Finding the sigma pole by analytic extrapolation of $\pi\pi$ scattering data

Irinel Caprini
National Institute of Physics and Nuclear Engineering, POB MG 6, Bucharest, R-077125, Romania
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I investigate the determination of the σ pole from $\pi\pi$ scattering data below the $K\bar{K}$ threshold, including the new precise results obtained from K_{e4} decay by NA48/2 Collaboration. I discuss also the experimental status of the threshold parameters a_0^0 and b_0^0 and the phase shift δ_0^0. In order to reduce the theoretical bias, I use a large class of analytic parametrizations of the isoscalar S wave, based on expansions in powers of conformal variables. The σ pole obtained with this method is consistent with the prediction based on ChPT and Roy equations. However, the theoretical uncertainties are now larger, reflecting the sensitivity of the pole position to the specific parametrizations valid in the physical region. I conclude that Roy equations offer the most precise method for the determination of the σ pole from $\pi\pi$ elastic scattering.

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I. INTRODUCTION

The determination of the pole associated with the σ resonance (or $f_0(600)$) is known to be a difficult problem. The pole is situated deep in the complex plane, its influence in the physical region is masked to a certain extent by the nearby Adler zero, and, until recently, the experimental data on $\pi\pi$ scattering at low energies were quite poor. This explains why the values reported by PDG [1] for the mass and width of σ cover a very large interval.

During the last years, chiral perturbation theory (ChPT) and Roy equations led to an accurate description of $\pi\pi$ scattering at low energies [2,3]. In particular, the scattering length a_0^0 and the effective range parameter b_0^0 of the isoscalar S wave given in [3]

$$a_0^0 = 0.220 \pm 0.005, \qquad b_0^0 = 0.276 \pm 0.006, \tag{1}$$

have remarkably small uncertainties.

The formalism based on Roy equations was shown recently [4] to control also the analytic extrapolation of the $\pi\pi$ amplitude in the complex plane, leading to precise values for the mass and width of σ:

$$M_\sigma = 441^{+16}_{-8}\ \mathrm{MeV}, \qquad \Gamma_\sigma/2 = 272^{+9}_{-12.5}\ \mathrm{MeV}. \tag{2}$$

In the standard method of detecting resonances, the experimental data on the partial wave with the quantum numbers of the resonance play an important role. Unlike this, the prediction (2) was obtained without using experimental data on the isoscalar S wave at low energies: the amplitude was calculated below 800 MeV, and also in the complex plane, from Roy equations, using experimental input at higher energies and theoretical results on the pion-pion scattering [3]. Roy equations provide a very suitable framework in this case, compensating the lack of experimental data on $\pi\pi$ scattering at low energies by theoretical information.

Recently [5], NA48/2 Collaboration measured the phase shift difference $\delta_0^0 - \delta_1^1$ at low energies from K_{e4} decay, with a precision much greater than that of the older experiments [6,7]. This revived the interest in the determination of the scattering length a_0^0 and the pole associated with σ by direct analytic extrapolation of the $\pi\pi$ scattering data. In [8] the authors propose a representation of the isoscalar S wave $t_0^0(s)$ based on an expansion in powers of a conformal mapping variable. To account for the theoretical uncertainties related to analytic continuation, two parametrizations were considered, the difference between them being interpreted as a systematic theoretical uncertainty of the method. In the framework discussed in [8], the mass and width of σ are obtained with an accuracy comparable to that quoted in (2).

In the present work I focus on the problem of systematic uncertainties within this approach. I note that the class of functions used in [8], although based on a convergent expansion, is still quite narrow when the expansion is restricted to a few terms. By enlarging the class of admissible analytic functions used for fitting the data, the theoretical bias is reduced and a more realistic estimate of the uncertainties in the position of the σ pole is obtained. In the present work I apply this idea, by using a large sample of analytic parametrizations of the $\pi\pi$ amplitude, suitable at low energies. A short description of the method and some results were given already in [9].

In the next section I discuss several parametrizations of the $\pi\pi$ isoscalar S wave, which satisfy analyticity and elastic unitarity. In the next two sections I apply these parametrizations for fitting the data on the phase shift δ_0^0: in Sec. III I consider only the data from K_{e4} decay, and in Sec. IV I include data up to the $K\bar{K}$ threshold. From the admissible parametrizations of the isoscalar S wave I find the threshold parameters a_0^0 and b_0^0 and the location of the σ pole. My conclusions are summarized in Sec. V.

II. THE ISOSCALAR S WAVE AT LOW ENERGY

I consider the $\pi\pi$ isoscalar S wave $t_0^0(s)$, which is an analytic function in the s-plane cut along $s \geq 4M_\pi^2$ and $s \leq 0$. I assume that $t_0^0(s)$ is the pure strong amplitude, where all the isospin breaking corrections are neglected. As in [2–4], I take for M_π and M_K the masses of the charged pion and charged kaon, respectively.

Neglecting the inelasticity due to the 4π channel below 1 GeV, unitarity implies that the relation

$$\mathrm{Im}\left[\frac{1}{t_0^0(s+i\epsilon)}\right] = -i\rho(s), \qquad \rho(s) = \sqrt{1-4M_\pi^2/s}, \tag{3}$$

is valid up to the threshold for $K\bar{K}$ production, $s = 4M_K^2$. From (3) it follows that the function $\psi(s)$, defined by

$$t_0^0(s) = \frac{1}{\psi(s) - i\rho(s)}, \tag{4}$$

is real in the elastic region:

$$\mathrm{Im}\,\psi(s+i\epsilon) = 0, \qquad 4M_\pi^2 \le s < 4M_K^2, \tag{5}$$

and is related to the phase shift $\delta_0^0(s)$ by

$$\psi(s) = \rho(s)\cot\delta_0^0(s). \tag{6}$$

Since the amplitudes are analytic functions of real type, Eq. (5) means that $\psi(s)$ has no discontinuity across the elastic unitarity cut. The definition (4) shows also that $\psi(s)$ has poles at the points where $t_0^0(s)$ has zeros. The amplitude is expected to vanish below threshold at a point s_A, related to the so-called Adler zeros. ChPT to lowest order predicts $s_A = M_\pi^2/2$. Assuming that $t_0^0(s)$ does not have other zeros in the complex plane, the product $(s - s_A)\psi(s)$ is analytic in the s-plane cut for $s \le 0$ and $s \ge 4M_K^2$. The effective range expansion amounts to expanding the function $\psi(s)$ in powers of $q^2 = (s/4 - M_\pi^2)$ around the threshold $q^2 = 0$, where it is regular. However, the branch point $s = 0$ limits the convergence of this expansion to the circle $|q^2| < M_\pi^2$.

A. Method of conformal mappings

The domain of convergence of a power series can be enlarged by expanding the function in powers of a variable which conformally maps a part of the holomorphy domain onto the interior of a disk. The use of conformal mappings in particle physics was first discussed in [10,11]; in the context of the effective range expansion for partial waves a conformal mapping was used in [12]; more recently, the method was applied for the description of exclusive semileptonic B decays [13] and in perturbative QCD [14]. As shown in [10], the asymptotic rate of convergence of the series in the physical region is optimal if the amplitude is expanded in powers of the variable which maps the entire holomorphy domain onto a disk. Since the disk is the natural convergence domain of the power series, the new expansion will converge in the whole analyticity domain, up to its boundary.

Consider the variable

$$w(s,\alpha) = \frac{\sqrt{s} - \alpha\sqrt{4M_K^2 - s}}{\sqrt{s} + \alpha\sqrt{4M_K^2 - s}}, \tag{7}$$

where $\alpha > 0$ is arbitrary. The function $w(s,\alpha)$ transforms the s-plane cut along $s \le 0$ and $s \ge 4M_K^2$ onto the unit disk $|w| < 1$ in the complex plane $w = w(s,\alpha)$, such that $w(4M_K^2,\alpha) = 1$ and $w(0,\alpha) = -1$. In [8] the authors adopt the expansion

$$\psi(s) = \frac{M_\pi^2}{s - s_A}\left[\frac{2s_A}{M_\pi\sqrt{s}} + B_0 + B_1 w(s,\alpha) + B_2 w(s,\alpha)^2 + \ldots\right] \tag{8}$$

with the particular choice $\alpha = 1$. In Eq. (8), the first term in parentheses, added to the expansion in powers of $w(s,\alpha)$, compensates the singularity of $\rho(s)$ at $s = 0$ in the denominator of (4), removing an unphysical singularity of $t_0^0(s)$ on the real axis which would appear otherwise.

A slightly different form was also used in [8]:

$$\psi(s) = \frac{M_\pi^2}{s - s_A}\,\frac{\mu_0^2 - s}{\mu_0^2}\left[\frac{2s_A}{M_\pi\sqrt{s}} + B_0 + B_1 w(s,1) + B_2 w(s,1)^2 + \ldots\right], \tag{9}$$

where the factor $(\mu_0^2 - s)$ displays explicitly the energy where the phase shift δ_0^0 passes through $\pi/2$, according to (6). This factor is useful for fitting narrow resonances but, as I shall show, it is not suitable for broad resonances like σ.

The power expansions in (8) and (9) converge in the disk $|w| < 1$ and, for a large number of terms, these parametrizations are equivalent. However, when the series are truncated at a finite number of terms, (8) and (9) lead to different results. This difference is interpreted in [8] as a systematic uncertainty of theoretical nature, which should be added to the statistical errors. In the present work I develop this idea, presenting other admissible analytic parametrizations of the amplitude.

A first generalization is to expand $\psi(s)$ in powers of $w = w(s,\alpha)$, for an arbitrary α, as in (8). By varying α, one changes the point mapped to the origin of the w plane and the position of the intervals where experimental data are available. Some examples are shown in Fig. 1. As we shall see, the flexibility offered by the parameter α allows one to describe the peculiar structure of the isoscalar S wave near the inelastic $K\bar{K}$ threshold.

B. Alternative procedure for ghost removal

The singularity at $s = 0$ of the phase space factor $\rho(s)$ in (4) can be alternatively eliminated if the term $i\rho(s)$ is replaced by a function which is analytic in the s-plane

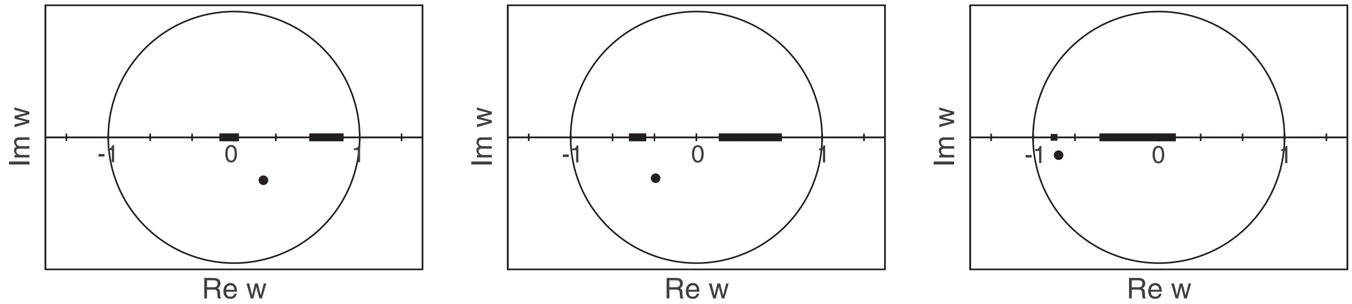

FIG. 1. The disk $|w| < 1$ in the complex plane $w = w(s, \alpha)$ defined in (7), for $\alpha = 0.36$ (left), $\alpha = 1$ (center), and $\alpha = 4$ (right). The thick segments indicate the regions where experimental data are available from K_{e4} decay [5–7] and the process $\pi N \to \pi\pi N$ (cf. the compilation of data made in [20]), respectively; the circle shows the σ pole on the second Riemann sheet from [4].

cut along $s \geq 4M_\pi^2$ and has the imaginary part equal to $\rho(s)$ on the upper edge of the cut. In the context of effective range approximation, Chew and Mandelstam [15] defined such a function, vanishing at threshold, by a once subtracted dispersion relation. For convenience, I consider the loop function of ChPT, $\bar{J}(s, M_\pi^2)$, written as

$$\bar{J}(s, M_\pi^2) = \frac{2}{\pi} + \frac{\rho(s)}{\pi} \ln\left[\frac{\rho(s) - 1}{1 + \rho(s)}\right], \qquad (10)$$

which vanishes at the origin, $J(0, M_\pi^2) = 0$, and satisfies the relation

$$\mathrm{Im}\, \bar{J}(s + i\epsilon, M_\pi^2) = \rho(s), \qquad s \geq 4M_\pi^2. \qquad (11)$$

If I define the function $\psi_1(s)$ by

$$t_0^0(s) = \frac{1}{\psi_1(s) - \bar{J}(s, M_\pi^2)}, \qquad (12)$$

the unitarity relation (3) and Eq. (11) show that $\psi_1(s)$ is real for $4M_\pi^2 \leq s < 4M_K^2$, where it is related to the phase shift δ_0^0 by

$$\psi_1(s) = \rho(s) \cot\delta_0^0(s) + \mathrm{Re}\bar{J}(s, M_\pi^2). \qquad (13)$$

The reality property implies also that $\psi_1(s)$ is analytic in the s-plane cut for $s \leq 0$ and $s \geq 4M_K^2$, except for the pole at $s = s_A$, and can be expanded as

$$\psi_1(s) = \frac{M_\pi^2}{s - s_A}[B_0 + B_1 w(s, \alpha) + B_2 w(s, \alpha)^2 + \ldots], \qquad (14)$$

in powers of the variable (7). I remark that the compensating term $2s_A/M_\pi^2\sqrt{s}$ appearing in (8) is no longer necessary in (14), since the function $\bar{J}(s, M_\pi^2)$ is by definition regular at $s = 0$.

C. S-matrix factorization

Other parametrizations of $t_0^0(s)$ are obtained by including some information about its behavior near the $K\bar{K}$ threshold. I do this by expressing the S-matrix element

$$S_0^0(s) = 1 + 2i\rho(s)t_0^0(s) \qquad (15)$$

as a product

$$S_0^0(s) = S_{\mathrm{rest}}(s)S_{f_0}(s), \qquad (16)$$

where each factor satisfies elastic unitarity ($|S_{\mathrm{rest}}(s)| = |S_{f_0}(s)| = 1$) below the $K\bar{K}$ threshold. The multiplication of the two S matrices amounts to the following addition rule for the corresponding amplitudes:

$$t_0^0(s) = t_{\mathrm{rest}}(s) + t_{f_0}(s) + 2i\rho(s)t_{\mathrm{rest}}(s)t_{f_0}(s), \qquad (17)$$

where

$$t_{\mathrm{rest}}(s) = \frac{S_{\mathrm{rest}}(s) - 1}{2i\rho(s)}, \qquad t_{f_0}(s) = \frac{S_{f_0}(s) - 1}{2i\rho(s)}. \qquad (18)$$

Crossing symmetry implemented by Roy equations [3] implies that the expansion of the partial wave amplitude around $s = 0$ starts with

$$t_0^0(s) = t_0 + t_1 s + t_2 s^{3/2} + O(s^2), \qquad (19)$$

where t_0 is nonzero. In order to cancel the singularity of the factor $\rho(s)$ at $s = 0$ in (17), either $t_{\mathrm{rest}}(s)$ or $t_{f_0}(s)$ must vanish at $s = 0$, but not both [since, cf. (19), the full amplitude does not have a zero there]. I choose to set $t_{f_0}(0) = 0$, taking for this amplitude the expression

$$t_{f_0}(s) = \frac{k_1 s}{\kappa - s - k_1 s\bar{J}(s, M_\pi^2) - (k_2 + k_3 s)\bar{J}(s, M_K^2)}, \qquad (20)$$

where $\bar{J}(s, M_\pi^2)$ is defined in (10) and $\bar{J}(s, M_K^2)$ is obtained replacing M_π in $\rho(s)$ by M_K. I note that by taking

$$\kappa = 1.01, \qquad k_1 = 0.08, \qquad k_2 = -1.09, \qquad k_3 = 1.16, \qquad (21)$$

the modulus of the corresponding S matrix, $S_{f_0}(s)$, is close, in the range $2M_K < \sqrt{s} < 1.16$ GeV, to the elasticity $\eta_0^0(s)$ measured in [16], while for

$$\kappa = 1.15, \qquad k_1 = 0.11, \qquad k_2 = 0.39, \qquad k_3 = 0.03, \qquad (22)$$

$$\kappa = 1.41, \qquad k_1 = 0.24, \qquad k_2 = -0.73, \qquad k_3 = 1.72, \tag{23}$$

$|S_{f_0}(s)|$ is close, in the same range, to the upper/lower edges of the band of the elasticity η_0^0 extracted from the decay $J/\psi \to \phi\pi\pi$ [17].

For my purpose, the specific form adopted for $t_{f_0}(s)$ is not a limitation, since the total amplitude contains the additional term $t_{\rm rest}(s)$. Elastic unitarity, $|S_{\rm rest}(s)| = 1$, implies that $t_{\rm rest}(s)$ can be written, for instance, as

$$t_{\rm rest}(s) = \frac{1}{\psi_{\rm rest}(s) - i\rho(s)}, \tag{24}$$

with $\psi_{\rm rest}$ analytic in the s-plane cut along $s \leq 0$ and $s \geq 4M_K^2$, except for a pole at $s = s_1$, where $t_{\rm rest}(s_1) = 0$ [from (17) and (20) it follows that s_1 is close to the Adler zero s_A]. Therefore, I can write $\psi_{\rm rest}(s)$ as

$$\psi_{\rm rest}(s) = \frac{M_\pi^2}{s - s_1}\left[\frac{2s_1}{M_\pi\sqrt{s}} + B_0 + B_1 w(s, \alpha) + B_2 w(s, \alpha)^2 + \ldots\right], \tag{25}$$

where $w(s, \alpha)$ is defined in (7). Alternatively, I can use for $t_{\rm rest}(s)$ an expression similar to (12), involving the function $\bar{J}(s, M_\pi^2)$.

Other admissible parametrizations are obtained if one assumes that $t_{\rm rest}(s)$ is almost regular near $s = 4M_K^2$. Since the next branch point, at $s = 4M_\eta^2$, is known to have a weak effect, one can neglect at low energies the right-hand cut of $\psi_{\rm rest}(s)$, and expand it as

$$\psi_{\rm rest}(s) = \frac{M_\pi^2}{s - s_1}\left[\frac{2s_1}{M_\pi\sqrt{s}} + B_0 + B_1 w_1(s, \alpha) + B_2 w_1(s, \alpha)^2 + \ldots\right], \tag{26}$$

where the variable

$$w_1(s, \alpha) = \frac{\sqrt{s} - \alpha}{\sqrt{s} + \alpha}, \qquad \alpha > 0, \tag{27}$$

maps the s-plane cut only for $s \leq 0$ onto the disk $|w_1| < 1$ of the complex plane $w_1 = w_1(s, \alpha)$.

The expressions given in this subsection are examples of possible analytic parametrizations of the amplitude at low energies. Following an idea of Dalitz and Tuan [18], in some phenomenological analyses [19] the individual S matrices in the product (16) are associated with specific resonances. However, in my work I use the factorization (16) only for mathematical purposes: by isolating a factor with a rapid variation near the $K\bar{K}$ threshold, I expect a better convergence for the expansion of the remaining part, $t_{\rm rest}(s)$. This part, which is fixed by the low energy data, contributes to both the elasticity $\eta_0^0(s) = |S_0^0(s)|$ and the phase shift δ_0^0 above the $K\bar{K}$ threshold. So, the behavior of t_0^0 above this point is left free in my fits.

III. FITS OF THE DATA FROM K_{e4} DECAY

I consider first the data on the difference $\delta_0^0 - \delta_1^1$ measured below 0.4 GeV from K_{e4} decay [5–7]. The P-wave phase shift δ_1^1 is known with precision in this energy range [3,20], allowing an accurate extraction of δ_0^0. As shown recently [21], the phase shift measured in K_{e4} decay differs from the pure strong phase shift $\delta_0^0(s)$ by an isospin correction overlooked so far, accounting for the differences between the masses of the charged and neutral mesons, and between the quark masses m_u and m_d. The correction evaluated in ChPT to one-loop reads [21]

$$\Delta[\delta_0^0(s)] = \frac{1}{32\pi F_0^2}\left\{(4\Delta_\pi + s)\rho(s) + (s - M_{\pi^0}^2)\left(1 + \frac{3}{2R}\right) \times \rho_0(s) - (2s - M_\pi^2)\rho(s)\right\}, \tag{28}$$

where $\rho(s)$ is defined in (3) and

$$\Delta_\pi = M_\pi^2 - M_{\pi^0}^2, \qquad \rho_0(s) = \sqrt{1 - 4M_{\pi^0}^2/s}, \qquad R = \frac{m_s - \hat{m}}{m_d - m_u}, \qquad \hat{m} = (m_u + m_d)/2. \tag{29}$$

With the estimate $R = 37 \pm 4$ given in [21], the correction $\Delta[\delta_0^0(s)]$ amounts to a fraction of a degree in the whole experimental range. This correction was subtracted from the phase shift derived from K_{e4} data, in order to obtain the pure strong phase shift $\delta_0^0(s)$.

The total number of points from the K_{e4} experiments is 21 (5 points from [6], 6 from [7] and 10 from [5]). As in [8], I increased the experimental error on the last point in [7] by 50%. For the 10 data from the NA48/2 experiment I used the covariance matrix published recently in [5]. I fitted these data with the parametrizations described in Sec. II.

In my analysis, the positive number α, specifying the conformal variables (7) and (27), together with the parameters κ and k_i appearing in (20), represent the input which defines an admissible class. To account for the uncertainty in the position of the Adler zero, s_A was varied between $0.4M_\pi^2$ and $0.6M_\pi^2$. In each admissible class, the coefficients B_i of the expansion in powers of the conformal variable are free. They are determined by fitting the low energy data.

I investigated a large class of combination of input parameters, from which I retained 16 admissible parametrizations: the first three are based on Eqs. (4) and (8), with $\alpha = 1$, $\alpha = 0.36$, and $\alpha = 4$, respectively. The value $\alpha = 0.36$ is special, since, as seen in Fig. 1, it maps the experimental range relevant in K_{e4} decay onto a symmetric interval around the origin $w = 0$. According to general theorems [12], this variable gives the best approximation of data in the experimental range. On the other hand, for α

TABLE I. Results of the fits of the data from K_{e4} decay [5–7], using the 16 parametrizations described in the text.

Number	χ^2	B_0	B_1	a_0^0	b_0^0	$\sqrt{s_\sigma}$ (MeV)
1.	21.7	7.5	−15.1	0.216	0.278	$459 + 259i$
2.	21.5	14.6	−12.4	0.214	0.282	$445 + 259i$
3.	21.9	−16.4	−37.1	0.217	0.275	$473 + 261i$
4.	20.9	7.7	−20.2	0.212	0.287	$412 + 237i$
5.	20.6	17.2	−16.5	0.210	0.292	$401 + 231i$
6.	21.2	−35.0	−60.3	0.214	0.284	$422 + 246i$
7.	21.5	14.6	−14.8	0.214	0.281	$443 + 262i$
8.	21.7	5.8	−18.8	0.215	0.278	$455 + 261i$
9.	21.8	−25.3	−47.9	0.216	0.276	$465 + 260i$
10.	21.6	15.2	−12.8	0.215	0.280	$451 + 264i$
11.	21.8	7.8	−15.5	0.216	0.277	$466 + 264i$
12.	21.5	15.0	−15.3	0.214	0.281	$448 + 264i$
13.	21.7	5.9	−19.5	0.216	0.277	$459 + 262i$
14.	21.8	7.8	−15.5	0.216	0.277	$465 + 263i$
15.	21.6	15.1	−12.8	0.215	0.280	$450 + 263i$
16.	21.4	14.5	−14.8	0.214	0.282	$443 + 261i$

greater than 1 (for instance, $\alpha = 4$) the variable $w(s, \alpha)$ maps the region close to the branch point $s = 4M_K^2$ near the origin of the w plane (see right panel of Fig. 1). As we shall see, the expansion (8) is then more suitable at higher energies.

The next three parametrizations are based on Eqs. (12) and (14), with the same choices $\alpha = 1$, $\alpha = 0.36$, and $\alpha = 4$. The difference from the previous fits is the way of eliminating the singularity at $s = 0$ of the phase space $\rho(s)$ in the denominator of $t_0^0(s)$.

The remaining ten parametrizations are based on the S-matrix factorization discussed in Sec. II C. In three of them I take the parameters κ and k_i from (22) and the expansion (26), with $\alpha = 0.36$, $\alpha = 1$, and $\alpha = 4$, and in the next two I use the same parameters κ and k_i and the expansion (25), with $\alpha = 0.36$ and $\alpha = 1$. In the other two cases I use the parameters κ and k_i from (23) and the expansion (26) with $\alpha = 0.36$ and $\alpha = 1$, while in the last three parametrizations I take the parameters κ and k_i from (21), using either the expansion (25) with $\alpha = 1$ and $\alpha = 0.36$, or the expansion (26) with $\alpha = 0.36$.

I obtained good fits of the 21 experimental points with 2 free parameters, B_0 and B_1, in the expansion in powers of the conformal variables. The values of χ^2 are very similar for all the fits, although the parametrizations are quite different. The values of χ^2 and the optimal parameters are given in Table I, for s_A (or s_1) fixed at $0.5M_\pi^2$ (note that if $s_1 = 0.5M_\pi^2$, the position s_A of the Adler zero, resulting from the fits, is slightly different: for the last 10 fits given in Table I, s_A varied between $0.42M_\pi^2$ and $0.47M_\pi^2$). The values of χ^2 decrease by about 0.4 if I take into account the theoretical uncertainty associated with the isospin correction [21]. For simplicity I indicate only the central values of the parameters, omitting the statistical errors.

The quality of the fits is seen in Fig. 2. Although the fits are almost indistinguishable in the experimental range, they exhibit large differences when extrapolated to higher energies. This illustrates the well-known phenomenon of instability of analytic extrapolation [22]. I note that the lowest curves in the right panel of Fig. 2 correspond to the fits 4, 5, and 6 in Table I, obtained with the parametrization (12)–(14). In particular, the fits 4 and 5, corresponding to the choices $\alpha = 1$ and $\alpha = 0.36$ in (14), exhibit a plateau at low values of $\delta_0^0(s)$. The increase of the phase shift required by the high energy data (see below) is obtained, for instance, with the choice $\alpha = 4$ in the expansion (8), or by using parametrizations based on the S-matrix factorization described in Sec. II C.

In Table I I give for each fit the central values of a_0^0 and b_0^0 and the position s_σ of the σ pole on the second Riemann

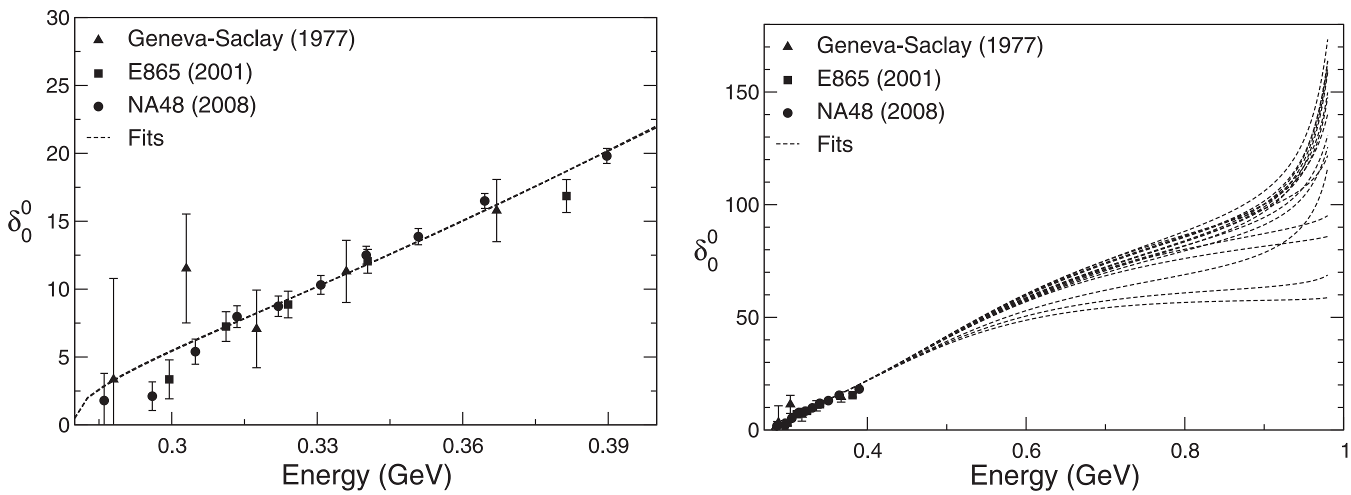

FIG. 2. Left: phase shift δ_0^0 derived from K_{e4} decay, fitted with the 16 parametrizations described in the text. Right: extrapolation of the fits above the experimental range.

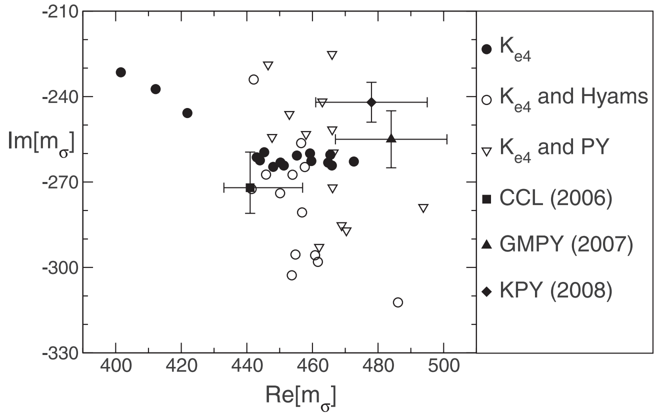

FIG. 3. Positions of the σ pole obtained by the analytic extrapolation of the parametrizations used for fitting various sets of data, compared with Refs. [4,8,23] (from the last reference I show the value obtained with isospin corrected K_{e4} data).

sheet, obtained by analytic extrapolation to the threshold and into the complex plane (as shown in [4], s_σ is the solution of the equation $S_0^0(s_\sigma) = 0$ on the first sheet).

Taking the average of the 16 admissible values for $m_\sigma = \sqrt{s_\sigma} = M_\sigma - i\Gamma_\sigma/2$, I obtain

$$\begin{aligned} M_\sigma &= 447 \pm 7(\text{stat})^{+25}_{-46}(\text{syst})\ \text{MeV}, \\ \Gamma_\sigma/2 &= 258 \pm 6(\text{stat})^{+10}_{-26}(\text{syst})\ \text{MeV}, \end{aligned} \tag{30}$$

where the systematic error is defined, as in [8], such as to cover all the admissible fits (the uncertainty in the Adler zero produces a small error, of about 4 MeV in M_σ and 3 MeV in $\Gamma_\sigma/2$).

The pole positions given in Table I are shown in Fig. 3, together with the results reported in [4,8,23]. Note that the three isolated points, with small values of M_σ and Γ_σ, correspond to the fits 4, 5, and 6 in Table I, which are based on the parametrization (12)–(14). As I mentioned, they lead to bad behavior of the phase shift at higher energies, in spite of the fact that they provide very good fits of the data from K_{e4} decay. From Table I it is seen that these parametrizations (especially 4 and 5) give low values for a_0^0 and large values for b_0^0.

In this section I obtained good fits of the phase shift measured from K_{e4} decay. However, the extrapolation of the phase shift above the experimental region is not acceptable for many of them. Therefore, I cannot take the results of this section as final. In particular, the narrow range spanned by most of the widths Γ_σ in Fig. 3 may signal a bias. In the next section I shall improve the description of $t_0^0(s)$ by including data on the phase shift at higher energies.

IV. INCLUSION OF HIGH ENERGY DATA

The difference $\delta_0^0(s) - \delta_0^2(s)$ is measured at $s = M_K^2$ from the decay $K \to \pi\pi$ [24]. However, in this case the radiative corrections are very large [25] and the extraction of the strong phase shift δ_0^0 is still uncertain [26]. For this reason, I shall not use this datum as input in my analysis.

Experimental data at higher energies are available from the $\pi N \to \pi\pi N$ process [27,28]. I considered two sets of data below the $K\bar{K}$ threshold:

(i) set I, which consists of 40 data points: 21 from K_{e4} decay [5–7] and 19 from the CERN-Munich experiment [27];

(ii) set II, which consists of 32 data points: 21 from K_{e4} decay [5–7] and a collection of 11 data points from $\pi N \to \pi\pi N$ [28], given in Eq. (2.13) of [20].

As in the previous section, I investigated a large number of parametrizations, but rejected many of them since they gave bad fits. For instance, the choice $\alpha = 0.36$ (or other values $\alpha < 1$) in the expansions (8) and (14) was not admissible, leading to high values of χ^2 (such parametrizations can not exhibit the rapid increase of the phase shift above 900 MeV). Also, the parametrization (9), considered in [8], proved to be not acceptable: with 3 free parameters, μ_0, B_0, and B_1, I obtained $\chi^2 = 45.6$ for the 40 points of set I, and $\chi^2 = 34.6$ for the 32 points of set II. We recall that expressions which display the energy where the phase shift passes through $\pi/2$ are often used for fitting narrow resonances. However, they are not suitable for broad resonances like σ.

I finally retained 13 admissible parametrizations: the first three are based on Eqs. (4) and (8), with $\alpha = 1$, $\alpha = 4$, and $\alpha = 6$, respectively. The next two are based on Eqs. (12) and (14), with the choices $\alpha = 1$ and $\alpha = 4$. The remaining eight parametrizations are based on the S-matrix factorization discussed in Sec. II C: in four cases I use the values κ and k_i from (21), and either the expansion (25) with $\alpha = 1$, $\alpha = 4$, and $\alpha = 0.5$, respectively, or the expansion (26) with $\alpha = 1$. In the next three cases I use the parameters κ and k_i from (22), and either the expansion (25) with $\alpha = 0.2$, or the expansion (26), with $\alpha = 0.5$ and $\alpha = 1$. Finally, in the last parametrization I take the values of κ and k_i from (23) and the expansion (26) with $\alpha = 4$.

For the first five parametrizations I used 3 nonzero coefficients, B_0, B_1, and B_2, in the expansion in powers of the conformal variables, and for the last eight I obtained good fits with 2 nonzero coefficients, B_0 and B_1. As in the previous section, I took into account the uncertainty in the position of the Adler zero, by allowing in each case s_A to vary between $0.4M_\pi^2$ and $0.6M_\pi^2$.

The results of the fits (for s_A or s_1 fixed at $0.5M_\pi^2$) are presented in Tables II and III, for the sets I and II, respectively. For completeness, I show also in column 3 the contribution to the χ^2 of the 21 points from K_{e4} decay. By comparing Tables II and III with Table I one notices that the description of the K_{e4} data, measured by their contribution to the total χ^2, is now slightly worse than in the fits restricted to the data from K_{e4} decay: constraining the behavior at high energies leads to a small deterioration of the description of the low energy data. The quality of the

TABLE II. Results of the fits of the 40 data points of set I, using the 13 parametrizations described in the text.

Number	χ^2	$\chi^2_{K_{e4}}$	B_0	B_1	B_2	a_0^0	b_0^0	$\sqrt{s_\sigma}$ (MeV)
1.	37.7	24.3	7.8	−23.5	−20.6	0.233	0.261	$486 + 312i$
2.	32.9	22.8	−20.3	−61.5	−23.9	0.226	0.271	$462 + 298i$
3.	32.6	22.6	−37.3	−84.6	−29.6	0.225	0.272	$461 + 296i$
4.	32.7	22.2	1.3	−52.3	−39.9	0.222	0.271	$458 + 265i$
5.	33.9	21.8	−52.8	−105.8	−26.0	0.207	0.283	$442 + 234i$
6.	32.4	22.1	9.3	−12.0	⋯	0.220	0.274	$457 + 281i$
7.	33.7	22.9	−2.2	−20.3	⋯	0.228	0.272	$454 + 303i$
8.	33.9	21.8	13.1	−12.2	⋯	0.215	0.278	$454 + 267i$
9.	38.0	21.8	5.2	−19.9	⋯	0.213	0.278	$456 + 256i$
10.	32.1	21.5	18.8	−12.9	⋯	0.216	0.281	$441 + 272i$
11.	31.9	21.6	12.4	−13.9	⋯	0.216	0.279	$446 + 267i$
12.	32.1	21.9	7.4	−15.5	⋯	0.219	0.276	$450 + 274i$
13.	32.7	22.6	−9.5	−29.3	⋯	0.225	0.272	$455 + 295i$

fits is shown in Fig. 4, and in Fig. 5 I show an expanded view of the energy region covered by K_{e4} decay. As expected from the previous discussion, the various parametrizations are no longer indistinguishable, as were those obtained from fitting only the K_{e4} data in Fig. 2.

Taking the average of the values of the threshold parameters a_0^0 and b_0^0 given in Tables II and III, weighted with the ratio $\chi^2/N_{\rm dof}$, I obtain, for the two sets

$$a_0^0 = 0.220 \pm 0.005(\text{stat}) \pm 0.013(\text{syst}) \pm 0.003(s_A) \quad (\text{I}),$$
$$a_0^0 = 0.215 \pm 0.005(\text{stat}) \pm 0.011(\text{syst}) \pm 0.003(s_A) \quad (\text{II}), \tag{31}$$

and

$$b_0^0 = 0.275 \pm 0.006(\text{stat})^{+0.009}_{-0.014}(\text{syst}) \pm 0.004(s_A) \quad (\text{I}),$$
$$b_0^0 = 0.277 \pm 0.006(\text{stat})^{+0.007}_{-0.010}(\text{syst}) \pm 0.004(s_A) \quad (\text{II}). \tag{32}$$

If I combine these determinations I obtain the values

$$a_0^0 = 0.218 \pm 0.014, \qquad b_0^0 = 0.276 \pm 0.013, \tag{33}$$

which are fully consistent with the results obtained from ChPT and low energy theorems for $\pi\pi$ scattering [3], quoted in (1).

Before discussing the results for the σ pole, let me make a few comments on the phase shift $\delta_0^0(s)$ shown in Fig. 4. It was advocated in some papers, for instance [23], that the phase shift of the isoscalar S wave exhibits a "hump" at energies around 800 MeV, before starting the rapid increase near the $K\bar{K}$ threshold. As seen from Fig. 4, no hump is seen in the fits of the data in the set I, while a weak hump appears only in a few fits of the data in set II. This proves that the hump seems to be an artifact of special parametrizations used for fitting the data (in particular, the expression (9) displays a pronounced hump if a small number of coefficients B_i is kept in the expansion). As

TABLE III. Results of the fits of the 32 data points of set II, using the 13 parametrizations described in the text.

Number	χ^2	$\chi^2_{K_{e4}}$	B_0	B_1	B_2	a_0^0	b_0^0	$\sqrt{s_\sigma}$ (MeV)
1.	28.6	22.8	5.4	−29.0	−20.9	0.225	0.267	$494 + 279i$
2.	24.8	22.4	−26.6	−71.8	−27.1	0.223	0.272	$470 + 287i$
3.	24.4	22.3	−45.9	−97.4	−33.0	0.223	0.273	$469 + 285i$
4.	25.1	22.0	−0.3	−56.6	−41.3	0.218	0.273	$466 + 251i$
5.	24.6	22.2	−56.6	−111.1	−26.9	0.205	0.284	$446 + 229i$
6.	23.9	21.8	7.4	−16.3	⋯	0.215	0.277	$466 + 259i$
7.	24.3	22.5	−6.2	−25.3	⋯	0.224	0.272	$462 + 293i$
8.	24.7	22.1	12.5	−16.9	⋯	0.208	0.281	$463 + 242i$
9.	26.6	22.9	1.0	−28.8	⋯	0.204	0.282	$465 + 225i$
10.	23.2	21.6	20.1	−16.8	⋯	0.210	0.284	$448 + 254i$
11.	23.2	21.8	11.6	−18.7	⋯	0.209	0.283	$453 + 246i$
12.	23.1	21.8	4.8	−20.8	⋯	0.213	0.279	$458 + 253i$
13.	23.5	22.0	−21.1	−43.3	⋯	0.219	0.274	$466 + 272i$

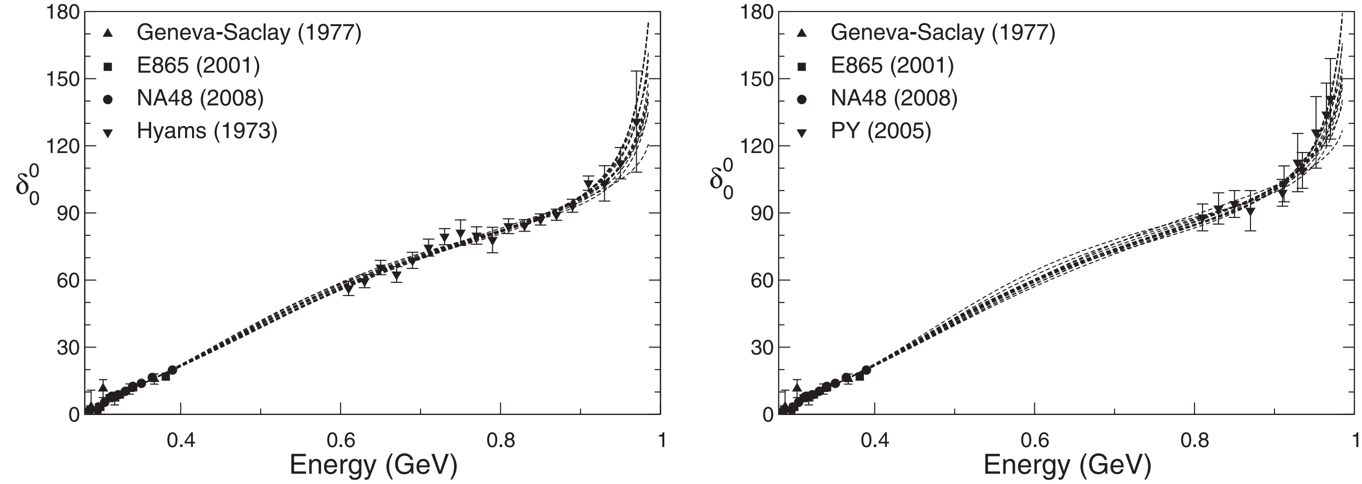

FIG. 4. Left: fits of the data in set I (K_{e4} data [5–7] plus the CERN-Munich data [27] below the $K\bar{K}$ threshold). Right: fits of the data in set II (K_{e4} data plus a selection of data from $\pi N \to \pi\pi N$, given in [20]).

discussed in [29], this shape is in conflict with the forward dispersion relation for the $\pi\pi$ amplitude of isospin $I = 0$.

It is of interest to calculate with my parametrizations the value of δ_0^0 at $\sqrt{s} = M_K$. We recall that the phase shift difference $\delta_0^0(s) - \delta_0^2(s)$ at this energy can be extracted from the decay $K \to \pi\pi$. However, as I mentioned in Sec. IV, in this case the isospin breaking corrections are large and the extraction of the strong phase shift δ_0^0 is still unclear. For this reason I did not use this information as input in my fits. Using the parameters given in Tables II and III and taking the averages of the admissible values in the two sets I obtain

$$\delta_0^0(M_K^2) = 38.9° \pm 0.6°(\text{stat})^{+1.7°}_{-1.4°}(\text{syst}) \qquad (\text{I}),$$
$$\delta_0^0(M_K^2) = 40.4° \pm 0.8°(\text{stat})^{+2.6°}_{-1.8°}(\text{syst}) \qquad (\text{II}). \tag{34}$$

In [8] the authors used as input in their fits the value $\delta_0^0(M_K^2) = 48.7 \pm 4.9°$, which is significantly larger than the output values given in (34).

The phase shift at 0.8 GeV is also of interest, since it is a key input in solving Roy equations [2]. We recall that in [2] the range is $\delta_0^0(0.8\ \text{GeV}) = 82.3 \pm 3.4°$ was adopted as input, while in [3] the more conservative choice $\delta_0^0(0.8\ \text{GeV}) = 82.3^{+10°}_{-4°}$ was made. From the parametrizations discussed above and the parameters B_i given in Tables II and III, I obtained the average values

$$\delta_0^0(0.8\ \text{GeV}) = 81.8° \pm 0.6°(\text{stat}) \pm 1.3°(\text{syst}) \qquad (\text{I}),$$
$$\delta_0^0(0.8\ \text{GeV}) = 85.9° \pm 0.7°(\text{stat})^{+3.3°}_{-2.6°}(\text{syst}) \qquad (\text{II}), \tag{35}$$

which are consistent with the ranges adopted in [2,3].

I turn now to the predictions for the σ pole obtained by the analytic continuation of the parametrizations considered above. The σ pole positions given in Tables II and III

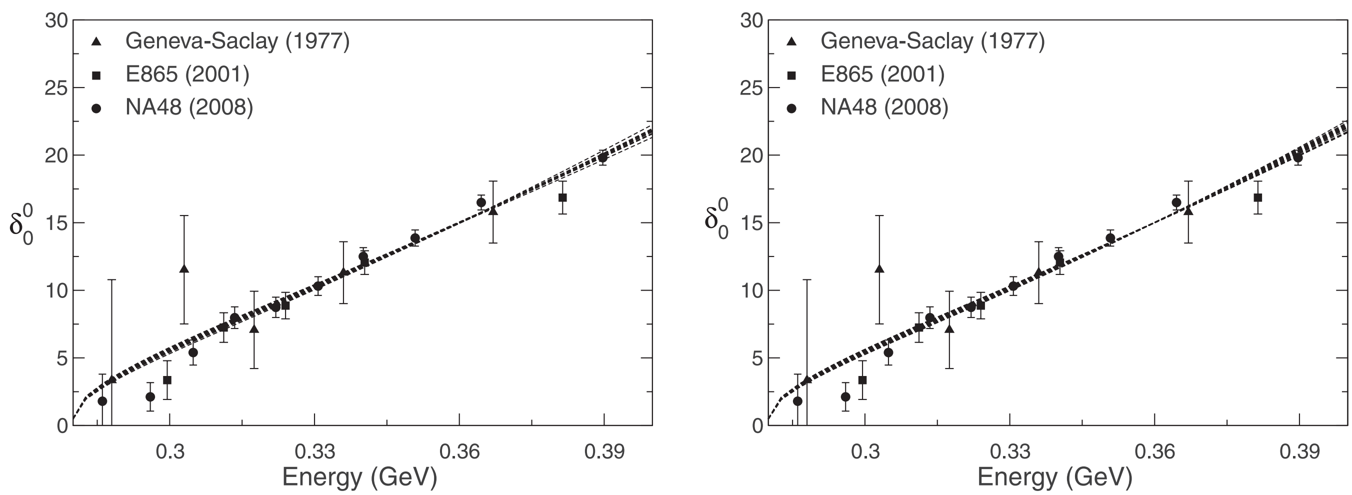

FIG. 5. Expanded view of the low energy region from Fig. 4.

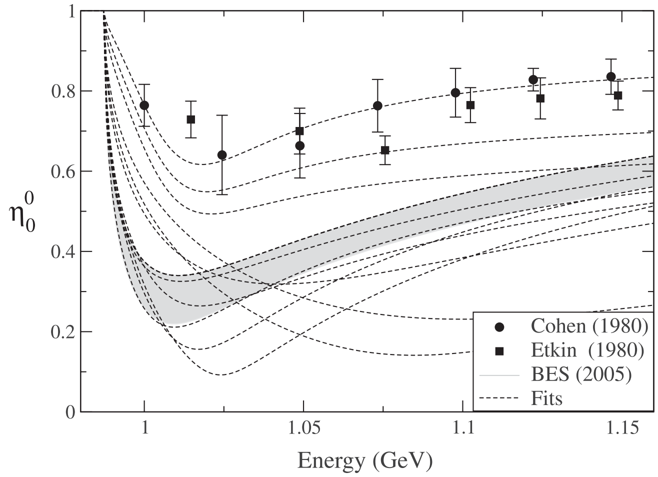

FIG. 6. Elasticity η_0^0 obtained with the 13 parametrizations of the data in set I (cf. Table II), compared with the experimental data from [16,17].

are shown in Fig. 3. The three isolated points obtained with some fits of the K_{e4} data are no longer allowed, but in the same time the tight correlation exhibited by the other fits of the K_{e4} data is now softened. This is due to the fact that the description of the K_{e4} data, measured by their contribution to the total χ^2, is slightly worse than in the fits restricted only to the K_{e4} data, and the various parametrizations are not as indistinguishable at low energies as in Fig. 2.

The averages of the values given in Tables II and III, respectively, weighted with the corresponding $\chi^2/N_{\rm dof}$, give

$$\begin{aligned} M_\sigma &= 455 \pm 6({\rm stat})^{+31}_{-13}({\rm syst})\ {\rm MeV},\\ \Gamma_\sigma/2 &= 278 \pm 6({\rm stat})^{+34}_{-43}({\rm syst})\ {\rm MeV} \qquad ({\rm I}),\\ M_\sigma &= 463 \pm 6({\rm stat})^{+31}_{-17}({\rm syst})\ {\rm MeV},\\ \Gamma_\sigma/2 &= 259 \pm 6({\rm stat})^{+33}_{-34}({\rm syst})\ {\rm MeV} \qquad ({\rm II}). \end{aligned} \tag{36}$$

As above, the systematic errors cover the values in the admissible samples. The uncertainty in the position of the Adler zero s_A has now a smaller effect, of 2 MeV for M_σ and 1 MeV for $\Gamma_\sigma/2$. Alternatively, I can define the central values from the optimal fits with the lowest χ^2 (fit 11 in Table II and fit 12 in Table III). This procedure gives

$$\begin{aligned} M_\sigma &= 446 \pm 6({\rm stat})^{+40}_{-4}({\rm syst})\ {\rm MeV},\\ \Gamma_\sigma/2 &= 267 \pm 6({\rm stat})^{+44}_{-33}({\rm syst})\ {\rm MeV} \qquad ({\rm I}),\\ M_\sigma &= 458 \pm 6({\rm stat})^{+36}_{-11}({\rm syst})\ {\rm MeV},\\ \Gamma_\sigma/2 &= 253 \pm 6({\rm stat})^{+39}_{-28}({\rm syst})\ {\rm MeV} \qquad ({\rm II}). \end{aligned} \tag{37}$$

However, this definition is not very sharp since, as seen from Tables II and III, there are several fits with very close values of χ^2, which lead to different values for M_σ and Γ_σ.

Equations (36) and (37) represent my final results for the σ pole position, obtained using the data on K_{e4} decay and two, rather complementary, sets of scattering data at higher energies. The differences between them indicate the sensitivity of the pole location to the behavior of the phase shift near the $K\bar{K}$ threshold.

The comparison of (36) and (37) with (2) shows that the analytic extrapolation of experimental data leads to values for the mass and width of σ which are consistent with ChPT and Roy equations, but have larger theoretical uncertainties. The errors are produced by the well-known instability of analytic continuation [22]: functions very close along a limited part of the boundary may differ drastically outside the initial range. I illustrate this feature in Fig. 6, where I show the elasticity η_0^0 calculated with the 13 parametrizations fitting the data of set I, given in Table II, extrapolated above the inelastic threshold. Also, in Fig. 7, I show the real and the imaginary parts of the same parametrizations along a range covering a part of the left-hand cut. Note that I am calculating now the amplitude on the cuts of the s plane, while, strictly speaking, the expansions in powers of conformal variables converge only

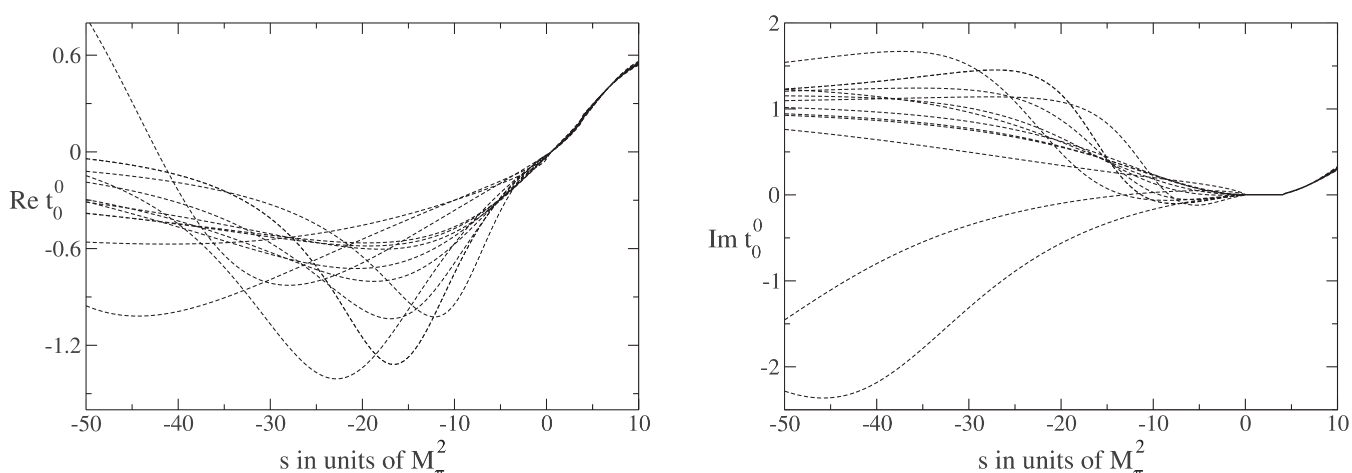

FIG. 7. $\mathrm{Re}\,t_0^0(s)$ (left) and $\mathrm{Im}\,t_0^0(s)$ (right) obtained by the extrapolation of the 13 parametrizations of the data in set I.

at points inside the analyticity domain [10]. However, the expansions are truncated at low orders and are far from the asymptotic regime, so we may view them as effective parametrizations which have a meaning also on the boundary.

We recall that information about $t_0^0(s)$ along the left-hand cut (rigorously speaking, only on a part of it) is obtained from crossing symmetry. The dominant contribution is given by the ρ resonance, which does not make a narrow peak. Of course, in the present approach crossing symmetry is not explicitly implemented. Figure 7 shows the differences in the values on the left-hand cut of amplitudes which are almost indiscernible in the physical region. Since the σ pole is rather close to the left-hand cut, the spread in the positions of the σ pole shown in Fig. 3 is not surprising.

V. CONCLUSIONS

The new accurate data on $\pi\pi$ scattering at low energies obtained from K_{e4} decay by the NA48/2 Collaboration [5] revived the interest in finding the σ pole by the standard method used for narrow resonances, i.e., by the analytic extrapolation of a suitable parametrization of the partial wave with the quantum numbers of the resonance.

In the present work I extended the investigation done in [8], by using a larger class of analytic functions for the parametrization of the $\pi\pi$ isoscalar S wave at low energies. The purpose was to reduce the theoretical bias and to provide a more realistic estimate of the systematic uncertainties on the pole position.

My analysis shows that, in spite of the remarkable accuracy of the new data obtained from K_{e4} decay [5], the inclusion of data at higher energies is necessary in order to reduce the theoretical bias and to exclude parametrizations which do not have a suitable behavior above the experimental range.

The values (36) represent my prediction for the mass and width of σ, obtained by the analytic extrapolation of a large number of admissible parametrizations of the isoscalar S wave. I present separately the results obtained by fitting with the same parametrizations the two sets of data from the process $\pi N \to \pi\pi N$, in order to illustrate the sensitivity of the pole position to the behavior near the $K\bar{K}$ threshold. I emphasize that, although I used a large number of parametrizations, the admissible sample is still limited. Therefore, the procedure is not entirely model independent (for a parametrization-free method for the detection of resonances from error-affected data given along a finite range see [30]).

My results (36) are consistent with the mass and width of σ obtained from ChPT and Roy equations, quoted in (2). However, the method employed here has larger theoretical uncertainties due to the phenomenon of instability of analytic extrapolation from a part on the boundary [22]: the differences between the various parametrizations are amplified by the extrapolation from the physical region to a distant point in the complex plane.

The extrapolation error can be kept under control by using additional information about the physical amplitude, besides the low energy experimental data. In the method based on Roy equations, this information is provided mainly by crossing symmetry and low energy theorems for $\pi\pi$ scattering [3]. As shown in [4], this tames the instability of the extrapolation to the σ pole, leading to the small errors quoted in (2) (for a detailed discussion see also [31]). I conclude that Roy equations provide the most precise determination of σ from $\pi\pi$ elastic scattering.

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