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Pre-holography

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We construct a symplectic isomorphism $\mathfrak{h}$ from classical Klein Gordon solutions of mass m on $(d+1)$-dimensional Lorentzian anti–de Sitter space (equipped with the usual symplectic form) to a certain symplectic space of functions on its conformal boundary (only) for all integer and half-integer Δ $(=\frac{d}{2}+\frac{1}{2}(d^2+4m^2)^{1/2})$. $\mathfrak{h}$ induces a large family of new examples of Rehren's *algebraic holography* in which the net of local quantum Klein Gordon algebras in AdS is seen to map to a suitably defined net of local algebras for the (generalized free) scalar conformal field with anomalous dimension Δ on d-dimensional Minkowski space (the AdS boundary). Relatedly, we show for these models that Bertola *et al.*'s *boundary-limit holography* becomes a quantum duality (only) if the test functions for boundary Wightman distributions are restricted in a particular way.

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The conjecture [1] in 1997 of a holography-like correspondence between a certain type of string theory on the bulk of anti–de Sitter space (AdS) (in 5-dimensions and producted with a 5-sphere) and a certain limit of a certain family of conformal field theories (CFT) on its (conformal) boundary (or between supergravity on the bulk and the same limit of CFT on the boundary [2,3]) has led to many new and surprising conjectured interrelationships between quantum gravity and Minkowskian quantum field theory. One spinoff of this conjecture was that a number of authors (see especially [4,5]) began to investigate the related, but distinct and simpler, question: In what sense can a correspondence be established between an "ordinary" (e.g. scalar) quantum field theory on a [Lorentzian, $(d+1)$-dimensional] AdS background and a suitable "ordinary" (conformal) field theory on its conformal boundary? This is a simpler question because it concerns not full quantum gravity but quantum field theory in curved spacetime [6]. Two different sorts of answer to this question were proposed, the *algebraic holography* of Rehren [4] and the *boundary-limit holography* of Bertola, Bros, Moschella, and Schaeffer [5]—both in the context of axiomatic quantum field theory [7].

Rehren's algebraic holography [4] is formulated in terms of the algebraic version of axiomatic quantum field theory. In this framework, the specification of a given quantum field theory on a given background spacetime is tantamount [6] to the specification of a *net of local* $*$*-algebras*. In other words, the specification, for each (suitable) region $\mathcal{O}$ of the background spacetime, of a $*$-algebra $\mathcal{A}(\mathcal{O})$—the collection of the latter algebras being *isotonous* which means that when one region sits inside another, then its algebra is a subalgebra of the algebra of the larger region.

The basic idea of algebraic holography is to map a given spacelike wedge (defined as in [4]) in AdS to its intersection with the boundary. As Rehren points out, this sets up a bijection between the set of all wedges in the bulk and the set of all double-cones on the boundary which moreover maps spacelike related bulk wedges to spacelike related boundary double-cones [8]. If we are then given a net of local algebras on the bulk (where, in our definition above, "region" is interpreted to mean wedge) then algebraic holography consists of the definition of a net of local algebras on the boundary (where, in our definition above, "region" is interpreted to mean double-cone) by identifying the algebra for a given boundary double-cone with the bulk wedge algebra which restricts to it [8,9].

Bertola *et al.*'s boundary-limit holography is formulated in terms of the Wightman version of axiomatic quantum field theory. In this framework, and assuming the theory involves only a single scalar field, the specification of a given quantum field theory on a given background spacetime is tantamount to the specification of a family of Wightman distributions $W_n(f_1, \ldots, f_n)$ for each integer n, each of which may roughly be interpreted as the result of smearing the (singular) n-point "expectation value" $\langle 0|\phi(x_1), \ldots \phi(x_n)|0\rangle$ in a suitable "vacuum state" $|0\rangle$, with (smooth, compactly supported) test functions $f_1 \ldots f_n$. In an oversimplified description, where one ignores the need to smear, what Bertola *et al.* show may be described by saying that, for a given family of Wightman functions $W((t_1, \rho_1, \Omega_1), \ldots, (t_n, \rho_n, \Omega_n))$ in the bulk of AdS, if one chooses Δ suitably, then the limit $\lim_{\rho_1,\ldots,\rho_n\to\pi/2}(\cos\rho_1 \ldots \cos\rho_n)^{-\Delta} W((t_1, \rho_1, \Omega_1), \ldots, (t_n, \rho_n, \Omega_n))$ will exist and define a family of Wightman functions $W((t_1, \Omega_1), \ldots, (t_n, \Omega_n))$ on the conformal boundary which belong to a CFT. What they actually show is that a correct distributional counterpart to this limiting procedure maps any Wightman theory in the bulk to a Wightman theory for the appropriate CFT on the boundary.

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Above, we have used the usual global coordinates in which the AdS_{d+1} metric takes the form $ds^2 = \sec^2\rho dt^2 - \sec^2\rho d\rho^2 - \tan^2\rho d\Omega_{d-1}^2$ with $0 \leq \rho \leq \pi/2$, $-\infty \leq t \leq \infty$, and Ω denotes the usual angular coordinates on the $(d-1)$-sphere. In the special case of the Klein Gordon (KG) equation, $(\cos^2\rho\partial_t^2 - \cos^2\rho\partial_\rho^2 - (d-1)\times \cot\rho\partial_\rho - \cot^2\rho\nabla_{S^{d-1}}^2 + m^2)\phi(t, \rho, \Omega) = 0$ quantized on AdS according to the scheme of Avis, Isham, and Storey [10] for vanishing boundary conditions, one finds that the two-point distribution in the bulk has a nontrivial boundary limit when Δ takes the value (cf. [3])

$$\Delta = \frac{d}{2} + \frac{1}{2}(d^2 + 4m^2)^{1/2} \qquad (1)$$

which, when one identifies the appropriate part of the boundary (see endnote [8]) with d-dimensional Minkowski space, turns out to transform to the standard two-point function $W_b(x, x') = (1/2\pi^{d/2})[\Gamma(\Delta)/\Gamma(\Delta - d/2 + 1)][-(t - t' - i\epsilon)^2 + (\mathbf{x} - \mathbf{x}')^2]^{-\Delta}$ for a conformal scalar field $\hat{\phi}_d^\Delta$ of anomalous dimension Δ (and other n-point functions will be those of a generalized free field with this 2-point function).

The work we report here had two interrelated purposes: to use the bulk KG model to construct examples of algebraic holography and to clarify the relation between boundary-limit and algebraic holography. As we shall see below, whenever Δ (1) is an integer or half-integer, we have found a way to fulfill both of these purposes and we will show first that, for such Δ, if one starts with the net of local algebras for a bulk KG field, then the net of local algebras defined on the boundary by algebraic holography coincides with the subnet of local algebras for $\hat{\phi}_d^\Delta$ which results when one replaces the usual test functions by a certain smaller family of test functions and "localizes" them in a suitable way as we will explain and discuss below. Second, we show that, for the same Δ, if one restricts the range of the Bertola *et al.* projection to the Wightman functions of $\hat{\phi}_d^\Delta$ smeared only with the same smaller family of test functions, then the resulting quantum theory is dual (i.e. isomorphic) to the bulk quantum theory.

In order to obtain these results, we import into, and adapt to this AdS-CFT context, the mathematical formalism (see [6]) which has been successful in constructing and analyzing the properties of linear quantum fields in other curved spacetime contexts. The key to everything we do is the construction, for the KG equation on AdS, whenever Δ is an integer or half-integer, of a classical counterpart to quantum holography, which we call the *pre-holography* map $\mathfrak{h}$.

To construct this, we first introduce the space S of smooth classical solutions to KG on AdS_{d+1} which vanish on the conformal boundary. We recall that (for $d \geq 2$ [11]) any such classical solution may be expanded [12] as

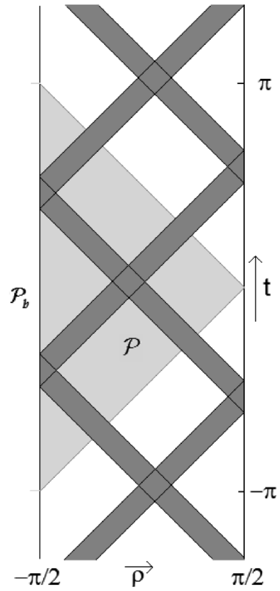

FIG. 1. The support (dark shading) of a "typical" classical solution for the bulk massless scalar field on AdS_{1+1} and a choice (light shading) of chart $\mathcal{P}$ for Poincaré coordinates. (See endnote [11].)

$$\phi(t, \rho, \Omega) = \sum_{nl\vec{m}} \left(\frac{\Gamma(1+n)\Gamma(\Delta+l+n)}{\Gamma(l+\frac{d}{2}+n)\Gamma(1+\Delta-\frac{d}{2}+n)}\right)^{1/2} \times \sin^l\rho\cos^\Delta\rho P_n^{(l+(d/2)-1,\Delta-(d/2))}(\cos 2\rho) \times (a_{nl\vec{m}}e^{-i(\Delta+l+2n)t}Y_{l\vec{m}}(\Omega) + \text{c.c.}) \qquad (2)$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials [13], $Y_{l,\vec{m}}$ are the (L^2-normalized) spherical harmonics on the $(d-1)$-sphere and the sum is over n from 0 to ∞ and the usual ranges of l and $\vec{m}$. We equip S with the (standard [6]) symplectic form $\sigma(\phi_1, \phi_2) = \int_{t=\text{const}}(\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2)g^{00}\sqrt{g}d^dx = \sum_{nl\vec{m}} i(a^1_{nl\vec{m}}a^{2*}_{nl\vec{m}} - a^{1*}_{nl\vec{m}}a^2_{nl\vec{m}})$ where the integral is over any $t =$ const surface. We then define our pre-holography map $\mathfrak{h}$ to be the map which sends such a classical solution to the function on the conformal boundary which has the expansion

$$\phi_b^s(t, \Omega) = \sum_{nl\vec{m}} \frac{i\Gamma(\Delta - \frac{d}{2} + 1)}{\pi} \times \left(\frac{\Gamma(l+\frac{d}{2}+n)n!}{\Gamma(\Delta+l+n)\Gamma(n+\Delta-\frac{d}{2}+1)}\right)^{1/2} \times (a_{nl\vec{m}}e^{-i(\Delta+l+2n)t}Y_{l\vec{m}}(\Omega) - \text{c.c.}) \qquad (3)$$

and we equip the range of $\mathfrak{h}$, which we call $\mathcal{F}_b$, with the antisymmetric bilinear form

$$\sigma_b(\phi_{b1}^s, \phi_{b2}^s) = \iint dt_1 dt_2 \iint d\Omega_1 d\Omega_2 E_b(t_1, \Omega_1; t_2, \Omega_2) \times \phi_{b1}^s(t_1, \Omega_1)\phi_{b2}^s(t_2, \Omega_2), \qquad (4)$$

where the integration is over a choice [8] (it obviously does not matter which) of $\mathcal{P}_b$ region and E_b is the boundary limit of the bulk Lichnérowicz (advanced minus retarded) fundamental solution E:

$$E_b(t_1, \Omega_1; t_2, \Omega_2) = \lim_{\rho_1,\rho_2\to\pi/2} (\cos\rho_1 \cos\rho_2)^{-\Delta} E(t_1, \rho_1, \Omega_1; t_2, \rho_2, \Omega_2)$$
$$= 2\mathrm{Im}\sum_{nl\vec{m}} \frac{\Gamma(\Delta + l + n)\Gamma(n + \Delta - \frac{d}{2} + 1)}{\Gamma(l + \frac{d}{2} + n)n!\Gamma(\Delta - \frac{d}{2} + 1)^2} e^{-i(\Delta+l+2n)(t_1-t_2)} Y_{l\vec{m}}(\Omega_1) Y^*_{l\vec{m}}(\Omega_2). \quad (5)$$

We note in passing that $iE_b(x, x')I = 2i\mathrm{Im}(\mathcal{W}_b(x, x')) =$ (when restricted to a $\mathcal{P}_b$ region [8]) $[\hat{\phi}^{\Delta}_d(x), \hat{\phi}^{\Delta}_d(x')]$.

One can check, for any pair of classical solutions, ϕ_1, ϕ_2, in S with coefficients in their mode expansions denoted $a^1_{nl\vec{m}}$, $a^2_{nl\vec{m}}$, that, defining ϕ^s_{b1}, ϕ^s_{b2} as in Eq. (3), we have, when (and only when) Δ is an integer or half-integer,

$$\sigma_b(\phi^s_{b1}, \phi^s_{b2}) = \sum_{nl\vec{m}} i(a^1_{nl\vec{m}} a^{2*}_{nl\vec{m}} - a^{1*}_{nl\vec{m}} a^2_{nl\vec{m}}) = \sigma(\phi_1, \phi_2) \quad (6)$$

and thus, by the equality of the first and last expressions here, we conclude both that σ_b is nondegenerate, and hence a symplectic form, and $\mathfrak{h}$: $S \to \mathcal{F}_b$ is a symplectic isomorphism. The origin of the restriction to integer or half-integer Δ lies in the calculation which is needed to show the first equality in (6): As may easily be seen, this calculation involves integrals of form $\int_{-\pi}^{\pi} \exp(\pm i(N + 2\Delta)t)dt$ where N is a positive integer and, for the equality to hold, these integrals have to vanish and therefore 2Δ has to be an integer.

We remark that the formula (4) may be written

$$\sigma_b(\phi^s_{b1}, \phi^s_{b2}) = \langle \phi^s_{b1} | E_b * \phi^s_{b2} \rangle, \quad (7)$$

where $\langle\cdot|\cdot\rangle$ denotes the L^2 inner product on our choice of $\mathcal{P}_b$ region on the conformal boundary and $*$ denotes convolution (i.e. smearing E_b in its second argument).

Apropos of σ_b being nondegenerate, we remark that, when restricted to a choice of $\mathcal{P}_b$, our space $\mathcal{F}_b$ falls short [11] of being the set of all smooth functions on $\mathcal{P}_b$ due to the incompleteness of the set of modes in terms of which ϕ^s_b is expanded in (3). Concomitantly, if we were to extend σ_b (restricted to $\mathcal{P}_b$) from the range $\mathcal{F}_b|_{\mathcal{P}_b}$ to the full set of smooth functions on $\mathcal{P}_b$, then it would be degenerate since, due to the incompleteness of the set of modes in terms of which it is expanded in (5), the operator E_b* has a nontrivial kernel.

Our purpose next is to exploit our just-defined preholography map $\mathfrak{h}$ to construct a mathematical object which corresponds to the quantum boundary limit $\hat{\phi}_b$ of the quantum bulk field $\hat{\phi}$ defined by the formal relation

$$\hat{\phi}_b(t, \Omega) = \lim_{\rho\to\pi/2} (\cos\rho)^{-\Delta} \hat{\phi}(t, \rho, \Omega). \quad (8)$$

We know [6] the quantum bulk field $\hat{\phi}$ can be defined in terms of quantities "$\sigma(\hat{\phi}, \psi)$" which deserve to be considered the "quantum bulk field $\hat{\phi}$, *symplectically smeared* with a classical test solution ψ" and which satisfy the commutation relations

$$[\sigma(\hat{\phi}, \psi_1), \sigma(\hat{\phi}, \psi_2)] = i\sigma(\psi_1, \psi_2)I, \quad (9)$$

and what we will do is to define, in terms of these $\sigma(\hat{\phi}, \psi)$, a quantity which deserves to be called "$\langle \hat{\phi}_b | \psi^s_b \rangle$" for each ψ^s_b in $\mathcal{F}_b$. To do this, we first observe that, if we replace $\hat{\phi}$ in (8) by a classical solution ϕ and expand ϕ as in (2), then, by (3) and (5), we have

$$\phi_b = E_b * \phi^s_b \quad (10)$$

and hence, for all $\phi \in S$ with $\mathfrak{h}(\phi) = \phi^s_b$ and with boundary limit ϕ_b and for any $\psi \in S$ with $\mathfrak{h}(\psi) = \psi^s_b$, we have, by (10) and the fact that $\mathfrak{h}$ is a symplectic isomorphism, that $\langle \phi_b | \psi^s_b \rangle = \langle E_b * \phi^s_b | \psi^s_b \rangle = -\sigma_b(\phi^s_b, \psi^s_b) = -\sigma(\phi, \psi)$, in view of which the appropriate definition is clearly

$$\langle \hat{\phi}_b | \psi^s_b \rangle = -\sigma(\hat{\phi}, \psi). \quad (11)$$

If we now choose [8] a Poincaré chart $\mathcal{P}$ and temporarily adopt the convention of equating any $\psi^s_b \in \mathcal{F}_b$ with its restriction to $\mathcal{P}_b$, (11) amounts to saying: *The boundary-limit quantum field $\hat{\phi}_b$, "spacetime smeared" on $\mathcal{P}_b$ with the test function ψ^s_b, is equal to minus the "symplectic smearing" of the bulk quantum field $\hat{\phi}$ with the bulk test solution ψ.*

In view of the fact that $\mathfrak{h}$ is a symplectic isomorphism, the algebra $\mathcal{A}_b$ of "smeared boundary fields" generated by the $\langle \hat{\phi}_b | \psi^s_b \rangle$ as ψ^s_b ranges over $\mathcal{F}_b$ is isomorphic to the bulk field algebra $\mathcal{A}_B$ generated by the $\sigma(\hat{\phi}, \psi)$ as ψ ranges over S. (For more details, see the definition of the "minimal field algebra" in [6] and note also the options discussed there for technically different alternatives.) Moreover, by (7) and (9), we have $[\langle \hat{\phi}_b | \psi^s_{b1} \rangle, \langle \hat{\phi}_b | \psi^s_{b2} \rangle] = i\langle \psi^s_{b1} | E_b * \psi^s_{b2} \rangle I$ and thus [cf. the note after (5)] the subalgebra of $\mathcal{A}_b$ generated by test functions in $\mathcal{F}_b|_{\mathcal{P}_b}$ may be naturally identified, when $\mathcal{P}_b$ is identified with d-dimensional Minkowski space, as the subalgebra of the usual field algebra $\mathcal{A}^{\Delta}_d$ for the conformal field $\hat{\phi}^{\Delta}_d$ obtained by restricting smearing functions from all of $C^\infty(\mathcal{P}_b)$ to $\mathcal{F}_b$.

Next we notice that, still for our models (i.e. involving the bulk KG equation and integer or half-integer Δ) one can, as usual (cf. [6]), define a subalgebra $\mathcal{A}_B(\mathcal{O})$ of our bulk field algebra for each open region $\mathcal{O}$ of bulk AdS by (cf. [6]) taking the algebra generated by the $\sigma(\hat{\phi}, \psi)$ where $\psi \in S$ takes the form $E * F$ where F ranges over smooth functions with compact support in $\mathcal{O}$. So, in particular, we obtain an algebra $\mathcal{A}_B(\mathcal{W})$ for each bulk wedge $\mathcal{W}$ in AdS (and similarly we obtain an algebra for each bulk double-cone). Next we observe that, by (11), each such bulk algebra $\mathcal{A}_B(\mathcal{O})$ is equal to the subalgebra of the boundary algebra $\mathcal{A}_b$ generated by $\langle\hat{\phi}_b|\psi_b^s\rangle$ for $\psi_b^s = \mathfrak{h}\psi$, $\psi = E * F$, $F \in \mathcal{O}$. If one makes a choice of Poincaré chart $\mathcal{P}$ then, when $\mathcal{O}$ is a wedge $\mathcal{W} \subset \mathcal{P}$, we call the latter subalgebra $\mathcal{A}_b(\mathcal{I})$ where $\mathcal{I} \subset \mathcal{P}_b$ is the double-cone to which $\mathcal{W}$ bijects under the Rehren bijection [8]. In other words, $\mathcal{A}_b(\mathcal{I})$ coincides with the element labeled by the region $\mathcal{I}$ of the net of local boundary algebras which gets identified with the element labeled by the region $\mathcal{W}$ of the net of local bulk algebras by the algebraic holography identification mentioned in our introductory paragraphs (and similarly for bulk double-cones in $\mathcal{P}$ and the boundary regions in $\mathcal{P}_b$ to which they biject [9]). So in this way our models provide concrete examples of algebraic holography. Moreover, in view of the above identification of $\mathcal{A}_b$ with $\mathcal{A}_d^\Delta$, this net of local boundary algebras may be regarded as a net of local algebras for the conformal field $\hat{\phi}_d^\Delta$, but we emphasize [14] that this differs from the usual net of local algebras for this theory, not only because the smearing functions are restricted to elements of $\mathcal{F}_b$ but also because these elements are differently "localized".

Turning to the connection with boundary-limit holography, if $|0\rangle$ is the Avis *et al.* [10] ground state for the bulk theory, i.e. the quasi-free state with symplectically smeared two-point function $\langle 0|\sigma(\hat{\phi}, \psi_1)\sigma(\hat{\phi}, \psi_2)|0\rangle = \sum_{nl\vec{m}} a^{1*}_{nl\vec{m}} a^2_{nl\vec{m}}$ [where $a^1_{nl\vec{m}}$ is related to ψ_1 as in (2) etc.] then one can show by (3) and (11) (again choosing a $\mathcal{P}$ and readopting our convention [see after (11)] and moreover identifying $\mathcal{P}_b$ with Minkowski space) that the "spacetime-smeared 2-point function" on the boundary $\langle 0|(\langle\hat{\phi}_b|\psi^s_{b1}\rangle\langle\hat{\phi}_b|\psi^s_{b2}\rangle)|0\rangle$ for a pair of test functions, ψ^s_{b1}, $\psi^s_{b2} \in \mathcal{F}_b$, is equal to $W_b(\psi^s_{b1}, \psi^s_{b2})$ and similarly for all n-point functions.

In view of the fact [6] that the covariantly smeared bulk field $\hat{\phi}(F)$, $F \in C_0^\infty(\mathrm{AdS})$ is equal to the symplectically smeared field $\sigma(\hat{\phi}, E * F)$, we conclude from (11) that, in our KG models and for Δ an integer or half-integer, the bulk smeared Wightman function $W(F_1, \ldots, F_n)$ is equal to the boundary smeared Wightman function $W_b(\mathfrak{h}(-E * F_1), \ldots, \mathfrak{h}(-E * F_n))$. Thus we see that the test function map $F \mapsto \mathfrak{h}(-E * F)$ induces a "quantum duality" between the sets of Wightman functions in bulk and boundary which are related by Bertola *et al.*'s boundary-limit holography. But in this duality, the test functions with which one smears the boundary Wightman functions are restricted to belong to our family $\mathcal{F}_b(= \mathrm{ran}(\mathfrak{h}))$.

Aside from its applications, given in this paper, to providing examples of algebraic holography and to clarifying its relationship to boundary-limit holography, we expect that our pre-holography map will be of use in elucidating other aspects of the AdS/CFT correspondence, albeit it is only of immediate relevance to the case of bulk theories which are linear. Furthermore, there are two specific further conclusions which immediately flow from our results which may be of relevance to less trivial holography models. First, if one wishes to construct models on the AdS boundary by requiring them to be related to the bulk theory by algebraic holography, then this may lead to a more restricted family of models (in the case of bulk KG, we found only models with integer or half-integer Δ) than the family one would obtain by requiring only that the boundary theory be related to the bulk theory by boundary-limit holography (which, for our bulk KG, have unrestricted Δ). The second conclusion concerns the sometimes-expressed expectation that it is unlikely there could be a duality between "ordinary" QFTs in bulk and boundary because (it is sometimes said) the boundary having lower dimensions, one would expect it only to be able to support "fewer degrees of freedom." Surprisingly, we have found that essentially the opposite to the above expectation holds true. Indeed we found that our bulk theory (i.e. AdS_{d+1} KG for an appropriately tuned mass) is dual to a *subtheory* of our boundary theory—i.e. to the theory of $\hat{\phi}_d^\Delta$ after its test functions have been restricted to the space $\mathcal{F}_b$. Our result thus shows us that there is, in fact, no simple correlation between dimension and "degrees of freedom."

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[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] K.-H. Rehren, Ann. Inst. Henri Poincaré **1**, 607 (2000); Phys. Lett. B **493**, 383 (2000).
[5] M. Bertola, J. Bros, U. Moschella, and R. Schaeffer, Nucl. Phys. **B587**, 619 (2000).
[6] B. S. Kay, *Encyclopedia of Mathematical Physics*, edited by J.-P. Françoise, G. Naber, and S. T. Tsou (Academic, New York, 2006), Vol. 4, p. 202.
[7] R. Haag, *Local Quantum Physics* (Springer, New York, 1996), 2nd ed..
[8] Actually, what we shall call the Rehren bijection here is slightly different from the definition in [4]. Reference [4]

works on the $\mathbb{Z}_2$ quotient of wrapped AdS and the definition of the bijection in [4] is adapted to this spacetime. We prefer to work on the unwrapped unquotiented AdS (i.e. the covering space of wrapped unquotiented AdS) because we do not want to have to get involved with quantum field theory either on spacetimes with closed timelike curves or on nonorientable spacetimes and also so as not to preclude from the outset noninteger Δ. With this preference, we find it necessary to alter the definitions slightly and regard the Rehren bijection as a bijection between bulk wedges belonging to a single choice of chart $\mathcal{P}$ for Poincaré coordinates and the double-cones belonging to the boundary $\mathcal{P}_b$ of that chart (which is conformal to a single copy of d-dimensional Minkowski space) and we shall here actually regard algebraic holography as a mapping from a net of local algebras for such a bulk chart to a net of local algebras on its boundary (i.e. on Minkowski space). Even though the resulting version of algebraic holography refers to a single choice of Poincaré chart, we still wish to privilege global coordinates. So, in particular, we define $\mathfrak{h}$ as a map from all solutions S on our (unwrapped, unquotiented) AdS to a set of functions $\mathcal{F}_b$ on our full boundary (albeit σ_b is defined by an integral over a choice of $\mathcal{P}_b$—it does not matter which). One could, alternatively, work instead throughout on a single Poincaré chart $\mathcal{P}$ (and use Poincaré coordinates) and define S to be the space of solutions on $\mathcal{P}$ (note that this would be the same space as our S under an obvious identification) and $\mathfrak{h}$ to be a map from this latter S to a space of functions on the corresponding $\mathcal{P}_b$. However, to do so would result in considerable loss of global perspective. By defining our pre-holography map globally, we are able, via (11) [see the second full paragraph after (11)] to identify the bulk algebra for any bounded open region in our AdS with a subalgebra of the algebra $\mathcal{A}_b$ of its boundary albeit the resulting subalgebras will not necessarily be identifiable as localized in specific regions and/or (in the case of bulk wedges and bulk double-cones not contained in $\mathcal{P}$) may be so identifiable but not in a unique way. We should also point out that our altered definition of Rehren bijection misses out a counterpart to the important property [4] that (when the notion of boundary double-cone is generalized by adding a conformal boundary to Minkowski space) causally complementary bulk wedges map to causally complementary boundary double-cones and, related to this, our notion of algebraic holography here is not taken to include a suitable counterpart to Rehren's property that algebras for causally complementary bulk wedges get identified with algebras for such causally complementary boundary double-cones. Nevertheless it is clear from the second full paragraph after (11) that the algebras for causally complementary bulk wedges do get identified with commuting subalgebras of $\mathcal{A}_b$ albeit the identification (cf. the last sentence of the previous paragraph) of the latter with the algebras for particular regions of the boundary will be ambiguous (and/but the identification will be choosable so that the regions are causally complementary).

[9] The Rehren bijection as we define it (see [8]) extends to a map from the set of all wedges and bulk double-cones in a choice of Poincaré chart $\mathcal{P}$ on the bulk to the set of double-cones and certain regions of its boundary $\mathcal{P}_b$, the image of a given bulk double-cone being defined to be the set of points in the boundary which can be reached by null-geodesics emanating from it. We note that such an extension of the Rehren bijection is not made in [4] but, if a given bulk double-cone (in a given choice of $\mathcal{P}$) is regarded as an intersection of bulk wedges (some of which will of course not lie in $\mathcal{P}$) then the algebra for the boundary region to which the given bulk double-cone bijects is, on our definition, easily seen to coincide, in the class of examples we obtain here, with the intersection of the algebras of all those bulk wedges. By virtue of this together with what is written in [4] one can see that, in our examples, the resulting extension of the notion of algebraic holography from the set of bulk wedges to the set of bulk wedges together with bulk double-cones is the appropriate counterpart (i.e. when one unwraps and unidentifies as explained in [8]) to the extension performed in [4].

[10] S. J. Avis, C. J. Isham, and D. Storey, Phys. Rev. D **18**, 3565 (1978).

[11] Our results also hold in $d = 1$ although this case needs separate treatment. See P. Larkin, Ph.D. thesis, University of York, 2008. ρ now goes from $-\pi/2$ to $\pi/2$ and the boundary consists of two lines. In place of (2), (3), and (5), one has expansions involving Gegenbauer polynomials. Another difference is that, in $d = 1$, the boundary modes are incomplete [see paragraph after Eq. (7)] by only a finite number of missing modes, whereas for $d \geq 2$ there are a countable number of missing modes. The $d = 1$, $m = 0$ ($\Delta = 1$) case is particularly simple and instructive. Here (see Fig. 1) it is useful to change coordinates to (t, x) where $x = \rho + \pi/2$. Then S consists of functions of form $\phi(t, x) = f(t + x) - f(t - x)$, where f is periodic with period 2π and is fixed uniquely by demanding that it be the derivative of another periodic function, $\mathfrak{h}$ has the simple closed-form action $\phi \mapsto f$, and $\sigma_b(f_1, f_2) = 2\int_{\mathcal{P}_b} f_1(t) f_2'(t) dt$ where $\mathcal{P}_b$ is coordinatized as the interval $(-\pi, \pi)$. Moreover, $E_b = 2\delta'$ and $\mathcal{F}_b(= \mathrm{ran}(\mathfrak{h}))$ consists of smooth periodic functions with period 2π which arise as derivatives of other such functions.

[12] V. Balasubramanian, S. B. Giddings, and A. E. Lawrence, J. High Energy Phys. 03 (1999) 001.

[13] *Higher Transcendental Functions (Bateman Manuscript Project)*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vols. I–III.

[14] In our net of local algebras for the conformal field $\hat{\phi}_d^\Delta$, our criterion for a smeared field $\langle \hat{\phi}_b | \psi_b^s \rangle$ to be localized in a given boundary double-cone $\mathcal{I}$ entails that ψ_b [which by (10) is equal to $E_b * \psi_b^s$] is supported in $\mathcal{I}$ rather than the smearing function ψ_b^s itself. We have shown that, for certain values of d and Δ, ψ_b^s will not be supported in $\mathcal{I}$. It is hoped to discuss this further elsewhere. (We conjecture that it will always be supported in the union of the causal future and the causal past of $\mathcal{I}$.) Nevertheless, our net always satisfies commutation at spacelike separation, as can be seen by combining (6) and (7) together with the fact that boundary double-cones are spacelike related whenever the bulk wedges to which they Rehren-biject are spacelike related, and the fact [6] that $\sigma(E * F_1, E * F_2)(= \langle F_1 | E * F_2 \rangle_{L^2(\mathcal{P}_b)}$ vanishes whenever the supports of F_1 and F_2 are spacelike related.