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DBI-essence

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Models where the dark energy is a scalar field with a nonstandard Dirac-Born-Infeld (DBI) kinetic term are investigated. Scaling solutions are studied and proven to be attractors. The corresponding shape of the brane tension and of the potential is also determined and found to be, as in the standard case, either exponentials or power law of the DBI field. In these scenarios, in contrast to the standard situation, the vacuum expectation value of the field at small redshifts can be small in comparison to the Planck mass which could be an advantage from the model building point of view. This situation arises when the present-day value of the Lorentz factor is large, this property being *per se* interesting. Serious shortcomings are also present such as the fact that, for simple potentials, the equation of state appears to be too far from the observational favored value -1. Another problem is that, although simple stringy-inspired models precisely lead to the power-law shape that has been shown to possess a tracking behavior, the power index turns out to have the wrong sign. Possible solutions to these issues are discussed.

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I. INTRODUCTION

Since the discovery of the fact that the expansion of the Universe is presently accelerated [1–10], various suggestions have been made in order to explain this observational fact. Among them is the hypothesis of dark energy, a fluid with a negative pressure representing about 70% of the total energy density in the Universe. The question of the physical nature of the dark energy has, of course, been widely discussed. The most natural candidate, still perfectly compatible with all the data available, is the cosmological constant. However, the difficulty to reconcile the value of Λ deduced from the observations with the value calculated theoretically [11] (maybe too naively?) has prompted the study of alternatives. Clearly, a simple scalar field, a "quintessence" field, is a natural candidate for such an alternative [12–16]. Among all the possibilities, scalar fields with inverse power-law potentials have attracted a lot of interest because, in this case, there is a solution of the equations of motion that is an attractor [12]. This means that the present-day behavior of the Universe is insensitive to the initial conditions. Usually, the attractor solution is a scaling solution, i.e. a solution for which the energy density scales as a power of the scale factor [17].

If the above mentioned route is correct, then another interesting issue is whether a candidate for quintessence in high energy physics can be identified. Clearly, this cannot be done without going beyond the standard model of particle physics. In particular, it would be very interesting to achieve this goal in string theory since it is presently our best candidate as a unified theory [18].

Recently, there have been many works aiming at connecting string theory with inflation which is also a phase of accelerated expansion (but taking place in the very early Universe at a much higher energy scale). For this purpose, new ideas in string theory based on the concept of branes have revealed themselves especially fruitful. In particular, scenarios where the inflaton is interpreted as the distance between two branes moving in the extra dimensions along a warped throat have given rise to many interesting studies [19–23]. In this article, we want to investigate whether the same kind of ideas can lead to sensible dark energy scenarios.

At the technical level, scenarios of the type mentioned above lead to scalar field models where the kinetic term is noncanonical. More precisely, the kinetic term has a Dirac-Born-Infeld (DBI) form. Physically, this originates from the fact that the action of the system is proportional to the volume traced out by the brane during its motion. This volume is given by the square root of the induced metric which automatically leads to a DBI kinetic term. Therefore, as the first step toward a scenario of "DBI-essence," it is first necessary to understand whether scaling and attractor solutions are still present when the scalar field has a DBI kinetic term. This question constitutes the main target of the present article.

This paper is organized as follows. In Sec. II, we briefly review the scaling properties of a quintessence field with a standard kinetic term. Then, in Sec. III, we reconsider this question but with a DBI kinetic term. In particular, we compare the DBI results with the standard ones. In Sec. IV, we study the behavior of DBI-essence at small redshifts.

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Since, in this case, the scalar field is no longer a test field, this requires numerical computations. Finally, in Sec. V, we present our conclusions and discuss the open issues that should be studied in the future.

II. SCALING SOLUTIONS WITH A STANDARD KINETIC TERM

We consider a spatially flat Friedmann-Lemaitre-Robertson-Walker universe containing a perfect fluid and a scalar field ϕ. Assuming that the scalar field and the perfect fluid are separately conserved, the equations of motion are given by

$$H^2 = \frac{\kappa}{3}(\rho + \rho_\phi), \tag{1}$$

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{2}$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0, \tag{3}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and ρ and p are, respectively, the energy density and the pressure of the perfect fluid. The dot denotes a derivative with respect to cosmic time and the quantity κ is defined by $\kappa \equiv 8\pi/m_{\rm Pl}^2$. In the following, we assume that the perfect fluid has a constant equation of state parameter $w \equiv p/\rho$, the two cases of main interest being $w = 1/3$ for the radiation-dominated era and $w = 0$ for the matter-dominated era. In this case, the conservation equation (2) can be integrated exactly and leads to the familiar behavior $\rho \propto a^{-3(1+w)}$. Moreover, if we further assume that the scalar field is a test field and that the evolution of the background geometry is mainly controlled by the perfect fluid, then one has $a(t) \propto t^{2/[3(1+w)]}$ or, for the Hubble parameter, $H = 2/[3(1+w)t]$.

Let us first briefly recall the scaling solutions in the simple case where the scalar field has a standard kinetic term. In this situation, the energy density and the pressure are given by the familiar expressions

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \qquad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{4}$$

If one inserts these expressions into the conservation equation for the scalar field (3), then one obtains the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{5}$$

where the prime denotes a derivative with respect to ϕ. Then, we seek potentials $V(\phi)$ such that the energy density of the test scalar field scales as a power law of the scale factor, namely $\rho_\phi \propto a^{-3(1+w_\phi)}$ where $w_\phi \equiv p_\phi/\rho_\phi$ is a constant. It has been established in Ref. [12] that scaling solutions exist if the potential has an exponential shape,

$$V(\phi) = M^4 e^{-\lambda\phi}, \tag{6}$$

where λ is a constant or is of the Ratra-Peebles type (i.e. inverse power law of the field), namely

$$V(\phi) = M^{4+\alpha}\phi^{-\alpha}. \tag{7}$$

In the first case, the particular solution leading to the scaling behavior reads

$$\phi(t) = \frac{2}{\lambda}\ln\left(\frac{t}{t_0}\right), \tag{8}$$

the constant λ and the mass scale M being linked by the relation

$$\lambda^2 M^4 t_0^2 = \frac{2(1-w)}{1+w}. \tag{9}$$

As is well known, the quintessence equation of state parameter is just given by the equation of state of the background perfect fluid, $w_\phi = w$. The particular solution (8) is important because it is an attractor. This means that the final (i.e. present-day) evolution of the field is in fact independent of the initial conditions. At the technical level, this can be seen by studying small (linear) perturbations around the attractor. The eigenvalues of the perturbations around the critical point can be expressed as

$$\lambda_\pm = \frac{1}{2m}\left[(m-6) \pm \sqrt{(m-6)^2 + 8m(m-6)}\right], \tag{10}$$

where we have defined $m \equiv 3(1+w)$. Since $w < 1$, one has $m < 6$ and the eigenvalues are negative and one has a stable spiral point. Moreover, for the particular solution (8), one has

$$\frac{d^2V}{d\phi^2} = \frac{9}{2}(1 - w_\phi^2)H^2. \tag{11}$$

This is an important formula because it implies that $\phi \sim m_{\rm Pl}$ today. Indeed, $V'' \sim V/\phi^2$ and $H^2 \sim V/m_{\rm Pl}^2$ when the field starts dominating the energy density content of the Universe; equating these two quantities leads to the above mentioned conclusion. For this reason, if one considers the supersymmetric extensions of the standard model of particle physics, a sensible model building of quintessence is only possible in a supergravity (SUGRA) framework [14,15]. Indeed, the SUGRA corrections (that are ignored in global supersymmetry) are here of order $\langle Q\rangle/m_{\rm Pl} \sim 1$ and cannot be neglected at small redshifts. However, a well-known difficulty of the exponential case is that the property $w_\phi = w$ implies that the scalar field cannot drive an accelerated expansion. This is why the inverse power law case seems to be more interesting.

In the case of the Ratra-Peebles potential (7), there also exists an exact particular solution of the Klein-Gordon equation that is an attractor. It reads

$$\phi = \phi_0\left(\frac{t}{t_0}\right)^{2/(\alpha+2)}, \tag{12}$$

where the quantity ϕ_0 is linked to the mass scale M by the formula

$$M^{4+\alpha}\phi_0^{-\alpha-2}t_0^2 = \frac{2}{\alpha(\alpha+2)}\left(\frac{2}{1+w} - \frac{\alpha}{\alpha+2}\right). \quad (13)$$

For this particular solution, the equation of state parameter can be expressed as

$$w_\phi = \frac{\alpha w - 2}{\alpha + 2}. \quad (14)$$

As expected, in the limit $\alpha \rightarrow +\infty$, one recovers the exponential case, $w_\phi = w$. However, the crucial difference with the exponential potential is that one can now have $w_\phi < w$, that is to say, the scalar field energy density can now scale more slowly than the background fluid and, hence, eventually dominates, causing the Universe to accelerate. Moreover, the solution (12) is also an attractor as revealed by a dynamical system analysis. Indeed, the eigenvalues of small perturbations around the critical point read

$$\lambda_\pm = \frac{(2n - m - 6) \pm \sqrt{(2n - m - 6)^2 + 8m(n-6)}}{2m}, \quad (15)$$

where we have defined $n \equiv 3(1 + w_\phi)$. Again, one can show that there exists a stable spiral point as long as both eigenvalues are negative, which is equivalent to $2n - m - 6 < 0$. On this attractor, the evolution of the second order derivative of the potential is given by

$$\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2} = \frac{9}{2}\frac{\alpha+1}{\alpha}(1 - w_\phi^2)H^2, \quad (16)$$

and one can check that this last equation reproduces the corresponding equation in the exponential case when $\alpha \rightarrow +\infty$. Again, this prompts a SUGRA treatment of the model building issue since one still has $\phi \sim m_{\rm Pl}$.

III. SCALING SOLUTIONS WITH A DBI KINETIC TERM

Let us now consider that the dark energy scalar field is a DBI scalar field. In this case, the action of the field can be written as

$$S_{\rm DBI} = -\int \mathrm{d}^4 x a^3(t)\left[T(\phi)\sqrt{1 - \frac{\dot\phi^2}{T(\phi)}} + V(\phi) - T(\phi)\right], \quad (17)$$

where $T(\phi)$ is the tension and $V(\phi)$ is the potential. In this article, gravity is assumed to obey four-dimensional general relativity with a standard Einstein-Hilbert Lagrangian. Then, it is easy to obtain the corresponding energy density and pressure of the scalar field by varying the action with respect to the metric tensor. They read

$$\rho_\phi = (\gamma - 1)T(\phi) + V(\phi),$$
$$p_\phi = \frac{\gamma - 1}{\gamma}T(\phi) - V(\phi), \quad (18)$$

where the quantity γ is reminiscent of the usual relativistic Lorentz factor and is given by

$$\gamma \equiv \frac{1}{\sqrt{1 - \dot\phi^2/T(\phi)}}. \quad (19)$$

The expressions (18) of the energy density and pressure of the DBI field should be compared to their standard counterpart; see Eqs. (4). As usual, if one inserts Eqs. (18) in the conservation equation (3), one obtains the DBI Klein-Gordon equation, namely

$$\ddot\phi - \frac{3T'(\phi)}{2T(\phi)}\dot\phi^2 + T'(\phi) + \frac{3H}{\gamma^2}\dot\phi + \frac{1}{\gamma^3}[V'(\phi) - T'(\phi)] = 0. \quad (20)$$

We notice that the equation of motion for ϕ is quite complicated compared to Eq. (5) despite the fact that the conservation equation has retained its standard form.

Let us also compare with other works in the literature. Let us start with K-essence where the action can be written as

$$S = \int \mathrm{d}^4 x\sqrt{-g}\,p(\phi, X), \quad (21)$$

where $X = (\nabla\phi)^2/2$. Clearly the action (17) is a special case of the above action. However, as first discussed in Ref. [24], K-essence usually means that the potential term vanishes and the negative pressure of the scalar field is realized only by considering the kinetic term [25–28]. On the other hand, our model cannot realize the negative pressure without the potential term, as shown below. Therefore, this class of models cannot encompass Eq. (17). Another model related to the present study is the case where the dark energy field is a tachyon for which the action is given by [29–31]

$$S = -\int \mathrm{d}^4 x\sqrt{-g}V(T)\sqrt{1 + \frac{1}{M^4}g^{\mu\nu}\partial_\mu T\partial_\nu T}, \quad (22)$$

where M is a fundamental scale and $V(T)$ is a potential which, of course, needs not to be the same function as $V(\phi)$ in Eq. (17). This class of theory is equivalent to the case studied here (through a redefinition of the field) only when the potential in Eq. (17) vanishes. Let us also notice that when $V(T)$ is constant the model is in fact equivalent to the Chaplygin gas with the equation of state $p \propto -1/\rho$ [32,33]. Therefore, beside the fact that the search of scaling solutions has not yet been investigated in this type of models, we conclude that the class of scenarios under scrutiny in this paper was not considered before.

Let us now return to the DBI case. As a warm-up, let us find the scaling solutions in the simple case where the potential vanishes. As already mentioned before, this means that we now seek tensions $T(\phi)$ such that the energy density of the test DBI scalar field scales as a $\rho_\phi \propto a^{-3(1+w_\phi)}$. From Eqs. (18), it is easy to show that γ is, in this case, constant and given by $\gamma = 1/w_\phi$. Then, the formula of the energy density, $\rho_\phi = (\gamma - 1)T(\phi) \propto a^{-3(1+w_\phi)}$ immediately gives the scaling in time of $T(\phi)$ which in turn, combined with $\dot{\phi}^2/T(\phi) = (\gamma^2 - 1)/\gamma^2$ and the fact that γ is constant, implies that

$$\dot{\phi} \propto t^{-(1+w_\phi)/(1+w)}. \tag{23}$$

This equation is easily solved. Let us start with $w_\phi = w$. In this case, one has $\phi \propto \ln t$ and, as a consequence,

$$T(\phi) = M^4 \mathrm{e}^{-\lambda\phi}, \tag{24}$$

where λ is a constant. Again, this case is very similar to the situation where we have a standard kinetic term and an exponential potential. As a consequence, this model suffers from the standard phenomenological problems. Since the scalar fields exactly track the background matter, one cannot have a large enough contribution of dark energy density today without spoiling big bang nucleosynthesis. Moreover, the scalar field behaves as matter today and, therefore, cannot cause the acceleration of the Universe.

On the other hand, if $w_\phi \neq w$, then the scalar field is just a power law of the cosmic time which implies that $T(\phi)$ can be expressed as

$$T(\phi) = M^{4+\alpha}\phi^{-\alpha}, \tag{25}$$

where M is a mass scale and w_ϕ is related to α and the background equation of state parameter w through the relation

$$w_\phi = \frac{\alpha w - 2}{\alpha + 2} = \frac{1}{\gamma}. \tag{26}$$

Interestingly enough, as discussed before, this is exactly the equation obtained when there is a standard kinetic term with a Ratra-Peebles potential; see Eq. (14). However, in the present case, $w_\phi = 1/\gamma > 0$. This means that the solution is physically relevant only if $w > 2/\alpha$ which excludes the case $w = 0$, at least for $\alpha > 0$.

Since it appears that the previously described situation is not satisfactory, we now envisage the case where the potential $V(\phi)$ is nonvanishing. In order to deal with this problem, we assume that γ is a constant. Without this hypothesis, the problem is technically very complicated but the actual convincing argument in favor of this assumption is that the corresponding scaling solutions (with γ constant) are attractors; see below. Then, the crucial observation is that Eq. (23) is still valid because, in its derivation, one has never assumed that $V = 0$. This implies that, as in the case of a vanishing potential, scaling solutions exist for tensions $T(\phi)$ given by Eq. (24) or Eq. (25). Then, the Klein-Gordon equation (20) can be used to determine the potential. Straightforward manipulations lead to

$$\frac{V(\phi)}{T(\phi)} = \frac{\gamma^2 - 1}{\gamma}\left(\frac{1}{1 + w_\phi} - \frac{\gamma}{1 + \gamma}\right), \tag{27}$$

that is to say the potential is proportional to the tension and has also the exponential shape or inverse power-law shape. It is interesting to notice that, when the field is on tracks in the standard kinetic case, the potential term is also proportional to the kinetic term, that is to say the ratio of the potential term to the kinetic term $K \equiv \dot{\phi}^2/2$ is a constant given by $V(\phi)/K(\phi) = 2/(1 + w_\phi) - 1$.

In the case of an exponential potential, the exact solution (from now on, we use a subscript "e" to denote the quantities that are evaluated with the exact particular solution of the Klein-Gordon equation) reads $\phi_\mathrm{e}(t) = 2/\lambda \ln(t/t_0)$ with $\lambda^2 M^4 t_0^2 = 4\gamma_\mathrm{e}^2/(\gamma_\mathrm{e}^2 - 1)$ and the equation of state parameter is $w_\phi = w$. In the Ratra-Peebles case, one has $\phi_\mathrm{e}(t) = \phi_0 (t/t_0)^{2/(\alpha+2)}$ with

$$M^{4+\alpha}\phi_0^{-\alpha-2} t_0^2 = \frac{4\gamma_\mathrm{e}^2}{(\alpha + 2)^2(\gamma_\mathrm{e}^2 - 1)}, \tag{28}$$

and the equation of state has the standard form given by Eq. (14). It is important to notice that, because we deal with a modified Klein-Gordon equation, the expressions of $\lambda^2 M^4 t_0^2$ and $M^{4+\alpha}\phi_0^{-\alpha-2} t_0^2$ are different from the ones obtained previously; see Eqs. (9) and (13). Let us also remark that these formulas can either be obtained from the requirement that the Lorentz factor is constant or by brute force calculation using the Klein-Gordon equation.

Let us now study the behavior of small perturbations around the particular solutions. Let us first start with the exponential case. For this purpose, we rewrite the equation of motion in terms of $u(\tau)$ defined by $u \equiv \lambda(\phi - \phi_\mathrm{e})$ and $t \equiv \mathrm{e}^\tau$. If we write $p \equiv u'$, then one obtains the system

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}\tau} = &-\left(5 + \frac{2}{\gamma^2}\frac{1}{1 + w}\right)p - \frac{3}{2}p^2 - \frac{4}{1 + w}\left(\frac{1}{\gamma^2} - \frac{1}{\gamma_\mathrm{e}^2}\right) \\ &+ \lambda^2 M^4(\mathrm{e}^{-u} - 1) \\ &+ \lambda^2 M^4\left[\frac{\gamma_\mathrm{e}^2 - 1}{\gamma_\mathrm{e}}\left(\frac{1}{1 + w} - \frac{\gamma_\mathrm{e}}{1 + \gamma_\mathrm{e}}\right) - 1\right] \\ &\times\left(\frac{\mathrm{e}^{-u}}{\gamma^3} - \frac{1}{\gamma_\mathrm{e}^3}\right) = 0,\end{aligned} \tag{29}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = p, \tag{30}$$

where, now, the quantity γ is no longer a constant and can be written as

$$\gamma = \left[1 - \frac{\gamma_e^2 - 1}{\gamma_e} e^u \left(1 + \frac{p}{2}\right)^2\right]^{-1/2}. \tag{31}$$

As a consequence, we see that the critical point is $(p, u) = (0, 0)$. Notice that, for the critical point, one checks that $\gamma = \gamma_e$. We now consider the behavior of small perturbations $(\delta p, \delta u)$ around the critical point $(0, 0)$. It is straightforward to establish that

$$\frac{d}{d\tau}\begin{pmatrix} \delta p \\ \delta u \end{pmatrix} = \begin{pmatrix} 1 - \frac{2}{1+w} & 2 - \frac{2}{1+w}\frac{1+\gamma_e^2}{\gamma_e^2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \delta p \\ \delta u \end{pmatrix}. \tag{32}$$

Then, the eigenvalues of this matrix can be expressed as

$$\lambda_\pm = \frac{1}{2m}\left[(m-6) \pm \sqrt{(m-6)^2 + 8m\left(m - 3\frac{\gamma_e^2+1}{\gamma_e^2}\right)}\right]. \tag{33}$$

This expression should be compared with Eq. (10). The only difference is the presence of the factor γ_e in the last term inside the square root. Otherwise, and this is quite remarkable, the expression is the same. Let us also notice that the condition where the kinetic energy cannot exceed the total energy is equivalent to $n \le 3(\gamma_e + 1)/\gamma_e$. Since $\gamma_e \ge 1$, the condition $n \le 3(\gamma_e^2 + 1)/\gamma_e^2$ is stronger than the condition $n \le 3(\gamma_e + 1)/\gamma_e$ and, therefore, is not automatically satisfied in our case. We conclude that there is a stable spiral point if $n \le 3(\gamma_e^2 + 1)/\gamma_e^2$.

Let us now turn to the inverse power-law case. This time, the dimensionless function $u(\tau)$ is defined by $u(\tau) \equiv \phi/\phi_e$, the definition of the time τ remaining the same. Then, a straightforward calculation leads to the following system of equations:

$$\begin{aligned}\frac{dp}{d\tau} = &-\left(\frac{5\alpha+2}{\alpha+2} + \frac{2}{\gamma^2}\frac{1}{1+w}\right)p - \frac{3\alpha}{2}\frac{p^2}{u} - \frac{4}{1+w}\frac{u}{\alpha+2} \\ &\times\left(\frac{1}{\gamma^2} - \frac{1}{\gamma_e^2}\right) + \alpha M^{4+\alpha}\phi_0^{-\alpha-2}(u^{-\alpha-1} - u) \\ &+ \alpha M^{4+\alpha}\phi_0^{-\alpha-2}\left[\frac{\gamma_e^2-1}{\gamma_e}\left(\frac{1}{1+w} - \frac{\gamma_e}{1+\gamma_e}\right) - 1\right] \\ &\times\left(\frac{u^{-\alpha-1}}{\gamma^3} - \frac{u}{\gamma_e^3}\right) = 0,\end{aligned} \tag{34}$$

$$\frac{du}{d\tau} = p, \tag{35}$$

where, this time, the Lorentz factor can be written as

$$\gamma = \left\{1 - \frac{u^\alpha}{M^{4+\alpha}\phi_0^{-\alpha-2}}\left[p^2 + \frac{4up}{\alpha+2} + \frac{4u^2}{(\alpha+2)^2}\right]\right\}^{-1/2}. \tag{36}$$

It is clear from the above system that the critical point is now given by $(p, u) = (0, 1)$. The next step is to study the behavior of small perturbations $(\delta p, 1 + \delta u)$ around the critical point. One arrives at

$$\frac{d}{d\tau}\begin{pmatrix} \delta p \\ \delta u \end{pmatrix} = \begin{pmatrix} \frac{\alpha-2}{\alpha+2} - \frac{2}{1+w} & \frac{2\alpha}{\alpha+2} - \frac{2}{1+w}\frac{1+\gamma_e^2}{\gamma_e^2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \delta p \\ \delta u \end{pmatrix}. \tag{37}$$

As expected, in the limit $\alpha \to +\infty$, the above matrix exactly reproduces the matrix obtained in the exponential case; see Eq. (32). Then, the next step is to determine the eigenvalues. The result reads

$$\lambda_\pm = \frac{2n-m-6}{2m} \pm \frac{1}{2m} \times \sqrt{(2n-m-6)^2 + 8m\left(n - 3\frac{\gamma_e^2+1}{\gamma_e^2}\right)}. \tag{38}$$

This expression should be compared with Eq. (15). As was the case before, the modification introduced by the DBI kinetic term is only apparent in the last term inside the square root. Therefore, there is a stable spiral point if $n \le 3(\gamma_e^2 + 1)/\gamma_e^2$ and $2n - m - 6 < 0$ are satisfied.

Finally, on the attractor, in the exponential case, it is easy to establish that the following relation holds:

$$\frac{d^2V}{d\phi^2} = 9\gamma_e(1+w)\left[1 - (1+w)\frac{\gamma_e}{1+\gamma_e}\right]H^2. \tag{39}$$

This formula is the generalization of Eq. (11). Obviously, one can also establish the corresponding expression in the inverse power-law case. It reads

$$\frac{d^2V}{d\phi^2} = 9\frac{\alpha+1}{\alpha}\gamma_e(1+w_\phi)\left[1 - (1+w_\phi)\frac{\gamma_e}{1+\gamma_e}\right]H^2, \tag{40}$$

and this is equivalent to Eq. (16). This has important consequences for model building. Indeed, if one repeats the discussion after Eq. (11), then one arrives at the conclusion that

$$\phi \sim \frac{m_{Pl}}{\sqrt{\gamma_e}} \tag{41}$$

because the second term in the bracket in Eq. (40) cannot exceed unity. Therefore, if $\gamma_e \gg 1$, then the vacuum expectation value of the field is not necessarily large in Planck units. This is certainly an important advantage of the DBI models over the standard ones with respect to model building issues.

Finally, it is also worth commenting about the shape of the tension $T(\phi)$. From a stringy point of view, the inverse of $T(\phi)$ represents the warp factor of the throat in which the branes are living. A natural choice [21] is $T(\phi) \propto \phi^4$, that is to say $\alpha = -4$. Therefore, this case belongs to the class of tracking models considered here which is a nontrivial result. Unfortunately, the sign of the exponent is not the correct one. Indeed, for $\alpha = -4$, one has $w_\phi = 2w + 1 > w$ which means that, despite the presence of an attractor, the scalar field drops faster than the background fluid and, hence, can never dominate the matter content of the Universe.

IV. NUMERICAL CALCULATIONS

In this section, we investigate the behavior of the DBI scalar field at small redshifts, when it starts dominating the matter content of the Universe. In this situation, the assumption that it is a test field breaks down and numerical calculations are required.

We first check that the attractor is observed numerically. As a representative example, we have chosen to investigate the case $\alpha = 4$. In Fig. 1, we have represented the evolution of the DBI energy density for three different initial conditions (more precisely, the initial velocity is always the same and corresponds to an initial value of the Lorentz factor $\gamma_{\rm ini} = 5$ but different initial vacuum expectation values $\phi_{\rm ini}$ are considered). In order to have a DBI energy density today equal to 70% of the critical energy density, we have tuned the scale M of the brane tension $T(\phi)$; see Eq. (25). The mass scale of the potential is determined by Eq. (27) which implies that $V(\phi) = CM^{4+\alpha}\phi^{-\alpha}$ where C is defined by

$$C \equiv \frac{\gamma_{\rm e}^2 - 1}{\gamma_{\rm e}}\left(\frac{1}{1+w_\phi} - \frac{\gamma_{\rm e}}{1+\gamma_{\rm e}}\right) = \frac{\gamma_{\rm e}^2 - 1}{\gamma_{\rm e}}\left[\frac{\alpha+2}{\alpha(1+w)} - \frac{\gamma_{\rm e}}{1+\gamma_{\rm e}}\right]. \quad (42)$$

Choosing a value of C is in fact equivalent to choosing the value of the Lorentz factor on the attractor, $\gamma_{\rm e}$, during a phase of evolution characterized by the background equation of state w. So, for instance, in Fig. 1, we have chosen $\gamma_{\rm e} = 20$ and $w = 1/3$. This means that the attractor solution should be such that $\gamma_{\rm e} = 20$ during the radiation-dominated era. Given Eq. (42) and $\alpha = 4$, this choice implies that $C \sim 3.443$. In this case, there is also an attractor during the matter-dominated era but the corresponding value of the Lorentz factor is different. It is easy to show that it reads

$$\gamma_{\rm e}^{\rm cdm} = \frac{\alpha(C-1)}{4}\left[1 + \sqrt{1 + \frac{8(\alpha+2)}{\alpha^2(C-1)^2}}\right]. \quad (43)$$

In the present case, this gives $\gamma_{\rm e}^{\rm cdm} \sim 5.438$.

The attractor behavior is clearly seen in Fig. 1. For initial conditions $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-9}$ and $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-8}$, the attractor is joined during the radiation-dominated era while for $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-10}$, it is reached during the matter-dominated era.

In Fig. 2, we have represented the evolution of the equation of state for the same situation. Again, the attractor behavior is clearly noticed. We can even check numerically that, on the attractor, Eq. (14) is valid. Since we consider a model with $\alpha = 4$, the DBI equation of state during the radiation-dominated era should be $w_\phi \simeq -0.11$. Clearly, this is what is obtained in Fig. 2. The present-day value of

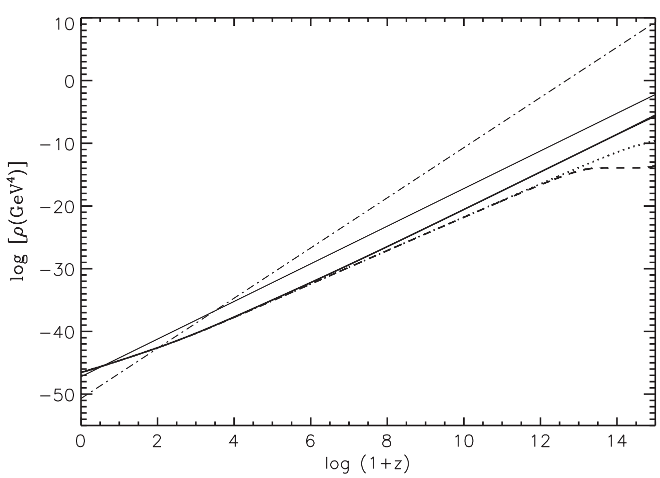

FIG. 1. Evolution of the DBI energy density for different initial conditions in the case where the potential is of the Ratra-Peebles type with $\alpha = 4$. The value of C is chosen to be $C \sim 3.443$ which corresponds to $\gamma_{\rm e} = 20$; see Eq. (42). The initial velocity $\dot\phi_{\rm ini}$ is always chosen such that $\gamma_{\rm ini} = 5$. The solid line corresponds to an initial vacuum expectation value of $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-10}$, the dotted line to $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-9}$ and the dashed line to $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-8}$. The energy density of radiation (dot-dashed line) and cold dark matter (dot-dot-dashed line) are also represented.

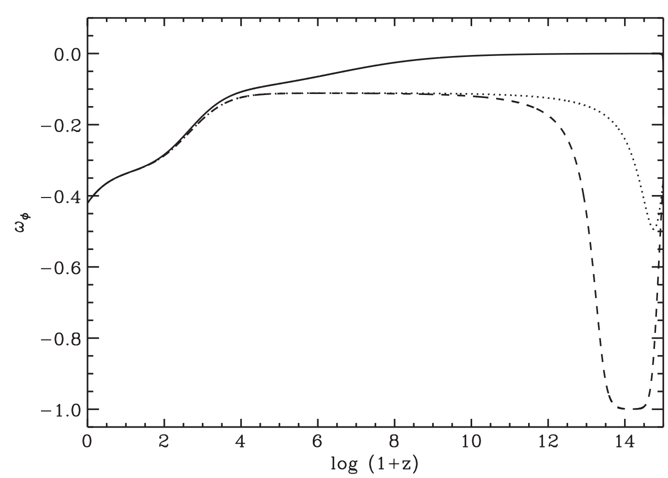

FIG. 2. Evolution of the DBI equation of state for different initial conditions in the case where the potential is of the Ratra-Peebles potential with $\alpha = 4$. As in Fig. 1, the value of C is chosen to be $C \sim 3.443$ which corresponds to $\gamma_{\rm e} = 20$ and the initial velocity $\dot\phi_{\rm ini}$ is such that $\gamma_{\rm ini} = 5$. The solid line corresponds to an initial vacuum expectation value of $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-10}$, the dotted line to $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-9}$ and the dashed line to $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-8}$. The final (present-day) value of γ is $\gamma_0 \sim 3.96$ and $\phi_0/m_{\rm Pl} \sim 1.37$. Finally, the equation of state is such that $w_0 \sim -0.42$, $w_1 \sim 6.68 \times 10^{-2}$.

the equation of state is $w_0 \simeq -0.42$ (the derivative of the equation of state at vanishing redshift being $w_1 \simeq 6.68 \times 10^{-2}$). The corresponding value for a scalar field with a standard kinetic term and the same Ratra-Peebles potential is $w_0 \simeq -0.487$. Firstly, and contrary to a naive expectation, the equation of state is not pushed toward -1. Therefore, it seems that we do not gain anything in comparison with the model with a standard kinetic term. Secondly, the value obtained seems to be too large given the constraints available on w_0. Even if one should put a damper on these constraints since they have not been obtained for the model under considerations here (usually, a simple law of the form $w = w_0 + w_1 z$ are used, which is clearly not valid for the model under consideration here, and this can cause a "bias problem," see Ref. [34]), the value is so far from -1 that the model is probably in trouble from the observational point of view. This is clearly a very serious problem for the class of models studied in the present article. One possibility is to decrease the value of α. For instance, $\alpha = 0.3$ implies $w_\phi \sim -0.9$. Of course, the corresponding model seems contrived and, in addition, in this case, a small value of the equation of state would also be obtained with a standard kinetic term. Another possibility would be to consider other shapes for the tension and the potential. The new shapes of $T(\phi)$ and $V(\phi)$ should approximatively reduce to the inverse power-law shape at large redshifts such that the attractor behavior is preserved and should differ from it at small redshifts in order to obtain an equation of state closer to -1. In the standard non-DBI case, such a mechanism can be realized, for instance, with the SUGRA potential [14,15].

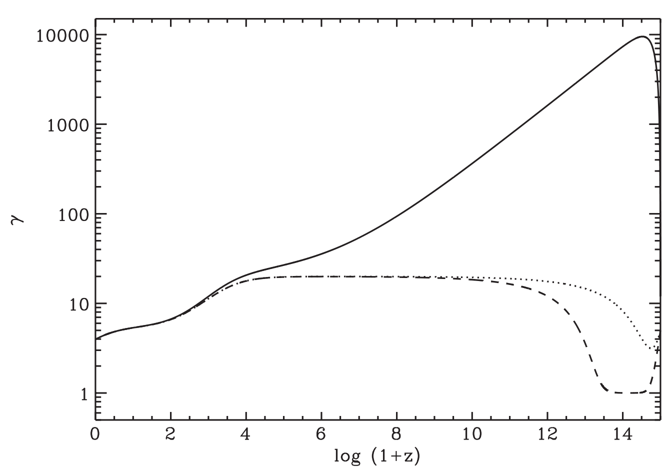

FIG. 3. Evolution of the Lorentz factor for different initial conditions for the Ratra-Peebles potential with $\alpha = 4$. The value of C is chosen to be $C \sim 3.443$ which corresponds to $\gamma_e = 20$ and the initial velocity $\dot{\phi}_{\rm ini}$ is chosen such that $\gamma_{\rm ini} = 5$. The solid line corresponds to an initial vacuum expectation value of $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-10}$, the dotted line to $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-9}$ and the dashed line to $\phi_{\rm ini}/m_{\rm Pl} \sim 10^{-8}$. The final value of the Lorentz factor is $\gamma_0 \sim 3.96$

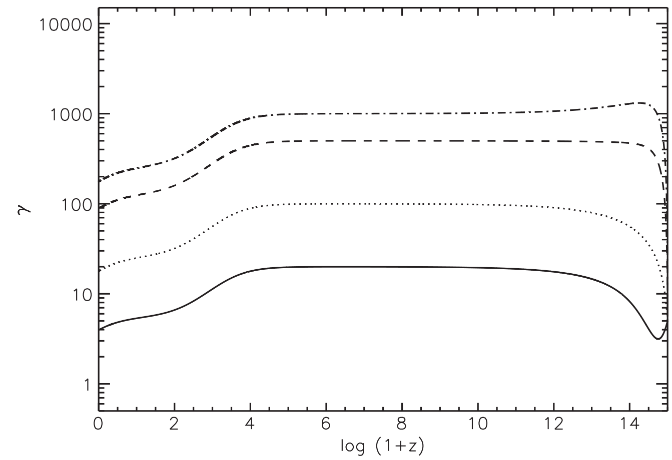

FIG. 4. Evolution of the Lorentz factor for different values of γ_e in the case where the potential is of the Ratra-Peebles type with $\alpha = 4$. The values considered are $\gamma_e = 20$ (solid line), $\gamma_e = 100$ (dotted line), $\gamma_e = 500$ (dashed line) and $\gamma_e = 1000$ (dot-dashed line). The corresponding values of the Lorentz factor today are $\gamma_0 \sim 3.96$, $\gamma_0 \sim 17.87$, $\gamma_0 \sim 88.56$ and $\gamma_0 \sim 176.17$ respectively. The final vacuum expectation values of the DBI field are $\phi_0/m_{\rm Pl} \sim 1.37$, $\phi_0/m_{\rm Pl} \sim 0.66$, $\phi_0/m_{\rm Pl} \sim 0.29$ and $\phi_0/m_{\rm Pl} \sim 0.2$ respectively.

Let us now study the evolution of the Lorentz factor γ. It is represented in Fig. 3 for different initial conditions, similar to the ones considered in the previous figures. In particular, the attractor value, valid during the radiation-dominated era, $\gamma_e = 20$, is clearly seen on this plot. The value $\gamma_e^{\rm cdm} \sim 5.4$, valid during the matter-dominated era, can also be noticed. An interesting point is that the present value of the Lorentz factor is far from 1. For the case under consideration, it is $\gamma_0 \sim 3.96$. This means that the non-standard kinetic term still plays a role even today. As noticed earlier, this can have important implications for model building issues since a large γ_e implies a small vacuum expectation value of the field. We have studied this point in more detail in Fig. 4 where the evolution of the Lorentz factor is represented for different values of γ_e. We notice that the larger γ_e, the larger the present-day value γ_0 and the smaller ϕ_0. For instance, for $\gamma_e = 1000$, one obtains $\gamma_0 \sim 176$ and $\phi_0/m_{\rm Pl} \sim 0.2 < 1$. This is certainly a desirable feature since, usually, vacuum expectation values of the order of the Planck mass are at the origin of many serious problems as, for instance, a coupling with the observable sector which violates the constraint on the presence of a fifth force and/or on the weak equivalence principle [35–38]. On the other hand, if the vacuum expectation value remains small in comparison with the Planck mass, then it could be difficult to use the SUGRA potential model to push the equation of state toward -1. Therefore, we face again the "no-go theorem" discussed recently in Refs. [36–38]: what is interesting from the cosmological point of view (a large vacuum expectation value in order to have w_0 close to -1) seems to be incompatible with local tests of gravity (a large vacuum expectation value usually means a strong coupling with ordinary matter).

V. CONCLUSIONS

In this section, we recap our main findings and discuss further issues that should be investigated. We have studied scenarios where the dark energy is a scalar field with a DBI kinetic term. We have shown that, if the brane tension and the potential possess either an exponential or a power-law shape, then there exist scaling solutions that are attractors. Moreover, if the Lorentz factor is large today, then the vacuum expectation value of the field can be small in comparison with the Planck mass. Let us also notice that the fact that the scaling solutions obtained in this article correspond to brane tensions of the power-law form, $T(\phi) \propto \phi^{-\alpha}$, is fairly remarkable since this general class of solutions encompasses the simple string-inspired models; see Ref. [20]. Unfortunately, for these stringy models, one typically finds $\alpha = -4 < 0$ for which the dark energy density drops faster than that of the background. This means that, despite the presence of an attractor, the model is not realistic because the dark energy can never dominate the energy density budget of our universe. Finally, maybe the most problematic aspect of the scenario is the fact that the equation of state today is too far from -1.

In order to improve the above described situation, one probably needs more complicated string-inspired models. In particular, one needs shapes of $T(\phi)$ and $V(\phi)$ that, for $\phi \ll m_{\rm Pl}/\sqrt{\gamma_{\rm e}}$ are of the power-law form (with $\alpha > 0$) in order to preserve the attractor, and, for $\phi \gg m_{\rm Pl}/\sqrt{\gamma_{\rm e}}$, deviate from this form in order to push the equation of state toward -1. Let us recall at this stage that this is exactly what the SUGRA model does, the characteristic scale being the Planck mass instead of $m_{\rm Pl}/\sqrt{\gamma_{\rm e}}$.

The fact that the present-day value of $\gamma_{\rm e}$ can be large is also an interesting feature of the models under scrutiny. For example, this implies that the sound velocity squared $c_{\rm s}^2$ can significantly deviate from 1 in contrast to the standard case. Indeed, in the DBI case, the sound velocity squared $c_{\rm s}^2$ is given by the following expression:

$$c_{\rm s}^2 = \frac{\partial p}{\partial X}\left(\frac{\partial \rho}{\partial X}\right)^{-1} = \frac{1}{\gamma_{\rm e}^2}, \tag{44}$$

where $p = p(X, \phi)$, $\rho = \rho(X, \phi)$ (and $X = \dot{\phi}^2/2$). A dark energy component with $c_{\rm s}^2 \ll 1$ implies less power on large scales and, hence, could account for the low multipoles of the cosmic microwave background anisotropies. Moreover, this would also produce peculiar features in the matter power spectrum as discussed in Refs. [39–41]. All these properties could be used to distinguish the DBI models from the standard ones. On more general grounds, it is clear that a complete calculation of the dark energy perturbations could bring new insights to the model.

It is also interesting to compare the case of dark energy with that of inflation. It seems that the DBI inflationary solution is also an attractor as studied in Ref. [42]. However, there are also important differences. Firstly, the energy scales involved are completely different: quintessence is a low-energy phenomenon contrary to inflation. Secondly, the inflaton field is never a test field (recall that the attractor solutions are found when the backreaction of the quintessence on the background geometry is neglected). Thirdly, the potentials have different shapes. In DBI inflation, one typically deals with potentials of the form $V_0 \pm m^2\phi^2$ and not with inverse power-law shapes. An important exception is the Coulomb potential, $V = M^4[1 - (\mu/\phi)^4]$, but one can show [22] that, if inflation is possible with such a potential, DBI-inflation is not. In other words, in this last case, inflation always occurs in the regime where $\gamma \simeq 1$, i.e. in a regime where the kinetic term can be considered as standard.

Finally, another interesting issue is that of the coupling of dark energy with the rest of the world. As already mentioned, this is usually a problem for quintessence because a small mass means a force with a very long range; see Refs. [36–38]. In some scenarios, this also implies variation of the constants, as, for instance, the fine structure constant. However, in the present context, the couplings are totally different. For example, the coupling with the electromagnetic field is of the form $\sqrt{\det(g_{\mu\nu} + F_{\mu\nu})}$, where, here, $g_{\mu\nu}$ is the induced metric on the brane [43]. The role of the quintessence field is usually played by one of the coordinates the metric $g_{\mu\nu}$ depends on. One sees that, because the formula expressing a determinant is quite complicated, the coupling can be different than the ones generally considered, for instance $f(\phi)F_{\mu\nu}F^{\mu\nu}$. Therefore, one can maybe expect this issue to be less problematic than in the standard case. More work is clearly needed in order to draw definitive conclusions on these matters.

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