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Spectral properties of ghost Neumann matrices

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We continue the analysis of the ghost wedge states in the oscillator formalism by studying the spectral properties of the ghost matrices of Neumann coefficients. We show that the traditional spectral representation is not valid for these matrices and propose a new heuristic formula that allows one to reconstruct them from the knowledge of their eigenvalues and eigenvectors. It turns out that additional data, which we call boundary data, are needed in order to actually implement the reconstruction. In particular our result lends support to the conjecture that there exists a ghost three strings vertex with properties parallel to those of the matter three strings vertex.

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I. INTRODUCTION

This paper is complementary to the analysis, started in [1], of the conjectured equivalence

$$e^{-(n-2/2)(\mathcal{L}_0^{(g)}+\mathcal{L}_0^{(g)\dagger})}|0\rangle = \mathcal{N}_n e^{c^\dagger S_n b^\dagger}|0\rangle \equiv |n\rangle, \qquad (1)$$

which is a crucial one in the recent developments in open string field theory [2–14]. Here $|n\rangle$ are the ghost wedge states in the oscillator formalism [15–19], which of course, must coincide with the corresponding surface wedge states. In [1] we dealt with the left-hand side (LHS) of this equation. We showed that, if we understand it ordered according to the natural normal ordering, it can be cast into the midterm form in (1), and we diagonalized the matrix S_n in such a squeezed state. Then we proved that, *if* we are allowed to star-multiply the squeezed states representing the ghost wedge states $|n\rangle$ the same way we do for the matter wedge states and diagonalize the corresponding matrices, the eigenvalue we obtain in the two cases are the same. In this paper we focus on the spectral properties of such operators like S_n and, among other things, we exhibit evidence that the above *if* is justified.

To be more precise, in order to fully prove (1) we have two possibilities. The most direct alternative is to define the three strings vertex for the ghost part, and thus the star product, pertinent to the natural normal ordering in the oscillator formalism; then to construct the wedge states appropriate for this vertex; finally to diagonalize the latter and show that they coincide with the midterm of (1) (with some additional specifications that will be clarified in due course). Unfortunately, the construction of the ghost vertex is not so straightforward as one would hope. Relying on the common lore on this subject, we face a large number of possibilities, which are mostly linked to the ghost zero mode insertions and our attempts in this direction so far have been unfruitful. Before continuing in such a challenging program it is wise to gather some evidence that the vertex one is looking for does exist and some indirect information about it. This is the original motivation of the present paper, which relies on the second alternative.

Having diagonalized the matrices S_n in the midterm of (1) in the basis of weight 2 differentials, see [1], one may wonder whether one can reconstruct the original matrices. For the matter part this is a standard procedure, simply one uses the spectral representation of the infinite matrices involved. But for the ghost sector we are interested in here things are more complicated (it should be recalled that the infinite matrices S_n are not square but *lame*, i.e., infinite rectangular). Ultimately, the answer is: yes, we can reconstruct the S_n matrices; in other words, we can derive the right-hand side (RHS) of (1) from the LHS, but the procedure is more involved than in the matter case. In fact, the traditional spectral representation is not valid for lame matrices and we have to figure out a new heuristic formula that allows us to reconstruct them from their eigenvalues and eigenvectors. It turns out that additional data, which we call boundary data, are needed in order to actually implement the reconstruction. Once this is done we can extract from them basic information about the Neumann coefficients matrices of the ghost three strings vertex.

The main results of our paper are the study of the spectral properties of the infinite matrices S_n in the $b-c$

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ghost bases, the reconstruction recipe for such infinite matrices (which is an interesting result in itself), and the evidence for the existence of the three-strings vertex we need for the ghost sector in the natural normal ordering.

The paper is organized as follows. In Sec. II, which is essentially pedagogical, we present an example of a three strings vertex which is not the one we are looking for as it cannot be diagonalized in the weight 2 basis, but has all the other good properties we expect of the true vertex. This example also illustrates the problems one comes across in constructing a ghost three strings vertex. In Sec. III we make contact with the results of [1] and give a more detailed proof that the squeezed states in the midterm of (1) have the same eigenvalue as the ghost wedge states in the oscillator formalism. We clarify that this is not enough to prove that (1) holds, and, in Sec. IV, we show where the problem lies and propose a new heuristic formula for the reconstruction of infinite lame matrices. Finally, Sec. V is devoted to our conclusions. Three appendices contain details of the calculations needed in the text. In particular, Appendix C presents a new proof of the fundamental Eq. (43).

Notation.—Any infinite matrix we meet in this paper is either square short, long legged, or lame. In this regard we will often use a compact notation: a subscript s will represent an integer label n running from 2 to ∞, while a subscript l will represent a label running from -1 to $+\infty$. So Y_{ss}, Y_{ll} will denote square short and long legged, respectively; Y_{sl}, Y_{ls} will denote short-long and long-short lame matrices, respectively. With the same meaning we will say that a matrix is (ll), (ss), (sl), or (ls). In a similar way, we will denote by V_s and V_l a short and long infinite vector, to which the above matrices naturally apply. Moreover, while n, m represent generic matrix indices, at times we will use $\bar{n}$, $\bar{m}$ to stress that they are short, i.e., $\bar{n}, \bar{m} \geq 2$.

II. THE THREE STRINGS VERTEX

This section is mostly pedagogical. We would like here to explain what the problems are with defining a three strings vertex for the ghost sector that fits the purposes of proving Eq. (1). The first problem we have to face is normal ordering. We will have in mind two main cases of normal orderings, those we have called *natural* and *conventional normal ordering* in [1]. The former is the obvious normal ordering required when the vacuum is $|0\rangle$, the latter is instead appropriate to the vacuum state $c_1|0\rangle$. A consistent vertex for conventional normal ordering exists, is the one explicitly computed by Gross and Jevicki, [18], who used the vacuum $c_0c_1|0\rangle$ (for general problems connected with the ghost sector, see [20–23]). But it is not what we need in the natural normal ordering case. A second problem is generated by the ghost insertions, which are free and there is no *a priori* principle to fix them. We know, however, that a certain number of conditions should be satisfied. One is the Becchi-Rouet-Stora-Tyutin (BRST) invariance of the three strings vertex. This is unfortunately hard to translate into a practical recipe for construction. Other conditions, i.e., cyclicity, bpz-compatibility, and commutativity of the Neumann coefficients matrices are more useful from a constructive point of view.

In the sequel we will consider a definite example. Even though it turns out not to be the right vertex we are looking for, it will allow us to illustrate many questions which would sound rather abstruse in the abstract.

To start with we first recall the relevant anticommutator and bpz rules

$$[c_n, b_m]_+ = \delta_{n+m,0}, \qquad bpz(c_n) = -(-1)^n c_{-n}, \qquad bpz(b_n) = (-1)^n b_{-n}.$$

Then we define the state $|\hat{0}\rangle = c_{-1}c_0c_1|0\rangle$, where $|0\rangle$ is the SL(2,R)-invariant vacuum, the tensor product of states

$$_{123}\langle\hat{\omega}| = {}_1\langle\hat{0}|_2\langle\hat{0}|_3\langle 0| \tag{2}$$

carrying total ghost number 6, and

$$|\omega\rangle_{123} = |0\rangle_1|0\rangle_2|\hat{0}\rangle_3 \tag{3}$$

carrying total ghost number 3. They satisfy $_{123}\langle\hat{\omega}|\omega\rangle_{123} = 1$. Finally we write down the general form of the three strings vertex

$$\langle\hat{V}_3| = \mathcal{K}\,_{123}\langle\hat{\omega}|e^{\hat{E}}, \qquad \hat{E} = -\sum_{r,s=1}^{3}\sum_{n,m}^{\infty} c_n^{(r)}\hat{V}_{nm}^{rs}b_m^{(s)}. \tag{4}$$

The dual vertex is

$$|V_3\rangle = \mathcal{K}e^{E}|\hat{\omega}\rangle_{123}, \qquad E = \sum_{r,s=1}^{3}\sum_{n,m}^{\infty} c_n^{(r)\dagger}V_{nm}^{rs}b_m^{(s)\dagger}. \tag{5}$$

The range of m, n is not specified. However, for reasons that will become clear later, we would like to interpret the matrices $\hat{V}_{nm}^{rs}$ and V_{nm}^{rs} as square long-legged matrices (ll). But, as soon as we try to evaluate, for instance, contractions like $\langle\hat{V}_3|\omega\rangle_{123}$ in order to compute the constant $\mathcal{K}$, a problem arises linked to the presence in the exponent (4) of couples of conjugate operators c_0, b_0, c_{-1}, b_1, and c_1, b_{-1}. In order to appreciate this problem let us consider the simple case of $e^{c_0V_{00}b_0}$. Interpreting this expression literally one gets

$$\begin{aligned} e^{c_0V_{00}b_0} &= 1 + c_0V_{00}b_0 + \tfrac{1}{2}c_0V_{00}b_0c_0V_{00}b_0 + \ldots \\ &= 1 + c_0(V_{00} + \tfrac{1}{2}V_{00}^2 + \ldots)b_0 \\ &= 1 + c_0(e^{V_{00}} - 1)b_0. \end{aligned} \tag{6}$$

It follows that, when inserted in $\langle\hat{V}_3|\omega\rangle_{123}$ a term like this does not yield 1, as one would expect. Moreover if, instead of the single zero mode we have considered here for simplicity, we had three, the result would be even more complicated. All this is not natural. Let us recall that the meaning of $\hat{V}^{rs}_{nm}$ (see [24] and below) is the coefficient of the monomial $z^{m+1}w^{n-2}$ in the expansion of $\langle\hat{V}_3|R(c^{(s)}(z)b^{(r)}(w))|\omega_{123}\rangle$ in powers of z and w (with opposite sign). Therefore interpreting the exponentials in (4) as in (6) is misleading. It is clear that what they really mean is something else. To adapt the oscillator formalism to the desired meaning we proceed as follows.

Let us introduce new conjugate operators η_a, $\xi^\dagger_a$, $a = -1, 0, 1$, in addition to c_n, b_m, such that

$$[\eta_a, \xi^\dagger_b]_+ = \delta_{ab} \tag{7}$$

and they anticommute with all the other oscillators. Moreover we require them to satisfy

$$\eta_a|0\rangle = 0, \qquad \langle 0|\xi^\dagger_a = 0, \tag{8}$$

while

$$\langle 0|\eta_a \neq 0, \qquad \xi^\dagger_a|0\rangle \neq 0. \tag{9}$$

Now let us replace in the exponent of (4) c_a with η_a (but not $c^\dagger_a$ in the exponent of (5) with $\eta^\dagger_a$) and $b^\dagger_a$ in the exponent of (4) with $\xi^\dagger_a$ (but not b_a in the exponent of (4) with ξ_a—in fact $c^\dagger_a$ and b_a will not be needed). With these rules $\langle\hat{V}_3|\omega\rangle_{123} = \mathcal{K}$ straightforwardly. The matrices $\hat{V}^{rs}_{nm}$ and V^{rs}_{nm} are naturally square long legged. The interpretation of $\hat{V}^{rs}_{nm}$ as the negative coefficient of order z^{m+1} and w^{n-2} in the expansion of $\langle\hat{V}_3|R(c^{(s)}(z)b^{(r)}(w))|\omega\rangle_{123}$ in powers of z and w, remains valid provided one replaces $b^{(r)\dagger}_{-1}$, $b^{(r)\dagger}_0$, $b^{(r)\dagger}_1$ in $b^{(r)}(w)$ with $\xi^{(r)\dagger}_{-1}$, $\xi^{(r)\dagger}_0$, $\xi^{(r)\dagger}_1$.

We stress again that the substitution of c_a with η_a and $b^\dagger_a$ with $\xi^\dagger_a$ is dictated by the requirement of consistency of the interpretation of the Neumann coefficient as expansion coefficients of the b- c propagator.

A. Ghost Neumann coefficients and their properties

It is time to go to a concrete example. To this end one has to explicitly compute $\hat{V}^{rs}_{nm}$ and V^{rs}_{nm} in (4) and (5). The method is well known: one expresses the propagator with zero mode insertions $\langle\!\langle c(z)b(w)\rangle\!\rangle$ in two different ways, first as a conformal field theory (CFT) correlator and then in terms of $\hat{V}_3$ and one equates the two expressions after mapping them to the disk via the maps

$$f_i(z_i) = \alpha^{2-i}f(z_i), \qquad i = 1, 2, 3, \tag{10}$$

where

$$f(z) = \left(\frac{1+iz}{1-iz}\right)^{2/3}. \tag{11}$$

Here $\alpha = e^{2\pi i/3}$ is one of the three third roots of unity. However, this recipe leaves several uncertainties due especially to the ghost insertions. For concreteness in Appendix A we make a specific choice of these insertions, in a way the simplest one: we set the insertions at infinity. Even so there remain some uncertainties which we fix by requiring certain properties, in particular, cyclicity, consistency with the bpz operation, and commutativity of the twisted matrices of Neumann coefficients (the motivation for the latter will become clear further on). With this (arbitrary) choice, the ghost Neumann coefficients worked out in Appendix A satisfy the following set of properties:

(i) *cyclicity*

$$\hat{V}^{rs}_{nm} = \hat{V}^{r+1,s+1}_{nm}, \tag{12}$$

(ii) *bpz consistency*

$$(-1)^{n+m}V^{rs}_{nm} = \hat{V}^{rs}_{nm}, \tag{13}$$

(iii) *commutativity*

Its meaning is the following. Defining $X = \hat{C}V^{rr}$, $X^+ = \hat{C}V^{12}$, $X^- = \hat{C}V^{21}$, we have

$$X^{rs}X^{r's'} = X^{r's'}X^{rs} \tag{14}$$

for all r, s, r', s'. In addition, we have

$$X + X^+ + X^- = 1 \tag{15}$$

and

$$X^+X^- = X^2 - X, \qquad X^2 + (X^+)^2 + (X^-)^2 = 1. \tag{16}$$

It should be stressed that all the X^{rs} matrices are (ll).

B. Formulas for wedge states

Our next goal is to define recursion relations for the ghost wedge states. To start with we define the star product of squeezed ghost states of the form

$$|S\rangle = \mathcal{N}\exp(c^\dagger S b^\dagger)|0\rangle. \tag{17}$$

We notice that since the vacuum is $|0\rangle$ we are implicitly referring to the natural normal ordering. The star product of two such states $|S_1\rangle$ and $|S_2\rangle$ is the bpz of the state

$$\langle\hat{V}_3||S_1\rangle_1|S_2\rangle_2. \tag{18}$$

However, this formula needs some specifications. We remark that the problem pointed out above, linked to the presence of couples of conjugate oscillators in the exponents, is present both in (17) and (18). We solve it as we did in Sec. II, with the help of additional oscillators η_a, $\xi^\dagger_b$. We interpret, for instance, (17) as follows. We replace the new oscillators in it as in Sec. II, then we exploit the anticommutativity properties of the latter to move them to the right and apply them to $|0\rangle$, then we substitute back $b^\dagger_a$ in the place of $\xi^\dagger_a$. The upshot of this operation is that no $b^\dagger_a$ oscillator will survive and the state (17) takes the form

$$|S\rangle = \mathcal{N}\exp\left(\sum_{n=-1}\sum_{m=2} c^\dagger_n S_{nm} b^\dagger_m\right)|0\rangle. \tag{19}$$

That is, the matrix S_{nm} in the exponent is lame ls. This is the precise meaning we attach to (17). Let us notice that the bpz dual expression of (19) is

$$\langle S| = \mathcal{N}\langle 0| \exp(-c\hat{C}S\hat{C}b). \tag{20}$$

The matrix S here is ll.

After this specification let us define the star product of $|S_1\rangle$ and $|S_2\rangle$. Let us recall the three strings vertex (4) and (5). Remembering the discussion before (19) we conclude that $\hat{V}^{rs}_{nm}$ is sl for $r = 1, 2$ and ll for $r = 3$, while V^{rs}_{nm} is ls for $r = 1, 2$ and ll for $r = 3$.

In evaluating this product we will have to evaluate vacuum expectation values (vev's) of the type

$$\langle\hat{0}| \exp(cFb + c\mu + \lambda b) \exp(c^\dagger G b^\dagger + \theta b^\dagger + c^\dagger \zeta)|0\rangle. \tag{21}$$

Here we are using an obvious compact notation: F, G denotes matrices F_{nm}, G_{nm}, and λ, μ, θ, ζ are anticommuting vectors λ_n, μ_n, θ_n, ζ_n. We expect the result of this evaluation to be

$$\begin{aligned}\langle\hat{0}| &\exp(cFb + c\mu + \lambda b) \exp(c^\dagger G b^\dagger + \theta b^\dagger + c^\dagger \zeta)|0\rangle \\ &= \det(1 + FG) \exp\left(-\theta \frac{1}{1 + FG} F\zeta - \lambda \frac{1}{1 + GF} G\mu \right. \\ &\quad \left. - \theta \frac{1}{1 + FG}\mu + \lambda \frac{1}{1 + GF}\zeta\right). \end{aligned} \tag{22}$$

In order for this formula to hold in (21) the operator denoted b, c must be creation operators with respect to $\langle\hat{0}|$ and annihilation operators with respect to the $|0\rangle$ vacuum. Vice versa, the oscillators denoted $c^\dagger$, $b^\dagger$ must be all creation operators with respect to $|0\rangle$, and annihilation operators with respect to $\langle\hat{0}|$. But this is precisely what happens if we assume the definition (19) for the squeezed states and (4) for the vertex with the summation over n starting from 2 (which is consistent with the interpretation by means of $\xi^\dagger_a$ and η_a, as before (19)).

Therefore it is correct to use formulas like (22) in order to evaluate the star product (18), but in this case the matrices F and G will be lame (ls or sl as the case be), while analogous considerations apply to the vectors λ, μ, θ, ζ (λ, ζ are long vectors, while μ, θ are short). The star product of two squeezed states like (17) is

$$|S_1\rangle \star |S_2\rangle = |S_{12}\rangle,$$

where the state in the RHS has the same form as (17), with the matrix S replaced by $S_{12} = \hat{C}T_{12}$. The latter is given by the familiar formula

$$T_{12} = X + (X^+, X^-)\frac{1}{1 - \Sigma_{12}\mathcal{V}}\Sigma_{12}\begin{pmatrix} X^- \\ X^+ \end{pmatrix}, \tag{23}$$

where

$$\Sigma_{12} = \begin{pmatrix} \hat{C}S_1 & 0 \\ 0 & \hat{C}S_2 \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} X & X^+ \\ X^- & X \end{pmatrix}. \tag{24}$$

The normalization of $|S_{12}\rangle$ is given by

$$\mathcal{N}_{12} = \mathcal{N}_1\mathcal{N}_2 \det(1 - \mathcal{V}\Sigma_{12}). \tag{25}$$

Notice that in this formula the four matrices in $\mathcal{V}\Sigma_{12}$ are ss.

These expressions are well defined. However, since they are expressed in terms of lame matrices we cannot operate with them in the same way we usually do with the analogous matrices of the matter sector. For that one needs the identities proved in the previous section, which are only valid for long square matrices. Luckily in the case of the wedge states it is possible to overcome this difficulty.

When computing a star product we would like to be able to apply the formulas of subsection II A, which are expressed in terms of long square matrices. To this end we would like (22) to be expressed in terms of long square matrices, rather than of lame matrices. This is possible at the price of some modifications.

Let us introduce the new conjugate operators η_a, $\xi^\dagger_a$, $a = -1, 0, 1$, as above, see (7)–(9) and let us replace in (22) c_a (but not $c^\dagger_a$) with η_a and $b^\dagger_a$ (but not b_a) with $\xi^\dagger_a$. Then in the RHS long square matrices and long vectors will feature (instead of lame matrices and short or long vectors). In the sequel we will use (22) in this sense. Such modifications, of course, are not for free. We have to justify them.[1] We will show later on that in the case of the wedge states such a move is justified.

Once this is done the calculation of the star product works smoothly without any substantial difference with respect to the matter case. The formulas are the same Eqs. (23)–(25) above, but expressed in terms of long square matrices to which we can apply the identities of subsection II A. This allows us to treat the ghost squeezed states in a way completely similar to the matter squeezed states. Of course, it remains for us to comply with the promise we made of showing that we are allowed to use long square matrices.

The wedge states are now defined to be squeezed states $|n\rangle \equiv |S_n\rangle$ that satisfy the recursive star product formula

$$|n\rangle \star |m\rangle = |n + m - 1\rangle. \tag{26}$$

This implies that the relevant matrices $T_n = \hat{C}S_n$ satisfy the recursion relation

$$T_{n+m-1} = \frac{X - (T_n + T_m)X + T_nT_m}{1 - (T_n + T_m)X + T_nT_mX} \tag{27}$$

[1] In the previous cases the introduction of the new oscillators was simply an auxiliary tool to help us interpret such formulas as (19). We could have done without them by *ad hoc* definitions. But now we are tampering with vev's, therefore we have to make sure that we do not modify anything essential.

or

$$T_{n+1} = X\frac{1-T_n}{1-T_nX}, \tag{28}$$

and the normalization constants are given by

$$\mathcal{N}_{n+1} = \mathcal{N}_n \mathcal{K} \det(1 - T_nX). \tag{29}$$

These relations are derived under the hypothesis that T_n and X, X^+, X^- commute and by using the identities of subsection II A. The solution to (28) is well known, [23,25]. We repeat the derivation in order to stress its uniqueness. We require that $|2\rangle$ coincide with the vacuum $|0\rangle$, both for the matter and the ghost sector.[2]

This implies, in particular, that $T_2 = 0$ and $\mathcal{N}_2 = 1$, which entails from (28) that $T_3 = X$, $T_4 = \frac{X}{1+X}$, etc. That is, T_n is a uniquely defined function of X. But X can be uniquely expressed in terms of the sliver matrix T

$$X = \frac{T}{T^2 - T + 1}, \tag{30}$$

a formula whose inverse is well known [26,27],

$$T = \frac{1}{2X}(1 + X - \sqrt{(1-X)(1+3X)}). \tag{31}$$

Therefore T_n can be expressed as a uniquely defined function of T. Now consider the formula

$$T_n = \frac{T + (-T)^{n-1}}{1-(-T)^n}.$$

It satisfies (28) as well as the condition $T_2 = 0$, therefore it is the unique solution to (28) we were looking for.

So far the states $|n\rangle$ have been defined solely in terms of the three strings vertex. One might ask what is their connection with the wedge states defined as surface states, [2,28–30]. This connection can be established: it can be shown that, with the appropriate insertion of the zero modes, the surface wedge matrix S_3 is actually V^{rr}, i.e., $T_3 = X$.

It is simple to see that similarly (29) has a unique solution satisfying $\mathcal{N}_2 = 1$ and $\mathcal{K} = \mathcal{N}_3$.

C. Commutation relations with K_1

What we have done so far is all very good, but the concrete example of vertex constructed in Appendix A is only academical, as the following remark shows. In [1] we diagonalized the LHS of (1) on the basis of weight 2 differentials, in which the operator K_1 is diagonal. In order to be able to compare this result with the wedge states defined above we have to make sure that also the matrices T_n, X, X_+, X_- can be diagonalized in the same basis. In this subsection we will discuss this problem.

Let us recall the definition of K_1:

$$K_1 = \sum_{p,q\geq -1} c_p^\dagger G_{pq} b_q + \sum_{p,q\geq 2} b_{\bar p}^\dagger H_{\bar p \bar q} c_{\bar q} - 3c_2 b_{-1}, \tag{32}$$

where

$$\begin{aligned} G_{pq} &= (p-1)\delta_{p+1,q} + (p+1)\delta_{p-1,q}, \\ H_{\bar p \bar q} &= (\bar p + 2)\delta_{\bar p+1,\bar q} + (\bar p - 2)\delta_{\bar p-1,\bar q}, \end{aligned} \tag{33}$$

G is a square long-legged matrix, and H a square short-legged one. In the common overlap we have $G = H^T$. We notice immediately that K_1 annihilates the vacuum

$$K_1|0\rangle = 0. \tag{34}$$

What is important for us is that the action of K_1 commutes with the matrices we want to diagonalize. Now let $T_n = \hat{C}S_n$, where S_n is the matrix of the squeezed state representing $|n\rangle$. We have seen that T_n can be either lame or square (ll). Since we want to diagonalize (28) we must consider the second alternative. But in order to arrive at square (ll) matrices, at the beginning of this section we introduced into the game the conjugate oscillators η_a, $\xi_a^\dagger$, $a = -1, 0, 1$. Therefore, to be consistent, they must appear also in the oscillator representation of K_1. This can be done as follows.

We write down K_1 as

$$K_1 = \sum_{p,q\geq -1} c_p^\dagger G_{pq} b_q + \sum_{p,q\geq -1} b_p^\dagger H_{pq} c_q, \tag{35}$$

where G and H have the same expression as before, but now also H is square long legged and $H = G^T$. What is important is that in the expression $b^\dagger H c$ we understand that $b_a^\dagger$ is replaced by $\xi_a^\dagger$ and c_a is replaced by η_a (for simplicity we dispense with writing the new K_1 explicitly). If we write

$$b(z) = \sum_{n\geq 2} b_n z^{-n-2} + \sum_{-1\leq a\leq 1} \xi_a^\dagger z^{-a-2} + \sum_{n\geq 2} b_n^\dagger z^{n-2}, \tag{36}$$

$$c(z) = \sum_{n\geq 2} c_n z^{n-1} + \sum_{-1\leq a\leq 1} \eta_a z^{-a+1} + \sum_{n\geq 2} c_n^\dagger z^{n+1}, \tag{37}$$

we find the expected conformal action of K_1 on these fields. For instance,

$$\begin{aligned} [K_1, b(z)] &= -\sum_n ((n-1)b_{n+1} + (n+1)b_{n-1})z^{-n-2} \\ &= (1+z^2)\partial b(z) + 4zb(z) \end{aligned}$$

after replacing back $b_a^\dagger$ for $\xi_a^\dagger$.

[2] It is worth recalling that our purpose in this paper is to complete the proof started in [1] of (1)

$$e^{-(n-2/2)(\mathcal{L}_0^{(g)} + \mathcal{L}_0^{(g)\dagger})}|0\rangle = \mathcal{N}_n e^{c^\dagger S_n b^\dagger}|0\rangle \equiv |n\rangle,$$

that is, that the LHS does represent the ghost wedge states. In this light the requirement that the wedge state $|n\rangle$ with $n = 2$ coincide with the vacuum state is natural.

On the basis of this discussion we expect therefore that

$$[G, T_n] = 0 \tag{38}$$

as square long-legged matrices. In particular we should find that G commutes with X. One can, however, show that this is not the case for the vertex explicitly constructed in Appendix A. Therefore that vertex has many good properties but not this one.

However we will show below it is very plausible that a three strings vertex that satisfies also (38) exists. Therefore in the sequel we imagine that we have done everything with this vertex and will try to justify its existence *a posteriori*.

III. THE DIAGONAL RECURSIVE RELATIONS FOR WEDGE STATES

So far we have worked, so to speak, on the RHS of Eq. (1). It is now time to make a comparison with the LHS. In [1] we showed that

$$e^{-(n-2/2)(\mathcal{L}_0^{(g)}+\mathcal{L}_0^{(g)\dagger})}|0\rangle = e^{\eta(t)}e^{c^\dagger\alpha(t)b^\dagger}|0\rangle, \tag{39}$$

where $t = (2-n)/2$,

$$\alpha(t) = A\frac{\sinh(\sqrt{(D^T)^2 - BA}t)}{\sqrt{(D^T)^2 - BA}\cosh(\sqrt{(D^T)^2 - BA}t) - D^T\sinh(\sqrt{(D^T)^2 - BA}t)} \tag{40}$$

and

$$\eta(t) = -\int_0^t dt' \mathrm{tr}(B\alpha(t')) \tag{41}$$

and A, B, D^T are matrices extracted from $\mathcal{L}_0^{(g)} + \mathcal{L}_0^{(g)\dagger}$. In particular, D^T as well as the combination $(D^T)^2 - BA$ are (ss) matrices, while A is lame (ls). The purpose of the paper was to show that the RHS of (40), multiplied by the twist matrix $\hat{C}$, does satisfy the recursion relations (28) and (29). This was achieved by diagonalizing the matrices $\tilde{A} = \hat{C}A$, D^T and $(D^T)^2 - BA$ on the weight 2 basis $V_n^{(2)}(\kappa)$, with $n = 2, 3, \ldots$. We concluded that if we are allowed to replace in (28) and (29) the matrices by their eigenvalues in such a basis, the recursion relations can be shown to be true. What remained to be proved was precisely the correctness of replacing in such formulas the matrices by their eigenvalues. We are now in the position to do it.

Let us examine first (40), multiplied from the left by the twist matrix. The RHS is the product of $\tilde{A}$ by a matrix which is diagonal in the weight 2 basis and is of type (ss). Therefore when we apply the latter to a vector $V_s^{(2)}$ with components $(V_2^{(2)}(\kappa), V_3^{(2)}(\kappa), \ldots)$, we obtain the same vector multiplied by the eigenvalue. When we next apply $\tilde{A}$ from the left to the resulting vector, things are a little bit more complicated because A is an (ls) matrix. The vector ensuing from the operation would seem to have three additional entries with $n = -1, 0, 1$, therefore making meaningless even the idea of eigenvalue and eigenvector. However, it was shown in [1] that

$$\sum_{q=2}^{\infty} \tilde{A}_{pq}V_q^{(2)}(\kappa) = \mathfrak{a}(\kappa)V_p^{(2)}(\kappa), \qquad p = 2, 3, \ldots, \tag{42}$$

$$\sum_{q=2}^{\infty} \tilde{A}_{aq}V_q^{(2)}(\kappa) = 0, \qquad a = -1, 0, 1 \tag{43}$$

(for a new proof of (43) see Appendix C); i.e., not only is the (ss) submatrix of $\tilde{A}$ diagonal in the weight 2 basis with eigenvalue $\mathfrak{a}(\kappa)$, but the potential additional vector elements vanish. This allows us to conclude that the same property is shared by the matrix $\alpha(t)$. That is, when applying $\alpha(t)$ to the weight 2 basis vector $V_s^{(2)}$ as above, we obtain the corresponding eigenvalue multiplying the same vector (without additional components): $\alpha(t)V_s^{(2)} = \alpha(\kappa, t)V_s^{(2)}$.

Now let us apply the above to (28). The latter is formulated in terms of long square matrices whose (ls) part has the form $\alpha(t)$. Now we can interpret (28) as an infinite series expansion, in which each term is a monomial of (possibly different) matrices whose (ls) part has the form $\alpha(t)$. Let us consider the weight 2 basis vector $V_s^{(2)}$ extended by adding three 0 components in position $-1, 0, 1$, and let us call it $V_l^{(2)}$. When we apply any of the above matrices to it, we get the same extended vector multiplied by the matrix eigenvalue: for instance, $\alpha(t)V_l^{(2)} = \alpha(\kappa, t)V_l^{(2)}$. Therefore we can repeat the operation as many times as needed for any monomial and obtain the same vector multiplied by the monomial in which each matrix is replaced by its eigenvalue. Resumming the series we obtain that the relation (28) applied to the weight 2 basis vector becomes a relation of the same form with the matrices replaced by the corresponding eigenvalues. But in [1] we checked that this relation for the eigenvalues is true. This, which is the main argument in [1] and the present paper, and is intended to lead to the proof of (1), has been laid out so far in a rather patchy way due to its complexity. For the sake of clarity it is worth reviewing it in full, even at the price of some repetitions.

Proof of the diagonal recursive relations for wedge states

We wish to show that the eigenvalues of the matrices S_n in (1) satisfy (28) and (29). This sentence has to be unambiguously understood. First we notice that $T_2 = 0$, which is consistent with $|2\rangle$ being identified with the vacuum $|0\rangle$ and $\mathcal{N}_2 = 1$. Now proving (28) means proving two things:

$$T_3 = X \tag{44}$$

and

$$T_{n+1} = T_3 \frac{1 - T_n}{1 - T_n T_3}. \tag{45}$$

This second equation is demonstrated by setting

$$T_n \equiv \hat{C} S_n = \tilde{\alpha}\left(-\frac{n-2}{2}\right) \tag{46}$$

and using $\tilde{\alpha}$ given by Eq. (40). This gives the explicit expression

$$T_n = -\tilde{A} \frac{\sinh(\sqrt{\Delta}\frac{n-2}{2})}{\sqrt{\Delta}\cosh(\sqrt{\Delta}\frac{n-2}{2}) + D^T \sinh(\sqrt{\Delta}\frac{n-2}{2})}, \tag{47}$$

where $\Delta = (D^T)^2 - BA$. On the basis of the remarks made at the beginning of this section, we replace everywhere the matrices by their eigenvalues

$$\sqrt{\Delta} = \frac{\pi|\kappa|}{2}, \qquad \tilde{A} = \frac{\kappa\pi}{2\sinh(\frac{\kappa\pi}{2})}, \qquad D^T = \frac{\kappa\pi}{2}\coth\left(\frac{\kappa\pi}{2}\right). \tag{48}$$

By inserting (46) into (45) one can see that the latter is satisfied (see Sec. 2.5 of [1] for details) if

$$D^T + \tilde{A} = \sqrt{\Delta}\coth\left(\frac{\kappa\pi}{2}\right).$$

This is immediately verified using (48).

Next, in order to prove (44), we recall that (45) can be solved by

$$T_n = \frac{T + (-T)^{n-1}}{1 - (-T)^n} \tag{49}$$

for some matrix T. This matrix is easily identified to be $T \equiv T_\infty$ (this makes sense because the absolute value (of the eigenvalue) of T turns out to be <1: $T = -e^{-(|\kappa|\pi/2)}$). But, from the defining Eq. (26), T represents the sliver [26,27,31]. Therefore it is related to X by Eq. (31) or by its inverse

$$X = \frac{T}{T^2 - T + 1}. \tag{50}$$

This is precisely (49) for $n = 3$. Therefore (44) is satisfied and in addition this tells us that the eigenvalue of X is

$$X = -\frac{1}{1 + 2\cosh(\frac{\kappa\pi}{2})}. \tag{51}$$

Since the recursive constraints propagates this identification to all the wedge states this completes our proof.[3]

Let us come to the normalization constants $\mathcal{N}_n$. They must satisfy a recursion relation

$$\mathcal{N}_n \mathcal{K} \det(1 - T_n X) = \mathcal{N}_{n+1}, \tag{53}$$

where $\mathcal{K}$ is some constant to be determined. We fix it by requiring that $\mathcal{N}_2 = 1$ so that the wedge state $|2\rangle$ coincides with the vacuum $|0\rangle$. We have

$$\eta_n = -\int_0^{t_n} dt \mathrm{tr}(B\alpha) = -\int_0^{t_n} dt \mathrm{tr}(\tilde{A}\tilde{\alpha}), \tag{54}$$

where the trace is over the weight 2 basis. Now identifying

$$\mathcal{N}_n = e^{\eta_n},$$

plugging in the relevant eigenvalues and proceeding as in Sec. (2.5) of [1] one can easily verify that (53) is satisfied.

This completes our proof that the squeezed states in the midterm of (1) have the same eigenvalue as the ghost wedge states in the oscillator formalism.

To complete this argument we must show that our choice of enlarging the Fock space at the beginning of this section is justified in the case of the wedge states. Since this requires the same type of arguments as in the previous subsection and is somewhat repetitious, we will account for it in Appendix B.

Finally, let us remark that without the commutativity property of the twisted Neumann coefficients matrices spelled out in Sec. II, it would be impossible to reproduce the results of [1] where the matrices A, B, C, D^T commute (in the appropriate way).

The results we have obtained in this section consolidate the result obtained in [1], however is not yet the end our proof of (1). In the next section we explain why.

IV. MATRIX RECONSTRUCTION FROM THE SPECTRUM

So far our argument has been carried out by replacing the matrices involved with their eigenvalues. It would seem that we are done with the proof of (1). However what we

[3] It might seem at first sight that Eq. (51) contradicts the well-known formula found by Gross and Jevicki [17,18,32],

$$X = -E\frac{M}{1 + 2M}E^{-1}, \tag{52}$$

which relates the twisted ghost Neumann coefficients matrix X with the corresponding matter matrix of Neumann coefficients M. If we naively diagonalize X and M on the matter basis of eigenvectors and use the result of [33], we obtain a value for X different from (51). However this is an "optical" effect: Eq. (52) is certainly true numerically, but X and M act on different spaces, therefore they are different operators. Each must be diagonalized in its own space. They cannot be diagonalized using the same basis. There is thus no room for contradiction between (51) and (52).

have to show is not only that the eigenvalues of the matrices featuring in the RHS of Eq. (39) coincide with the eigenvalues of the matrix S_n in (17), where S_n satisfies the recursion relation (28), but *that the matrices themselves coincide*. Now, in general, if one knows eigenvalues and eigenvectors of a matrix operator one can reconstruct the original matrix. This is true for the matter sector of (1), but in the ghost sector this is not the case. In the ghost sectors things are unfortunately more complicated due to the existence of zero modes. This section is devoted to explaining this additional complication.

So far our argument has consisted in applying the matrices involved such as $\tilde{A}$, D^T, and, in particular, $\alpha(t)$ to the weight 2 basis vector $V^{(2)}$. As shown in [1], the exponent $c^\dagger \alpha b^\dagger$ in (39) can be written as follows:

$$\begin{aligned} c^\dagger \alpha b^\dagger &= \sum_{n=-1,m=2} c_n^\dagger \alpha_{nm}(t) b_m^\dagger \\ &= \sum_{n=-1,m=2}^{\infty} \int d\kappa d\kappa' \tilde{c}^\dagger(\kappa) \tilde{V}_n^{(-1)}(\kappa) \tilde{\alpha}_{nm}(t) \\ &\quad \times \tilde{V}_m^{(2)}(\kappa') b^\dagger(\kappa') \\ &= \sum_{n=2}^{\infty} \int d\kappa d\kappa' \tilde{c}^\dagger(\kappa) \tilde{V}_n^{(-1)}(\kappa) \tilde{\alpha}(\kappa, t) \tilde{V}_n^{(2)}(\kappa') b^\dagger(\kappa') \\ &= \int d\kappa \tilde{c}^\dagger(\kappa) \tilde{\alpha}(\kappa, t) b^\dagger(\kappa), \end{aligned} \tag{55}$$

where we have introduced

$$\begin{aligned} (-1)^n c_n^\dagger &= \int d\kappa \tilde{c}^\dagger(\kappa) \tilde{V}_n^{(-1)}(\kappa), \\ b_n^\dagger &= \int d\kappa b^\dagger(\kappa) \tilde{V}_n^{(2)}(\kappa), \qquad n \geq 2. \end{aligned} \tag{56}$$

It is clear that if the LHS of (55) is exactly equal to the RHS, our proof is complete. However the question is: in (55) we went from left to right, i.e., from the LHS we derived the RHS. Can we go the other way? In other words, given that we know the eigenvalue of some matrix in the weight 2 basis (or, for that matter, in the weight -1 basis) can we reconstruct the original matrix? For instance, we notice that in the intermediate steps of (55) the summation over $n = -1, 0, 1$ has disappeared. The obvious question is: how can we reconstruct these modes when we run the argument from right to left?

To start with let us recall the definitions of the two bases. The non-normalized basis (weight 2 basis) is given by

$$f_\kappa^{(2)}(z) = \sum_{n=2} V_n^{(2)}(\kappa) z^{n-2} \tag{57}$$

in terms of the generating function

$$\begin{aligned} f_\kappa^{(2)}(z) &= \left(\frac{1}{1+z^2}\right)^2 e^{\kappa \arctan(z)} \\ &= 1 + \kappa z + \left(\frac{\kappa^2}{2} - 2\right) z^2 + \ldots \end{aligned} \tag{58}$$

Following [34,35] (see also Appendix B of [1]), we normalize the eigenfunctions as follows:

$$\tilde{V}_n^{(2)}(\kappa) = \sqrt{A_2(\kappa)} V_n^{(2)}(\kappa), \tag{59}$$

where

$$A_2(\kappa) = \frac{\kappa(\kappa^2+4)}{2\sinh(\frac{\pi\kappa}{2})}.$$

The non-normalized weight -1 basis is given by

$$f_\kappa^{(-1)}(z) = \sum_{n=-1} V_n^{(-1)}(\kappa) z^{n+1} \tag{60}$$

in terms of the generating function

$$\begin{aligned} f_\kappa^{(-1)}(z) &= (1+z^2) e^{\kappa \arctan(z)} \\ &= 1 + \kappa z + \left(\frac{\kappa^2}{2} + 1\right) z^2 + \ldots \end{aligned} \tag{61}$$

The normalized one is

$$\begin{aligned} \tilde{V}_n^{(-1)}(\kappa) &= \sqrt{A_{-1}(\kappa)} V_n^{(-1)}(\kappa), \\ \sqrt{A_{-1}(\kappa)} &= \mathcal{P}\frac{1}{\kappa}\frac{\sqrt{A_2(\kappa)}}{\kappa^2+4}, \end{aligned} \tag{62}$$

where $\mathcal{P}$ denotes the principal value. We reported in [1] the bi-orthogonality

$$\int_{-\infty}^{\infty} d\kappa \tilde{V}_n^{(-1)}(\kappa) \tilde{V}_m^{(2)}(\kappa) = \delta_{n,m}, \qquad n \geq 2 \tag{63}$$

and "bi-completeness" relation

$$\sum_{n=2}^{\infty} \tilde{V}_n^{(-1)}(\kappa) \tilde{V}_n^{(2)}(\kappa') = \delta(\kappa, \kappa') \tag{64}$$

taking them from [36]. These relations can be formally proved, but it is evident that they have to be handled with care. Let us recall again Eqs. (42) and (43), which turned out to be crucial in the previous sections, and let us do the following. We multiply (43) by $V_n^{(2)}(\kappa)$ and integrate over κ: we get $A_{an} = 0$ for $n \geq 2$, which is evidently false. On the other hand, Eq. (43) is correct (we present a new demonstration of it in Appendix C). Therefore it is apparent that in the above exercise we did something illegal. This can only be the exchange between the (infinite) summation and the integration over κ. We remark that the same kind of exchange occurs also in the intermediate steps of (55). We are therefore warned that in doing so we may lose some information. The question is: is there a way to repair the illegality we commit in this way and recover the full relevant information?

In mathematical terms this involves the problem of the spectral representation for lame operators. Unfortunately we have not been able to find any treatment of this problem in the mathematical literature. We proceed therefore in a heuristic way.

A. The problem

Let us analyze the reconstruction of the matrix $\tilde{A}$. Since $\sum_{l=2}^{\infty}\tilde{A}_{nl}\tilde{V}_l^{(2)}(\kappa)=\mathfrak{a}(\kappa)\tilde{V}_n^{(2)}(\kappa)$, we might argue as follows:

$$\int_{-\infty}^{\infty}d\kappa\tilde{V}_m^{(-1)}(\kappa)\mathfrak{a}(\kappa)\tilde{V}_n^{(2)}(\kappa)=\sum_{l=2}^{\infty}\tilde{A}_{nl}\int_{-\infty}^{\infty}\tilde{V}_m^{(-1)}(\kappa)V_l^{(2)}(\kappa)=\tilde{A}_{nm} \quad (65)$$

using the bi-orthogonality relations (63). Therefore we should be able to reconstruct the $\tilde{A}$ matrix starting from

$$\mathfrak{a}(\kappa)=\frac{\pi\kappa}{2}\frac{1}{\sinh(\frac{\pi\kappa}{2})}$$

and the bases. Here are the first few basis elements

$$V_2^{(2)}(\kappa)=1,\qquad V_3^{(2)}(\kappa)=\kappa,$$
$$V_4^{(2)}(\kappa)=\frac{\kappa^2-4}{2},\qquad V_5^{(2)}(\kappa)=\frac{1}{6}\kappa(\kappa^2-14), \quad (66)$$
$$V_6^{(2)}(\kappa)=\frac{1}{24}(\kappa^4-32\kappa^2+72),\qquad \ldots$$

and

$$V_2^{(-1)}(\kappa)=\frac{1}{6}\kappa(\kappa^2+4),$$
$$V_3^{(-1)}(\kappa)=\frac{\kappa^2(\kappa^2+4)}{24},$$
$$V_4^{(-1)}(\kappa)=\frac{\kappa}{120}(\kappa^4-16), \quad (67)$$
$$V_5^{(-1)}(\kappa)=\frac{\kappa^2}{720}(\kappa^4-10\kappa^2-56),$$
$$V_6^{(-1)}(\kappa)=\frac{\kappa}{5040}(\kappa^6-28\kappa^4-56\kappa^2+288),\qquad \ldots$$

These must be multiplied by the normalization factor

$$\sqrt{A_2(\kappa)}=\sqrt{\frac{\kappa(\kappa^2+4)}{2\sinh(\frac{\pi\kappa}{2})}}$$

and

$$\sqrt{A_{-1}(\kappa)}=\mathcal{P}\frac{1}{\kappa}\sqrt{\frac{\kappa}{2(\kappa^2+4)\sinh(\frac{\pi\kappa}{2})}}$$

so that

$$\sqrt{A_2(\kappa)A_{-1}(\kappa)}=\frac{1}{2\sinh(\frac{\pi\kappa}{2})}.$$

Using these formulas we get

$$\tilde{A}_{22}=\frac{\pi}{24}\int_{-\infty}^{\infty}d\kappa\frac{\kappa^2(\kappa^2+4)}{(\sinh(\frac{\pi\kappa}{2}))^2}\approx 0.5332, \quad (68)$$

$$\tilde{A}_{24}=\frac{\pi}{480}\int_{-\infty}^{\infty}d\kappa\frac{\kappa^2(\kappa^4-16)}{(\sinh(\frac{\pi\kappa}{2}))^2}\approx -0.0762, \quad (69)$$

$$\tilde{A}_{33}=\frac{\pi}{96}\int_{-\infty}^{\infty}d\kappa\frac{\kappa^4(\kappa^2+4)}{(\sinh(\frac{\pi\kappa}{2}))^2}\approx 0.1524, \quad (70)$$

$$\tilde{A}_{35}=\frac{\pi}{2880}\int_{-\infty}^{\infty}d\kappa\frac{\kappa^4(\kappa^2-14)(\kappa^2+4)}{(\sinh(\frac{\pi\kappa}{2}))^2}\approx -0.0508. \quad (71)$$

From the definition of the matrix A_{nm} one should get instead

$$\tilde{A}_{22}=-\tfrac{12}{15}=-0.8,\qquad \tilde{A}_{24}=\tfrac{16}{35}\approx 0.457$$
$$\tilde{A}_{33}=-\tfrac{18}{35}\approx -0.514,\qquad \tilde{A}_{35}=\tfrac{22}{63}\approx 0.349. \quad (72)$$

As we can see, the reconstructed matrix elements are far apart from the expected values.

Let us do the same for D^T:

$$D_{22}^T=\frac{\pi}{24}\int_{-\infty}^{\infty}d\kappa\,\kappa^2(\kappa^2+4)\frac{\cosh(\frac{\pi\kappa}{2})}{(\sinh(\frac{\pi\kappa}{2}))^2}\approx 2.665, \quad (73)$$

$$D_{24}^T=\frac{\pi}{480}\int_{-\infty}^{\infty}d\kappa(\kappa^2(\kappa^4-16))\frac{\cosh(\frac{\pi\kappa}{2})}{(\sinh(\frac{\pi\kappa}{2}))^2}\approx 0.5333, \quad (74)$$

$$D_{33}^T=\frac{\pi}{96}\int_{-\infty}^{\infty}d\kappa(\kappa^4(\kappa^2+4))\frac{\cosh(\frac{\pi\kappa}{2})}{(\sinh(\frac{\pi\kappa}{2}))^2}\approx 5.3333, \quad (75)$$

$$D_{35}^T=\frac{\pi}{2880}\int_{-\infty}^{\infty}d\kappa(\kappa^4(\kappa^2-14)(\kappa^2+4))\frac{\cosh(\frac{\pi\kappa}{2})}{(\sinh(\frac{\pi\kappa}{2}))^2}\approx 1.0667. \quad (76)$$

We should have instead

$$D_{22}=4,\qquad D_{42}=0,\qquad D_{33}=6,\qquad D_{53}=\tfrac{2}{3}. \quad (77)$$

Also here we are far apart from the true values.

We remark that if we do the same exercise for the A and C matrices of the matter sector (see [1]), we find a perfect coincidence between the original matrices and the ones reconstructed by means of the spectrum.

B. The solution

The idea is to apply the matrix $\tilde{A}$ not to the weight 2 basis, but to the weight -1 basis, i.e.,

$$\sum_{l=-1}^{\infty} \tilde{V}_l^{(-1)}(\kappa)\tilde{A}_{l\bar{m}} = \mathfrak{a}(\kappa)\tilde{V}_{\bar{m}}^{(-1)}. \tag{78}$$

This formula was proved in Appendix D3 of [1]. Then

$$\begin{aligned}\int_{-\infty}^{\infty} d\kappa \tilde{V}_{\bar{m}}^{(-1)}(\kappa)\mathfrak{a}(\kappa)\tilde{V}_{\bar{n}}^{(2)}(\kappa) &= \int_{-\infty}^{\infty} d\kappa \sum_{l=-1}^{\infty} V_l^{(-1)}(\kappa)\tilde{A}_{l\bar{m}}\tilde{V}_{\bar{n}}^{(2)}(\kappa) \\ &= \sum_{a=-1,0,1} \tilde{A}_{a\bar{m}} \int_{-\infty}^{\infty} d\kappa \tilde{V}_a^{(-1)}(\kappa)\tilde{V}_n^{(2)}(\kappa) \\ &\quad + \sum_{l=2}^{\infty} \tilde{A}_{l\bar{m}} \int_{-\infty}^{\infty} d\kappa \tilde{V}_l^{(-1)}(\kappa)\tilde{V}_{\bar{n}}^{(2)}(\kappa),\end{aligned} \tag{79}$$

where barred indices denote "short" indices, i.e., $\bar{m}, \bar{n} \geq 2$. Now use the decomposition (see [36] and Appendix B of [1])

$$\tilde{V}_a^{(-1)}(\kappa) = \sum_{n=2}^{\infty} b_{a\bar{n}} \tilde{V}_{\bar{n}}^{(-1)}(\kappa). \tag{80}$$

One can easily obtain

$$b_{-1,2n+3} = (-1)^n(n+1), \qquad b_{0,2n+2} = (-1)^n, \qquad b_{1,2n+3} = (-1)^n(n+2). \tag{81}$$

Inserting these into (79) we get

$$\tilde{A}_{\bar{n}\bar{m}} = \int_{-\infty}^{\infty} d\kappa \tilde{V}_{\bar{m}}^{(-1)}(\kappa)\mathfrak{a}(\kappa)\tilde{V}_{\bar{n}}^{(2)}(\kappa) - \sum_{a=-1,0,1} b_{a\bar{n}}\tilde{A}_{a\bar{m}}. \tag{82}$$

Now the corrections to the values obtained in (68)–(71) are easy to compute. For instance

$$\tilde{A}_{24} = -0.076 - \tilde{A}_{04}b_{02} = -0.076 + \tfrac{8}{15} \approx 0.457, \tag{83}$$

$$\begin{aligned}\tilde{A}_{33} &= 0.1524 - \tilde{A}_{-1,3}b_{-1,3} - \tilde{A}_{1,3}b_{1,3} = 0.1524 - \tfrac{2}{3} \\ &= -0.5142,\end{aligned} \tag{84}$$

and so on.

As for B the answer is easy since $B_{\bar{n}\bar{m}} = A_{\bar{n}\bar{m}}$. Notice that the terms $B_{\bar{n}a}$ are different from $A_{a\bar{n}}$. These terms should also be considered as known terms.

We can reconstruct in a similar way also D^T. For this we must apply C to the -1 basis. This amounts to the same formulas above, with the substitution of $\tilde{A}$ with C and $\mathfrak{a}(\kappa)$ with $\mathfrak{c}(\kappa)$. Remember that $C_{\bar{n}\bar{m}} = D^T_{\bar{n}\bar{m}}$ for $\bar{n}, \bar{m} \geq 2$. In particular

$$\begin{aligned}D^T_{\bar{n}\bar{m}} &= C_{\bar{n}\bar{m}} \\ &= \int_{-\infty}^{\infty} d\kappa \tilde{V}_{\bar{m}}^{(-1)}(\kappa)\mathfrak{c}(\kappa)\tilde{V}_{\bar{n}}^{(2)}(\kappa) - \sum_{a=-1,0,1} b_{a\bar{n}}C_{a\bar{m}}.\end{aligned} \tag{85}$$

For instance,

$$D^T_{22} = C_{22} = 2.6665 - C_{02}b_{02} = 2.6665 + 2\tfrac{2}{3} \approx 4, \tag{86}$$

$$\begin{aligned}D^T_{33} &= C_{33} = 5.3333 - (C_{-1,3}b_{-1,3} + C_{1,3}b_{1,3}) \\ &= 5.3333 - \tfrac{2}{3} + 2\tfrac{2}{3} \approx 6,\end{aligned} \tag{87}$$

$$\begin{aligned}D^T_{35} &= C_{35} = 1.0667 - (C_{-1,3}b_{-1,3} + C_{1,3}b_{1,3}) \\ &= 1.0667 - \tfrac{2}{5} \approx 0.6667 = \tfrac{2}{3},\end{aligned} \tag{88}$$

and so on. $\tilde{A}_{a,\bar{n}}$, $\tilde{B}_{\bar{n},a}$, and $C_{a,\bar{n}}$ and $C_{\bar{n},a}$ will be referred to from now on as *boundary terms*. Notice that $A_{a,n} = -C_{-a,n}$ and $C_{\bar{n},-a} = B_{\bar{n},a}$.

In the absence of a mathematical theorem we formulate the following: Heuristic rule. In order to reconstruct any matrix $A_{\bar{n}\bar{m}} = B_{\bar{n}\bar{m}}$ and $C_{\bar{n}\bar{m}} = D^T_{\bar{n}\bar{m}}$ from its eigenvalues, apply A and C to the weight -1 basis, separate the $\bar{n}, \bar{m} \geq 2$ part from the zero mode part and operate as in Eq. (79) above. As for the matrix elements $A_{a,n}$ and $B_{\bar{n},a}$ they cannot be reconstructed from the eigenvalues, but they have to be rather considered as known terms of the problem. We will refer to them as boundary data.

Usually (for instance in the matter sector) we start from a matrix (for instance the matrices $\tilde{A}$ or C of [1]), diagonalize it, and determine the spectrum, i.e., eigenvalues and eigenvectors. Vice versa, starting from the latter, we can reconstruct the initial matrix using its spectral representation.

In the present case the situation is somewhat different. Given the matrices we can compute the spectrum (see Sec. 5 of [1]). Vice versa, given the spectrum *and* the *boundary data* $A_{a,n}$ and $B_{\bar{n},a}$ we can compute the matrices $\tilde{A}$ and C and the related ones. This also means that, in order to determine the eigenvalue of a given diagonalizable matrix over the weight 2 basis, the *ss* part of that matrix contains all the necessary information, but in order to reconstruct even its *ss* part we have to know the action of that matrix over the weight -1 basis, i.e., we need the information stored in the latter.

It is clear that, with the above heuristic rule, it is possible to reconstruct, at least in principle, any matrix which can be expressed as a series of products of $\tilde{A}$, B, C, D^T, in particular $\tilde{\alpha}(t)$. Unfortunately so far we have not been able to produce a simple, manageable reconstruction algorithm.

V. CONCLUSION

Let us return to the validity of (1) and reformulate the question raised at the beginning of the previous section. In [1] we wrote the LHS of this equation in the form (39). We have shown above that the RHS of the latter has the form of a wedge state, and in fact we proved that once the squeezed state matrix $\tilde{\alpha}(\kappa, \frac{2-n}{2}) \equiv \tilde{\alpha}_n$ there is diagonalized in the weight 2 basis, it coincides with the (diagonalized) matrix that represents the nth wedge state $|n\rangle$, defined by the squeezed state (19) whose matrix T_n satisfies the recursion relations (28).

The next question to be answered is: in view of the discussion of the previous section, do also the matrices $\tilde{\alpha}_n$ coincide with the matrices T_n and, in particular, the matrix elements $(\tilde{\alpha}_n)_{a,m}$ with $(T_n)_{a,m}$ with $a = -1, 0, 1$? Remember that our reconstruction algorithm tells us that $(\tilde{\alpha}_n)_{a,\bar{m}}$ (beside the other matrix elements) is uniquely determined by the spectrum of $\tilde{\alpha}_n$ *and* by the boundary data. This is true, in particular, for $\tilde{\alpha}_3$, which was interpreted above as X. We therefore expect that $(\tilde{\alpha}_3)_{n,m}$ coincides with $X_{n,m}$. If this is so, solving (15) and (16), for $X^\pm$, *we can, in principle, reconstruct the three strings vertex from the* A, B, C, *and* D^T *matrices*. This vertex has precisely the features we have hypothesized in Secs. II and III, in particular, the commutativity of the twisted matrices of Neumann coefficients (otherwise they would not be simultaneously diagonalized).

However the reconstruction of $X^\pm$ is not on the same footing as the reconstruction of X (or T). For the latter, as we have seen, there exists a precise (though unwieldy) procedure to obtain it from the A, B, C, and D^T matrices. For $X^\pm$ instead we have to proceed on the basis of (15) and (16). To discuss this point let us introduce the following notation: for any matrix M we represent by M_{ee} the part of M with both even entries, M_{oo} the part with both odd entries, and accordingly M_{eo}, M_{oe} with obvious meaning. From [1] we know that all the matrices A, B, C, D have vanishing eo and oe parts. All matrices T_n, and, in particular, X will therefore share the same property. We expect instead that $X^\pm_{eo}$ and $X^\pm_{oe}$ be nonvanishing. Remember that $X^+ = \hat{C}X^-\hat{C}$. Therefore

$$X^+_{ee} = X^-_{ee}, \qquad X^+_{oo} = X^-_{oo}, \tag{89}$$

$$X^+_{eo} = -X^-_{eo}, \qquad X^+_{oe} = -X^-_{oe}. \tag{90}$$

Substituting these relations into (15) we find

$$X^+_{ee} = X^-_{ee} = \tfrac{1}{2}(1_{ee} - X_{ee}), \tag{91}$$

$$X^+_{oo} = X^-_{oo} = \tfrac{1}{2}(1_{oo} - X_{oo}). \tag{92}$$

Therefore both $X^\pm_{ee}$ and $X^\pm_{oo}$ are immediately derived from X. From (16) we get instead

$$X^\pm_{eo}X^\pm_{oo} = X^\pm_{ee}X^\pm_{eo}, \tag{93}$$

$$X^\pm_{oe}X^\pm_{ee} = X^\pm_{oo}X^\pm_{oe}, \tag{94}$$

and

$$X^\pm_{eo}X^\pm_{oe} = \tfrac{1}{4}(1_{ee} + 3X_{ee})(1_{ee} - X_{ee}) \tag{95}$$

and a parallel equation with o exchanged everywhere with e. This means that $X^\pm_{eo}$ and $X^\pm_{oe}$ are not determined algorithmically like T_n, but only by solving the quadratic equations (95) subject to the commutativity relations (93) and (94).

If the solution to such equations, as we expect, exists, this is a proof of the validity of (1). In fact the analysis in Sec. III was carried out under the hypothesis that a vertex, with the properties illustrated in Sec. II A *and* with the twisted matrices of Neumann coefficients commuting with K_1, existed. But we have just shown that such a vertex can be deduced (reconstructed) directly from the LHS of (1), in the sense we have just specified.

The information we have extracted from the reconstruction path taken up in this paper is not conclusive. The existence proof of the three strings vertex, as we have just seen, is not complete. On the other hand the missing part in the proof is rather marginal and, what is more important, our general characterization of the three strings vertex (Sec. II A) has not met any inconsistencies. This is reassuring in the prospect of coping with the task of explicitly constructing the three strings vertex endowed with the above properties.

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APPENDIX A: THE GHOST NEUMANN COEFFICIENTS

In this appendix we explicitly compute $\hat{V}^{rs}_{nm}$ and V^{rs}_{nm}. We use the definitions (4) and (5). The method is well known: we express the propagator $\langle\langle c(z)b(w)\rangle\rangle$ in two different ways, first as a CFT correlator and then in terms of $\hat{V}_3$ and we equate the two expressions after mapping them to the disk via the maps (10). However, this recipe leaves several uncertainties. We will fix them by requiring certain properties, in particular, cyclicity, consistency with the bpz operation, and commutativity of the twisted matrices of Neumann coefficients (the reason for the latter will become clear later on).

First we have to insert the three c zero modes. One way is to insert them at different points t_i and use the correlator (A1)

$$\langle\langle c(z)b(w)\rangle\rangle_{(t_1,t_2,t_3)} = \langle 0|c(z)b(w)c(t_1)c(t_2)c(t_3)|0\rangle = \frac{1}{z-w}\prod_{i=1}^{3}\frac{t_i-z}{t_i-w}(t_1-t_2)\times(t_1-t_3)(t_2-t_3). \tag{A1}$$

So we have to compare

$$\langle f_1\circ c(t_1)f_2\circ c(t_2)f_3\circ c(t_3)f_r\circ c^{(r)}(z)f_s\circ b^{(s)}(w)\rangle \tag{A2}$$

with

$$\langle\hat V_3|R(c^{(r)}(z)b^{(s)}(w))|\omega\rangle_{123}, \tag{A3}$$

where R denotes radial ordering. If :: denotes the natural normal ordering, we have, for instance,

$$R(c(z)b(w)) = \sum_{n,k}:c_nb_k:z^{-n+1}w^{-k-2}+\frac{1}{z-w}. \tag{A4}$$

This should be inserted inside (A3). Let us refer to the last term in (A4) as the *ordering term*. We notice that the choice we have made for this term is rather arbitrary. What precisely has to be inserted in (A3) depends also on the definition of the three strings vertex, therefore it should be decided on the basis of a consistent definition of the latter. For the time being we continue on the basis of (A4), later on we will introduce the necessary modifications.

To start with let us compute the $\mathcal{K}$ constant. We have

$$\langle\hat V_3|\omega\rangle_{123}=\mathcal{K}=\langle f_1\circ c(t_1)f_2\circ c(t_2)f_3\circ c(t_3)\rangle = \frac{(f_1(t_1)-f_2(t_2))(f_1(t_1)-f_3(t_3))(f_2(t_2)-f_3(t_3))}{f_1'(t_1)f_2'(t_2)f_3'(t_3)}. \tag{A5}$$

Now

$$\langle\hat V_3|R(c^{(r)}(z)b^{(s)}(w))|\omega\rangle_{123} = \langle\hat V_3|\sum_{n,k}:c_n^{(r)}b_k^{(s)}:z^{-n+1}w^{-k-2}+\frac{1}{z-w}|\omega\rangle_{123} = \mathcal{K}\Big(-\hat V_{kn}^{sr}z^{n+1}w^{k-2}+\frac{\delta^{rs}}{z-w}\Big). \tag{A6}$$

On the other hand, from direct computation,

$$\langle f_1\circ c(t_1)f_2\circ c(t_2)f_3\circ c(t_3)f_r\circ c^{(r)}(z)f_s\circ b^{(s)}(w)\rangle = \frac{(f_s'(w))^2}{f_r'(z)}\frac{1}{f_r(z)-f_s(w)}\times\frac{(f_1(t_1)-f_2(t_2))(f_1(t_1)-f_3(t_3))(f_2(t_2)-f_3(t_3))}{f_1'(t_1)f_2'(t_2)f_3'(t_3)}\cdot\prod_{i=1}^{3}\frac{f_i(t_i)-f_r(z)}{f_i(t_i)-f_s(w)}. \tag{A7}$$

Comparing the last two equations and using (A5) we get

$$\hat V_{kn}^{sr} = -\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+2}}\frac{1}{w^{k-1}}\cdot\Big(\frac{(f_s'(w))^2}{f_r'(z)}\times\frac{1}{f_r(z)-f_s(w)}\prod_{i=1}^{3}\frac{f_i(t_i)-f_r(z)}{f_i(t_i)-f_s(w)}-\frac{\delta^{rs}}{z-w}\Big). \tag{A8}$$

After obvious changes of indices and variables we end up with

$$\hat V_{nm}^{rs} = \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n-1}}\frac{1}{w^{m+2}}\cdot\Big(\frac{(f_r'(z))^2}{(f_s'(w))}\times\frac{1}{f_r(z)-f_s(w)}\prod_{i=1}^{3}\frac{f_s(w)-f_i(t_i)}{f_r(z)-f_i(t_i)}-\frac{\delta^{rs}}{z-w}\Big). \tag{A9}$$

Now we make a definite choice for the insertions, that is we take $t_i\to\infty$. We remark that this choice leads to simple formulas but remains anyhow arbitrary.[4]

Since $f_i(\infty)=\alpha^{-i}$ we get

$$\prod_{i=1}^{3}\frac{f_i(t_i)-f_s(w)}{f_i(t_i)-f_r(z)} = \frac{f(w)^3-1}{f(z)^3-1}. \tag{A10}$$

It is straightforward to check cyclicity

$$\hat V_{nm}^{rs} = \hat V_{nm}^{r+1,s+1}. \tag{A11}$$

Moreover (by letting $z\to -z$, $w\to -w$)

$$\hat V_{nm}^{rs} = (-1)^{n+m}\hat V_{nm}^{sr}. \tag{A12}$$

Now let us consider the decomposition

$$\hat V_{nm}^{rs} = \tfrac{1}{3}(E_{nm}+\bar\alpha^{r-s}U_{nm}+\alpha^{r-s}\bar U_{nm}), \tag{A13}$$

where

$$E_{nm} = \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\mathcal{N}_{nm}(z,w)\mathcal{E}(z,w),\quad U_{nm} = \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\mathcal{N}_{nm}(z,w)\mathcal{U}(z,w),\quad \bar U_{nm} = \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\mathcal{N}_{nm}(z,w)\bar{\mathcal{U}}(z,w) \tag{A14}$$

and

$$\mathcal{E}(z,w) = \frac{3f(z)f(w)}{f^3(z)-f^3(w)},\quad \mathcal{U}(z,w) = \frac{3f^2(z)}{f^3(z)-f^3(w)},\quad \bar{\mathcal{U}}(z,w) = \frac{3f^2(w)}{f^3(z)-f^3(w)},$$

$$\mathcal{N}_{nm}(z,w) = \frac{1}{z^{n-1}}\frac{1}{w^{m+2}}(f'(z))^2(f'(w))^{-1}\frac{f^3(w)-1}{f^3(z)-1}.$$

[4] We recall that zero mode insertions can be introduced also by means of the operator $Y(t)=\frac{1}{2}\partial^2c(t)\partial c(t)c(t)$ instead of three different $c(t_i)$. This has not lead so far to better results.

After some elementary algebra, using $f'(z) = \frac{4i}{3}\frac{1}{1+z^2}f(z)$, one finds

$$\begin{aligned} E_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left(\frac{1}{1+zw} - \frac{w}{w-z} - \frac{z^2}{w}\frac{1}{z-w}\right) \\ &= (-1)^n\delta_{nm} - \delta_{n,0}\delta_{m,0} - \delta_{n,1}\delta_{m,-1}, \\ U_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left[\frac{f(z)}{f(w)}\left(\frac{1}{1+zw} - \frac{w}{w-z}\right) - \frac{z^2}{w}\frac{1}{z-w}\right], \\ \bar{U}_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left[\frac{f(w)}{f(z)}\left(\frac{1}{1+zw} - \frac{w}{w-z}\right) - \frac{z^2}{w}\frac{1}{z-w}\right]. \end{aligned} \tag{A15}$$

In this way the ambiguities are eliminated. The $\delta_{n,m}$ in (A15) is for $n, m \geq 0$.

In a similar way one can compute the dual vertex, the right one. One gets

$$\begin{aligned} {}_{123}\langle\omega|R(I\circ c(z)I\circ b(w))|V_3\rangle &= {}_{123}\langle\omega|\sum_{n,k}(-1)^{k+n+1}\colon c_n^{(r)}b_k^{(s)}\colon z^{n+1}w^{k-2} + \frac{z^3}{w^3}\frac{\delta^{rs}}{z-w}|V_3\rangle \\ &= \mathcal{K}\left(\sum_{n,k}V_{kn}^{sr}(-1)^{n+m+1}z^{n+1}w^{k-2} + \frac{z^3}{w^3}\frac{\delta^{rs}}{z-w}\right). \end{aligned} \tag{A16}$$

Equating now to (A2) and repeating the same procedure as above we finally obtain

$$(-1)^{n+m}V_{nm}^{rs} = \tfrac{1}{3}(E'_{nm} + \bar{\alpha}^{r-s}U'_{nm} + \alpha^{r-s}\bar{U}'_{nm}), \tag{A17}$$

where

$$\begin{aligned} E'_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left(\frac{1}{1+zw} - \frac{w}{w-z} - \frac{w^2}{z}\frac{1}{z-w}\right), \\ U'_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left[\frac{f(z)}{f(w)}\left(\frac{1}{1+zw} - \frac{w}{w-z}\right) - \frac{w^2}{z}\frac{1}{z-w}\right], \\ \bar{U}'_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left[\frac{f(w)}{f(z)}\left(\frac{1}{1+zw} - \frac{w}{w-z}\right) - \frac{w^2}{z}\frac{1}{z-w}\right]. \end{aligned} \tag{A18}$$

As we see, we have

$$(-1)^{n+m}V_{nm}^{rs} = \hat{V}_{nm}^{rs} \tag{A19}$$

except perhaps for the values of the labels both involving zero modes. That the relation (13) should hold for the full range of the labels is instead a basic requirement. We will use also this, besides cyclicity and commutativity, in order to guess the final form of the vertex.

Motivated by these requirements we introduce minor modifications in the previous definitions. We start from the basic (A15) without the last term (the ordering term)

$$E_{nm} = \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left(\frac{1}{1+zw} - \frac{w}{w-z}\right), \tag{A20}$$

$$U_{nm} = \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\frac{f(z)}{f(w)}\left(\frac{1}{1+zw} - \frac{w}{w-z}\right), \tag{A21}$$

$$\bar{U}_{nm} = \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\frac{f(w)}{f(z)}\left(\frac{1}{1+zw} - \frac{w}{w-z}\right). \tag{A22}$$

Then we define the ordering term

$$Z_{nm} = \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left(\frac{w}{w-z} - \frac{1}{zw}\right). \tag{A23}$$

Next we define the matrices

$$\mathcal{E} = E + Z, \qquad \mathcal{U} = U + Z, \qquad \bar{\mathcal{U}} = \bar{U} + Z, \tag{A24}$$

which will be our basic ingredients. The choice of Z is made in such a way that $\mathcal{E} = \hat{C}$. In fact

$$\mathcal{E}_{nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left(\frac{1}{1+zw} - \frac{1}{zw} \right) = (-1)^n \delta_{nm} \tag{A25}$$

for $n, m \geq -1$.

The double integrals in (A20)–(A22) are ambiguous in the range $-1 \leq n, m \leq 1$, [37]. However, after the addition of the ordering term (A23) all ambiguities disappear.

In conclusion, we define the three strings ghost vertex as follows. With reference to (4) and (5) we set

$$\hat{V}^{rs}_{nm} = \tfrac{1}{3}(\mathcal{E}_{nm} + \bar{\alpha}^{r-s}\mathcal{U}_{nm} + \alpha^{r-s}\bar{\mathcal{U}}_{nm}) \tag{A26}$$

and

$$V^{rs}_{nm} = (-1)^{n+m}\hat{V}^{rs}_{nm}, \qquad V^{rs} = \hat{C}\hat{V}^{rs}\hat{C}. \tag{A27}$$

From the definition of $\mathcal{U}$ and $\bar{\mathcal{U}}$ it is easy to verify that $\hat{C}\mathcal{U} = \bar{\mathcal{U}}\hat{C}$, where $\hat{C}$ denotes the twist matrix. We have seen above that $\mathcal{E} \equiv \hat{C}$.

Now using the method of [37] it is possible to show that $\mathcal{U}^2 = 1$ for $n, m \geq -1$. This implies that, besides

$$X + X^+ + X^- = 1,$$

where $X = \hat{C}V^{rr}$, $X^+ = \hat{C}V^{12}$, $X^- = \hat{C}V^{21}$, we have the commutativity property

$$X^{rs}X^{r's'} = X^{r's'}X^{rs}$$

and

$$X^+X^- = X^2 - X, \qquad X^2 + (X^+)^2 + (X^-)^2 = 1.$$

It should be stressed that all the X^{rs} matrices are (ll).

APPENDIX B: WHY WE CAN USE LONG SQUARE MATRICES

Let us return to Eqs. (23)–(25), applied to wedge states, that is let us suppose $S_1 = S_n$ and $S_2 = S_m$. Let us concentrate on Eq. (23): the RHS can be understood in terms of a series expansion in which each monomial is the product of alternating lame matrices X, $X^{\pm}$, T_n, T_m, the rightmost and leftmost ones being (ls). These matrices cannot be assumed to satisfy the identities of Sec. II A, in particular, they cannot be assumed to commute. However, let us apply any such monomial from the left to the above introduced weight 2 basis vector $V^{(2)}_s$:

$$\ldots Y_{sl}Z_{ls}V^{(2)}_s. \tag{B1}$$

Since the rightmost matrix Z in the monomial is (ls), whatever matrix it is it is obvious that we can simply replace it with corresponding long square matrices and replace $V^{(2)}_s$ by $V^{(2)}_l$. According to the discussion in Sec. III, the result of the application is the same extended vector multiplied by the matrix eigenvalue. This is obvious if the matrix in question is X, T_n, or T_m, as has been discussed above. If the rightmost matrix in the monomial is $X^{\pm}$ the same conclusion requires some comment. Since $X^{\pm}_{ls}$ can be trivially replaced by a long square matrice applied to $V^{(2)}_l$, we are entitled to apply to $X^{\pm}_{ll}$ the identities of subsection II A. Therefore, using a well-known result, $X^{\pm}_{ll}$ can be expressed in terms of X_{ll}, and the result of the application of $X^{\pm}_{ll}$ to $V^{(2)}_l$ is $V^{(2)}_l$ multiplied by the matrix eigenvalue.

The next to the rightmost matrix in the monomial we have picked up is of the type Y_{sl}. If Y is X, T_n, or T_m we can apply to it the argument used in Sec. III for $\alpha(t)$: we can replace them with Y_{ll} since, due to (43) and the consequences thereof, the initial three elements of $Y_{ll}V^{(2)}_l$ are zero. The result once again is the product of the eigenvalues of Y_{ll} and of Z_{ll} multiplying $V^{(2)}_l$.

If, on the other hand, Y_{sl} is $X^{\pm}_{sl}$, we can argue as follows. The result of replacing $X^{\pm}_{sl}$ by $X^{\pm}_{ll}$ in front of $V^{(2)}_l$ is a vector with three more entries (corresponding as always to $n = -1, 0, 1$, if n is the left label of $X^{\pm}$). However, we can use the same argument as above, remarking that $X^{\pm}_{ll}$ can be expressed in terms of X_{ll}. Therefore, due to (43) and its consequences, we can conclude that these three additional entries are 0. Therefore writing $X^{\pm}_{sl}V^{(2)}_l$ is tantamount to writing $X^{\pm}_{ll}V^{(2)}_l$, and the result is once again $V^{(2)}_l$ multiplied by the product of the eigenvalues of Y_{ll} and of Z_{ll}.

From this point on the argument is recursive and there is no need to repeat it again. Resumming the series we can conclude that in Eq. (23) we can everywhere replace the matrices by the corresponding long square ones. Analogous things can be repeated for Eq. (25). This is our justification for enlarging the Fock space at the beginning of this section.

APPENDIX C: PROOF OF EQ. (3.5)

We want to show that

$$X = \sum_{n=2}^{\infty} \tilde{A}_{-1,n}V^{(2)}_n(\kappa) = 0. \tag{C1}$$

Set $n = 2l + 1$. Then

$$\tilde{A}_{-1,2l+1} = \frac{2(-1)^l}{2l+1}$$

are the only nonvanishing matrix elements. Define

$$F(z) = \sum_{l=1}^{\infty} \frac{2(-1)^l}{2l+1} V^{(2)}_{2l+1} z^{2l+1} \tag{C2}$$

so that $X = F(1)$ and $F(0) = 0$. We get

$$\frac{dF}{dz} = \sum_{l=1}^{\infty} 2(-1)^l V_{2l+1}^{(2)} z^{2l} = iz(f_k^{(2)}(iz) - f_k^{(2)}(-iz))$$
$$= \frac{iz}{(1-z^2)^2}\left(\left(\frac{1+z}{1-z}\right)^{\xi} - \left(\frac{1+z}{1-z}\right)^{-\xi}\right),$$
$$\xi = \frac{i\kappa}{2}.$$

Therefore

$$F(1) = \int_0^1 dz(iz)((1+z)^{\xi-2}(1-z)^{-\xi-2} - (1+z)^{-\xi-2}(1-z)^{\xi-2})$$
$$= \frac{i}{\xi(1+\xi)} F(2-\xi, 2, 1-\xi, -1) + \frac{i}{\xi(1-\xi)} F(2+\xi, 2, 1+\xi, -1).$$

Using Eq. (C.2) one gets

$$F(2-\xi, 2, 1-\xi, -1) = \frac{1}{4}\frac{\xi}{\xi-1}.$$

Therefore one can easily show that

$$\frac{i}{\xi(1+\xi)} F(2-\xi, 2, 1-\xi, -1) = -\frac{i}{4(1-\xi^2)},$$
$$\frac{i}{\xi(1-\xi)} F(2+\xi, 2, 1+\xi, -1) = \frac{i}{4(1-\xi^2)},$$

and $F(1) = 0$.

Next we want to show

$$Y = \sum_{n=2}^{\infty} \tilde{A}_{0,n} V_n^{(2)}(k) = 0. \tag{C3}$$

This time put $n = 2l$

$$\tilde{A}_{0,2l} = (-1)^{l+1}\left(\frac{1}{2l+1} + \frac{1}{2l-1}\right).$$

Define

$$F(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2l+1} V_{2l}^{(2)} z^{2l+1}, \tag{C4}$$

$$G(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2l-1} V_{2l}^{(2)} z^{2l-1}, \tag{C5}$$

so that $F(1) + G(1) = Y$ and $F(0) + G(0) = 0$. We get

$$\frac{dF}{dz} = \sum_{l=1}^{\infty} (-1)^{l+1} V_{2l}^{(2)} z^{2l} = \frac{z^2}{2}(f_k^{(2)}(iz) + f_k^{(2)}(-iz))$$
$$= \frac{z^2}{2(1-z^2)^2}\left(\left(\frac{1+z}{1-z}\right)^{\xi} + \left(\frac{1+z}{1-z}\right)^{-\xi}\right)$$

and

$$\frac{dG}{dz} = \sum_{l=1}^{\infty} (-1)^{l+1} V_{2l}^{(2)} z^{2l-2} = \frac{1}{2}(f_k^{(2)}(iz) + f_k^{(2)}(-iz))$$
$$= \frac{1}{2(1-z^2)^2}\left(\left(\frac{1+z}{1-z}\right)^{\xi} + \left(\frac{1+z}{1-z}\right)^{-\xi}\right),$$

which give

$$F(1) = \int_0^1 dz \frac{z^2}{2}((1+z)^{\xi-2}(1-z)^{-\xi-2} + (1+z)^{-\xi-2}(1-z)^{\xi-2})$$
$$= \frac{1}{\xi(1+\xi)(1-\xi)} F(2-\xi, 3, 2-\xi, -1) - \frac{1}{\xi(1-\xi)(1+\xi)} F(2+\xi, 3, 2+\xi, -1) = 0 \tag{C6}$$

and

$$G(1) = \int_0^1 dz \frac{1}{2}((1+z)^{\xi-2}(1-z)^{-\xi-2} + (1+z)^{-\xi-2}(1-z)^{\xi-2})$$
$$= -\frac{1}{(1+\xi)} F(2-\xi, 1, -\xi, -1) - \frac{1}{(1-\xi)} F(2+\xi, 1, \xi, -1) = 0. \tag{C7}$$

These identities can be obtained by means of well-known relations valid for the hypergeometric functions, such as those in Appendix C of [1].

Lastly, we want to show that

$$Z = \sum_{n=2}^{\infty} \tilde{A}_{1,n} V_n^{(2)}(k) = 0. \tag{C8}$$

Setting $n = 2l + 1$ one realizes that

$$\tilde{A}_{1,2l+1} = \frac{2(-1)^{l+1}}{2l+1} = -\tilde{A}_{-1,2l+1}. \tag{C9}$$

So $Z = -X = 0$.

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