Remark on orbital precession due to central-force perturbations

O. I. Chashchina
Department of physics, Novosibirsk State University, 630 090, Novosibirsk, Russia

Z. K. Silagadze
Budker Institute of Nuclear Physics and Novosibirsk State University, 630 090, Novosibirsk, Russia
(Received 19 February 2008; published 12 May 2008)

We show that the main result of the recent paper by G. S. Adkins and J. McDonnell, Phys. Rev. D **75**, 082001 (2007), the formula for the precession of Keplerian orbits induced by central-force perturbations, can be obtained very simply by the use of Hamilton's vector.

DOI: 10.1103/PhysRevD.77.107502 PACS numbers: 04.80.Cc

In a recent paper, G. S. Adkins and J. McDonnell reconsidered the old problem of perihelion precession of Keplerian orbits under the influence of arbitrary central-force perturbations. Their main result is the formula for perihelion precession in the form of a one-dimensional integral convenient for numerical calculations.

The reason why this well studied and essentially textbook problem [1] came into the focus of the current research is the recent usage of this classical effect to constrain hypothetical modifications of Newtonian gravity from higher dimensional models [2], as well as the density of dark matter in the solar system [3].

Traditionally, the simplest way to study the perihelion motion is the use of the Runge-Lenz vector [4,5]. The Runge-Lenz vector

$$\vec{A} = \vec{v} \times \vec{L} - \alpha \vec{e}_r \tag{1}$$

is the extra constant of motion originated from the hidden symmetry of the Coulomb/Kepler problem [6]. Here, $\alpha = GmM$, $\vec{L}$ is the angular momentum vector, and $\vec{v}$ is the relative velocity of a planet of mass m with respect to the Sun of mass M. Geometrically, the Runge-Lenz vector points towards the perihelion. Therefore, its precession rate is just the precession rate of the perihelion [7].

However, we will use not the Runge-Lenz vector, but its less known cousin, the Hamilton vector [8–11]

$$\vec{u} = \vec{v} - \frac{\alpha}{L}\vec{e}_\varphi, \tag{2}$$

where φ is the polar angle in the orbit plane. This very useful vector constant of motion of the Kepler problem was well known in the past, but mysteriously disappeared from textbooks after the first decade of the twentieth century [8,10–12].

Of course, $\vec{A}$ and $\vec{u}$ are not independent constants of motion. The relation between them is

$$\vec{A} = \vec{u} \times \vec{L}. \tag{3}$$

Remembering that the magnitude of the Runge-Lenz vector is $A = \alpha e$, where e is the eccentricity of the orbit [4], we get from (3) the magnitude of the Hamilton vector

$$u = \frac{\alpha e}{L}. \tag{4}$$

If the potential $U(r)$ contains a small central-force perturbation $V(r)$ besides the Coulomb binding potential,

$$U(r) = -\frac{\alpha}{r} + V(r),$$

the Hamilton vector (as well as the Runge-Lenz vector) ceases to be conserved and begins to precess at the same rate as the Runge-Lenz vector, because according to (3) the two vectors are perpendicular.

To calculate the precession rate of the Hamilton vector we first find its time derivative

$$\dot{\vec{u}} = -\frac{1}{\mu}\frac{dV(r)}{dr}\vec{e}_r, \tag{5}$$

where $\mu = \frac{mM}{m+M}$ is the reduced mass. To get (5), we have used Newton's equation of motion for $\dot{\vec{v}}$ and the equation $\dot{\vec{e}}_\varphi = -\dot{\varphi}\vec{e}_r$.

Now the precession rate of the vector $\vec{u}$ can be found as [5]

$$\vec{\omega} = \frac{\vec{u} \times \dot{\vec{u}}}{u^2}. \tag{6}$$

Because of (2), (5), and (6) only the tangential component $r\dot{\varphi}$ of the velocity vector $\vec{v} = \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi$ contributes and we get

$$\vec{\omega} = \frac{1}{\mu u^2}\left(r\dot{\varphi} - \frac{\alpha}{L}\right)\frac{dV(r)}{dr}\vec{e}_r \times \vec{e}_\varphi = \frac{p}{\alpha e^2}\left(r\dot{\varphi} - \frac{\alpha}{L}\right)\frac{dV(r)}{dr}\vec{k},$$

where $\vec{k}$ is the unit vector in the z direction, and

$$p = \frac{L^2}{\mu\alpha} \tag{7}$$

is the semilatus rectum of the unperturbed orbit.

Therefore, under the complete orbital cycle the Hamilton vector and hence the perihelion of the orbit revolves by the angle

$$\Delta\Theta_p = \int_0^T \omega dt = \frac{p}{\alpha e^2}\int_0^{2\pi}\left(r - \frac{\alpha}{L\dot{\varphi}}\right)\frac{dV(r)}{dr}d\varphi. \quad (8)$$

However, $L = \mu r^2\dot{\varphi}$ and, therefore,

$$\frac{\alpha}{L\dot{\varphi}} = \frac{r^2}{p}.$$

Besides, to first order in the perturbation potential $V(r)$ we can use the unperturbed orbit equation

$$\frac{p}{r} = 1 + e\cos\varphi$$

while integrating (8) and get

$$\Delta\Theta_p = \frac{p^2}{\alpha e}\int_0^{2\pi}\frac{\cos\varphi}{(1+e\cos\varphi)^2}\frac{dV(r)}{dr}d\varphi. \quad (9)$$

If now we introduce the new integration variable $z = \cos\varphi$, Eq. (9) transforms into

$$\Delta\Theta_p = -\frac{2p}{\alpha e^2}\int_{-1}^{1}\frac{z}{\sqrt{1-z^2}}\frac{dV(\frac{p}{1+ez})}{dz}dz, \quad (10)$$

and this is just Eq. (30) from [13] up to the applied notations.

The extreme simplicity of this back-of-envelope derivation demonstrates clearly that the real backbone behind the Adkins and McDonnell perihelion precession formula is the Hamilton's vector, the lost sparkling diamond of introductory level mechanics.

After this work had been completed, we became aware of the paper [14], in which the author rediscovers the Hamilton vector and advocates essentially the same treatment of the perihelion precession problem as described in this paper. "Everything has been said before, but since nobody listens we have to keep going back and beginning all over again" [15].

An occasional conversation with I. B. Khriplovich triggered our interest to this problem. The work of Z. K. S. is supported in part by Grants No. Sci.School-905.2006.2 and No. RFBR 06-02-16192-a.

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