$D^6\mathcal{R}^4$ term in type IIB string theory on T^2 and U-duality

Anirban Basu*

Institute for Advanced Study, Princeton, New Jersey 08540, USA

(Received 9 February 2008; published 12 May 2008)

We propose a manifestly U-duality invariant modular form for the $D^6\mathcal{R}^4$ interaction in the effective action of type IIB string theory compactified on T^2. It receives perturbative contributions up to genus three, as well as nonperturbative contributions from D-instantons and (p, q) string instantons wrapping T^2. Our construction is based on constraints coming from string perturbation theory, U-duality, the decompactification limit to ten dimensions, and the equality of the perturbative part of the amplitude in type IIA and type IIB string theories. Using duality, parts of the perturbative amplitude are also shown to match exactly the results obtained from 11 dimensional supergravity compactified on T^3 at one loop. We also obtain parts of the genus one and genus k amplitudes for the $D^{2k}\mathcal{R}^4$ interaction for arbitrary $k \geq 4$. We enhance a part of this amplitude to a U-duality invariant modular form.

DOI: 10.1103/PhysRevD.77.106004 PACS numbers: 11.25.Sq, 11.25.Db, 11.25.Yb

1. INTRODUCTION

It is an important problem to construct the low energy effective action of string theory. Not only does it yield valuable information about the perturbative and nonperturbative structure of string theory, but it also elucidates the role of U-duality. The effective action of string theory can be constructed perturbatively in α', the inverse of the string tension. Of course there are also expected to be corrections which are nonperturbative in α'. Constructing certain interactions in the effective action is sometimes tractable in theories with maximal supersymmetry. These special interactions are Bogomol'nyi-Prasad-Sommerfield (BPS), and receive only a finite number of perturbative contributions, as well as corrections due to various instantons. We shall consider the special case of toroidal compactification of type IIB superstring theory to eight dimensions, such that it preserves all the 32 supersymmetries.

Certain classes of BPS interactions in the low energy eight dimensional effective action are expected to satisfy nonrenormalization theorems. For example, the $D^{2k}\mathcal{R}^4$ interactions (at least for sufficiently low values of k), where k is a non-negative integer, are expected to receive only a finite number of perturbative contributions, as well as non-perturbative corrections from D-instantons, and (p, q) string instantons wrapping T^2. Here $\mathcal{R}^4$ stands for the $t_8t_8R^4$ interaction [1–3], and can be expressed entirely in terms of four powers of the Weyl tensor. The U-duality symmetry and maximal supersymmetry imposes strong constraints on these interactions.

Type IIB superstring theory compactified on T^2 has a conjectured U-duality symmetry group $SL(2, \mathbb{Z})_U \times SL(3, \mathbb{Z})_M$ [4,5]. The complex structure modulus U of T^2 transforms nontrivially under $SL(2, \mathbb{Z})_U$ as

$$U \to \frac{aU + b}{cU + d}, \tag{1}$$

where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$.

The $SL(3, \mathbb{Z})_M$ factor of the U-duality group arises in a somewhat involved way. The theory has an $SL(2, \mathbb{Z})_\tau$ (S-duality) symmetry under which the complexified coupling

$$\tau = \tau_1 + i\tau_2 = C_0 + ie^{-\phi} \tag{2}$$

transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{3}$$

while the combination $B_R + \tau B_N$ transforms as

$$B_R + \tau B_N \to \frac{B_R + \tau B_N}{c\tau + d}, \tag{4}$$

where $B_N(B_R)$ is the modulus from the Neveu-Schwarz–Neveu-Schwarz (NS-NS) Ramond-Ramond (R-R) two-form on T^2. It also has an $SL(2, \mathbb{Z})_T$ (T-duality) symmetry under which the Kahler structure modulus of T^2

$$T = B_N + iV_2, \tag{5}$$

transforms as

$$T \to \frac{aT + b}{cT + d}, \tag{6}$$

where V_2 is the volume of T^2 in the string frame. It also acts on the complex scalar ρ defined by

$$\rho = -B_R + i\tau_1 V_2, \tag{7}$$

as

$$\rho \to \frac{\rho}{cT + d}, \tag{8}$$

while leaving the eight dimensional dilaton invariant. The $SL(2, \mathbb{Z})_\tau$ and $SL(2, \mathbb{Z})_T$ symmetries can be intertwined

*abasu@ias.edu

and embedded into the $SL(3,\mathbb{Z})_M$ factor of the U-duality group.

The part of the supergravity action involving the scalars can be written in the Einstein frame as (we are following the conventions of [6])

$$S \sim \frac{1}{l_s^6}\int d^8x\sqrt{-\hat{g}_8} \times \Big(-\frac{\partial_\mu U\hat{\partial}^\mu \bar{U}}{2U_2^2} + \frac{1}{4}\mathrm{Tr}(\partial_\mu M\hat{\partial}^\mu M^{-1}) + \ldots\Big), \quad (9)$$

where the hat denotes quantities in the eight dimensional Einstein frame. In (9), M is a symmetric matrix with determinant one given by

$$M = \nu^{1/3}\begin{pmatrix} 1/\tau_2 & \tau_1/\tau_2 & \mathrm{Re}(B)/\tau_2 \\ \tau_1/\tau_2 & |\tau|^2/\tau_2 & \mathrm{Re}(\bar{\tau}B)/\tau_2 \\ \mathrm{Re}(B)/\tau_2 & \mathrm{Re}(\bar{\tau}B)/\tau_2 & 1/\nu + |B|^2/\tau_2 \end{pmatrix}, \quad (10)$$

where $B = B_R + \tau B_N$, and $\nu = (\tau_2 V_2^2)^{-1}$.

In the Einstein frame, where the metric is U-duality invariant, the coefficients of these protected $D^{2k}\mathcal{R}^4$ interactions should be given by modular forms of the U-duality group, which are invariant under $SL(2,\mathbb{Z})_U \times SL(3,\mathbb{Z})_M$ transformations. Constructing these modular forms for toroidal compactifications of type II string theory and M theory that preserve maximal supersymmetry, and analyzing their nonrenormalization properties have been worked out for some of these operators in various dimensions [6–23] (see [24,25] for reviews). In eight dimensions, a modular form for the $D^4\mathcal{R}^4$ interaction has been proposed recently [26]. In this work, we shall propose a manifestly U-duality invariant modular form for the $D^6\mathcal{R}^4$ interaction in the effective action. By this, we actually mean the

$$(s^3 + t^3 + u^3)\mathcal{R}^4 \quad (11)$$

interaction involving the elastic scattering of two gravitons.

To summarize, we propose that modular form is given by

$$\mathcal{E}_{(3/2,3/2)}(M) + \tfrac{20}{3}E_3(M^{-1})^{SL(3,\mathbb{Z})}E_3(U,\bar{U})^{SL(2,\mathbb{Z})} + f(U,\bar{U}) + \tfrac{1}{2}E_{3/2}(M)^{SL(3,\mathbb{Z})}E_1(U,\bar{U})^{SL(2,\mathbb{Z})}, \quad (12)$$

where $E_s(M)^{SL(3,\mathbb{Z})}$ ($E_s(M^{-1})^{SL(3,\mathbb{Z})}$) is the nonholomorphic modular invariant Eisenstein series of $SL(3,\mathbb{Z})_M$ of order s in the fundamental (antifundamental) representation. Also $E_s(U,\bar{U})^{SL(2,\mathbb{Z})}$ is the nonholomorphic modular invariant Eisenstein series of $SL(2,\mathbb{Z})_U$. These Einstein series satisfy the Laplace equation on the fundamental domain of moduli space. On the other hand, $f(U,\bar{U})$ and $\mathcal{E}_{(3/2,3/2)}(M)$ are $SL(2,\mathbb{Z})_U$ and $SL(3,\mathbb{Z})_M$ invariant modular forms, respectively, that satisfy the Poisson equation on the fundamental domain of moduli space given by

$$\Delta_{SL(2,\mathbb{Z})_U} f(U,\bar{U}) = 12 f(U,\bar{U}) - 6(E_1(U,\bar{U}))^2, \quad (13)$$

and

$$\Delta_{SL(3,\mathbb{Z})}\mathcal{E}_{(3/2,3/2)}(M) = 12\mathcal{E}_{(3/2,3/2)}(M) - \tfrac{3}{2}(E_{3/2}(M))^2. \quad (14)$$

We begin by constructing the perturbative part of the modular form. Constraints coming from string perturbation theory, U-duality, the decompactification limit to ten dimensions, and the equality of the perturbative part of the amplitude in type IIA and type IIB string theories, lead us to propose the complete perturbative part of the modular form.[1] This receives contributions only up to genus three in string perturbation theory. Using duality, we next provide evidence for some of these contributions by analyzing the one-loop four graviton scattering amplitude in 11 dimensional supergravity compactified on T^3.

We next propose the exact expression for the modular form based on constraints of supersymmetry and the ten dimensional $SL(2,\mathbb{Z})_\tau$ invariant answer. This provides the nonperturbative completion of the perturbative part of the modular form and involves contributions from D-instantons, as well as from (q,q) string instantons wrapping T^2. Analyzing one-loop 11 dimensional supergravity compactified on T^3, we also obtain parts of the genus one and genus k amplitudes for the $D^{2k}\mathcal{R}^4$ interaction for arbitrary $k \geq 4$. We enhance a part of this amplitude to a U-duality invariant modular form. We also make some comments about generalizing our construction to toroidal compactifications with maximal supersymmetry to lower dimensions. In the appendices, relevant details for the Eisenstein series of $SL(2,\mathbb{Z})$ and $SL(3,\mathbb{Z})$ and the torus amplitude are summarized. They also contain a discussion about possible contributions to the modular form we might have missed, where we provide arguments that they should vanish.

II. THE PERTURBATIVE PART OF THE PROPOSED MODULAR FORM

We begin by constructing the perturbative part of the proposed modular form. The low energy effective action for type IIB superstring theory in ten dimensions includes the interaction (in the string frame) [21]

$$S \sim l_s^4\int d^{10}x\sqrt{-g}\big(\zeta(3)^2 e^{-2\phi} + 2\zeta(3)\zeta(2) + 6\zeta(4)e^{2\phi} + \frac{2}{9}\zeta(6)e^{4\phi} + \ldots\big)D^6\mathcal{R}^4, \quad (15)$$

[1] Since the $\mathcal{R}^4$ interaction involves the even-even spin structures only, the perturbative contributions have to be the same in the two type II string theories. Thus this part of the amplitude must be symmetric under the interchange of U and T, while the eight dimensional IIA dilaton goes to the IIB dilaton and vice versa.

where the ... involve contributions from D-instantons. Thus from (15), we see that the $D^6\mathcal{R}^4$ interaction receives perturbative contributions only up to genus three. Compactifying on T^2 of volume $V_2 l_s^2$ in the string frame, this leads to an interaction in the eight dimensional Einstein frame given by

$$S \sim l_s^6 \int d^8x\sqrt{-\hat{g}_8}(V_2e^{-\phi})^2(\zeta(3)^2e^{-2\phi} + 2\zeta(3)\zeta(2) + 6\zeta(4)e^{2\phi} + \tfrac{2}{9}\zeta(6)e^{4\phi} + \ldots)\hat{D}^6\hat{\mathcal{R}}^4. \quad (16)$$

Thus the modular form for the $D^6\mathcal{R}^4$ interaction must include, among other terms,

$$(V_2e^{-\phi})^2(\zeta(3)^2e^{-2\phi} + 2\zeta(3)\zeta(2) + 6\zeta(4)e^{2\phi} + \tfrac{2}{9}\zeta(6)e^{4\phi}). \quad (17)$$

We first construct the perturbative part of the modular form.

A. Constraints using string perturbation theory

Let us consider the perturbative contributions to the $D^6\mathcal{R}^4$ interaction. As mentioned before, by this interaction, we actually mean the term

$$(s^3 + t^3 + u^3)\mathcal{R}^4 \quad (18)$$

in the four graviton scattering amplitude.

Consider the tree-level and one-loop amplitudes for this interaction using string perturbation theory. The sum of the contributions to the four graviton amplitude at tree level [1,3] and at one loop [3,27] in type II string theory compactified on T^2 is proportional to[2]

$$\left[-V_2e^{-2\phi}\frac{\Gamma(-l_s^2s/4)\Gamma(-l_s^2t/4)\Gamma(-l_s^2u/4)}{\Gamma(1+l_s^2s/4)\Gamma(1+l_s^2t/4)\Gamma(1+l_s^2u/4)} + 2\pi I\right]\mathcal{R}^4, \quad (19)$$

where V_2 is the volume of T^2 in the string frame, s, t, u are the Mandelstam variables, and I is obtained from the one-loop amplitude. We are looking at the part of the amplitude involving the even-even spin structures, and hence the amplitude is the same for type IIA and type IIB string theories. Now I is given by

$$I = \int_{\mathcal{F}} \frac{d^2\Omega}{\Omega_2^2} Z_{\rm lat} F(\Omega, \bar{\Omega}), \quad (20)$$

where $\mathcal{F}$ is the fundamental domain of $SL(2,\mathbb{Z})$, and $d^2\Omega = d\Omega d\bar{\Omega}/2$. The relative coefficient between the tree level and the one-loop terms in (19) is fixed using unitarity [28]. In (20), the lattice factor $Z_{\rm lat}$ which depends on the moduli is given by [29]

$$Z_{\rm lat} = V_2 \sum_{m_1,m_2,n_1,n_2\in\mathbb{Z}} e^{-(\pi/\Omega_2)\sum_{i,j}(G+B_N)_{ij}(m_i+n_i\Omega)(m_j+n_j\bar{\Omega})}$$
$$= V_2 \sum_{A\in{\rm Mat}(2\times2,\mathbb{Z})} \exp\left[-2\pi i T(\det A) - \frac{\pi T_2}{\Omega_2 U_2}\left|\begin{pmatrix}1 & U\end{pmatrix} A \begin{pmatrix}\Omega \\ 1\end{pmatrix}\right|^2\right], \quad (21)$$

where

$$G_{ij} = \frac{T_2}{U_2}\begin{pmatrix}1 & U_1 \\ U_1 & |U|^2\end{pmatrix}. \quad (22)$$

Also the dynamical factor $F(\Omega, \bar{\Omega})$ in (20), which is independent of the moduli, is given by

$$F(\Omega, \bar{\Omega}) = \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{d^2\nu_i}{\Omega_2}(\chi_{12}\chi_{34})^{l_s^2 s}(\chi_{14}\chi_{23})^{l_s^2 t}(\chi_{13}\chi_{24})^{l_s^2 u}. \quad (23)$$

In (23), ν_i $(i = 1, \ldots, 4)$ are the positions of insertions of the four vertex operators on the toroidal world sheet, and ν_4 has been set equal to Ω using conformal invariance. Also $d^2\nu_i = d\nu_i^R d\nu_i^I$, where ν_i^R (ν_i^I) are the real (imaginary) parts of ν_i. The integral over $\mathcal{T}$ is over the domain $\mathcal{T} = \{-1/2 \leq \nu_i^R < 1/2, 0 \leq \nu_i^I < \Omega_2\}$. Finally, $\ln\chi(\nu_i - \nu_j; \Omega)$ is the scalar Green function between the points ν_i and ν_j on the toroidal world sheet.

Expanding (20) to sixth order in the momenta, we get that

$$I = \frac{l_s^6}{3}(s^3 + t^3 + u^3)[\hat{I}_1 + \hat{I}_2], \quad (24)$$

where

$$\hat{I}_1 = 4\int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_{\rm lat} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{d^2\nu_i}{\Omega_2} \ln\hat{\chi}(\nu_1 - \nu_2; \Omega) \times \ln\hat{\chi}(\nu_1 - \nu_3; \Omega)\hat{\chi}(\nu_2 - \nu_3; \Omega), \quad (25)$$

and

$$\hat{I}_2 = \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_{\rm lat} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{d^2\nu_i}{\Omega_2}[\ln\hat{\chi}(\nu_1 - \nu_2; \Omega)]^3, \quad (26)$$

which can be depicted diagrammatically as in Fig. 1.

In the expressions above, we have defined

$$\hat{\chi}(\nu_i - \nu_i; \Omega) = \chi(\nu_i - \nu_j; \Omega) - \tfrac{1}{2}\ln|(2\pi)^{1/2}\eta(\Omega)|^2. \quad (27)$$

Thus we have removed the zero mode part of the scalar propagator, which does not contribute to the on-shell amplitude using $s + t + u = 0$.

In (25) and (26), note that the one-loop contribution has been integrated over the restricted fundamental domain $\mathcal{F}_L$ of $SL(2,\mathbb{Z})$, which is obtained from $\mathcal{F}$ by restricting

[2] The calculation actually yields $\mathcal{R}^4$ at the linearized level.

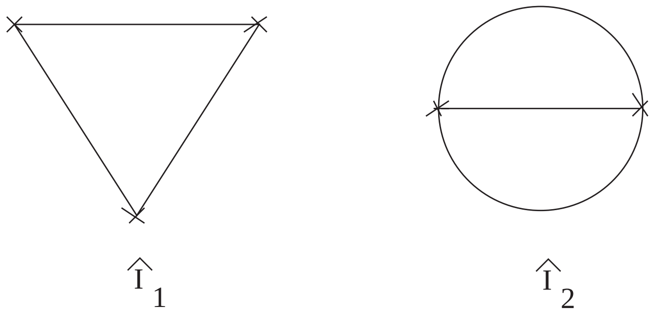

FIG. 1. Schematics of the torus amplitude.

to $\Omega_2 \leq L$. This is necessary to separate the analytic parts of the amplitude from the nonanalytic parts (see [30] for a detailed discussion). The integral over $\mathcal{F}_L$ gives both finite and divergent terms to the amplitude in the limit $L \to \infty$. The terms which are finite in this limit are the analytic parts of the amplitude. The parts which diverge in this limit cancel in the whole amplitude when the contribution from the part of the moduli space $\mathcal{F}$ with $\Omega_2 > L$ is also included. In addition to these divergences which cancel, the contribution from $\mathcal{F}$ with $\Omega_2 > L$ also gives the various nonanalytic terms in the amplitude. Keeping this in mind, we shall consider only the contributions which are finite in the limit $L \to \infty$ and drop all divergent terms. In the calculations, we shall see that the domain of integration $\mathcal{F}$ shall often be changed to the upper half-plane or a strip. Then truncating to $\mathcal{F}_L$ to calculate the analytic terms cannot be done when the integration over $\mathcal{F}_L$ produces divergences of the form $\ln L$ [30]. However, for our case there are no logarithmic divergences, and so this is not a problem for us.

In calculating both $\hat{I}_1$ and $\hat{I}_2$, we need to add the contributions from the zero orbit, the nondegenerate orbits, and the degenerate orbits of $SL(2, \mathbb{Z})$, respectively [29].

(i) The contribution from the zero orbit involves setting $A = 0$ in (21).

(ii) The contribution from the nondegenerate orbits involves setting

$$A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \tag{28}$$

where $k > j \geq 0$, $p \neq 0$ in (21), and changing the domain of integration to be the double cover of the upper half-plane.

(iii) The contribution from the degenerate orbits involves setting

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix} \tag{29}$$

such that $(j, p) \neq (0, 0)$ in (21), and changing the domain of integration to be the strip $0 < \Omega_2 < L$, $|\Omega_1| < 1/2$.

The details of the calculation of $\hat{I}_1$ and $\hat{I}_2$ are given in the appendix. This gives us

$$\begin{aligned} \hat{I}_1 &= \frac{1}{8\pi^6} E_3(U, \bar{U})^{SL(2,\mathbb{Z})} E_3(T, \bar{T})^{SL(2,\mathbb{Z})}, \\ \hat{I}_2 &= \frac{1}{32\pi^6} E_3(U, \bar{U})^{SL(2,\mathbb{Z})} E_3(T, \bar{T})^{SL(2,\mathbb{Z})} \\ &\quad + \frac{3}{32\pi^3} \zeta(2)\zeta(3)(E_1(U, \bar{U})^{SL(2,\mathbb{Z})} + E_1(T, \bar{T})^{SL(2,\mathbb{Z})}). \end{aligned} \tag{30}$$

Thus the total amplitude in (19) gives

$$\begin{aligned} &\bigg[\zeta(3)^2 e^{-2\phi} V_2 + \frac{10}{\pi^5} E_3(U, \bar{U})^{SL(2,\mathbb{Z})} E_3(T, \bar{T})^{SL(2,\mathbb{Z})} \\ &\quad + \zeta(3)(E_1(U, \bar{U})^{SL(2,\mathbb{Z})} + E_1(T, \bar{T})^{SL(2,\mathbb{Z})})\bigg] \\ &\quad \times l_s^6 (s^3 + t^3 + u^3)\mathcal{R}^4. \end{aligned} \tag{31}$$

B. Constraints using U-duality and the decompactification limit

Having obtained the tree level and the one-loop contributions to the scattering amplitude, we now show how U-duality and the decompactification limit constrains the perturbative structure of the modular form. Now (31) leads to the term in the effective action in the Einstein frame given by

$$\begin{aligned} &l_s^6 \int d^8x \sqrt{-\hat{g}_8} V_2 e^{-2\phi} \bigg[\zeta(3)^2 e^{-2\phi} V_2 + \frac{10}{\pi^5} E_3(U, \bar{U})^{SL(2,\mathbb{Z})} \\ &\quad \times E_3(T, \bar{T})^{SL(2,\mathbb{Z})} + \zeta(3)(E_1(U, \bar{U})^{SL(2,\mathbb{Z})} \\ &\quad + E_1(T, \bar{T})^{SL(2,\mathbb{Z})})\bigg] \hat{D}^6 \hat{\mathcal{R}}^4. \end{aligned} \tag{32}$$

Thus the tree level and the one-loop contributions to the modular form are given by

$$\begin{aligned} &\zeta(3)^2 (\tau_2^2 V_2)^2 + \tau_2^2 V_2 \bigg[\frac{10}{\pi^5} E_3(U, \bar{U})^{SL(2,\mathbb{Z})} E_3(T, \bar{T})^{SL(2,\mathbb{Z})} \\ &\quad + \zeta(3)(E_1(U, \bar{U})^{SL(2,\mathbb{Z})} + E_1(T, \bar{T})^{SL(2,\mathbb{Z})})\bigg]. \end{aligned} \tag{33}$$

Note that the U dependent parts of the modular form in (33) involving $E_3(U, \bar{U})^{SL(2,\mathbb{Z})}$ and $E_1(U, \bar{U})^{SL(2,\mathbb{Z})}$ are $SL(2, \mathbb{Z})_U$ invariant. Thus whatever multiplies these terms must be $SL(3, \mathbb{Z})_M$ invariant. Thus in (33), the two expressions

$$\tau_2^2 V_2 E_3(T, \bar{T})^{SL(2,\mathbb{Z})}, \tag{34}$$

which multiplies $E_3(U, \bar{U})^{SL(2,\mathbb{Z})}$, and

$$\tau_2^2 V_2, \tag{35}$$

which multiplies $E_1(U, \bar{U})^{SL(2,\mathbb{Z})}$, must both be enhanced to invariant modular forms of $SL(3, \mathbb{Z})_M$. Such modular forms need not be simple expressions involving Eisenstein series of $SL(3,\mathbb{Z})_M$. For example, the modular forms for the $\mathcal{R}^4$ and the $D^4\mathcal{R}^4$ interactions in ten dimensions are given by Eisenstein series of $SL(2,\mathbb{Z})_\tau$ which satisfies the Laplace equation on the fundamental domain of $SL(2,\mathbb{Z})_\tau$, however, the modular form for the $D^6\mathcal{R}^4$ interaction is more complicated, and satisfies a Poisson equation on the fundamental domain of $SL(2,\mathbb{Z})_\tau$. However, we now argue that there are simple and natural modular forms of $SL(3,\mathbb{Z})_M$ to which (34) and (35) can be ehanced to.

In order to motivate natural candidates for these modular forms, from (33) note that the genus g contribution to the perturbative part of the modular form involves $(\tau_2^2 V_2)^{2-g}$. Given the structure of the perturbative contributions to $E_s(M)^{SL(3,\mathbb{Z})}$ which follow from (A13), we see that the possible choices are severely restricted. In fact, there are only two possibilities:

(i) $E_{-3/2}(M)^{SL(3,\mathbb{Z})}$, which contributes at genus one and three, and

(ii) $E_{3/2}(M)^{SL(3,\mathbb{Z})}$, which contributes at genus one and two.

The only other possibility based on the $\tau_2^2 V_2$ dependence is $E_{-9/2}(M)^{SL(3,\mathbb{Z})}$, which contributes at genus zero and five. However the tree-level contribution is proportional to $(\tau_2^2 V_2)^2 E_6(T,\bar{T})^{SL(2,\mathbb{Z})}$, which is inconsistent with the known tree-level amplitude.

In fact, from (A13), we see that[3]

$$E_{-3/2}(M)^{SL(3,\mathbb{Z})}_{\text{pert}} = \frac{1}{60}(\tau_2^2 V_2)^{-1} + \frac{3}{2\pi^5}\tau_2^2 V_2 E_3(T,\bar{T})^{SL(2,\mathbb{Z})}, \tag{36}$$

which has a genus one contribution involving (34), where we have also used the relation (A5). Also we have that

$$E_{3/2}(M)^{SL(3,\mathbb{Z})}_{\text{pert}} = 2\zeta(3)\tau_2^2 V_2 + 2E_1(T,\bar{T})^{SL(2,\mathbb{Z})}, \tag{37}$$

which has a genus one contribution involving (35). This suggests a natural enhancement

$$\begin{aligned} &\frac{10}{\pi^5}\tau_2^2 V_2 E_3(T,\bar{T})^{SL(2,\mathbb{Z})} E_3(U,\bar{U})^{SL(2,\mathbb{Z})} \\ &\quad\to \frac{20}{3}E_{-3/2}(M)^{SL(3,\mathbb{Z})}_{\text{pert}} E_3(U,\bar{U})^{SL(2,\mathbb{Z})}, \\ &\zeta(3)\tau_2^2 V_2 E_1(U,\bar{U})^{SL(2,\mathbb{Z})} \to \frac{1}{2}E_{3/2}(M)^{SL(3,\mathbb{Z})}_{\text{pert}} E_1(U,\bar{U})^{SL(2,\mathbb{Z})}. \end{aligned} \tag{38}$$

[3] We use $\zeta(-3) = 1/120$.

Thus (33) gets enhanced to

$$\begin{aligned} &\zeta(3)^2(\tau_2^2 V_2)^2 + \zeta(3)\tau_2^2 V_2 E_1(T,\bar{T})^{SL(2,\mathbb{Z})} \\ &\quad + \tfrac{1}{9}(\tau_2^2 V_2)^{-1} E_3(T,\bar{T})^{SL(2,\mathbb{Z})} \\ &\quad + \tfrac{20}{3}E_{-3/2}(M)^{SL(3,\mathbb{Z})}_{\text{pert}} E_3(U,\bar{U})^{SL(2,\mathbb{Z})} \\ &\quad + \tfrac{1}{2}E_{3/2}(M)^{SL(3,\mathbb{Z})}_{\text{pert}} E_1(U,\bar{U})^{SL(2,\mathbb{Z})}, \end{aligned} \tag{39}$$

where we have added the term involving $(\tau_2^2 V_2)^{-1} E_3(T,\bar{T})^{SL(2,\mathbb{Z})}$ by hand. This is a genus three contribution and has to be added to ensure the perturbative equality of the type IIA and type IIB scattering amplitudes, for reasons explained before.

However as we shall explain below, (39) cannot be the complete perturbative part of the modular form, because it does not give the correct perturbative contributions on decompactifying to ten dimensions: the genus two contribution vanishes as we shall shortly explain, contradicting (15). We thus add a term

$$f(T,\bar{T}) + f(U,\bar{U}) \tag{40}$$

by hand to (39), where $f(T,\bar{T})$ ($f(U,\bar{U})$) is invariant under $SL(2,\mathbb{Z})_T$ ($SL(2,\mathbb{Z})_U$) transformations. This yields a genus two contribution, and is also manifestly symmetric under the interchange of T and U. We shall fix $f(T,\bar{T})$ later.

Thus, adding (39) and (40), we propose that the complete perturbative part of the modular form is given by

$$\begin{aligned} &\zeta(3)^2(\tau_2^2 V_2)^2 + \zeta(3)\tau_2^2 V_2 E_1(T,\bar{T})^{SL(2,\mathbb{Z})} + f(T,\bar{T}) \\ &\quad + \tfrac{1}{9}(\tau_2^2 V_2)^{-1} E_3(T,\bar{T})^{SL(2,\mathbb{Z})} + f(U,\bar{U}) \\ &\quad + \tfrac{20}{3}E_{-3/2}(M)^{SL(3,\mathbb{Z})}_{\text{pert}} E_3(U,\bar{U})^{SL(2,\mathbb{Z})} \\ &\quad + \tfrac{1}{2}E_{3/2}(M)^{SL(3,\mathbb{Z})}_{\text{pert}} E_1(U,\bar{U})^{SL(2,\mathbb{Z})}. \end{aligned} \tag{41}$$

Thus, converting to the string frame, we see that (41) yields the contributions

$$\begin{aligned} &\text{genus 0: } \zeta(3)^2, \\ &\text{genus 1: } \frac{10}{\pi^5}E_3(U,\bar{U})^{SL(2,\mathbb{Z})}E_3(T,\bar{T})^{SL(2,\mathbb{Z})} + \zeta(3) \\ &\quad \times (E_1(U,\bar{U})^{SL(2,\mathbb{Z})} + E_1(T,\bar{T})^{SL(2,\mathbb{Z})}), \\ &\text{genus 2: } E_1(U,\bar{U})^{SL(2,\mathbb{Z})}E_1(T,\bar{T})^{SL(2,\mathbb{Z})} + f(T,\bar{T}) \\ &\quad + f(U,\bar{U}), \\ &\text{genus 3: } \frac{1}{9}(E_3(U,\bar{U})^{SL(2,\mathbb{Z})} + E_3(T,\bar{T})^{SL(2,\mathbb{Z})}), \end{aligned} \tag{42}$$

and so the perturbative part of the amplitude is the same in type IIA and type IIB string theories.

We now show that in ten dimensions, (41) without the $f(T,\bar{T}) + f(U,\bar{U})$ term gives all the contributions in (15) except the genus two contribution. We first decompactify to nine dimensions by defining

$$T_2 = r_\infty r_B, \qquad U_2 = \frac{r_\infty}{r_B}, \tag{43}$$

where r_∞ is the direction that is being decompactified. Here r_∞ and r_B are the radii of T^2 in the string frame. Now let us take the limit $r_\infty \to \infty$, so that $T_2, U_2 \to \infty$. This leads to the nine dimensional interaction

$$l_s^5 \int d^9x\sqrt{-g_9}\bigg[(r_B e^{-2\phi})\zeta(3)^2 + \bigg\{\frac{15}{\pi^4}\zeta(5)\zeta(6)\bigg(r_B^5 + \frac{1}{r_B^5}\bigg) + 2\zeta(2)\zeta(3)\bigg(r_B + \frac{1}{r_B}\bigg)\bigg\} + 4\zeta(2)^2(r_B e^{-2\phi})^{-1} + \frac{2}{9}\zeta(6) \times (r_B e^{-2\phi})^{-2}\bigg(r_B^3 + \frac{1}{r_B^3}\bigg)\bigg]D^6\mathcal{R}^4, \tag{44}$$

where we have set $l_s \int d^8x\sqrt{-g_8}r_\infty = \int d^9x\sqrt{-g_9}$. We have dropped a term that diverges in the nine dimensional limit. This term comes from the genus one amplitude and is given by

$$\frac{40}{\pi^5}\zeta(6)^2 l_s^5 \int d^9x\sqrt{-g_9}r_\infty^5 D^6\mathcal{R}^4. \tag{45}$$

This term is only one of an infinite number of such diverging terms coming from the infinite number of analytic terms. These diverging terms as well as the nonanalytic terms must add up to give the massless threshold singularity in nine dimensions, and hence do not form a part of the $D^6\mathcal{R}^4$ interaction in nine dimensions. Clearly because the infinite number of divergent terms must add to give the threshold singularity, every divergent term must be independent of the dilaton, and hence must come from the decompactification limit of the genus one amplitude only. The fact that there are no divergent terms from the higher genus amplitudes is a consistency check of our proposal. Also, note that the one-loop amplitude in (44) precisely agrees with string perturbation theory [31], providing a nontrivial check for our proposed modular form.

Finally, taking the limit $r_B \to \infty$, we get the term in the ten dimensional effective action

$$l_s^4 \int d^{10}x\sqrt{-g}\bigg(\zeta(3)^2 e^{-2\phi} + 2\zeta(3)\zeta(2) + \frac{2}{9}\zeta(6)e^{4\phi}\bigg)D^6\mathcal{R}^4, \tag{46}$$

where we have set $l_s \int d^9x\sqrt{-g_9}r_B = \int d^{10}x\sqrt{-g}$. We have dropped a divergent term given by

$$\frac{15}{\pi^4}\zeta(5)\zeta(6)l_s^4 \int d^{10}x\sqrt{-g}r_B^4 D^6\mathcal{R}^4. \tag{47}$$

Apart from the genus two term, (46) precisely matches (15) providing some more evidence for the perturbative part of the modular form. Dropping the $f(T,\bar{T}) + f(U,\bar{U})$ term in (42), note that the ten dimensional contribution comes entirely from the terms which are independent of U in (42).

Finally, let us consider the divergent term (47). This has been computed directly in ten dimensions in [15], where it was shown that the divergent term and the genus two contribution together are proportional to

$$\tfrac{2}{3}\zeta(4)e^{4\phi^B} + \tfrac{1}{2}\zeta(5)r_B^4. \tag{48}$$

This is exactly what we get by adding the genus two contribution in (46) and the divergence in (47),[4] up to an overall irrelevant numerical factor of $\zeta(6)/3\zeta(4)$. This provides another strong check of our proposal.

III. EVIDENCE USING ELEVEN DIMENSIONAL SUPERGRAVITY AT ONE LOOP ON T^3

We now provide some evidence for the perturbative part of the proposed modular form by considering the four graviton scattering amplitude in 11 dimensional supergravity compactified on T^3. Of course 11 dimensional supergravity cannot give the complete answer. There are extra contributions due to membrane instantons wrapping the T^3. This will give contributions depending on the Kahler structure modulus in type IIA, and complex structure modulus in type IIB string theory. So the supergravity analysis will miss such contributions, and we shall see that it yields the leading U_2 behavior of some of the terms, which arise while going from the M theory to the string theory coordinates.

In order to look at the supergravity contributions to the $D^6\mathcal{R}^4$ interaction, we need to go beyond the one-loop amplitude.[5] Two and three-loop contributions (and possibly higher loops as well) also contribute to the amplitude [15,32–34] which we shall not discuss. We shall see that the one-loop supergravity amplitude coupled with the genus zero string theory amplitude will give us some of the terms in our proposed modular form.

So let us consider one-loop supergravity in 11 dimensions compactified on T^3. Apart from the overall kinematic factor which contains the spacetime dependence, the calculation simplifies and boils down to a box diagram calculation in scalar field theory with cubic interaction, essentially because of supersymmetry. The four graviton amplitude is given by [15,35–37]

$$A_4 = \frac{\kappa_{11}^4}{(2\pi)^{11}}\hat{K}[I(S,T) + I(S,U) + I(U,T)], \tag{49}$$

where $\hat{K}$ involves the $\mathcal{R}^4$ interaction at the linearized level, and

[4]We also use $\zeta(5) = \pi^4/90$.

[5]In this section, loops refer to spacetime loops in 11 dimensional supergravity on T^3. We shall refer to the world sheet expansion of string perturbation theory as the genus expansion.

$$I(S,T) = \frac{2\pi^4}{l_{11}^3 V_3}\int_0^\infty \frac{d\sigma}{\sigma}\int_0^1 d\omega_3 \int_0^{\omega_3} d\omega_2 \int_0^{\omega_2} d\omega_1 \times \sum_{\{l_1,l_2,l_3\}} e^{-G^{IJ}l_I l_J \sigma/l_{11}^2 - Q(S,T;\omega_r)\sigma}, \quad (50)$$

where $Q(S,T;\omega_r) = -S\omega_1(\omega_3-\omega_2) - T(\omega_2-\omega_1)(1-\omega_3)$.[6] Here V_3 is the volume of T^3 in the M theory metric. Denoting the torus directions as 1, 2, and 3, we choose $G_{11} = R_{11}^2$ to be the metric along the M theory circle, thus $R_{11} = e^{2\phi^A/3}$. Though we need the $(s^3+u^3+t^3)\mathcal{R}^4$ term, we shall later find it useful to extract a part of the momentum independent amplitude from (49) in order to fix normalizations. This is given by

$$\begin{aligned} A_4(S=T=U=0) &= \frac{\kappa_{11}^4\hat{K}}{(2\pi)^{11}}\cdot\frac{\pi^4}{l_{11}^3 V_3}\int_0^\infty\frac{d\sigma}{\sigma}\sum_{\{l_1,l_2,l_3\}} e^{-G^{IJ}l_I l_J\sigma/l_{11}^2} \\ &= \frac{\kappa_{11}^4\hat{K}}{(2\pi)^{11}}\cdot\pi^4\int_0^\infty\frac{d\sigma}{\sigma^{5/2}}\sum_{\{\hat{l}_1,\hat{l}_2,\hat{l}_3\}} e^{-\pi G_{IJ}\hat{l}_I\hat{l}_J l_{11}^2/\sigma}, \end{aligned} \quad (51)$$

where we have done Poisson resummation using (A16). Considering the $\hat{l}_1 \neq 0$, $\hat{l}_2 = \hat{l}_3 = 0$ piece, (51) gives [9]

$$A_4(S=T=U=0) = \frac{\kappa_{11}^4\hat{K}}{(2\pi)^{11}l_{11}^3}[\pi^3\zeta(3)e^{-2\phi^A}+\ldots]. \quad (52)$$

Let us now focus on the $(s^3+u^3+t^3)\mathcal{R}^4$ interaction, which is contained in the analytic part of (50). The relevant expression is given by [26]

$$\begin{aligned} I(S,T)_{\text{anal}} &= \frac{2\pi^4 \mathcal{G}_{ST}^3}{3!\,l_{11}^3 V_3}\sum_{(l_1,l_2,l_3)\neq(0,0,0)}\int_0^\infty d\sigma\sigma^2 e^{-G^{IJ}l_I l_J\sigma/l_{11}^2} \\ &= \frac{2\pi^7\mathcal{G}_{ST}^3}{3!}\sum_{(\hat{l}_1,\hat{l}_2,\hat{l}_3)\neq(0,0,0)}\int_0^\infty d\sigma\sqrt{\sigma}e^{-\pi G_{IJ}\hat{l}_I\hat{l}_J l_{11}^2/\sigma}, \end{aligned} \quad (53)$$

where

$$\begin{aligned} \mathcal{G}_{ST}^3 &= \int_0^1 d\omega_3\int_0^{\omega_3}d\omega_2\int_0^{\omega_2}d\omega_1(-Q(S,T;\omega_r))^3 \\ &= \frac{12}{9!}((s^2t+st^2)+3(s^3+t^3)). \end{aligned} \quad (54)$$

We are interested only in those terms in (53) that lead to the perturbative string contributions given in the previous section. There are two contributions to this:

(i) the $(\hat{l}_2,\hat{l}_3) = (0,0)$, $\hat{l}_1 \neq 0$ part of (53), which we call $I(S,T)^1_{\text{anal}}$ and

(ii) the $(\hat{l}_2,\hat{l}_3) \neq (0,0)$, $l_1 = 0$ part of (53), where we have undone the Poisson resummation over $\hat{l}_1$ to go to l_1, which we call $I(S,T)^2_{\text{anal}}$.

Proceeding along the lines of [26], we get that

$$I(S,T)^1_{\text{anal}} = \frac{\pi^9}{135}\mathcal{G}_{ST}^3 l_{11}^3 e^{2\phi^A}, \quad (55)$$

where we have used $\zeta(-3) = 1/120$, and

$$I(S,T)^2_{\text{anal}} = \frac{2\pi^7\mathcal{G}_{ST}^3}{3!R_{11}l_{11}}\sum_{(\hat{l}_2,\hat{l}_3)\neq(0,0)}\int_0^\infty d\sigma\sigma e^{-\pi l_{11}^2\hat{l}_i\hat{l}_j g_{ij}/(\sigma R_{11})}, \quad (56)$$

where we have used the IIA string frame metric

$$g^A_{i-1,j-1} = R_{11}\Big(G_{ij} - \frac{G_{1i}G_{1j}}{G_{11}}\Big), \quad (57)$$

where $i, j = 2, 3$. Using

$$g^A_{ij} = \frac{T_2^A}{U_2^A}\begin{pmatrix} 1 & U_1^A \\ U_1^A & |U^A|^2\end{pmatrix}, \quad (58)$$

we get that

$$I(S,T)^2_{\text{anal}} = \frac{4\pi^4}{3!}\Big(\frac{l_{11}}{R_{11}}\Big)^3 (T_2^A)^2\mathcal{G}_{ST}^3 E_3(U^A,\bar{U}^A)^{SL(2,\mathbb{Z})}. \quad (59)$$

Thus adding (55) and (59), we see that the perturbative part is given by

$$I(S,T)_{\text{anal}} = \Big[\frac{\pi^9}{135}l_{11}^3 e^{2\phi^A} + \frac{4\pi^4}{3!}(T_2^A)^2E_3(U^A,\bar{U}^A)^{SL(2,\mathbb{Z})} \times\Big(\frac{l_{11}}{R_{11}}\Big)^3\Big]\mathcal{G}_{ST}^3. \quad (60)$$

Finally, using

$$\mathcal{G}_{ST}^3 + \mathcal{G}_{SU}^3 + \mathcal{G}_{UT}^3 = \frac{60}{9!}(s^3+t^3+u^3), \quad (61)$$

we get that

$$\begin{aligned} A_4 = \frac{\kappa_{11}^4\hat{K}}{(2\pi)^{11}l_{11}^3}\Big[&\pi^3\zeta(3)e^{-2\phi^A} \\ &+\frac{60}{9!}\Big\{\frac{4\pi^4}{3!}(T_2^A)^2E_3(U^A,\bar{U}^A)^{SL(2,\mathbb{Z})} + \frac{\pi^9}{135}e^{4\phi^A}\Big\} \\ &\times l_s^6(s^3+t^3+u^3)\Big], \end{aligned} \quad (62)$$

where we have used $l_{11} = e^{\phi^A/3}l_s$.

In order to fix the genus zero contribution, we note that the tree-level amplitude is given by

$$T_2^A e^{-2\phi^A}\Big(\zeta(3) + \frac{\zeta(3)^2}{2\cdot 96}l_s^6(s^3+t^3+u^3)+\ldots\Big)\mathcal{R}^4. \quad (63)$$

Thus given the genus zero $\mathcal{R}^4$ interaction in (62), we can

[6] Note that σ has dimensions of (length)^2.

also deduce the precise coefficient of the $(s^3 + t^3 + u^3)\mathcal{R}^4$ interaction at genus zero. This contribution has to come from the two-loop four graviton amplitude.

This leads to terms in the IIB effective action in the string frame

$$l_s^6 \int d^8x\sqrt{-g_8}\Big[\frac{\pi^3\zeta(3)^2}{2\cdot 96}e^{-2\phi}V_2 + \frac{60}{9!} \times\Big\{\frac{4\pi^4}{3!}E_3(T,\bar{T})^{SL(2,\mathbb{Z})}U_2^3 + (e^{-2\phi}V_2)^{-2} \times\frac{\pi^9}{135}U_2^3\Big\}\Big]D^6\mathcal{R}^4. \quad (64)$$

These are contributions at genus zero, one, and three, respectively. Given the U_2 dependence and the perturbative equality of the type IIA and type IIB amplitudes, it is natural to guess that a part of the amplitude with the complete U dependence is

$$\frac{\pi^3 l_s^6}{2\cdot 96}\int d^8x\sqrt{-g_8}\Big[\zeta(3)^2 e^{-2\phi}V_2 + \frac{10}{\pi^5}E_3(T,\bar{T})^{SL(2,\mathbb{Z})}E_3(U,\bar{U})^{SL(2,\mathbb{Z})} + \frac{1}{9}(e^{-2\phi}V_2)^{-2}(E_3(U,\bar{U})^{SL(2,\mathbb{Z})} + E_3(T,\bar{T})^{SL(2,\mathbb{Z})})\Big] \times D^6\mathcal{R}^4, \quad (65)$$

where we have used $\zeta(6) = \pi^6/945$. This precisely matches some of the terms in (42).

IV. THE EXPRESSION FOR THE EXACT MODULAR FORM

Given the expression (41) for the perturbative part of the modular form, it is natural to propose that the exact expression for the modular form is given by

$$\mathcal{E}_{(3/2,3/2)}(M) + \tfrac{20}{3}E_{-3/2}(M)^{SL(3,\mathbb{Z})}E_3(U,\bar{U})^{SL(2,\mathbb{Z})} + f(U,\bar{U}) + \tfrac{1}{2}E_{3/2}(M)^{SL(3,\mathbb{Z})}E_1(U,\bar{U})^{SL(2,\mathbb{Z})}, \quad (66)$$

where[7]

$$\mathcal{E}_{(3/2,3/2)}(M)_{\rm pert} = \zeta(3)^2(\tau_2^2V_2)^2 + \zeta(3)\tau_2^2V_2E_1(T,\bar{T})^{SL(2,\mathbb{Z})} + f(T,\bar{T}) + \tfrac{1}{9}(\tau_2^2V_2)^{-1}E_3(T,\bar{T})^{SL(2,\mathbb{Z})}. \quad (67)$$

We now construct $f(T,\bar{T})$, and also obtain the nonperturbative completion of (67). Now, the modular form $\mathcal{E}_{(3/2,3/2)}(\tau,\bar{\tau})$ for the $D^6\mathcal{R}^4$ interaction in ten dimensions satisfies a Poisson equation

$$\Delta_{SL(2,\mathbb{Z})}\mathcal{E}_{(3/2,3/2)}(\tau,\bar{\tau}) = 12\mathcal{E}_{(3/2,3/2)}(\tau,\bar{\tau}) - 6(E_{3/2}(\tau,\bar{\tau}))^2 \quad (68)$$

on the fundamental domain of $SL(2,\mathbb{Z})_\tau$ [21]. The source term in (68) is the square of the modular form for the $\mathcal{R}^4$ interaction, which can be understood based on considerations of supersymmetry. Because $SL(2,\mathbb{Z})_\tau \subset SL(3,\mathbb{Z})_M$, and the U dependence in the expression (66) is already fixed, it is natural to propose that $\mathcal{E}_{(3/2,3/2)}(M)$ satisfies a Poisson equation on the fundamental domain of $SL(3,\mathbb{Z})_M$ given by

$$\Delta_{SL(3,\mathbb{Z})}\mathcal{E}_{(3/2,3/2)}(M) = \alpha\mathcal{E}_{(3/2,3/2)}(M) + \beta(E_{3/2}(M))^2, \quad (69)$$

where α and β are numbers. Again, the source term in (69) is the square of the modular form for the $\mathcal{R}^4$ interaction in eight dimensions [6].

Let us first consider the perturbative content of (69). We use the relation

$$\Delta^{\rm pert}_{SL(3,\mathbb{Z})} = \Delta_{SL(2,\mathbb{Z})_T} + 3\mu^2\frac{\partial^2}{\partial\mu^2}, \quad (70)$$

where $\mu = \tau_2^2 V_2$ is the eight dimensional dilaton. Now (70) can be obtained based on symmetries alone. From (67), we see that every term in the perturbative part of $\mathcal{E}_{(3/2,3/2)}(M)$ is of the form $\mu^k g_k(T,\bar{T})$, where $g_k(T,\bar{T})$ is $SL(2,\mathbb{Z})_T$ invariant. Thus $\Delta^{\rm pert}_{SL(3,\mathbb{Z})}$ must have the form

$$\Delta^{\rm pert}_{SL(3,\mathbb{Z})} = \xi_1\Delta_{SL(2,\mathbb{Z})_T} + \xi_2\mu^2\frac{\partial^2}{\partial\mu^2} + \xi_3\mu\frac{\partial}{\partial\mu}, \quad (71)$$

where ξ_1, ξ_2, and ξ_3 are numbers. In order to determine them, we act with $\Delta^{\rm pert}_{SL(3,\mathbb{Z})}$ on $E_s(M)^{\rm pert}_{SL(3,\mathbb{Z})}$, which is given by the first two terms in (A13), such that $\Delta^{\rm pert}_{SL(3,\mathbb{Z})}E_s(M)^{\rm pert}_{SL(3,\mathbb{Z})} = 2s(2s/3-1)E_s(M)^{\rm pert}_{SL(3,\mathbb{Z})}$. The first term in (A13) gives $\xi_2 = 3$, $\xi_3 = 0$, while using (A6), we see that the second term in (A13) gives $\xi_1 = 1$, leading to (70).

Using (67) and (70) and

$$E_{3/2}(M)_{\rm pert} = 2\mu\zeta(3) + 2E_1(T,\bar{T})^{SL(2,\mathbb{Z})}, \quad (72)$$

we see that (69) gives us the set of equations

$$\alpha + 4\beta = 6, \qquad \alpha + 8\beta = 0, \qquad \frac{\alpha}{9} = \frac{4}{3}, \quad (73)$$

and

$$\Delta_{SL(2,\mathbb{Z})_T}f(T,\bar{T}) = \alpha f(T,\bar{T}) + 4\beta(E_1(T,\bar{T}))^2. \quad (74)$$

Here we have used the relation (A6) for $s = 1$ and $s = 3$.[8]

[7] Using (A17), we could also use the relation $E_{-3/2}(M)^{SL(3,\mathbb{Z})} = E_3(M^{-1})^{SL(3,\mathbb{Z})}$ in (66).

[8] We use the relation $\Delta_{SL(2,\mathbb{Z})_T}E_1(T,\bar{T}) = 0$ for the unregularized expression.

So (73) is solved by

$$\alpha = 12, \qquad \beta = -\tfrac{3}{2}, \tag{75}$$

thus (74) reduces to

$$\Delta_{SL(2,\mathbb{Z})_T} f(T,\bar{T}) = 12 f(T,\bar{T}) - 6(E_1(T,\bar{T}))^2. \tag{76}$$

Thus (76) gives us the equation for $f(T,\bar{T})$ (and $f(U,\bar{U})$ as well), while (69) reduces to

$$\Delta_{SL(3,\mathbb{Z})}\mathcal{E}_{(3/2,3/2)}(M) = 12\mathcal{E}_{(3/2,3/2)}(M) - \tfrac{3}{2}(E_{3/2}(M))^2, \tag{77}$$

thus giving us an explicit equation satisfied by the modular form $\mathcal{E}_{(3/2,3/2)}(M)$. Note that the solution of the homogeneous equation $\Delta_{SL(3,\mathbb{Z})} h(M)_{SL(3,\mathbb{Z})}(M) = 12 h(M)_{SL(3,\mathbb{Z})} \times (M)$ (which is the Eisenstein series $E_s(M)_{SL(3,\mathbb{Z})}$ for $4s/3 = 1 \pm \sqrt{17}$) cannot be added to a particular solution of (77) simply because this is inconsistent with the structure of terms obtained using string perturbation theory.

We next understand the structure of $f(T,\bar{T})$ in more detail.

A. Understanding the structure of $f(T,\bar{T})$

The structure of (76) is very similar to (68), which has been analyzed in [21], and our analysis is along similar lines. In (76) we substitute

$$f(T,\bar{T}) = f_0(T_2) + \sum_{k\neq 0} f_k(T_2) e^{2\pi i k T_1}. \tag{78}$$

Here $f_0(T_2)$ receives perturbative contributions from the zero world sheet instanton sector, as well as nonperturbative contributions from world sheet instanton and anti-instanton pairs of equal and opposite NS-NS charge. On the other hand, the remaining part of (78) receives contributions from world sheet instantons of nonvanishing NS-NS charge. Substituting the regularized expression for $E_1(T,\bar{T})$ given by (A8), we get the equation satisfied by $f_0(T_2)$

$$\Big(T_2^2\frac{\partial^2}{\partial T_2^2} - 12\Big) f_0(T_2) = -6\Big[(2\zeta(2)T_2 - \pi\ln T_2)^2 + 4\pi^2\sum_{k\neq 0}\mu^2(k,1)e^{-4\pi|k|T_2}\Big]. \tag{79}$$

Now writing

$$f_0(T_2) = \hat{f}_0(T_2) + \sum_{k\neq 0}\hat{f}_k(T_2)e^{-4\pi|k|T_2}, \tag{80}$$

where $\hat{f}_0(T_2)$ is the contribution from the zero world sheet instanton sector, and $\hat{f}_k(T_2)$ is the contribution from the world sheet instanton anti-instanton sector with vanishing NS-NS charge, from (79) we get differential equations for $\hat{f}_0(T_2)$ and $\hat{f}_k(T_2)$. For $\hat{f}_0(T_2)$ we get

$$\Big(T_2^2\frac{\partial^2}{\partial T_2^2} - 12\Big)\hat{f}_0(T_2) = -6(2\zeta(2)T_2 - \pi\ln T_2)^2, \tag{81}$$

which has the solution

$$\hat{f}_0(T_2) = \frac{\pi^2}{720}[65 - 20\pi T_2 + 48\pi^2 T_2^2] + \pi^2\ln T_2\Big[-\frac{\pi T_2}{3} + \frac{1}{2}\ln T_2 - \frac{1}{12}\Big] + \lambda_1 T_2^4 + \frac{\lambda_2}{T_2^3}, \tag{82}$$

where λ_1 and λ_2 are arbitrary constants. We shall fix them soon.

For $\hat{f}_k(T_2)$, we get

$$\Big[T_2^2\Big(\frac{\partial^2}{\partial T_2^2} - 8\pi|k|\frac{\partial}{\partial T_2} + (4\pi|k|)^2\Big) - 12\Big]\hat{f}_k(T_2) = -24\pi^2\mu^2(k,1), \tag{83}$$

which has the solution

$$\hat{f}_k(T_2) = -\frac{\mu^2(k,1)}{448|k|^3\pi T_2^3}[24(4\pi|k|T_2+1)^2 + ((4\pi|k|T_2)^3 - 3)^2 + 15 + (4\pi|k|T_2)^4(2 - 4\pi|k|T_2) + (4\pi|k|T_2)^7 e^{4\pi|k|T_2}\mathrm{Ei}(-4\pi|k|T_2)], \tag{84}$$

where Ei(x) is the exponential integral function. Using the relation [38]

$$\mathrm{Ei}(-x) = e^{-x}\Big[-\frac{1}{x} + \int_0^\infty dt\frac{e^{-t}}{(t+x)^2}\Big], \qquad x > 0, \tag{85}$$

we see that the last term in (84) has the correct structure to be a world sheet instanton contribution.

For the world sheet instantons with nonvanishing NS-NS charge, we get the equation

$$\Big[T_2^2\Big(\frac{\partial^2}{\partial T_2^2} - 4\pi^2k^2\Big) - 12\Big]f_k(T_2) = -24\pi(2\zeta(2)T_2 - \pi\ln T_2)\mu(k,1)e^{-2\pi|k|T_2} - 24\pi^2\sum_{k_1\neq 0, k_2\neq 0, k_1+k_2=k}\mu(k_1,1)\mu(k_2,1)e^{-2\pi(|k_1|+|k_2|)T_2}, \tag{86}$$

which in principle can be solved iteratively by expanding in large T_2.

Substituting (82) and the corresponding expression for $\hat{f}_0(U_2)$ into (66), we can easily study the decompactification limit as before. Only the T_2^2 term in the expression for $\hat{f}_0(T_2)$ (and the U_2^2 term in the expression for $\hat{f}_0(U_2)$) contributes in this limit. In nine dimensions, in addition to (44) it also gives a term

$$6\zeta(4)l_s^5\int d^9x\sqrt{-g_9}(r_Be^{-2\phi})^{-1}\Big(r_B^2+\frac{1}{r_B^2}\Big)D^6\mathcal{R}^4, \quad (87)$$

where we have used $\zeta(4)=\pi^4/90$. However, it also gives a divergent contribution

$$\lambda_1 l_s^5\int d^9x\sqrt{-g_9}(r_Be^{-2\phi})^{-1}\Big(r_B^4+\frac{1}{r_B^4}\Big)r_\infty^2D^6\mathcal{R}^4 \quad (88)$$

which we shall return to soon.

Further decompactifying to ten dimension, this gives an additional contribution to (46) which is equal to

$$6\zeta(4)l_s^4\int d^{10}x\sqrt{-g}e^{2\phi}D^6\mathcal{R}^4, \quad (89)$$

which precisely gives the missing genus two contribution in (15). This is a nontrivial consistency check on our proposed modular form.

Note that we can send

$$f(T,\bar{T})\rightarrow f(T,\bar{T})+\lambda E_4(T,\bar{T})^{SL(2,\mathbb{Z})}, \quad (90)$$

for arbitrary λ in (76) because $E_4(T,\bar{T})^{SL(2,\mathbb{Z})}$ satisfies the homogeneous equation

$$\Delta_{SL(2,\mathbb{Z})}E_4(T,\bar{T})^{SL(2,\mathbb{Z})}=12E_4(T,\bar{T})^{SL(2,\mathbb{Z})}. \quad (91)$$

In the zero world sheet instanton sector, this involves shifting the coefficient of the T_2^4 term

$$\lambda_1\rightarrow\hat{\lambda}_1\equiv\lambda_1+2\lambda\zeta(8), \quad (92)$$

and the T_2^{-3} term

$$\lambda_2\rightarrow\hat{\lambda}_2\equiv\lambda_2+\frac{5\pi}{8}\lambda\zeta(7). \quad (93)$$

In the sector with world sheet instanton charge k, the extra terms are automatically solutions of the homogeneous equation in (86).

We now provide two arguments that we must set the coefficient of the T_2^4 term to zero, thus $\hat{\lambda}_1=0$. From (88), note that we get a divergent contribution with a nontrivial dilaton dependence. As discussed before, the divergences add to give threshold singularities, and hence must come only from the genus one amplitude. Thus it follows that $\hat{\lambda}_1=0$.

The vanishing of $\hat{\lambda}_1$ can also be argued based on the factorization properties of the amplitude. Stripping off the eight dimensional dilaton factor from the various loop amplitudes, from (42), (82), and (92), we see that for large T_2, the genus two amplitude goes as $T_2^2+\hat{\lambda}_1T_2^4$, while the genus one amplitude goes as T_2. Now considering the degeneration limit of the genus two surface into two genus one surfaces as in Fig. 2, we see that the large T_2 limit of the genus two amplitude should scale no larger than T_2^2, thus $\hat{\lambda}_1=0$.

Note that from (42), it follows that the genus three amplitude at large T_2 goes as T_2^3. This is consistent with the degeneration limits described in Fig. 2, when $\hat{\lambda}_1=0$.

We now proceed to calculate $\hat{\lambda}_2$ along the lines of [21]. Multiplying (76) by $E_4(T,\bar{T})^{SL(2,\mathbb{Z})}$ and integrating over the restricted fundamental domain of $SL(2,\mathbb{Z})_T$, we get that

$$\begin{aligned}\int_{\mathcal{F}_L}&\frac{d^2T}{T_2^2}E_4(T,\bar{T})^{SL(2,\mathbb{Z})}\Delta_{SL(2,\mathbb{Z})_T}f(T,\bar{T})\\ &=12\int_{\mathcal{F}_L}\frac{d^2T}{T_2^2}E_4(T,\bar{T})^{SL(2,\mathbb{Z})}f(T,\bar{T})\\ &\quad-6\int_{\mathcal{F}_L}\frac{d^2T}{T_2^2}E_4(T,\bar{T})^{SL(2,\mathbb{Z})}(E_1(T,\bar{T}))^2. \quad (94)\end{aligned}$$

We have restricted the integral to be over $\mathcal{F}_L$ as the integrals diverge and we regulate them and finally take $L\rightarrow\infty$. Integrating by parts, and using (91), from (94) we get that

$$\begin{aligned}\int_{-1/2}^{1/2}&dT_1\Big(E_4^{SL(2,\mathbb{Z})}\frac{\partial f}{\partial T_2}-f\frac{\partial E_4^{SL(2,\mathbb{Z})}}{\partial T_2}\Big)_{T_2=L}\\ &=-6\int_{\mathcal{F}_L}\frac{d^2T}{T_2^2}E_4(T,\bar{T})^{SL(2,\mathbb{Z})}(E_1(T,\bar{T}))^2. \quad (95)\end{aligned}$$

Using (82) with λ_2 replaced by $\hat{\lambda}_2$, the left-hand side of (95) yields

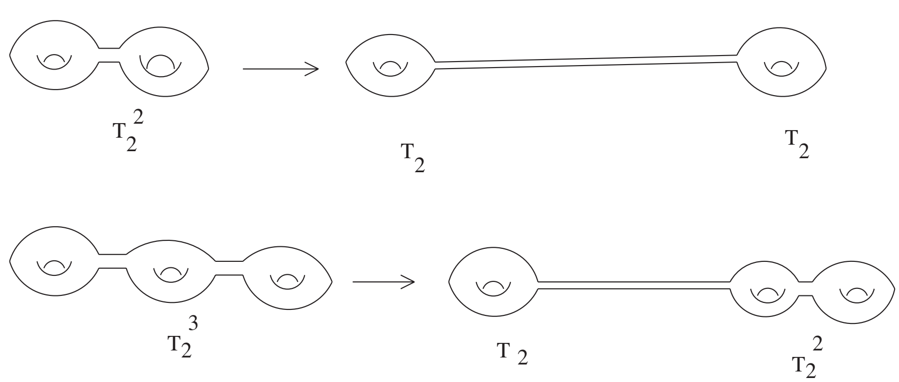

FIG. 2. Degeneration limits of the genus two and genus three surfaces.

$$\zeta(8)\Big(-14\hat{\lambda}_2 - \frac{4\pi^4}{15}L^5 - \frac{\pi^3}{2}L^4 - \frac{8\pi^2}{9}L^3 + 2\pi^3 L^4 \ln L - 4\pi^2 L^3(\ln L)^2 + \frac{8\pi^2}{3}L^3 \ln L\Big). \tag{96}$$

Using the Poincaré series representation for $E_4^{SL(2,\mathbb{Z})}$, and the Rankin-Selberg formula the right-hand side of (95) yields

$$\zeta(8)\Big(-\frac{48}{5}\zeta(2)^2 L^5 - 3\pi\zeta(2)L^4 - \frac{8\pi^2}{9}L^3 + 12\zeta(2)L^4\ln L - 4\pi^2 L^3(\ln L)^2 + \frac{8\pi^2}{3}L^3\ln L\Big) - 48\pi^2\zeta(8)\int_0^L dT_2 T_2^2 \sum_{k\neq 0}\mu^2(k,1)e^{-4\pi|k|T_2}, \tag{97}$$

leading to

$$\hat{\lambda}_2 = \frac{3}{14\pi}\sum_{k=1}^{\infty}\frac{\mu^2(k,1)}{k^3} = \frac{1}{4}\zeta(3)\zeta(5), \tag{98}$$

using an identity due to Ramanujan [39].

B. Understanding the nonperturbative structure of $\mathcal{E}_{(3/2,3/2)}(M)$

Having understood the perturbative part of $\mathcal{E}_{(3/2,3/2)}(M)$, let us focus on the nonperturbative part of $\mathcal{E}_{(3/2,3/2)}(M)$. From (77), we can see what are the various kinds of nonperturbative contributions $\mathcal{E}_{(3/2,3/2)}(M)$ receives. This allows us to write

$$\mathcal{E}_{(3/2,3/2)}(M)_{\rm nonpert} = \sum_{k\neq 0}(f_k(\phi_i)e^{2\pi ik\tau_1} + u_k(\phi_i)) + \sum_{k\neq 0}(g_k(\phi_i,\tau_1)e^{2\pi ikB_R} + v_k(\phi_i,\tau_1)) + \sum_{k\neq 0, l\neq 0} h_{k,l}(\phi_i,\tau_1)e^{2\pi i(k\tau_1 + lB_R)}, \tag{99}$$

where $\phi_i = \{B_N, V_2, \tau_2\}$. In (99), $f_k(\phi_i)$ involves charge k (single and double) D-instanton contributions, while $g_k(\phi_i,\tau_1)$ involves (single and double) (p,q) string instanton contributions carrying R-R charge k. The $h_{k,l}(\phi_i,\tau_1)$ term involves contributions from charge k D-instantons and R-R charge l (p,q) string instantons put together. Also $u_k(\phi_i)$ includes D-instanton anti-D-instanton contributions with total charge zero, which goes as $e^{-4\pi|k|\tau_2}$ for large τ_2. Finally, $v_k(\phi_i,\tau_1)$ includes (p,q) and (p',q') string instanton contributions with total R-R charge zero, which goes as $e^{-4\pi|k\tau|V_2}$ in the sector with only D-strings.

From (77), we obtain explicit differential equations satisfied by these nonperturbative contributions. Defining

$$\hat{\Delta} = \tau_2^2\frac{\partial^2}{\partial\tau_2^2} + V_2^2\partial_{B_N}^2 + 3\partial_\nu(\nu^2\partial_\nu), \tag{100}$$

we get that

$$(\hat{\Delta} - 4\pi^2k^2\tau_2^2 - 12)f_k(\phi_i) = -48\pi\tau_2V_2(\tau_2^2V_2\zeta(3) + E_1(T,\bar{T})^{SL(2,\mathbb{Z})})|k|\mu\Big(k,\frac{3}{2}\Big)K_1(2\pi|k|\tau_2) - 96(\pi\tau_2V_2)^2\sum_{k_i\neq 0, k_1+k_2=k}|k_1k_2|\mu\Big(k_1,\frac{3}{2}\Big)\mu\Big(k_2,\frac{3}{2}\Big)K_1(2\pi|k_1|\tau_2)K_1(2\pi|k_2|\tau_2). \tag{101}$$

Further defining

$$\mu(k,l,s) = \sum_{m>0, m|k,l}\frac{1}{m^{2s-1}}, \tag{102}$$

such that $\mu(k,0,s) = \mu(k,s)$, we also get that

$$\begin{aligned}(\hat{\Delta} + \tau_2^2\partial_{\tau_1}^2 - V_2^2[4\pi^2k^2|\tau|^2 + 4\pi ik\tau_1\partial_{B_N}] - 12)g_k(\phi_i,\tau_1) &= -24\pi(\tau_2^2V_2\zeta(3) + E_1(T,\bar{T})^{SL(2,\mathbb{Z})})\sum_l \mu(k,l,1)e^{-2\pi|l-k\tau|V_2 + 2\pi ilB_N} \\ &\quad - 24\pi^2\sum_{k_i\neq 0, l_i, k_1+k_2=k}\mu(k_1,l_1,1)\mu(k_2,l_2,1)e^{-2\pi(|l_1-k_1\tau| + |l_2-k_2\tau|)V_2 + 2\pi i(l_1+l_2)B_N},\end{aligned} \tag{103}$$

and

$$\begin{aligned}(\hat{\Delta} + \tau_2^2[\partial_{\tau_1}^2 + 4\pi ik\partial_{\tau_1} - 4\pi^2k^2] - V_2^2[4\pi^2k^2|\tau|^2 + 4\pi ik\tau_1\partial_{B_N}] - 12)h_{k,l}(\phi_i,\tau_1) \\ = -96\pi^2\tau_2V_2\sum_m|k|\mu\Big(k,\frac{3}{2}\Big)\mu(l,m,1)K_1(2\pi|k|\tau_2)e^{-2\pi|m-l\tau|V_2 + 2\pi imB_N}.\end{aligned} \tag{104}$$

The remaining two differential equations are given by

$$(\hat{\Delta} - 12)u_k(\phi_i) = -96(\pi\tau_2 V_2)^2|k|^2\mu^2(k, \tfrac{3}{2})K_1^2(2\pi|k|\tau_2), \quad (105)$$

and

$$(\hat{\Delta} + \tau_2^2\partial_{\tau_1}^2 - 12)v_k(\phi_i, \tau_1) = -24\pi^2\sum_{l_1,l_2}\mu(k, l_1, 1)\mu(k, l_2, 1)e^{-2\pi(|l_1-k\tau|+|l_2+k\tau|)V_2+2\pi i(l_1+l_2)B_N}. \quad (106)$$

V. MORE PREDICTIONS FROM ELEVEN DIMENSIONAL SUPERGRAVITY AT ONE LOOP ON T^3

We can generalize the calculations in Sec. III to make predictions for some of the perturbative contributions to the $D^{2k}\mathcal{R}^4$ interaction for arbitrary values of $k \geq 4$. We show below that we obtain parts of the genus one and genus k contributions to the amplitude. However, it need not be the case that the $D^{2k}\mathcal{R}^4$ interaction is protected for all values of k.

The analytic part of the amplitude relevant for the $D^{2k}\mathcal{R}^4$ interaction is given by [26]

$$\begin{aligned} I(S,T)_{\rm anal} &= \frac{2\pi^4 \mathcal{G}^k_{ST}}{k! l_{11}^3 V_3}\sum_{(l_1,l_2,l_3)\neq(0,0,0)}\int_0^\infty d\sigma\sigma^{k-1}e^{-G^{IJ}l_Il_J\sigma/l_{11}^2} \\ &= \frac{2\pi^{4+k}\mathcal{G}^k_{ST}}{k!}\sum_{(\hat{l}_1,\hat{l}_2,\hat{l}_3)\neq(0,0,0)}\int_0^\infty d\sigma\sigma^{k-5/2} \\ &\quad\times e^{-\pi G_{IJ}\hat{l}_I\hat{l}_J l_{11}^2/\sigma}, \end{aligned} \quad (107)$$

where

$$\mathcal{G}^k_{ST} = \int_0^1 d\omega_3\int_0^{\omega_3}d\omega_2\int_0^{\omega_2}d\omega_1(-Q(S,T;\omega_r))^k. \quad (108)$$

Following the same steps as in Sec. III, the two perturbative contributions are given by

$$I(S,T)^1_{\rm anal} = 4\pi^{2k+5/2}\Gamma\Big(\frac{3}{2}-k\Big)\zeta(3-2k)l_{11}^{2k-3}e^{2(2k-3)\phi^A/3}\times\frac{\mathcal{G}^k_{ST}}{k!}, \quad (109)$$

and

$$I(S,T)^2_{\rm anal} = \frac{2\pi^4 l_{11}^{2k-3}}{kR_{11}^k}(T_2^A)^{k-1}\mathcal{G}^k_{ST}E_k(U^A,\bar{U}^A)^{SL(2,\mathbb{Z})}. \quad (110)$$

This leads to

$$\begin{aligned} A_4 = \frac{\kappa_{11}^4\hat{K}}{(2\pi)^{11}l_{11}^3}\Big[&\frac{2\pi^4}{k}(T_2^A)^{k-1}E_k(U^A,\bar{U}^A)^{SL(2,\mathbb{Z})} \\ &+\frac{4\pi^{2k+5/2}}{k!}\Gamma\Big(\frac{3}{2}-k\Big)\zeta(3-2k)e^{2(k-1)\phi^A}\Big]l_s^{2k}\mathcal{W}^k, \end{aligned} \quad (111)$$

where

$$\mathcal{W}^k = \mathcal{G}^k_{ST} + \mathcal{G}^k_{SU} + \mathcal{G}^k_{UT}. \quad (112)$$

Now $\mathcal{W}^k$ contains all the possible $2k$th power of the derivatives acting on $\mathcal{R}^4$ consistent with the kinematical structure of the amplitude. This is unique up to $k = 5$, namely, for $k = 4$, $\mathcal{W}^4 \sim (s^2 + t^2 + u^2)^2$, while for $k = 5$, $\mathcal{W}^5 \sim (s^2 + t^2 + u^2)(s^3 + t^3 + u^3)$. For $k = 6$, there are two independent structures and so $\mathcal{W}^6 \sim (s^2 + t^2 + u^2)^3 + (s^3 + t^3 + u^3)^2$, leading to two different spacetime structures for the $D^{12}\mathcal{R}^4$ interaction. Thus when we mean the $D^{2k}\mathcal{R}^4$ interaction, we mean that these various possibilities have already been taken into account. Thus, (111) leads to terms in the IIB effective action given by

$$\begin{aligned} l_s^{2k}\int d^8x\sqrt{-g_8}\Big[&\frac{2\pi}{k}(U_2^B)^kE_k(T^B,\bar{T}^B)^{SL(2,\mathbb{Z})} \\ &+\frac{4\pi^{2k-1/2}}{k!}\Gamma\Big(\frac{3}{2}-k\Big)\zeta(3-2k)(e^{-2\phi^B}T_2^B)^{1-k} \\ &\times(U_2^B)^k\Big]D^{2k}\mathcal{R}^4. \end{aligned} \quad (113)$$

Given the perturbative equality of the amplitude in the two type II theories, and (A1), it is natural to enhance the $(U_2^B)^k$ factors to $E_k(U^B,\bar{U}^B)^{SL(2,\mathbb{Z})}$, and symmetrize in U^B and T^B. Thus (113) gets enhanced to

$$\begin{aligned} l_s^{2k}\int d^8x\sqrt{-g_8}\Big[&\frac{(2k)!}{(2\pi)^{2k-1}|B_{2k}|k}E_k(T^B,\bar{T}^B)^{SL(2,\mathbb{Z})} \\ &\times E_k(U^B,\bar{U}^B)^{SL(2,\mathbb{Z})}+\frac{4\Gamma(k+\frac{1}{2})\Gamma(k-1)\zeta(2k-2)}{\pi^{2k-3/2}|B_{2k}|} \\ &\times(e^{-2\phi^B}T_2^B)^{1-k}(E_k(T^B,\bar{T}^B)^{SL(2,\mathbb{Z})} \\ &+E_k(U^B,\bar{U}^B)^{SL(2,\mathbb{Z})})\Big]D^{2k}\mathcal{R}^4, \end{aligned} \quad (114)$$

where we have used the relations [38]

$$\zeta(2k) = \frac{2^{2k-1}\pi^{2k}|B_{2k}|}{(2k)!}, \quad (115)$$

where k is a positive integer, B_{2k} are the Bernoulli numbers, the identity (A3), and

$$\Gamma(2x) = \frac{2^{2x-1/2}}{\sqrt{2\pi}}\Gamma(x)\Gamma\Big(x+\frac{1}{2}\Big). \quad (116)$$

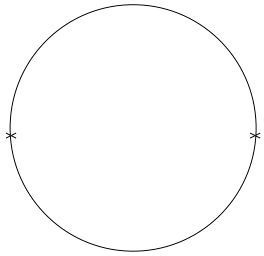

FIG. 3. Schematics of the $D^4\mathcal{R}^4$ torus amplitude.

Thus from (114), we see that 11 dimensional supergravity gives predictions for parts of the genus one and genus k amplitudes for the $D^{2k}\mathcal{R}^4$ interaction, for arbitrary k.

Thus from (114), we see that at genus one, there is a contribution proportional to $E_k(T^B, \bar{T}^B)^{SL(2,\mathbb{Z})} \times E_k(U^B, \bar{U}^B)^{SL(2,\mathbb{Z})}$. For low values of k, it is easy to see that there is such a contribution. For $k = 2$, as shown in [26], this arises from the only diagram that contributes to the torus amplitude given by Fig. 3.

For $k = 3$, from Fig. 1, we see that $\hat{I}_1$ gives such a contribution proportional to $E_3(T^B, \bar{T}^B)^{SL(2,\mathbb{Z})} \times E_3(U^B, \bar{U}^B)^{SL(2,\mathbb{Z})}$. However, there is also another contribution from $\hat{I}_2$.

For $k = 4$, again we can see that the part of the torus amplitude coming from the diagram in Fig. 4 is proportional to $E_4(T^B, \bar{T}^B)^{SL(2,\mathbb{Z})} E_4(U^B, \bar{U}^B)^{SL(2,\mathbb{Z})}$. This can be obtained by using the relation

$$\int_{\mathcal{T}} \prod_{i=1}^{3} \frac{d^2\nu_i}{\Omega_2} \ln\hat{\chi}(\nu_1 - \nu_2; \Omega) \ln\hat{\chi}(\nu_1 - \nu_3; \Omega) \times \hat{\chi}(\nu_2 - \nu_4; \Omega)\hat{\chi}(\nu_3 - \nu_4; \Omega) = \frac{1}{(4\pi)^4} \sum_{(m,n)\neq(0,0)} \frac{\Omega_2^4}{|m\Omega + n|^8} = \frac{1}{(4\pi)^4} E_4(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})}, \tag{117}$$

and generalizing the calculation of $\hat{I}_1$ summarized in Appendix B 1. However, just like in the $k = 3$ case, other parts of the torus amplitude should also give the same contribution, so the final numerical coefficient will be different.

So from the discussion above, one can see that when the k points form a polygon with no internal lines, the integral over the vertex operator insertions is proportional to $E_k(\Omega, \bar{\Omega})^{SL(2,\mathbb{Z})}$, while it leads to the contribution predicted from supergravity. However, this topology is no more possible for $k \geq 5$, and so there is no particularly simple contribution to the torus amplitude that gives the answer. The various contributions must add to give the answer predicted from supergravity. It would be interesting to see this explicitly coming out of the torus amplitude.

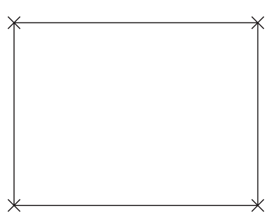

FIG. 4. Schematics of part of the $D^8\mathcal{R}^4$ torus amplitude.

After converting to the Einstein frame, let us consider the U^B dependent coefficient of the $\hat{D}^{2k}\hat{\mathcal{R}}^4$ interaction in (114).[9] Since it involves $E_k(U^B, \bar{U}^B)^{SL(2,\mathbb{Z})}$, which is $SL(2,\mathbb{Z})_U$ invariant, whatever multiplies it in the whole amplitude should be $SL(3,\mathbb{Z})_M$ invariant. In fact, this contribution is given by

$$\frac{2\Gamma(k+\frac{1}{2})}{\pi|B_{2k}|}(2\pi^{5/2-2k}(e^{-2\phi^B}T_2^B)^{1-2k/3}\Gamma(k-1)\zeta(2k-2) + \pi^{3/2-2k}\Gamma(k)(e^{-2\phi^B}T_2^B)^{k/3}E_k(T^B, \bar{T}^B)^{SL(2,\mathbb{Z})}) \times E_k(U^B, \bar{U}^B)^{SL(2,\mathbb{Z})} = \frac{2\Gamma(k+\frac{1}{2})\Gamma(\frac{3}{2}-k)}{\pi|B_{2k}|} E_{3/2-k}(M)^{SL(3,\mathbb{Z})}_{\rm pert} E_k(U^B, \bar{U}^B)^{SL(2,\mathbb{Z})} = \frac{2\Gamma(k+\frac{1}{2})\Gamma(\frac{3}{2}-k)}{\pi|B_{2k}|} E_k(M^{-1})^{SL(3,\mathbb{Z})}_{\rm pert} E_k(U^B, \bar{U}^B)^{SL(2,\mathbb{Z})}, \tag{118}$$

on using (A17). Extending it to the nonperturbative completion, we get the manifestly U-duality invariant modular form

$$\frac{2\Gamma(k+\frac{1}{2})\Gamma(\frac{3}{2}-k)}{\pi|B_{2k}|} E_k(M^{-1})^{SL(3,\mathbb{Z})} E_k(U^B, \bar{U}^B)^{SL(2,\mathbb{Z})}. \tag{119}$$

Thus one-loop supergravity and U-duality gives a prediction for a part of the complete modular form.

Decompactifying to nine dimensions, we see that (114) gives the interaction

$$l_s^{2k-1} \int d^9x\sqrt{-g_9}\Big[\frac{4\pi^{3/2}}{k!}\zeta(2k-1)\Gamma\Big(k-\frac{1}{2}\Big) \times \Big(r_B^{2k-1} + \frac{1}{r_B^{2k-1}}\Big) + 4\pi^2\frac{\zeta(2k-2)}{k(k-1)}(e^{-2\phi^B}r_B)^{1-k} \times \Big(r_B^k + \frac{1}{r_B^k}\Big)\Big]D^{2k}\mathcal{R}^4, \tag{120}$$

which contributes at genus one and at genus k. It also gives the divergent contribution

$$\frac{4\pi}{k}\zeta(2k)l_s^{2k-1}\int d^9x\sqrt{-g_9}r_\infty^{2k-1}D^{2k}\mathcal{R}^4 \tag{121}$$

[9] The remaining part which depends only on T^B must form part of an $SL(3,\mathbb{Z})_M$ invariant modular form.

which leads to the threshold singularities. Further decompactifying (120) to ten dimensions, this leads to the interaction

$$4\pi^2\frac{\zeta(2k-2)}{k(k-1)}l_s^{2k-2}\int d^{10}x\sqrt{-g}e^{-2(1-k)\phi^B}D^{2k}\mathcal{R}^4, \tag{122}$$

which contributes at genus k, while the genus one contribution vanishes. It also gives the divergent contribution

$$\frac{4\pi^{3/2}}{k!}\zeta(2k-1)\Gamma\Big(k-\frac{1}{2}\Big)l_s^{2k-2}\int d^{10}x\sqrt{-g}r_B^{2(k-1)}D^{2k}\mathcal{R}^4, \tag{123}$$

corresponding to the threshold singularities.

VI. DISCUSSION

We have made a proposal for the modular form for the $D^6\mathcal{R}^4$ interaction and showed that it satisfies several nontrivial consistency checks. Some parts of the torus amplitude, however, have been constructed based on the perturbative equality of the type IIA and type IIB amplitudes, and some heuristic arguments. Calculating the full amplitude explicitly would be useful in verifying the proposal we make.

Let us make some comments about the possible modular form for the $D^6\mathcal{R}^4$ interaction in toroidal compactifications preserving maximal supersymmetry to lower dimensions, where the U-duality group is no longer reducible. The scalars parametrize the coset manifold $\mathcal{M} = G/H$, where G is a noncompact group, and H is its maximal compact subgroup [40,41]. The conjectured U-duality group is $\hat{G}$, the discrete version of G. Thus in the Einstein frame the term in the supergravity action involving the scalars is given by

$$S \sim \frac{1}{l_s^{8-d}}\int d^{10-d}x\sqrt{-\hat{g}_{10-d}}\,\mathrm{Tr}(\partial_\mu M\hat{\partial}^\mu M^{-1}), \tag{124}$$

where M parametrizes $\mathcal{M}$. Based on the $D^6\mathcal{R}^4$ interaction in ten dimensions as well as the modular form we propose, it is conceivable that the U-duality invariant modular form in lower dimensions is given by the solution of the Poisson equation on the fundamental domain of $\hat{G}$ given by

$$\Delta_{\hat{G}}\mathcal{E}_{(3/2,3/2)}(M) = \lambda_1\mathcal{E}_{(3/2,3/2)}(M) - \lambda_2(E_{3/2}(M))^2, \tag{125}$$

where λ_1 and λ_2 are constants.

ACKNOWLEDGMENTS

I would like to thank Michael Green and Pierre Vanhove for useful comments. The work of A. B. is supported by NSF Grant No. PHY-0503584 and the William D. Loughlin membership.

APPENDIX A: EXPRESSIONS FOR THE EISENSTEIN SERIES

In the section below, we write down explicit expressions for the Eisenstein series of $SL(2,\mathbb{Z})$ and $SL(3,\mathbb{Z})$ that are useful in the main text.

1. The Eisenstein series for $SL(2,\mathbb{Z})$

The Eisenstein series of order s for $SL(2,\mathbb{Z})$ is defined by

$$\begin{aligned}E_s(T,\bar{T})^{SL(2,\mathbb{Z})} &= \sum_{(p,q)\neq(0,0)}\frac{T_2^s}{|p+qT|^{2s}}\\ &= 2\zeta(2s)T_2^s + 2\sqrt{\pi}T_2^{1-s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1) + \frac{2\pi^s\sqrt{T_2}}{\Gamma(s)}\sum_{m_1\neq0,m_2\neq0}\Big|\frac{m_1}{m_2}\Big|^{s-1/2}\\ &\quad\times K_{s-1/2}(2\pi T_2|m_1m_2|)e^{2\pi i m_1m_2T_1}\\ &= 2\zeta(2s)T_2^s + 2\sqrt{\pi}T_2^{1-s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1) + \frac{4\pi^s\sqrt{T_2}}{\Gamma(s)}\sum_{k\neq0}|k|^{s-1/2}\mu(k,s)K_{s-1/2}(2\pi T_2|k|)e^{2\pi ikT_1},\end{aligned} \tag{A1}$$

where

$$\mu(k,s) = \sum_{m>0,m|k}\frac{1}{m^{2s-1}}. \tag{A2}$$

Using the relations

$$\zeta(2s-1)\Gamma(s-\tfrac{1}{2}) = \pi^{2s-3/2}\zeta(2-2s)\Gamma(1-s) \tag{A3}$$

and

$$K_s(x) = K_{-s}(x), \tag{A4}$$

we see that

$$\Gamma(s)E_s(T,\bar{T})^{SL(2,\mathbb{Z})} = \pi^{2s-1}\Gamma(1-s)E_{1-s}(T,\bar{T})^{SL(2,\mathbb{Z})}. \tag{A5}$$

Now (A1) satisfies the Laplace equation

$$\begin{aligned}\Delta_{SL(2,\mathbb{Z})}E_s(T,\bar{T})^{SL(2,\mathbb{Z})} &= 4T_2^2\frac{\partial^2}{\partial T\partial\bar{T}}E_s(T,\bar{T})^{SL(2,\mathbb{Z})}\\ &= s(s-1)E_s(T,\bar{T})^{SL(2,\mathbb{Z})}\end{aligned} \tag{A6}$$

on the fundamental domain of $SL(2,\mathbb{Z})_T$.

We shall need the expression for $E_1(T,\bar{T})^{SL(2,\mathbb{Z})}$ in the main text. From (A1), note that this diverges because $\zeta(1)$ is infinite, and thus needs to be regularized. We regularize the second term in (A1) by setting $1-s=\epsilon$ and taking the limit $\epsilon\to 0$, where we also use

$$\zeta(\epsilon)=\frac{1}{\epsilon}+\gamma+O(\epsilon), \tag{A7}$$

where γ is the Euler constant. Using an $\overline{\rm MS}$-like regularization scheme, where we drop the $1/\epsilon$ pole term as well as terms involving the Euler constant, we get that (using $\zeta(2)=\pi^2/6$)

$$\begin{aligned}E_1(T,\bar{T})^{SL(2,\mathbb{Z})}&=\frac{\pi^2}{3}T_2-\pi\ln T_2\\&\quad+2\pi\sqrt{T_2}\sum_{m\neq 0,n\neq 0}\left|\frac{m}{n}\right|^{1/2}\\&\quad\times K_{1/2}(2\pi T_2|mn|)e^{2\pi imnT_1}\\&=-\pi\ln(T_2|\eta(T)|^4),\end{aligned} \tag{A8}$$

where we have used

$$K_{1/2}(x)=\sqrt{\frac{\pi}{2x}}e^{-x}, \tag{A9}$$

and the definition of the Dedekind eta function

$$\eta(T)=e^{\pi iT/12}\prod_{k=1}^{\infty}(1-e^{2\pi ikT}). \tag{A10}$$

This yields the same result as in [29].

2. The Eisenstein series for $SL(3,\mathbb{Z})$

The Eisenstein series of order s for $SL(3,\mathbb{Z})$ in the fundamental representation is defined by

$$\begin{aligned}E_s(M)^{SL(3,\mathbb{Z})}&=\sum_{m_i}{}'(m_iM_{ij}m_j)^{-s}\\&=\sum_{m_i}{}'\nu^{-s/3}\left(\frac{|m_1+m_2\tau+m_3B|^2}{\tau_2}+\frac{m_3^2}{\nu}\right)^{-s},\end{aligned} \tag{A11}$$

where m_i are integers, and the sum excludes $\{m_1,m_2,m_3\}=\{0,0,0\}$. The integers m_i transform in the antifundamental representation of $SL(3,\mathbb{Z})$, and the matrix M_{ij} is given by (10).

Using the integral representation

$$\begin{aligned}E_s(M)^{SL(3,\mathbb{Z})}&=\frac{\nu^{-s/3}\pi^s}{\Gamma(s)}\\&\quad\times\int_0^\infty\frac{dt}{t^{s+1}}\sum_{m_i}{}'e^{-\pi(|m_1+m_2\tau+m_3B|^2/\tau_2+m_3^2/\nu)/t},\end{aligned} \tag{A12}$$

we can evaluate (A12) to get that

$$\begin{aligned}E_s(M)^{SL(3,\mathbb{Z})}&=2(\tau_2^2V_2)^{2s/3}\zeta(2s)+\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}(\tau_2^2V_2)^{1/2-s/3}E_{s-1/2}(T,\bar{T})^{SL(2,\mathbb{Z})}\\&\quad+\frac{2\pi^s}{\Gamma(s)}\tau_2^{s/3+1/2}V_2^{2s/3}\sum_{m_1\neq 0,m_2\neq 0}\left|\frac{m_1}{m_2}\right|^{s-1/2}K_{s-1/2}(2\pi\tau_2|m_1m_2|)e^{2\pi im_1m_2\tau_1}\\&\quad+\frac{2\pi^s}{\Gamma(s)}\tau_2^{1-2s/3}V_2^{1-s/3}\sum_{m_1\neq 0,m_3\neq 0,m_2}\left|\frac{m_2-m_1\tau}{m_3}\right|^{s-1}K_{s-1}(2\pi|m_3(m_2-m_1\tau)|V_2)e^{2\pi im_3(m_1B_R+m_2B_N)}.\end{aligned} \tag{A13}$$

Now (A11) satisfies the Laplace equation [6]

$$\begin{aligned}\Delta_{SL(3,\mathbb{Z})}E_s(M)^{SL(3,\mathbb{Z})}&=\Big[4\tau_2^2\frac{\partial^2}{\partial\tau\partial\bar{\tau}}+\frac{1}{\nu\tau_2}|\partial_{B_N}-\tau\partial_{B_R}|^2\\&\quad+3\partial_\nu(\nu^2\partial_\nu)\Big]E_s(M)^{SL(3,\mathbb{Z})}\\&=\frac{2s(2s-3)}{3}E_s(M)^{SL(3,\mathbb{Z})}\end{aligned} \tag{A14}$$

on the fundamental domain of $SL(3,\mathbb{Z})_M$.

We can also define the Eisenstein series of order s in the antifundamental representation by

$$E_s(M^{-1})^{SL(3,\mathbb{Z})}=\sum_{\hat{m}_i}{}'(\hat{m}_iM^{ij}\hat{m}_j)^{-s}, \tag{A15}$$

where $\hat{m}_i$ transforms in the fundamental representation of $SL(3,\mathbb{Z})$. Now using the result

$$\sum_{\hat{l}_i}{}'e^{-\pi\sigma G^{ij}\hat{l}_i\hat{l}_j}=\sigma^{-3/2}\sqrt{\det G}\sum_{l_i}{}'e^{-\pi G_{ij}l_il_j/\sigma} \tag{A16}$$

for invertible matrices, which can be derived using Poisson resummation, we get that

$$E_s(M^{-1})^{SL(3,\mathbb{Z})}=E_{3/2-s}(M)^{SL(3,\mathbb{Z})}. \tag{A17}$$

Thus there is a simple relationship between the Eisenstein series for the fundamental and the antifundamental representations.

APPENDIX B: CALCULATING $\hat{I}_1$ AND $\hat{I}_2$

Here we provide various details of calculating $\hat{I}_1$ and $\hat{I}_2$, which are needed to calculate the torus amplitude.

1. Calculating $\hat{I}_1$

We first evaluate (25), for which we use the representation

$$\ln\hat{\chi}(\nu;\Omega) = \frac{1}{4\pi}\sum_{(m,n)\neq(0,0)}\frac{\Omega_2}{|m\Omega+n|^2} e^{\pi[\bar{\nu}(m\Omega+n)-\nu(m\bar{\Omega}+n)]/\Omega_2} \tag{B1}$$

for the scalar propagator on the torus. This leads to the relation [30]

$$\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu_i}{\Omega_2}\ln\hat{\chi}(\nu_1-\nu_2;\Omega)\ln\hat{\chi}(\nu_1-\nu_3;\Omega)\hat{\chi}(\nu_2-\nu_3;\Omega) = \frac{1}{(4\pi)^3}\sum_{(m,n)\neq(0,0)}\frac{\Omega_2^3}{|m\Omega+n|^6} = \frac{1}{(4\pi)^3}E_3(\Omega,\bar{\Omega})^{SL(2,\mathbb{Z})}, \tag{B2}$$

where we have used (A1).

Thus,

$$\frac{(4\pi)^3}{4}\hat{I}_1 = \int_{\mathcal{F}_L}\frac{d^2\Omega}{\Omega_2^2}Z_{\rm lat}E_3(\Omega,\bar{\Omega})^{SL(2,\mathbb{Z})} = \hat{I}_1^1+\hat{I}_1^2+\hat{I}_1^3, \tag{B3}$$

where $\hat{I}_1^1$, $\hat{I}_1^2$, and $\hat{I}_1^3$ are the contributions from the zero orbit, the nondegenerate orbits, and the degenerate orbits of $SL(2,\mathbb{Z})$, respectively, as mentioned in the main text.

In order to evaluate (B3), from (A1) we use the expression

$$E_3(\Omega,\bar{\Omega})^{SL(2,\mathbb{Z})} = 2\zeta(6)\Omega_2^3 + \frac{3\pi\zeta(5)}{4\Omega_2^2} + \pi^3\sqrt{\Omega_2}\sum_{m_1\neq0,m_2\neq0}\left|\frac{m_1}{m_2}\right|^{5/2} \times K_{5/2}(2\pi\Omega_2|m_1m_2|)e^{2\pi i m_1m_2\Omega_1}. \tag{B4}$$

In doing the integrals, we frequently make use of the definition

$$K_s(x) = \frac{1}{2}\left(\frac{x}{2}\right)^s\int_0^\infty\frac{dt}{t^{s+1}}e^{-t-x^2/4t}. \tag{B5}$$

Integrating over the restricted fundamental domain $\mathcal{F}_L$ of $SL(2,\mathbb{Z})$, we keep only the finite terms in the limit $L\to\infty$. The details of the calculation are very similar to [26] and so we only mention the results.

(i) The contribution from the zero orbit gives [30]

$$\hat{I}_1^1 = V_2\int_{\mathcal{F}_L}\frac{d^2\Omega}{\Omega_2^2}E_3(\Omega,\bar{\Omega})^{SL(2,\mathbb{Z})} = 0, \tag{B6}$$

up to L dependent terms.

(ii) The contribution from the nondegenerate orbits gives

$$\hat{I}_1^2 = 2V_2\int_{-\infty}^{\infty}d\Omega_1\int_0^\infty\frac{d\Omega_2}{\Omega_2^2}E_3(\Omega,\bar{\Omega})^{SL(2,\mathbb{Z})} \times\sum_{k>j\geq0,p\neq0}e^{-2\pi iTkp-(\pi T_2/\Omega_2U_2)|k\Omega+j+pU|^2} = 2\sqrt{T_2}E_3(U,\bar{U})^{SL(2,\mathbb{Z})}\sum_{p\neq0,k\neq0}\left|\frac{p}{k}\right|^{5/2} \times K_{5/2}(2\pi T_2|pk|)e^{2\pi ipkT_1}, \tag{B7}$$

where we have also used

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}, \qquad K_{3/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left(1+\frac{1}{x}\right), \qquad K_{5/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left(1+\frac{3}{x}+\frac{3}{x^2}\right), \tag{B8}$$

and the identity

$$\frac{K_{1/2}(x+y)}{\sqrt{x+y}} + \frac{3\sqrt{x+y}}{xy}K_{3/2}(x+y) + \frac{3(x+y)^{3/2}}{x^2y^2}K_{5/2}(x+y) = \sqrt{\frac{2xy}{\pi}}\cdot\frac{K_{5/2}(x)K_{5/2}(y)}{x+y}. \tag{B9}$$

(iii) The contribution from the degenerate orbits gives

$$\hat{I}_1^3 = V_2\int_{-1/2}^{1/2}d\Omega_1\int_0^\infty\frac{d\Omega_2}{\Omega_2^2}E_3(\Omega,\bar{\Omega})^{SL(2,\mathbb{Z})} \times\sum_{(j,p)\neq(0,0)}e^{-(\pi T_2/\Omega_2U_2)|j+pU|^2} = \frac{2}{\pi^3}\left(2\zeta(6)T_2^3+\frac{3\pi\zeta(5)}{4T_2^2}\right)E_3(U,\bar{U})^{SL(2,\mathbb{Z})}. \tag{B10}$$

Thus from (B3), we get that

$$\hat{I}_1 = \frac{1}{8\pi^6}E_3(U,\bar{U})^{SL(2,\mathbb{Z})}E_3(T,\bar{T})^{SL(2,\mathbb{Z})}. \tag{B11}$$

2. Calculating $\hat{I}_2$

We next evaluate (26), for which we use the representation

$$\ln\hat{\chi}(\nu;\Omega) = \frac{\Omega_2}{4\pi}\sum_{n\neq0}\frac{1}{n^2}e^{2\pi in({\rm Im}\nu)/\Omega_2} + \frac{1}{4}\sum_{m\neq0,k\in\mathbb{Z}}\frac{1}{|m|} \times e^{2\pi im(k\Omega_1+{\rm Re}\nu)-2\pi\Omega_2|m||k-({\rm Im}\nu)/\Omega_2|} \tag{B12}$$

for the scalar propagator on the torus. Again we write

$$\hat{I}_2 = \hat{I}_2^1+\hat{I}_2^2+\hat{I}_2^3, \tag{B13}$$

where $\hat{I}_2^1$, $\hat{I}_2^2$, and $\hat{I}_2^3$ are the contributions from the zero orbit, the nondegenerate orbits, and the degenerate orbits of $SL(2,\mathbb{Z})$, respectively.

(i) The contribution from the zero orbit gives [30]

$$\hat{I}_2^1 = V_2 \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} \int_{\mathcal{T}} \prod_{i=1}^3 \frac{d^2\nu_i}{\Omega_2} [\ln\hat{\chi}(\nu_1 - \nu_2;\Omega)]^3 = \frac{T_2}{32\pi}\zeta(2)\zeta(3), \tag{B14}$$

up to L dependent terms. This integral can be evaluated by using the Rankin-Selberg identity to unfold the integration over the fundamental domain to the upper half-plane, using the Poincaré series representation of the scalar propagator.

(ii) The contribution from the nondegenerate orbits gives

$$\hat{I}_2^2 = 2V_2 \int_{-\infty}^{\infty} d\Omega_1 \int_0^\infty \frac{d\Omega_2}{\Omega_2^2} \int_{\mathcal{T}} \prod_{i=1}^3 \frac{d^2\nu_i}{\Omega_2} [\ln\hat{\chi}(\nu_1 - \nu_2;\Omega)]^3 \sum_{k>j\geq 0, p\neq 0} e^{-2\pi i T k p - (\pi T_2/\Omega_2 U_2)|k\Omega + j + pU|^2} = \hat{I}_2^{2,1} + \hat{I}_2^{2,2} + \hat{I}_2^{2,3}, \tag{B15}$$

where

$$\begin{aligned}\hat{I}_2^{2,1} &= 2V_2 \sum_{m_1\neq 0, m_2\neq 0, m_3\neq 0} \frac{\delta(m_1+m_2+m_3)}{m_1^2 m_2^2 m_3^2} \int_{-\infty}^\infty d\Omega_1 \int_0^\infty \frac{d\Omega_2}{\Omega_2^2}\left(\frac{\Omega_2}{4\pi}\right)^3 \sum_{k>j\geq 0, p\neq 0} e^{-2\pi i T k p - (\pi T_2/\Omega_2 U_2)|k\Omega + j + pU|^2} \\ &= \frac{4T_2\zeta(6)}{(4\pi)^3} \int_{-\infty}^\infty d\Omega_1 \int_0^\infty d\Omega_2 \Omega_2 \sum_{k>j\geq 0, p\neq 0} e^{-2\pi i T k p - (\pi T_2/\Omega_2 U_2)|k\Omega + j + pU|^2} \\ &= \frac{\zeta(6) U_2^3 \sqrt{T_2}}{16\pi^3} \sum_{p\neq 0, k\neq 0} \left|\frac{p}{k}\right|^{5/2} K_{5/2}(2\pi T_2|pk|) e^{2\pi i p k T_1},\end{aligned} \tag{B16}$$

$$\begin{aligned}\hat{I}_2^{2,2} &= \frac{3V_2}{32\pi}\int_{-\infty}^\infty d\Omega_1 \int_0^\infty \frac{d\Omega_2}{\Omega_2}\int_0^1 dx \int_0^1 dy \sum_{m\neq 0}\frac{1}{m^2} e^{2\pi i m(x-y)} \sum_{n\neq 0}\frac{1}{n^2} \\ &\times \sum_{(r_1,r_2)\in\mathbb{Z}} e^{2\pi i n(r_1-r_2)\Omega_1 - 2\pi|n|\Omega_2(|r_1-(x-y)|+|r_2-(x-y)|)} \sum_{k>j\geq 0, p\neq 0} e^{-2\pi i T k p - (\pi T_2/\Omega_2 U_2)|k\Omega + j + pU|^2} \\ &= \frac{3V_2}{32\pi^2}\int_{-\infty}^\infty d\Omega_1 \int_0^\infty d\Omega_2 \sum_{n\neq 0}\frac{1}{n^2}\sum_{m\neq 0}\frac{1}{|m|^3}\frac{d}{d\Omega_2}\left\{\tan^{-1}\left(\frac{2\Omega_2|n|}{|m|}\right)\right\}\frac{(1-e^{-4\pi|n|\Omega_2})}{(1-2\cos(2\pi n\Omega_1)e^{-2\pi|n|\Omega_2} + e^{-4\pi|n|\Omega_2})} \\ &\times \sum_{k>j\geq 0, p\neq 0} e^{-2\pi i T k p - (\pi T_2/\Omega_2 U_2)|k\Omega + j + pU|^2}.\end{aligned} \tag{B17}$$

Now using the representation [38]

$$\tan^{-1} x = \frac{\pi}{2} - \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)x^{2k+1}} = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + \ldots, \tag{B18}$$

we see that only the $k=0$ and $k=1$ terms in (B18) contribute to (B17) while doing the sum over m. While the constant term trivially vanishes, the terms for $k\geq 2$ vanish on doing the sum over m, because $\zeta(-2p)=0$ for all positive integers p. Thus, using $\zeta(0)=-1/2$, we get that

$$\begin{aligned}\hat{I}_2^{2,2} &= \frac{3V_2}{32\pi^2}\int_{-\infty}^\infty d\Omega_1 \int_0^\infty d\Omega_2 \sum_{n\neq 0}\frac{1}{|n|^3}\left(\frac{\zeta(2)}{\Omega_2^2} + \frac{1}{8n^2\Omega_2^4}\right)\left\{1 + \frac{2e^{-2\pi|n|\Omega_2}(\cos(2\pi n\Omega_1) - e^{-2\pi|n|\Omega_2})}{1-2\cos(2\pi n\Omega_1)e^{-2\pi|n|\Omega_2} + e^{-4\pi|n|\Omega_2}}\right\} \\ &\times \sum_{k>j\geq 0, p\neq 0} e^{-2\pi i T k p - (\pi T_2/\Omega_2 U_2)|k\Omega + j + pU|^2}.\end{aligned} \tag{B19}$$

Now in (B19), the terms in $\{\ldots\}$ are 1, and another term which exponentially decreases as $\Omega_2\to\infty$. We call these two contributions $\hat{I}_2^{2,2}(1\text{ only})$ and $\hat{I}_2^{2,2}(\text{not }1)$, respectively.

The term involving 1 gives us

$$\hat{I}_2^{2,2}(1\ \text{only}) = \frac{3\sqrt{T_2}}{32\pi^2}\Big[2\zeta(2)\zeta(3)\sum_{p\neq 0,k\neq 0}\left|\frac{p}{k}\right|^{1/2}K_{1/2}(2\pi T_2|pk|)e^{2\pi i pkT_1} + \frac{\zeta(5)}{4U_2^2}\sum_{p\neq 0,k\neq 0}\left|\frac{p}{k}\right|^{5/2}K_{5/2}(2\pi T_2|pk|)e^{2\pi i pkT_1}\Big], \tag{B20}$$

while $\hat{I}_2^{2,2}(\text{not}1)$ can be expanded in a power series in $e^{-2\pi|n|\Omega_2}$ for large Ω_2, and integrated term by term. This gives a far more complicated expression which we shall return to later.

Finally, the remaining expression is given by

$$\hat{I}_2^{2,3} = \frac{V_2}{32}\int_{-\infty}^{\infty} d\Omega_1 \int_0^{\infty}\frac{d\Omega_2}{\Omega_2^2}\int_0^1 dx\int_0^1 dy \sum_{m_1,m_2,m_2\neq 0;l_1,l_2,l_3\in\mathbb{Z}}\frac{\delta(\sum_i m_i)}{|m_1m_2m_3|}e^{2\pi i m_i l_i\Omega_1 - 2\pi\Omega_2|m_i||l_i-(x-y)|} \times \sum_{k>j\geq 0,p\neq 0} e^{-2\pi i Tkp-(\pi T_2/\Omega_2 U_2)|k\Omega+j+pU|^2}. \tag{B21}$$

We shall also return to this expression later.

(iii) The contribution from the degenerate orbits gives

$$\hat{I}_2^{3} = V_2\int_{-1/2}^{1/2} d\Omega_1\int_0^L\frac{d\Omega_2}{\Omega_2^2}\int_{\mathcal{T}}\prod_{i=1}^3\frac{d^2\nu_i}{\Omega_2}[\ln\hat{\chi}(\nu_1-\nu_2;\Omega)]^3\sum_{(j,p)\neq(0,0)}e^{-(\pi T_2/\Omega_2U_2)|j+pU|^2} = \hat{I}_2^{3,1}+\hat{I}_2^{3,2}+\hat{I}_2^{3,3}, \tag{B22}$$

where

$$\hat{I}_2^{3,1} = V_2\int_0^L\frac{d\Omega_2}{\Omega_2^2}\int_0^1 dx\int_0^1 dy\Big(\frac{\Omega_2}{4\pi}\Big)^3\Big(\sum_{m\neq 0}\frac{1}{m^2}e^{2\pi i m(x-y)}\Big)^3\sum_{(j,p)\neq(0,0)}e^{-(\pi T_2/\Omega_2U_2)|j+pU|^2} = \frac{T_2^3E_3(U,\bar{U})^{SL(2,\mathbb{Z})}}{32\pi^6}\sum_{m\neq 0,n\neq 0,p\neq 0}\frac{\delta(m+n+p)}{m^2n^2p^2} = \frac{\zeta(6)}{16\pi^6}T_2^3E_3(U,\bar{U})^{SL(2,\mathbb{Z})}. \tag{B23}$$

Also

$$\hat{I}_2^{3,2} = \frac{3V_2}{64\pi}\int_0^L\frac{d\Omega_2}{\Omega_2}\int_0^1 dx\int_0^1 dy\sum_{m\neq 0,n\neq 0,k\in\mathbb{Z}}\frac{1}{m^2n^2}e^{2\pi i m(x-y)-4\pi\Omega_2|n||k-(x-y)|}\sum_{(j,p)\neq(0,0)}e^{-(\pi T_2/\Omega_2U_2)|j+pU|^2} = \frac{3V_2}{64\pi^2}\sum_{m\neq 0,n\neq 0}\frac{1}{|m|^3n^2}\int_0^\infty dx\frac{d}{dx}\Big\{\tan^{-1}\Big(\frac{2|n|x}{|m|}\Big)\Big\}\sum_{(j,p)\neq(0,0)}e^{-(\pi T_2/xU_2)|j+pU|^2}. \tag{B24}$$

Using the representation (B18), once again we see that only the $k=0$ and $k=1$ terms in (B18) contribute to (B24). Thus we get that

$$\hat{I}_2^{3,2} = \frac{3}{32\pi^3}\zeta(2)\zeta(3)E_1(U,\bar{U})^{SL(2,\mathbb{Z})} + \frac{3\zeta(5)}{128\pi^5T_2^2}E_3(U,\bar{U})^{SL(2,\mathbb{Z})}. \tag{B25}$$

Also we have that

$$\hat{I}_2^{3,3} = \frac{V_2}{64}\int_0^L\frac{d\Omega_2}{\Omega_2^2}\int_0^1 dx\int_0^1 dy\sum_{m_1,m_2,m_2\neq 0;l_1,l_2,l_3\in\mathbb{Z}}\frac{\delta(\sum_i m_i)\delta(\sum_i l_im_i)}{|m_1m_2m_3|}e^{-2\pi\Omega_2|m_i||l_i-(x-y)|}\sum_{(j,p)\neq(0,0)}e^{-(\pi T_2/\Omega_2U_2)|j+pU|^2}. \tag{B26}$$

Although (B26) is a complicated expression, it is not difficult to see that the integrand goes as $O(e^{-\Omega_2})$ as $\Omega_2\to\infty$, and does not involve any power law suppressed terms. Thus we have that

$$\hat{I}_2^{3,3} = T_2\sum_{M,N}g_{MN}\int_0^\infty d\Omega_2\Omega_2^{-M}e^{-N\Omega_2}\sum_{(j,p)\neq(0,0)}e^{-(\pi T_2/\Omega_2U_2)|j+pU|^2}, \tag{B27}$$

where $g_{M,N}$ are unspecified functions of M and N; M is an integer and N is nonzero.

Let us denote the terms independent of T_1 and U_1 in the various expressions as perturbative in T and U, respectively (not to be confused with string perturbation theory). Thus $\hat{I}_2^{3,3}$ is perturbative in T, but has a nontrivial dependence on U_1. First let us consider the terms in $\hat{I}_2^{3,3}$ which are perturbative in U as well. In order to do this, we use the relation

$$\sum_{(j,p)\neq(0,0)} e^{-(\pi T_2/\Omega_2 U_2)|j+pU|^2} = \sum_{j\neq 0} e^{-(\pi T_2 |j|^2/U_2\Omega_2)} + \sqrt{\frac{U_2\Omega_2}{T_2}}\sum_{p\neq 0} e^{-(\pi p^2 T_2 U_2/\Omega_2)} + \sqrt{\frac{U_2\Omega_2}{T_2}}\sum_{p\neq 0,\hat{j}\neq 0} e^{2\pi i p\hat{j}\bar{U} - (\pi U_2/T_2\Omega_2)(pT_2+\hat{j}\Omega_2)^2}. \tag{B28}$$

We now outline the principal steps to deduce the various terms on the right-hand side of (B28). The first term is obtained by setting $p=0$, while to obtain the remaining terms which have $p\neq 0$, we Poisson resum on j to go to the variable $\hat{j}$. The second term is given by the $\hat{j}=0$ contribution, while the third term has $\hat{j}\neq 0$. Thus the first two terms in (B28) give the perturbative contributions.

This gives

$$\hat{I}_2^{3,3}(\text{pert}) = 2\sum_{M,N}\frac{g_{MN}}{(\pi N^{-1})^{(M-1)/2} T_2^{(M-3)/2}}\sum_{j\neq 0}\left(\frac{U_2}{|j|^2}\right)^{(M-1)/2} K_{M-1}\left(2|j|\sqrt{\frac{\pi N T_2}{U_2}}\right) + 2\sum_{M,N}\frac{g_{MN}}{(\pi N^{-1})^{(2M-3)/4}(U_2T_2)^{(2M-5)/4}}\sum_{p\neq 0}\frac{1}{|p|^{M-3/2}} K_{M-3/2}(2|p|\sqrt{\pi N T_2 U_2}). \tag{B29}$$

We now fix $\hat{I}_2^{3,3}(\text{pert})$ using the constraint that the amplitude must be the same in the two type II string theories. Note that the perturbative parts come only from the zero orbit and the degenerate orbit contributions to the amplitude. Thus from the perturbative contributions already calculated in $\hat{I}_2^1$, $\hat{I}_2^{3,1}$, and $\hat{I}_2^{3,2}$ we see that $\hat{I}_2^{3,3}(\text{pert})$ must contain

$$-\frac{3\zeta(2)\zeta(3)}{32\pi^2}\ln T_2. \tag{B30}$$

We now argue that there are no other perturbative contributions to $\hat{I}_2^{3,3}(\text{pert})$. Suppose there are other such contributions apart from (B30): because these are the only remaining ones, and they must be symmetric under interchange of U_2 and T_2, they must be of the form

$$h(U_2) + h(T_2) + \sum_i r_i(U_2) r_i(T_2). \tag{B31}$$

Thus the derivative with respect to U_2 of the total perturbative contributions (B30) and (B31) is given by

$$\frac{\partial h(U_2)}{\partial U_2} + \sum_i \frac{\partial r_i(U_2)}{\partial U_2} r_i(T_2). \tag{B32}$$

Consider the large U_2 limit of (B32). Let $h(U_2)\sim U_2^\lambda$, and $r_i(U_2)\sim U_2^{\lambda_i}$ for large U_2.[10] Thus (B32) has to contain a term

$$\lambda U_2^{\lambda-1} + T_2\sum_i \lambda_i^2 (U_2T_2)^{\lambda_i-1} \tag{B33}$$

at large U_2. Now consider the large U_2 behavior of the U_2 derivative of (B29). For large x, using the relation

$$K_s(x)\sim\sqrt{\frac{\pi}{2x}}e^{-x}, \tag{B34}$$

we see that the second term does not contribute. On the other hand, for small x using the relations

$$K_0(x)\sim -\ln x; \qquad K_m(x)\sim\frac{\Gamma(m)}{2}\left(\frac{x}{2}\right)^{-m}, \qquad m>0, \tag{B35}$$

from the first term, we get that

$$\frac{\partial\hat{I}_2^{3,3}}{\partial U_2}(\text{pert}) \sim -\frac{T_2}{U_2}\sum_N g_{1N} + 2\sum_{M>1,N}\frac{g_{MN}\Gamma(M)\zeta(2M-2)}{\pi^{M-1}}\times\left(\frac{U_2}{T_2}\right)^{M-2}, \tag{B36}$$

which can never be of the form (B33). Thus

$$\hat{I}_2^{3,3}(\text{pert}) = -\frac{3\zeta(2)\zeta(3)}{32\pi^2}\ln T_2. \tag{B37}$$

This contribution has a logarithmic dependence on T_2 and must arise from the infinite sum over N in (B29). Any constant term in $\hat{I}_2^{3,3}(\text{pert})$ can be absorbed in the regularization of the infrared divergences. Thus we see that

[10] Assuming a more general behavior of the form $h(U_2)\sim U_2^\lambda(\ln U_2)^{\hat{\lambda}}$, $r_i(U_2)\sim U_2^{\lambda_i}(\ln U_2)^{\hat{\lambda}_i}$ does not change the conclusions below.

$$\hat{I}_2^{\rm pert} = \frac{1}{32\pi^6} E_3(U,\bar{U})^{SL(2,\mathbb{Z})}_{\rm pert} E_3(T,\bar{T})^{SL(2,\mathbb{Z})}_{\rm pert} + \frac{3}{32\pi^3}\zeta(2)\zeta(3)(E_1(U,\bar{U})^{SL(2,\mathbb{Z})}_{\rm pert} + E_1(T,\bar{T})^{SL(2,\mathbb{Z})}_{\rm pert}). \quad (B38)$$

Extending (B38) to its nonperturbative completion, we get that

$$\hat{I}_2 = \frac{1}{32\pi^6} E_3(U,\bar{U})^{SL(2,\mathbb{Z})} E_3(T,\bar{T})^{SL(2,\mathbb{Z})} + \frac{3}{32\pi^3}\zeta(2)\zeta(3)(E_1(U,\bar{U})^{SL(2,\mathbb{Z})} + E_1(T,\bar{T})^{SL(2,\mathbb{Z})}). \quad (B39)$$

Considering the various nonperturbative contributions that have already been calculated in $\hat{I}_2^{2,1}$, $\hat{I}_2^{2,2}$(1 only), $\hat{I}_2^{3,1}$, and $\hat{I}_2^{3,2}$, we get all the terms in (B39) with the precise coefficients apart from just one term. This term is

$$\frac{\sqrt{U_2 T_2}}{32}\sum_{p\neq 0, k\neq 0}\left|\frac{p}{k}\right|^{5/2} K_{5/2}(2\pi T_2|pk|)e^{2\pi i pk T_1} \times \sum_{m\neq 0, n\neq 0}\left|\frac{m}{n}\right|^{5/2} K_{5/2}(2\pi T_2|mn|)e^{2\pi i mn U_1}. \quad (B40)$$

Now (B40) depends on T_1, and thus cannot be obtained from $\hat{I}_2^{3,3}$. Thus

$$\hat{I}_2^{3,3}(\text{nonpert}) = 0. \quad (B41)$$

So (B40) must be obtained from the only remaining contributions leading to

$$\hat{I}_2^{2,2}(\text{not}1) + \hat{I}_2^{2,3} = \frac{\sqrt{U_2 T_2}}{32}\sum_{p\neq 0, k\neq 0}\left|\frac{p}{k}\right|^{5/2} \times K_{5/2}(2\pi T_2|pk|)e^{2\pi i pk T_1} \times \sum_{m\neq 0, n\neq 0}\left|\frac{m}{n}\right|^{5/2} \times K_{5/2}(2\pi T_2|mn|)e^{2\pi i mn U_1}. \quad (B42)$$

This concludes the calculation of the torus amplitude. We have obtained some parts of the amplitude based on consistency and heuristic arguments, but have not explicitly calculated those contributions. It would be nice to calculate them explicitly. In the next appendix, we provide some more evidence that the extra contributions in (B31) vanish.

APPENDIX C: A SELF-CONSISTENCY CHECK FOR THE TORUS AMPLITUDE

In the previous section, we have calculated the four graviton amplitude on the torus. Some parts of the amplitude were obtained using indirect arguments and not by explicit calculations. We now show that the answer we got is consistent with the structure we have proposed for the modular form.

We mentioned that there can be additional contributions to the torus amplitude given by (B31). Let $h(T,\bar{T})_{\rm pert} \equiv h(T_2)$. We now show that $h(T,\bar{T}) = 0$ based on very different considerations compared to the previous discussion. This contribution yields an additional term $\mu h(T,\bar{T})$ to (67). Repeating the arguments as before, we get back the results of Sec. IV, along with an extra equation given by

$$\Delta_{SL(2,\mathbb{Z})_T} h(T,\bar{T}) = 12 h(T,\bar{T}). \quad (C1)$$

This is, of course, solved by $E_4(T,\bar{T})^{SL(2,\mathbb{Z})}$. Thus adding $E_4(T,\bar{T})^{SL(2,\mathbb{Z})} + E_4(U,\bar{U})^{SL(2,\mathbb{Z})}$ to the torus amplitude, we see that in nine dimensions this leads to a divergent term

$$2\zeta(8) l_s^5 \int d^9x\sqrt{-g_9}\left(r_B^4 + \frac{1}{r_B^4}\right) r_\infty^3 D^6 \mathcal{R}^4. \quad (C2)$$

However, from (45) we see that the divergence needed to produce threshold singularities should go as r_∞^5, and is also independent of r_B. This has a different behavior than (C2), thus $h(T,\bar{T}) = 0$.

Also the T_2^4 dependence of the torus amplitude at large T_2 is inconsistent with the large T_2 scaling behavior of the genus two and three amplitudes based on considerations of degeneration limits of the Riemann surfaces as discussed before, which leads to the same conclusion.

The remaining terms in (B31) give an additional perturbative (in the string coupling) contribution to the proposed modular form

$$\mu \sum_i r_i(T,\bar{T}) r_i(U,\bar{U}), \quad (C3)$$

where $r_i(T,\bar{T})_{\rm pert} \equiv r_i(T_2)$. Thus $r_i(U,\bar{U})$ must be an $SL(2,\mathbb{Z})_U$ invariant modular form, while $\mu r_i(T,\bar{T})$ must get enhanced to an $SL(3,\mathbb{Z})_M$ invariant modular form $r_i(M)$. Now using the symmetry under interchange of U and T, we conclude that $r_i(M)$ receives only one perturbative contribution at genus zero, and instanton corrections.

On the other hand, we know that $r_i(M)$ must satisfy the Laplace equation, or a Poisson equation on moduli space. If it satisfies the Laplace equation, it will have two perturbative contributions, contradicting the statement above. If it satisfies a Poisson equation, considerations of supersymmetry constrain the source term to involve the modular form for the $\mathcal{R}^4$ interaction, namely, $E_{3/2}(M)^{SL(3,\mathbb{Z})}$, which has a genus zero and a genus one contribution. Thus the solution of the Poisson equation will have more than one perturbative contribution, again contradicting the statement above. Thus $r_i(M) = 0$, and (B31) vanishes.

[1] M. B. Green and J. H. Schwarz, Nucl. Phys. **B198**, 252 (1982).
[2] M. B. Green and J. H. Schwarz, Phys. Lett. B **109**, 444 (1982).
[3] E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60**, 917 (1988).
[4] C. M. Hull and P. K. Townsend, Nucl. Phys. **B438**, 109 (1995).
[5] E. Witten, Nucl. Phys. **B443**, 85 (1995).
[6] E. Kiritsis and B. Pioline, Nucl. Phys. **B508**, 509 (1997).
[7] M. B. Green and M. Gutperle, Nucl. Phys. **B498**, 195 (1997).
[8] M. B. Green and P. Vanhove, Phys. Lett. B **408**, 122 (1997).
[9] M. B. Green, M. Gutperle, and P. Vanhove, Phys. Lett. B **409**, 177 (1997).
[10] B. Pioline and E. Kiritsis, Phys. Lett. B **418**, 61 (1998).
[11] B. Pioline, Phys. Lett. B **431**, 73 (1998).
[12] M. B. Green and S. Sethi, Phys. Rev. D **59**, 046006 (1999).
[13] N. A. Obers and B. Pioline, arXiv:hep-th/9812139.
[14] N. A. Obers and B. Pioline, Commun. Math. Phys. **209**, 275 (2000).
[15] M. B. Green, H.-h. Kwon, and P. Vanhove, Phys. Rev. D **61**, 104010 (2000).
[16] N. A. Obers and B. Pioline, Classical Quantum Gravity **17**, 1215 (2000).
[17] B. Pioline, H. Nicolai, J. Plefka, and A. Waldron, J. High Energy Phys. 03 (2001) 036.
[18] N. Berkovits, J. High Energy Phys. 09 (2004) 047.
[19] E. D'Hoker and D. H. Phong, Nucl. Phys. **B715**, 3 (2005).
[20] E. D'Hoker, M. Gutperle, and D. H. Phong, Nucl. Phys. **B722**, 81 (2005).
[21] M. B. Green and P. Vanhove, J. High Energy Phys. 01 (2006) 093.
[22] N. Berkovits, Phys. Rev. Lett. **98**, 211601 (2007).
[23] M. B. Green, J. G. Russo, and P. Vanhove, J. High Energy Phys. 02 (2007) 099.
[24] N. A. Obers and B. Pioline, Phys. Rep. **318**, 113 (1999).
[25] M. B. Green, arXiv:hep-th/9903124.
[26] A. Basu, preceding Article, Phys. Rev. D **77**, 106003 (2008).
[27] M. B. Green and J. H. Schwarz, Nucl. Phys. **B198**, 441 (1982).
[28] N. Sakai and Y. Tanii, Nucl. Phys. **B287**, 457 (1987).
[29] L. J. Dixon, V. Kaplunovsky, and J. Louis, Nucl. Phys. **B355**, 649 (1991).
[30] M. B. Green and P. Vanhove, Phys. Rev. D **61**, 104011 (2000).
[31] M. B. Green, J. G. Russo, and P. Vanhove, arXiv:0801.0322v1.
[32] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein, and J. S. Rozowsky, Nucl. Phys. **B530**, 401 (1998).
[33] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein, and J. S. Rozowsky, arXiv:hep-th/9809163.
[34] Z. Bern *et al.*, Phys. Rev. Lett. **98**, 161303 (2007).
[35] M. B. Green, J. H. Schwarz, and L. Brink, Nucl. Phys. **B198**, 474 (1982).
[36] J. G. Russo and A. A. Tseytlin, Nucl. Phys. **B508**, 245 (1997).
[37] M. B. Green, Nucl. Phys. B, Proc. Suppl. **68**, 242 (1998).
[38] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 2007).
[39] T. M. Apostol, *Introduction to Analytic Number Theory* (Springer Verlag, New York, 1976).
[40] E. Cremmer, "Supergravities in 5 Dimensions," Invited paper at the Nuffield Gravity Workshop, Cambridge, Eng., 1980.
[41] B. Julia, "Group Disintegrations," Invited paper presented at Nuffield Gravity Workshop, Cambridge, Eng., 1980.