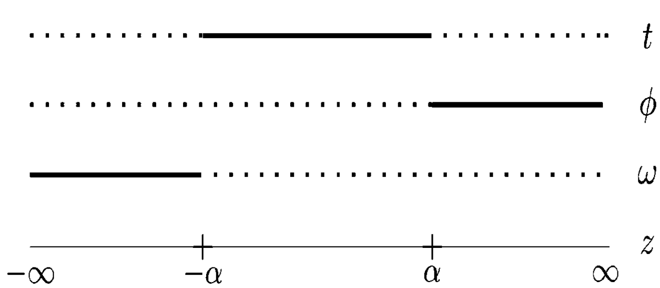

FIG. 3. Rod structure of Myers-Perry static 5-D black hole. For the static case $\alpha = m$.

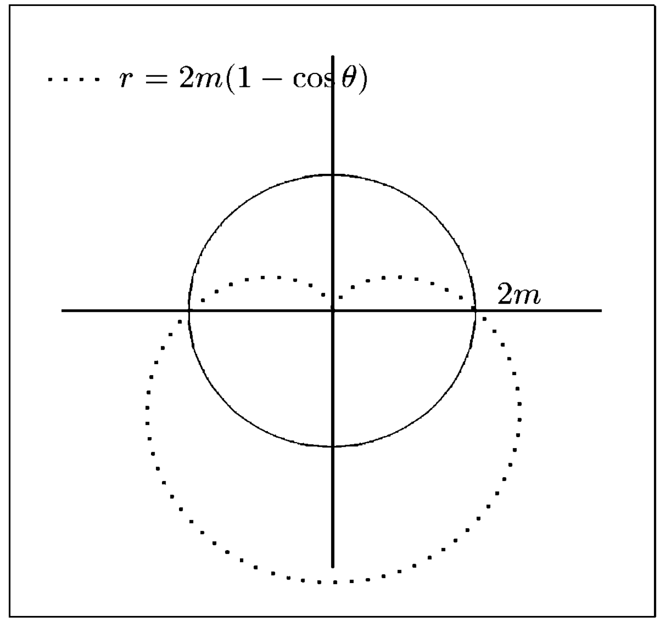

FIG. 4. Squeme of the horizons, for a fixed angle ϕ and sweeping θ for the 5D solution generated from Minkowski and adding the rod $(-\infty, -a_0)$ with $A = \frac{\sqrt{3}}{2}$. The solid circle represents the trapped surface $r = 2m$, while the dotted curve corresponds to $r = 2m(1 - \cos\theta)$.

$$\kappa_{M5} = -a_0\frac{(x-1)(x+2y-1)^2}{(x+1)^3(1+y)^2} = -a_0\frac{(r-2m)[r-2m(1-\cos\theta)]^2}{r^3(1+\cos\theta)^2}. \quad (62)$$

Both scalars exhibit singularity at $r = 0$ as well as the horizon at $r = 2m$. Note, however that the solution (59) presents an additional marginally trapped surface ($\kappa = 0$) defined by $r = 2m(1 - \cos\theta)$. The scalar κ also becomes singular at $\theta = \pi$. The profiles of the marginally trapped surfaces are shown in Fig. 4 and then rotated in Fig. 5.

2. Case $A = \frac{\sqrt{3}}{4}$

Another interesting case occurs if we choose $A = \frac{\sqrt{3}}{4}$ in (58). The generated solution then becomes

$$ds^2_{M5} = a_0^{-(1/2)}\left\{[(x+y)(1+y)(1+x)]^{1/4} \times \left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + \frac{(1-y)(x+1)^{1/2}}{(1+y)^{1/2}}d\phi^2 - \frac{(x-1)(1+y)^{1/2}}{(x+1)^{1/2}}dt^2\right\} + a_0(x+1)(1+y)d\omega^2. \quad (63)$$

This time the $g_{\omega\omega}$ component satisfies the condition for asymptotic flatness Eq. (42), but $g_{tt} \simeq x^{1/2}$ does not and the

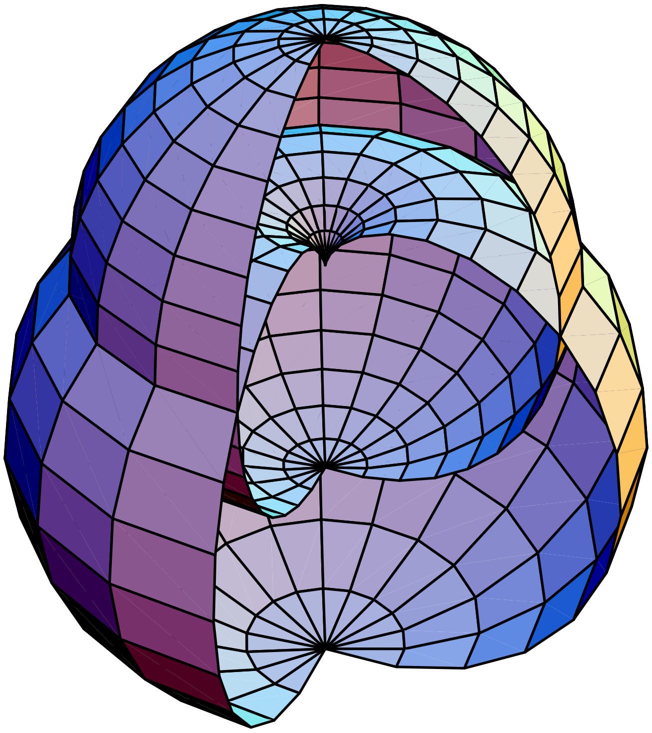

FIG. 5 (color online). Marginally trapped surfaces of the Minkowski 5D, generated with a Minkowski 4D seed plus a dilaton field $\Phi = \frac{\sqrt{3}}{2}\log[a_0(x+1)(1+y)]$, these surfaces are the rotated slices of Fig. 4.

solution is not globally asymptotically flat. The rod structure is shown in Fig. 6.

New marginally trapped surfaces also arise in this case. The scalar $\kappa_{\{t,r\}}$ that characterizes the trapping of a spacelike surface of constant t and r for (63) is

$$\kappa_{\{t,r\}} = -\sqrt{a_0}m^{3/4}\frac{(r-2m)[r-\frac{7}{8}m(1-\cos\theta)]^2}{(1+\cos\theta)^{1/4}[r-m(1-\cos\theta)]^{9/4}r^{5/4}}. \tag{64}$$

From Eq. (64) we learn that the solution has two marginally trapped surfaces, one situated at the Schwarzschild horizon, $r = 2m$, while the second apparent horizon is the surface defined by $r = \frac{7}{8}m(1-\cos\theta)$, with $r_{\max} = \frac{7}{4}m$. This second surface remains hidden inside $r = 2m$.

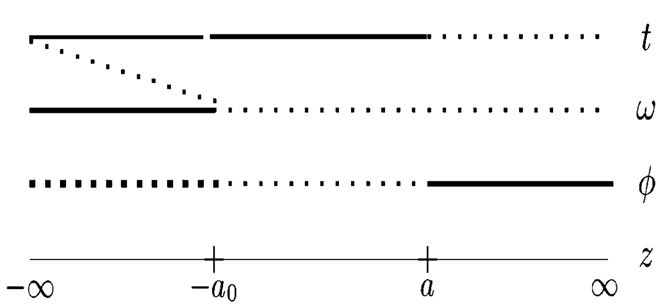

FIG. 6. Rod structure for the 5D solution generated from Minkowski and adding the rod $(-\infty, -a_0)$ with $A = \frac{\sqrt{3}}{4}$. The dotted line between t and ω indicates that the rod $(-\infty, -a_0)$ has components in both directions, ∂_t and ∂_ω.

B. Starting with a Schwarzschild seed

Consider now the Schwarzschild 4D line element in prolate spheroidal coordinates,

$$ds^2 = m^2(x+1)^2\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + m^2(1-y^2)(x+1)^2 d\phi^2 - \frac{(x-1)}{(x+1)}dt^2. \tag{65}$$

The rod structure of this solution is as follows: a finite rod of length $2m$ in the timelike direction, ∂_t, and two semi-infinite rods $(-\infty, -m)$, (m, ∞) in the spacelike direction ∂_ϕ, as shows Fig. 7.

Adding up the semi-infinite rod $(-\infty, -m)$ by taking $\Phi = \frac{\sqrt{3}}{4}\log[m(x+1)(1+y)]$ and lifting to 5D results in the following spacetime:

$$ds_5^2 = m^{3/2}\left\{(x+1)^{9/4}\frac{(1+y)^{1/4}}{(x+y)^{3/4}}\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x+1)^{3/2}(1-y)(1+y)^{1/2}d\phi^2 - \frac{1}{m^2}\frac{(x-1)}{(x+1)^{3/2}}\frac{dt^2}{(1+y)^{1/2}}\right\} + m(x+1)(1+y)d\omega^2. \tag{66}$$

This procedure introduces a singularity in g_{tt} due to the factor $(1+y)^{-(1/2)}$. The corresponding rod structure is shown in Fig. 8.

Several interesting properties of the spacetime (66) are worth to point out: the metric is asymptotically flat, it has a horizon at $x = 1$ and $g_{\phi\phi}$ is finite in all the domain of x and y. There is a divergence in g_{tt} at $y = -1$.

The spacetime possesses, besides the surface $r = 2m$, another marginally trapped surface that deforms the spherical symmetry of the horizon. To see it we calculate the scalar $\kappa_{\{t,r\}}$ for (66),

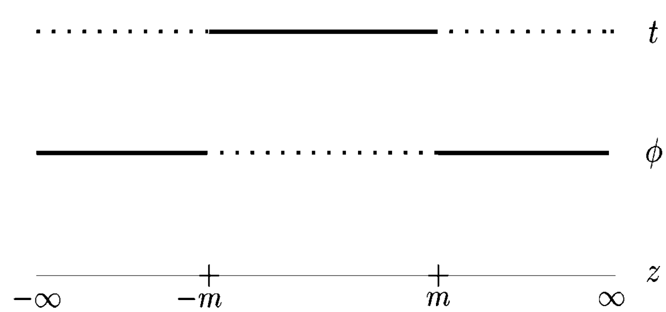

FIG. 7. Rod structure of the Schwarzschild 4D black hole with mass $2m$.

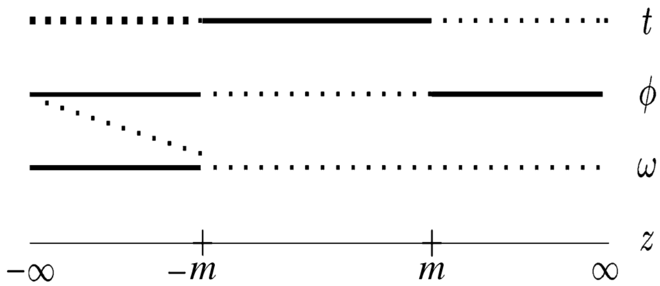

FIG. 8. Rod structure of the static 5D solution generated from Schwarzschild spacetime by adding a semi-infinite rod $(-\infty, -m)$ with $\Phi = \frac{\sqrt{3}}{4}\log[m(x+1)(1+y)]$. The bold dotted line indicates negative mass density in the interval $(-\infty, -m)$; the dotted line between ϕ and ω symbolizes that the semi-infinite rod $(-\infty, -m)$ has components in both directions ∂_ϕ and ∂_ω.

$$\kappa_{\mathrm{Schw}-5D} = -\frac{1}{4^3 m^{3/2}} \frac{(x-1)(16x+19y-3)^2}{(x+1)^{13/4}(1+y)^{1/4}(x+y)^{5/4}} = -4\frac{(r-2m)[r-\frac{19}{16}m(1-\cos\theta)]^2}{(1+\cos\theta)^{1/4} r^{13/4}[r-m(1-\cos\theta)]^{5/4}}. \quad (67)$$

The vanishing of the scalar κ indicates the presence of a marginally trapped surface that can be associated to a horizon. For the solution (66), κ vanishes on two surfaces: $r = 2m$ and $r = \frac{19}{16}m(1-\cos\theta)$, indicating the distortion of horizon that we have mentioned before. The new horizon "intermingles" with portions of the classical Schwarzschild horizon, presenting for an external observer a nonspherical horizon of a "peanut" shape. Also there is the singularity at $r = 0$ where κ diverges. Therefore the solution may be thought of a 5D black hole distorted by the presence of a string with a negative mass density, hence deforming its horizon. Slices of the horizons are shown in Fig. 9 and rotated in Fig. 10.

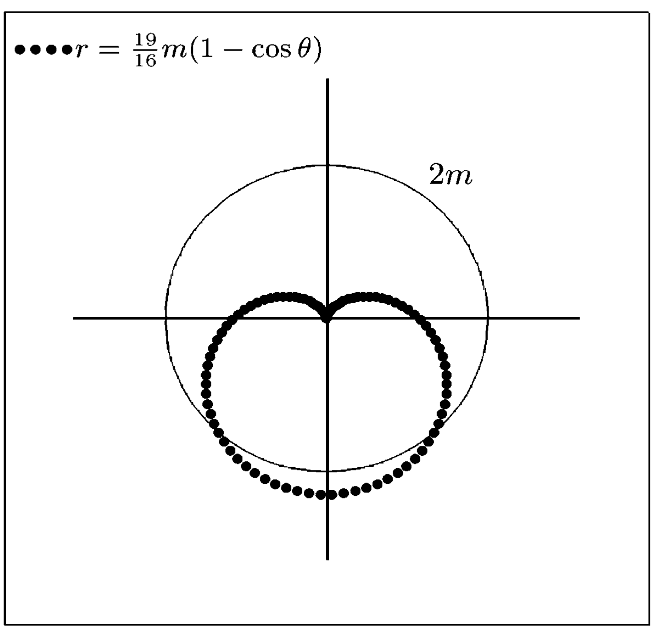

FIG. 9. Squeme of the horizons, for a fixed angle ϕ and sweeping θ, of the generated Schwarzschild 5D black hole, with $A = \frac{\sqrt{3}}{4}$; $r_{\max} = 2.37m$ at $\theta = \pi$.

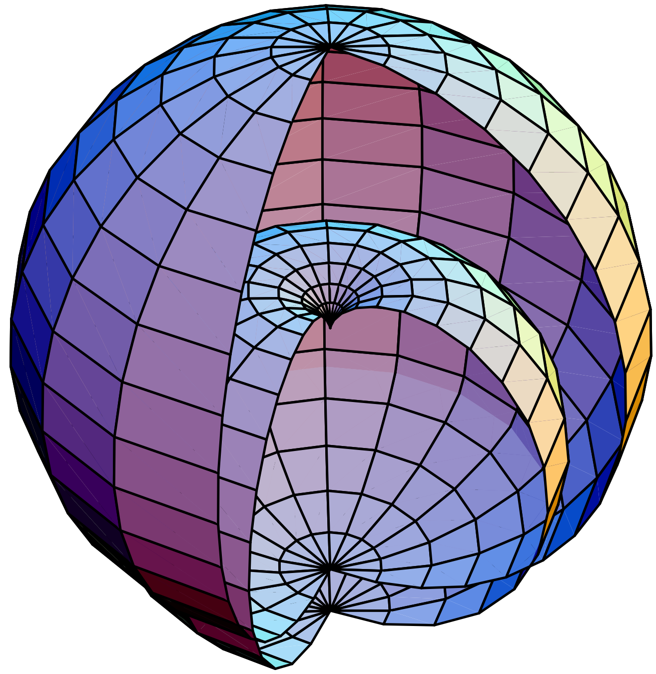

FIG. 10 (color online). Marginally trapped surfaces of the Schwarzschild 5D, generated with a Schwarzschild 4D seed plus a dilaton field $\Phi = \frac{\sqrt{3}}{4}\log[(x+1)(1+y)]$; these are the rotated slices of Fig. 9.

C. The static Myers-Perry solution

We note in passing that the static Myers-Perry solution (60) can be obtained by our lifting method using the appropriated seed

$$U = \frac{1}{4}\log\left[\frac{(r_- + z - \alpha)(r_+ + r_- - 2\alpha)}{\alpha(r_+ + r_- + 2\alpha)}\right] = \frac{1}{4}\log\left[\frac{(1+y)(x-1)^2}{(x+1)}\right], \quad (68)$$

where $r_\pm^2 = r^2 + (z \pm \alpha)^2$. Using U in the line element (28) and adding up a semi-infinite rod $(-\infty, -\alpha)$ as in (29), with the dilaton

$$\Phi = \frac{\sqrt{3}}{4}\log[(x+1)(1+y)] = \frac{\sqrt{3}}{4}\log[\sqrt{r^2 + (z+\alpha)^2} + z + \alpha]. \quad (69)$$

We do not discuss this case further for it was thoroughly done in the literature.

D. Changing the dilaton

We again start with the Schwarzschild seed, line element (65) or (C2). Taking the dilaton field as

$$\Phi = c\left\{\text{arc sinh}\frac{(m+z)}{r} + \text{arc sinh}\frac{(m-z)}{r}\right\}. \tag{70}$$

Now, changing to prolate spheroidal coordinates, (x, y),

$$x = (\sqrt{(z+m)^2 + r^2} + \sqrt{(z-m)^2 + r^2})/2m, \tag{71}$$

$$y = (\sqrt{(z+m)^2 + r^2} - \sqrt{(z-m)^2 + r^2})/2m, \tag{72}$$

the dilaton field (70) is written as

$$\Phi = c\log\left(\frac{x-1}{x+1}\right), \tag{73}$$

changing now to curvature coordinates (r, θ)

$$x = \frac{r}{m} - 1, \qquad y = \cos\theta, \tag{74}$$

we find the following 5D metric:

$$\begin{aligned} ds^2 = {}& \left(1 - \frac{2m}{r}\right)^{-(2c/\sqrt{3})}\left[-\left(1-\frac{2m}{r}\right)^{a} dt^2 \right.\\ & + \left(1-\frac{2m}{r}\right)^{-a} dr^2 + \left(1-\frac{2m}{r}\right)^{1-a} \\ & \left. \times r^2(d\theta^2 + \sin\theta^2 d\phi^2)\right] + \left(1-\frac{2m}{r}\right)^{4c/\sqrt{3}} dw^2, \end{aligned} \tag{75}$$

here $a = \sqrt{1-4c^2}$. We recognize the metric in square brackets as the 4D scalar solution derived in [16].

The corresponding scalar $\kappa_{\{t,r\}}$ is given by

$$\kappa_{\{t,r\}} = -\frac{4}{r^2}\left(1 - \frac{m(1+a)}{r}\right)^2\left(1-\frac{2m}{r}\right)^{a+(2c/\sqrt{3})-2}. \tag{76}$$

In general this spacetime does not possess a regular horizon, as can be seen from (76) since the exponent $a + \frac{2c}{\sqrt{3}} - 2 < 0$. There is a marginally trapped surface at $r = m(1+a)$ that is always hidden inside the singular surface $r = 2m$ since $a < 1$. However, an interesting situation occurs when $c = \sqrt{3}/4$ when the metric becomes

$$\begin{aligned} ds^2 = {}& -dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ & + \left(1-\frac{2m}{r}\right) dw^2. \end{aligned} \tag{77}$$

This spacetime is the Kaluza-Klein bubble and is asymptotically flat. Performing the Wick rotation $t \mapsto i\omega$, $\omega \mapsto it$ we finish with the black string $S^2 \times R$, Schwarzschild $\times$ R.

VII. GENERATION OF 5D STATIONARY SOLUTIONS

According to the above outlined method, to construct nonstatic solutions one must start with vacuum stationary solutions, since the method does not introduce nondiagonal elements into the generated 5D metric. The simplest stationary seed is the Kerr solution, whose line element in Boyer-Lindquist coordinates (t, r, θ, ϕ) is

$$\begin{aligned} ds^2_{\text{Kerr}} = {}& -\frac{\Delta - a^2\sin^2\theta}{\Sigma} dt^2 - 2a\sin^2\theta\frac{r^2 + a^2 - \Delta}{\Sigma} dt d\phi \\ & + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 \\ & + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right), \end{aligned} \tag{78}$$

where $\Delta = r^2 - 2mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. The corresponding rod structure is shown in Fig. 11.

Using the Kerr solution as seed we construct deformations to the Myers-Perry 5D rotating black hole. The generated solutions present a rod structure very close to the original undeformed one, except for some new singularities that can be avoided using a different seed. In what follows we explore two cases: the Kerr seed plus a semi-infinite rod $(-\infty, -\alpha)$ with scalar charge $A = \frac{\sqrt{3}}{2}$ and Kerr seed again with an extra distinct semi-infinite rod (a_0, ∞) with the same scalar charge.

A. Inserting a semi-infinite rod $(-\infty, -\alpha)$

Lifting the Kerr solution to a fifth dimension by adding the semi-infinite rod $(-\infty, -\alpha)$, corresponding to $\Phi = \frac{\sqrt{3}}{2}\log[\alpha(x+1)(1+y)]$, we obtain the following line element,

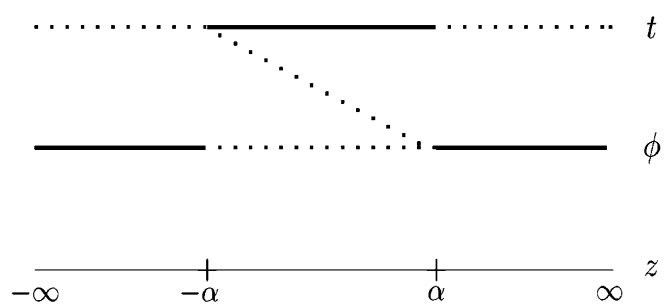

FIG. 11. Rod structure of the stationary Kerr black hole; the dotted line that intersects ∂_t and ∂_ϕ is intended to indicate that the orientation of the finite rod $(-\alpha, \alpha)$ has one component along ∂_t and other along ∂_ϕ.

$$ds^2 = e^{\sigma}\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + g_{ij}dx^i dx^j, \qquad i, j = t, \phi, \omega, \qquad e^{\sigma} = \frac{m^2}{\alpha}\frac{(x+1)^2(1+y)^2}{(x+y)^3}[(1+px)^2 + q^2y^2],$$

$$g_{tt} = -\frac{1}{\alpha(x+1)(1+y)}\frac{(p^2x^2 + q^2y^2 - 1)}{[(1+px)^2 + q^2y^2]}, \qquad g_{t\phi} = -\frac{2a(1-y)}{\alpha(x+1)}\frac{(1+xp)}{[(1+px)^2+q^2y^2]}, \tag{79}$$

$$g_{\phi\phi} = -\frac{(1-y)}{\alpha(x+1)}\frac{\{4a^2(1-y^2)(1+xp)^2 - \alpha^2(x^2-1)[(1+px)^2+q^2y^2]^2\}}{[(1+px)^2+q^2y^2][p^2x^2+q^2y^2-1]}, \qquad g_{\omega\omega} = [\alpha(x+1)(1+y)]^2,$$

where the parametrization is $p = \alpha/m = \sqrt{m^2 - a^2}/M$, $q = a/m$, and $p^2 + q^2 = 1$, a and m stand for the acceleration and mass, respectively. From the inspection of $g_{\omega\omega}$ it is apparent that the solution is not asymptotically flat and an extra source is introduced. The rod structure of (79) is analyzed in detail in Appendix B and it is shown in Fig. 12. The rod structure resembles the one of the Myers-Perry rotating black hole [24], except for the divergence of g_{tt} in the interval $(-\infty, -\alpha)$. In spite of the similarity in the rod structure, the corresponding metric functions are different and the reason is that using Kerr solution as seed, the 5D generated metric inherits second degree polynomials in x and y.

For the stationary solutions, it is interesting to compare the corresponding scalar $\kappa_{\{t,r\}}$ for the Kerr solution and the generated solutions. If we choose the fixed coordinates as $\{x^a\} = t, r$ and the coordinates describing the hypersurface as $\{x^A\} = \theta, \phi$, the scalar κ defining the trapping for such spacelike surface in the Kerr spacetime is given by

$$\kappa^{\mathrm{Kerr}}_{\{t,r\}} = -g^{rr}(U^{\mathrm{Kerr}}_r)^2 = -\frac{\Delta}{\Sigma}\frac{[r(2r^2 + a^2 + a^2\cos^2\theta) + a^2 m\sin^2\theta]^2}{[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]^2}, \tag{80}$$

where $\Delta = r^2 - 2mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$.

The terms in square brackets are always strictly positive, therefore $\Delta = 0$ determines the only marginally trapped surfaces or horizons described by the spheres $r_\pm = m \pm \sqrt{m^2 - a^2}$ corresponding to the inner and outer horizons in Kerr geometry. When lifted to five dimensions, the new solution (79) presents, besides the Kerr horizons, additional marginally trapped surfaces, as can be seen analyzing the invariant κ given by

$$\kappa^{\mathrm{Kerr5}}_{\{t,r\}} = -\frac{\Delta}{\Sigma}\frac{(r - m + \alpha\cos\theta)}{(r-m+\alpha)^4(1+\cos\theta)^2} \times \frac{[\mathcal{U}(r,\theta)]^2}{[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]^2},$$
$$\mathcal{U}(r,\theta) = \frac{3\alpha}{2}(\cos\theta - 1)[(r^2+a^2)^2 - \Delta a^2\sin^2\theta] + (r-m+\alpha)(r-m+\alpha\cos\theta) \times [r(2r^2 + a^2 + a^2\cos^2\theta) + a^2 m\sin^2\theta]. \tag{81}$$

The last expression (81) shows that $\kappa_{\{t,r\}}$ vanishes when $\Delta = 0$, coinciding with the Kerr inner and outer horizons; but it also vanishes when $r = r_h = m - \alpha\cos\theta$. This additional marginally trapped surface lies between the inner and outer Kerr horizons, touching them tangentially. For $r_h < m - \sqrt{m^2 - a^2}\cos\theta$, $\kappa_{\{t,r\}}$ becomes positive and the surface becomes trapped. Furthermore, another marginally trapped surface exists for those values of r such that

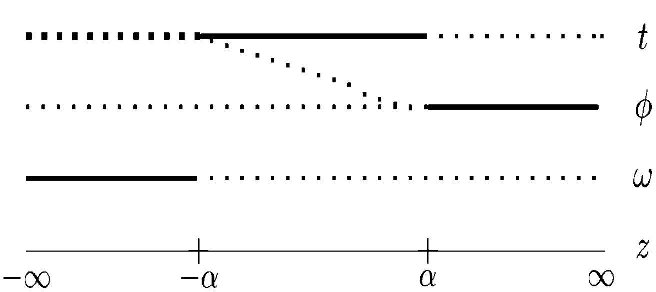

FIG. 12. Rod structure of the 5D stationary solution generated from Kerr spacetime by inserting in the direction ∂_ω a semi-infinite rod $(-\infty, -\alpha)$ with $\Phi = \frac{\sqrt{3}}{2}\log[\alpha(x+1)(1+y)]$. The bold dotted rod has negative density.

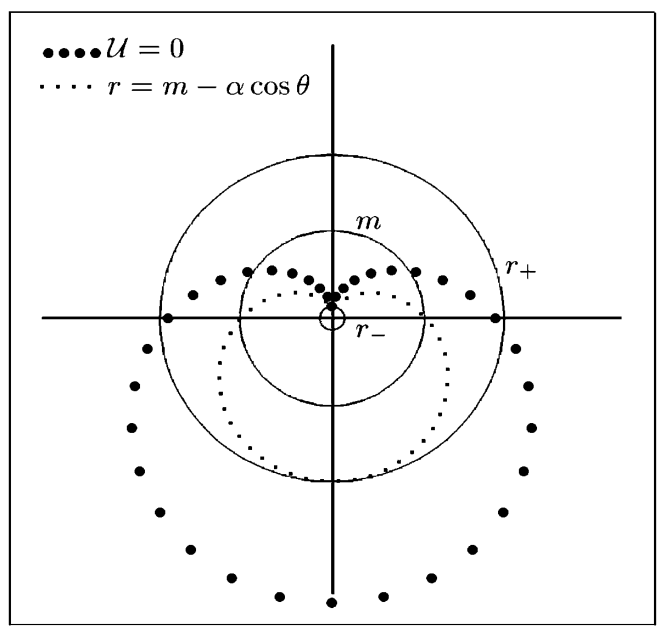

FIG. 13. Horizon squeme for the 5D stationary solution generated from Kerr spacetime adding a semi-infinite rod $(-\infty, -\alpha)$, with $\Phi = \frac{\sqrt{3}}{2}\log[\alpha(x+1)(1+y)]$.

$\mathcal{U}(r, \theta) = 0$. Since $\mathcal{U}$ is a fifth degree polynomial in r it must have at least one real root generating a marginally trapped surface. When $a = 0$, the surface $\mathcal{U} = 0$ corresponds to the one found in the static example of the previous section at $r = \frac{19}{16} m(1 - \cos\theta)$. In Fig. 13 a numerical profile of $\mathcal{U} = 0$ is shown along with slices of other horizons. The rotated slices are shown in Fig.14.

We now compare these results with those obtained for $\kappa_{\{t,r\}}$ corresponding to the Myers-Perry five-dimensional spinning black hole [24]. The Myers-Perry metric with one rotation $a_1 \neq 0$ in Boyer-Lindquist coordinates is

$$ds^2 = -dt^2 + \frac{r_0^2}{\Sigma}[dt - a_1 \sin^2\theta d\phi]^2 + (r^2 + a_1^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right), \quad (82)$$

where $\Sigma = r^2 + a_1^2\cos^2\theta$ and $\Delta = r^2 + a_1^2 - r_0^2$. While the scalar $\kappa_{\{t,r\}}$ is given by

$$\kappa_{\{t,r\}}^{MP} = -\frac{\Delta}{\Sigma} \frac{[r^2(r^2 + a_1^2) + (2r^2 + a_1^2)(r^2 + a_1^2\cos^2\theta) + r_0^2 a_1^2 \sin^2\theta]^2}{r^2[(r^2 + a_1^2\cos^2\theta)(r^2 + a_1^2) + r_0^2 a_1^2 \sin^2\theta]^2}. \quad (83)$$

The expression (83) clearly diverges at $r = 0$ which corresponds to a strong curvature singularity, while the unique horizon is given by $\Delta = r^2 + a_1^2 - r_0^2 = 0$, and is described by the sphere $r = \sqrt{r_0^2 - a_1^2}$.

B. Second stationary example

Another interesting example is obtained by taking the Kerr solution and introducing a semi-infinite rod (a_0, ∞) with $\Phi = \frac{\sqrt{3}}{2} \log[a_0(x + 1)(1 - y)]$. It follows that the rod

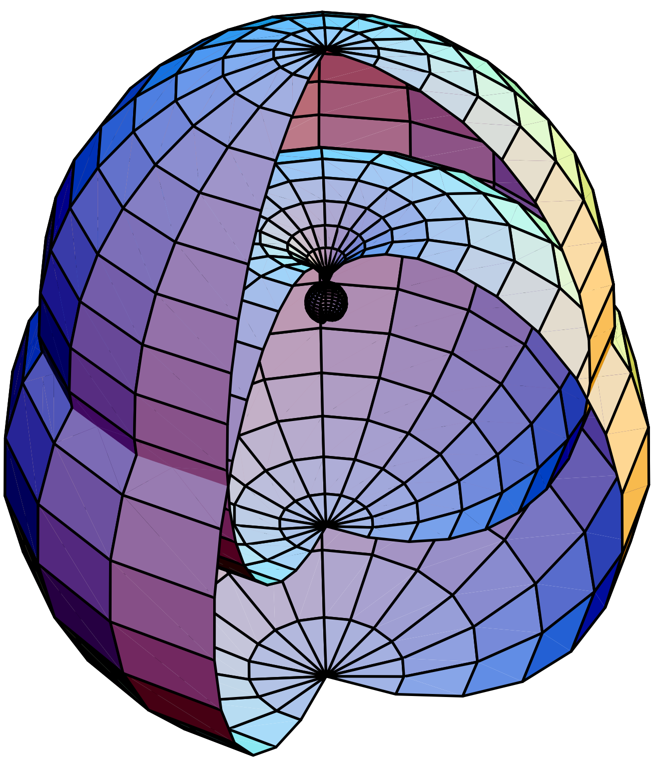

FIG. 14 (color online). Marginally trapped surfaces of Kerr 5D, generated with a Kerr 4-D seed plus a dilaton field $\Phi = \frac{\sqrt{3}}{2} \log[(x + 1)(1 + y)]$, these are the rotated slices of Fig. 13.

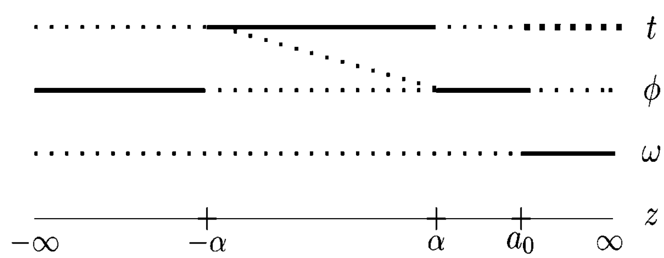

FIG. 15. Rod structure of the 5D stationary solution generated using Kerr spacetime as seed and inserting in the direction ∂_ω a semi-infinite rod (a_0, ∞).

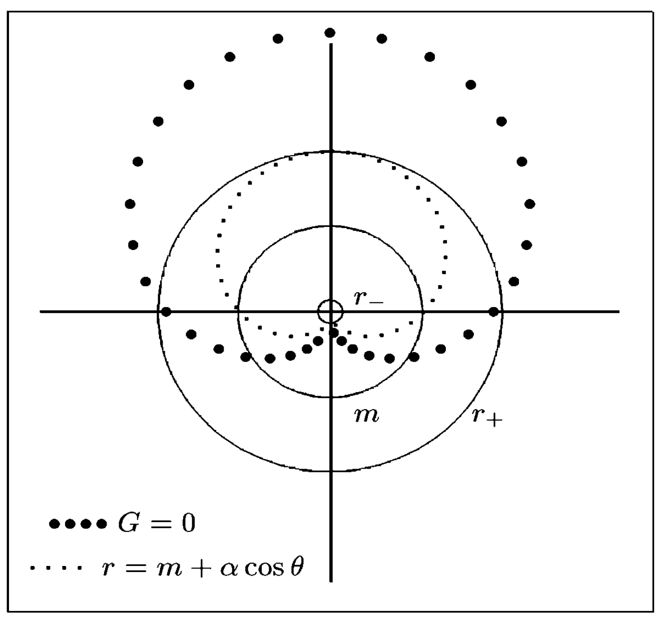

FIG. 16. Horizon slices for the stationary solution generated from Kerr spacetime adding a semi-infinite rod (a_0, ∞); $r_\pm = m \pm \sqrt{m^2 - a^2}$.

structure resembles the Emparan-Reall black ring [1], as shown in Fig. 15. The metric is given by the following expression

$$ds^2 = e^{\sigma}\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + \gamma_{ab}dx^a dx^b, \qquad a, b = t, \phi, \omega, \qquad e^{\sigma} = \frac{m^2}{a_0}\frac{(x+1)^2(1-y)^2}{(x-y)^3}[(1+px)^2 + q^2y^2],$$
$$g_{tt} = -\frac{1}{a_0(x+1)(1-y)}\frac{(p^2x^2+q^2y^2-1)}{[(1+px)^2+q^2y^2]}, \qquad g_{t\phi} = -\frac{2a(1+y)}{a_0(x+1)}\frac{(1+xp)}{[(1+px)^2+q^2y^2]}, \tag{84}$$
$$g_{\phi\phi} = -\frac{(1+y)}{a_0(x+1)}\frac{\{4a^2(1-y^2)(1+xp)^2 - \alpha^2(x^2-1)[(1+px)^2+q^2y^2]^2\}}{[(1+px)^2+q^2y^2][p^2x^2+q^2y^2-1]}, \qquad g_{\omega\omega} = [a_0(x+1)(1-y)]^2.$$

Finally the scalar $\kappa_{\{t,r\}}$ for the solution (84) amounts to

$$\kappa_{\{t,r\}} = -\frac{a_0 m^2}{\alpha}\frac{\Delta}{\Sigma}\frac{(r-m-\alpha\cos\theta)}{(r-m+\alpha)^4(1-\cos\theta)^2} \times \frac{G(r,\theta)^2}{[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]^2},$$
$$G(r,\theta) = \frac{3\alpha}{2}(\cos\theta+1)[(r^2+a^2)^2 - \Delta a^2\sin^2\theta] - (r-m+\alpha)(r-m-\alpha\cos\theta) \times [r(2r^2+a^2+a^2\cos^2\theta) + a^2 m\sin^2\theta]. \tag{85}$$

The corresponding rod structure resembles the one of a black ring due to Emparan-Reall [1], however, the horizons rather correspond to the 5D rotating black hole and not the ring. The marginally trapped surfaces are located at $\Delta = 0$ and $(r_h - m - \alpha\cos\theta) = 0$, or $r_\pm = m \pm \sqrt{m^2 - a^2}$ and $r_h = m + \sqrt{m^2 - a^2}\cos\theta$. A new marginally trapped surface arises from $G(r, \theta) = 0$ and has the shape of the previously studied $\mathcal{U} = 0$ (Fig. 13), but turned upside down. The profiles of horizons are shown in Fig. 16.

The generated stationary metrics (79) and (84) acquire a simpler form in Boyer-Lindquist coordinates, the expressions are presented in the Appendix C.

VIII. CONCLUSIONS

In this paper we addressed the construction in a simple and controllable way of higher dimensional vacuum solutions to Einstein equations which depend, at most, on two variables. The four-dimensional vacuum seed metrics with G_2 symmetry are generalized to include massless dilatons, which serve to lift the solutions to higher dimensions. The method works both, in the case when the two commuting Killing vectors of the seed are spacelike as well as when one of the Killing vectors is timelike. The "translation" of the algorithm to the case of one spacelike and one timelike Killing vector is given here for the first time.

The algorithm is illustrated with the generation of some static and stationary solutions. Starting with Minkowski, Schwarzschild, and Kerr seeds and "adding" rods to the fifth dimension, deformed 5D black hole solutions were generated. The corresponding rod structure of some of these solutions resembles the Myers-Perry black hole or the Emparan-Reall black ring, however the topology of the horizons is rather different. The generated solutions present distorted horizons due to the presence of extra sources.

We have seen that the rod structure alone does not reflect some important properties of spacetime, since it is insensitive to the exponents or powers of the metric functions. Neither rod directions are apparent from the metric expressions in the sense that even for static metrics the rods may have crossed components aligned with spacelike and timelike Killing directions. Therefore in order to characterize these spacetimes one must perform the singularity analysis and study their horizons. Nevertheless, by imposing conditions on asymptotic behavior of spacetimes as well as certain physical properties one may show [6] how the rod structure is important to single out such a spacetime.

In future works it would be interesting to consider five-dimensional solutions with double rotation. Such solutions can be obtained by lifting stationary solutions with both electromagnetic and scalar fields. In this case, the second rotation will be induced by the crossed terms which appear due to the lifting of the electromagnetic degrees of freedom. These solutions will be discussed elsewhere.

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APPENDIX A: EINSTEIN-SCALAR EQUATIONS IN WEYL COORDINATES

Here we present for completeness the Einstein-scalar coupled field equations in Weyl coordinates. The general stationary axisymmetric line element can be put

$$ds^2 = -e^U(dt + \mathrm{A}d\phi)^2 + e^{-U}r^2d\phi^2 + e^{\sigma}(dr^2 + dz^2). \tag{A1}$$

The Einstein field equations are:

(a) The U-A equations

$$U_{rr} + (1/r)U_r + U_{zz} + \frac{e^{2U}}{4r^2}[A_r^2 + A_z^2] = 0, \quad (A2)$$

$$\left(\frac{e^{2U}A_r}{r}\right)_r + \left(\frac{e^{2U}A_z}{r}\right)_z = 0. \quad (A3)$$

(b) The Φ equation

$$\Phi_{rr} + \frac{1}{r}\Phi_r + \Phi_{zz} = 0. \quad (A4)$$

(c) And finally, the σ equation

$$\sigma_r + U_r = r[\Phi_r^2 - \Phi_z^2] + \frac{r}{2}[U_r^2 - U_z^2] - \frac{e^{2U}}{2r}[A_r^2 - A_z^2], \quad (A5)$$

$$\sigma_z + U_z = 2r\Phi_r\Phi_z + rU_rU_z - \frac{e^{2U}}{r}A_rA_z. \quad (A6)$$

When $A = 0$ we deal with static solutions. Notice that the contribution of the function U to the nonlinear σ is identical to that of Φ. The solutions are defined by U, Φ, and A, while the function σ is obtained by quadratures.

APPENDIX B: ANALYSIS OF THE ROD STRUCTURE FOR THE KERR 5D SOLUTION (79)

To determine the direction of each rod we follow the steps of Sec. III in [3], for the solution (79). The equation to analyze for each interval on the z axis is $g_{ij}\vec{v} = 0$, explicitly

$$e^{-(2\Phi/\sqrt{3})}\begin{pmatrix} \gamma_{tt} & \gamma_{t\phi} & 0 \\ \gamma_{\phi t} & \gamma_{\phi\phi} & 0 \\ 0 & 0 & e^{(6\Phi/\sqrt{3})} \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = 0, \quad (B1)$$

where $\Phi = \frac{\sqrt{3}}{2}\log[\alpha(x+1)(1+y)]$ is the introduced rod, while $\vec{v}$ is the direction of the rod corresponding to the analyzed interval and γ_{ab} denote the seed metric functions. The analysis is performed in the limit $r \to 0$ that is

$$x = \frac{|z+\alpha| + |z-\alpha|}{2\alpha}, \qquad y = \frac{|z+\alpha| - |z-\alpha|}{2\alpha}. \quad (B2)$$

The rod structure is as follows:

(i) The semi-infinite spacelike rod $z \in (-\infty, -\alpha)$ corresponds to $x = -\frac{z}{\alpha}$ and $y = -1$. Substituting in (79), we get

$$\begin{aligned} g_{tt} &= \frac{p^2(1+\frac{z}{\alpha})}{\alpha(1+y)[(1-\frac{pz}{\alpha})^2 + q^2]}, \\ g_{t\phi} &= -\frac{4a}{\alpha}\frac{(1-\frac{pz}{\alpha})}{(1-\frac{z}{\alpha})[(1-\frac{pz}{\alpha})^2+q^2]}, \\ g_{\phi\phi} &= \frac{2\alpha}{p^2}\frac{[(1-\frac{pz}{\alpha})^2+q^2]}{(1-\frac{z}{\alpha})}, \\ g_{\omega\omega} &= \alpha^2\left(1-\frac{z}{\alpha}\right)^2(1+y)^2 = 0. \end{aligned} \quad (B3)$$

Note that $g_{\omega\omega} = 0$ and g_{tt} diverges in this interval. The vanishing of $g_{\omega\omega}$ means that the rod $(-\infty, -\alpha)$ is entirely located along the ∂_ω direction, i.e. $v^1 = v^2 = 0$, $v^3 = 1$.

The analysis is analogous for the rod $z \in (\alpha, \infty)$ that corresponds to $x = \frac{z}{\alpha}$ and $y = 1$. Substituting in (79) we obtain that $g_{tt} \neq 0$, $g_{t\phi} = g_{\phi\phi} = 0$, while $g_{\omega\omega} \neq 0$. Solving the system $g_{ij}\vec{v} = 0$ gives that the rod (α, ∞) has the direction $v^1 = v^3 = 0$ and $v^2 = 1$, i. e. it is situated along ∂_ϕ.

(ii) The finite timelike rod $(-\alpha, \alpha)$ corresponds to $y = \frac{z}{\alpha}$ and $x = 1$. Substituting in (79) we get

$$\begin{aligned} g_{tt} &= \frac{q^2(1-\frac{z}{\alpha})}{2\alpha[(1+p)^2 + q^2(\frac{z}{\alpha})^2]}, \\ g_{t\phi} &= -\frac{a}{\alpha}\frac{(1+p)(1-\frac{z}{\alpha})}{[(1+p)^2+q^2(\frac{z}{\alpha})^2]}, \\ g_{\phi\phi} &= \frac{2a^2(1+p)^2}{\alpha q^2}\frac{(1-\frac{z}{\alpha})}{[(1+p)^2+q^2(\frac{z}{\alpha})^2]}, \\ g_{\omega\omega} &= 4\alpha^2\left(1+\frac{z}{\alpha}\right)^2. \end{aligned} \quad (B4)$$

Solving the system $g_{ij}\vec{v} = 0$ gives that the rod (α, ∞) has components along the two Killing directions ∂_t and ∂_ϕ, $v^3 = 0$, $v^1 = 1$, and $v^2 = \Omega = \frac{q}{2m(1+p)}$. The corresponding rod structure is shown in Fig. 12.

APPENDIX C: 5D STATIONARY METRICS IN BOYER-LINDQUIST COORDINATES

The generated stationary metrics with Kerr as seed, expressed in Boyer-Lindquist coordinates, are given below. At the end we include the Schwarzschild 4D solution in Weyl coordinates. The transformation between prolate spheroidal coordinates and Boyer-Lindquist coordinates is $x = (r-m)/\alpha$, $y = \cos\theta$.

$$
\begin{aligned}
ds^2 &= e^{\sigma}\left(\frac{dr^2}{\Delta} + d\theta^2\right) + g_{ij}dx^i dx^j, \qquad i, j = t, \phi, \omega,\\
e^{\sigma} &= \frac{\alpha}{a_0}\Sigma\frac{(r-m+\alpha)^2(1\pm\cos\theta)^2}{(r-m\pm\alpha\cos\theta)^3},\\
g_{tt} &= -\frac{\alpha}{a_0}\frac{(\Delta - a^2\sin^2\theta)}{\Sigma(r-m+\alpha)(1\pm\cos\theta)},\\
g_{t\phi} &= -\frac{2am\alpha(1\mp\cos\theta)r}{a_0(r-m+\alpha)\Sigma},\\
g_{\phi\phi} &= -\frac{\alpha(1\mp\cos\theta)}{a_0(r-m+\alpha)\Sigma}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta],\\
g_{\omega\omega} &= \frac{a_0^2}{\alpha^2}(r-m+\alpha)^2(1\pm\cos\theta)^2,\\
\Delta &= r^2 - 2mr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta.
\end{aligned}
\tag{C1}
$$

The upper sign corresponds to a rod structure of Kerr when the semi-infinite rod $(-\infty, -a_0)$ is inserted, while the lower sign is for the spacetime generated from Kerr with the inserted semi-infinite rod (a_0, ∞), both with the scalar charge $A = \frac{\sqrt{3}}{2}$. In the first case we took $a_0 = \alpha$.

The Schwarzschild solution in Weyl coordinates is given by the line element

$$
\begin{aligned}
ds^2 &= -e^U dt^2 + e^{\sigma^{\rm vac}}(dr^2 + dz^2) + r^2 e^{-U} d\phi^2,\\
\sigma^{\rm vac} &= -U + \gamma, \qquad U = -\log\left[\frac{m - z + r_-}{-m - z + r_+}\right],\\
\gamma &= \log\left[\frac{(r_- + r_+)^2 - 4m^2}{4r_- r_+}\right],
\end{aligned}
\tag{C2}
$$

where $r_{\pm}^2 = (m \pm z)^2 + r^2$.

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