# Matrix model maps and reconstruction of AdS supergravity interactions

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We consider the question of reconstructing (cubic) SUGRA interactions in AdS/CFT. The method we introduce is based on the matrix model maps (MMP) which were previously successfully employed at the linearized level. The strategy is to start with the map for $1/2$ BPS configurations, which is exactly known (to all orders) in the Hamiltonian framework. We then use the extension of the matrix model map with the corresponding Ward identities to completely specify the interaction. A central point in this construction is the nonvanishing of off-shell interactions (even for highest-weight states).

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## I. INTRODUCTION

The question of reconstructing bulk supergravity (SUGRA) through the AdS/CFT [1,2] correspondence is of considerable interest. Initially much insight into the correspondence was gained through the GKP-W holographic relation which states [3,4] that correlators of Yang-Mills theory coincide with certain boundary to boundary amplitudes in supergravity. Indeed this was the scheme which provided some of the initial prescriptions for relating cubic supergravity interactions to gauge theory correlators [5–14]. The holographic relation, however, has elements of an $S$-matrix relation, and one can ask what set of correlators contains all the information for reconstructing the theory in the bulk. Although some studies [15] have been done along this direction, there are still some main questions left open. The issue/problem seems to be analogous to the question of reconstructing the off-shell theory from strictly on-shell data, a problem which is usually plagued by nonuniqueness. In addition there is the question of unitarity, namely, the issue of securing a unitary and local evolution of the bulk theory. An alternative is to develop the construction directly in the Hamiltonian framework, a method we consider in the present work. The basic building block of our construction will be the matrix model representation that was developed in the last few years beginning with the $1/2$ BPS sector of the theory. This approach came for studies of $1/2$ BPS correlators in gauge theory [16,17] and the dual exact configurations in SUGRA [18,19]. What emerged is a fermion droplet correspondence (see [20] and references therein). Its Hamiltonian version given through collective field theory [21,22] can serve as a starting point for reconstructing the full theory. Specifically, the strategy that we develop for the construction of the bulk interaction is then as follows: starting from the nonzero $c = 1$ collective field theory vertex we proceed with the action of raising operators to establish Ward identities that, as we argue, are capable of determining the full cubic vertex. The form of the raising and lowering operators can be deduced through the matrix model map (MMP) formulated in [23]. The map of [23] was given at the linearized level, and was shown to provide a mapping from eigenfuctions of matrix model equations to those of AdS. As such our work represents an extension to the nonlinear level of the mapping introduced in [23]. The outline of this paper is as follows. In Sec. II we discuss the form of cubic interactions in supergravity as well as for the $1/2$ BPS collective field theory. Here we also discuss and resolve issues that concern the comparison of the (vanishing) SUGRA vertex for the $1/2$ BPS sector with the (nonvanishing) matrix model vertex. In Sec. III we review the linearized MMP of [23] in terms of canonical transformations on phase space. This version turns out to be useful for the nonlinear extension that we give in Sec. IV, where we consider a simplified limit. Finally, in Sec. V we discuss the Ward identities and their ability to determine the vertex (from the initial highest-weight one). Throughout this paper we restrict our analysis to the $AdS_2$ case, where the method can be presented in the simplest possible way.

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## II. CUBIC INTERACTIONS IN THE 1/2 BPS SECTOR

Our starting point is the direct Hamiltonian level map that was accomplished in the (limited) 1/2 BPS sector of the theory. On the Yang-Mills side one has a (reduced) matrix model Hamiltonian established in [16] and its collective field theory Hamiltonian description. This is fully reproduced on the gravity side with the 1/2 BPS reduction of 10D SUGRA in [18]. In particular, the energy of the 10D geometries of the 1/2 BPS sector was shown [18] to be given by

$$E = \int dx_1 \int dx_2 (x_1^2 + x_2^2) u(x_1, x_2), \tag{1}$$

where $u(x_1, x_2)$ is a density function distinguishing between space-time regions having different boundary conditions (“black” and “white” regions). The expression (1) is recognizable as the energy of fermions (corresponding to matrix eigenvalues) in a harmonic oscillator potential. In this language, $u$ is responsible for differentiating between particles (*fermion droplets*) and holes. After identifying $x_1 = x$ and $x_2 = y$, and after performing the $x_2$ integration over a black region (fermion droplet), the energy (1) can be shown to be equivalent to the collective field theory Hamiltonian [24]

$$H = \int dx \left( \frac{y_+^3}{3} - \frac{y_-^3}{3} + x^2 (y_+ - y_-) \right) \tag{2}$$

of a one-matrix model described by

$$H = \tfrac{1}{2} \mathrm{Tr}(X^2 + P^2). \tag{3}$$

The Hermitian $N \times N$ matrix $X(t)$ depends only on time, and $P(t) = \dot{X}(t)$ denotes its conjugate momentum. The functions $y_+$ and $y_-$ describe the upper and lower profiles of the Fermi droplet. Furthermore, the matrix Hamiltonian is related to (2) via $X = x$ and $P = y$. It is important to emphasize that the collective field formalism describes well the *fully interacting* theory of chiral primaries on $\mathrm{AdS}_5 \times S^5$. To show this explicitly, we examine next the form of the cubic vertex as given by collective field theory. The dynamics of the resulting collective field theory can be directly induced from the much simpler dynamics of the one-dimensional matrix $X(t)$ (with eigenvalues $\lambda_i$), after a change to the density field obeying the following cubic collective Hamiltonian:

$$H_{\mathrm{coll}} = \int dx \left( \frac{1}{2} \partial_x \Pi \phi \partial_x \Pi + \frac{\pi^2}{6} \phi^3 + \frac{1}{2}(x^2 - \mu)\phi \right). \tag{4}$$

The static ground state equation yields the background value $\phi_0$ for the field $\phi$. One can then introduce small fluctuations about the background, letting

$$\phi(x, t) = \phi_0(x) + \frac{1}{\sqrt{\pi}} \partial_x \eta(x, t). \tag{5}$$

After expanding the Hamiltonian one finds

$$H = \int dx \bigg[ \pi \phi_0 \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_x \eta)^2 \right) + \frac{\pi^2}{6} (\partial_x \eta)^3 + \frac{\pi}{2} \Pi \partial_x \eta \Pi \bigg]. \tag{6}$$

Note that the corresponding quadratic Lagrangian takes the form

$$L_2 = \int dt \int dx \frac{1}{2} \left[ \frac{\dot{\eta}^2}{\pi \phi_0} - \pi \phi_0 \eta_{,x}^2 \right], \tag{7}$$

describing a massless particle in a gravitational background with metric

$$g_{\mu\nu}^0 = \left( \frac{1}{\pi \phi_0}, \pi \phi_0 \right). \tag{8}$$

The metric can be removed by an appropriate coordinate transformation. In terms of the “time of flight” coordinate $\tau$, the Hamiltonian then becomes

$$H = \int d\tau \bigg[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_\tau \eta)^2 + \frac{1}{6\pi^2 \phi_0^2} ((\partial_\tau \eta)^3 + 3 \Pi \partial_\tau \eta \Pi) \bigg]. \tag{9}$$

Furthermore, notice that this is the theory of a massless boson with a spatially dependent coupling, $g_{\mathrm{string}}(\tau) = \frac{1}{\pi^2 \phi_0^2}$. Since we are interested in studying the (cubic) interaction terms, let us concentrate on

$$H^{(3)} = \int d\tau \frac{1}{6\pi^2 \phi_0^2} ((\partial_\tau \eta)^3 + 3 \Pi \partial_\tau \eta \Pi). \tag{10}$$

If we recall how the (standard) fields $\alpha_\pm$ were introduced, $\alpha_\pm(x, t) = \partial_x \Pi \pm \pi \phi(x, t)$, it is clear that they could have been expanded about the background in a similar way:

$$\alpha_\pm = \pm \pi \phi_0 + \tilde{\alpha}_\pm. \tag{11}$$

The cubic Hamiltonian takes on a much simpler form in terms of $\tilde{\alpha}_\pm$:

$$H^{(3)} = \int_0^\pi \frac{d\tau}{\phi_0^2} (\tilde{\alpha}_+^3(\tau) - \tilde{\alpha}_-^3(\tau)). \tag{12}$$

This can be simplified even further by combining $\tilde{\alpha}_\pm$ into a single field $\alpha(\tau)$ in the following way:

$$\alpha(\tau) = \tilde{\alpha}_+(\tau) \quad \text{for } \tau > 0, = -\tilde{\alpha}_-(\tau) \quad \text{for } \tau < 0, \tag{13}$$

where we must now take $-\pi < \tau < \pi$. Finally, expressing the cubic Hamiltonian in terms of the new field $\alpha$, we find

$$H^{(3)} = \int_{-\pi}^{\pi} \frac{d\tau}{\phi_0^2} \alpha^3(\tau). \tag{14}$$

Expanding $\alpha$ into creation and annihilation operators gives

$$\alpha(\tau) = \sum_n \sqrt{n}(e^{in\tau}a_n + e^{-in\tau}a_n^\dagger). \tag{15}$$

Rewriting (14) in terms of creation and annihilation operators we find several terms, but we would like to restrict our attention to the one containing $a_1 a_2 a_3^\dagger$:

$$\begin{aligned} H^{(3)} &= \sqrt{n_1 n_2 n_3}\int_{-\pi}^{\pi}\frac{d\tau}{\sin^2\tau}e^{i(n_1+n_2-n_3)\tau}a_1 a_2 a_3^\dagger + \dots \\ &= -\sqrt{n_1 n_2 n_3}\int\left(\frac{d}{d\tau}\cot\tau\right)e^{i(n_1+n_2-n_3)\tau}a_1 a_2 a_3^\dagger + \dots \\ &= i(n_1+n_2-n_3) \\ &\quad\times\sqrt{n_1 n_2 n_3}\int d\tau\cot\tau e^{i(n_1+n_2-n_3)\tau}a_1 a_2 a_3^\dagger + \dots \end{aligned}$$

where we have implicitly used the fact that the boundary term cancels. Introducing $z = e^{i\tau}$ and letting $n \equiv n_1 + n_2 - n_3$, the integral above becomes

$$I = \int_{-\pi}^{\pi} d\tau\cot\tau e^{in\tau} = \int dz\frac{z^{n-1}}{(z-1)(z+1)}(z^2+1), \tag{16}$$

which has simple poles at $z_k = 0, \pm 1$. Evaluating the integral we find that the only nonzero contribution from the residues occurs for $n > 0$ and even, and is given by $\sum_k \mathrm{Res}(f, z_k) = 2$, yielding

$$H^{(3)} = -4\pi\sqrt{n_1 n_2 n_3}(n_1 + n_2 - n_3)a_1 a_2 a_3^\dagger + \dots. \tag{17}$$

We should note that the vertex vanishes when $(n_1 + n_2 - n_3) = 0$, which is the on-shell energy conservation condition. We mention here the relevance of an Euclidean picture which was established recently in [25]. It corresponds to the inverted harmonic oscillator model of the c = 1 string theory [24]. The analogue cubic Hamiltonian interaction was shown capable of reconstructing the noncritical string amplitudes at both tree and loop level. The relevance of this $S$-matrix to AdS/CFT (and comparison with 1/2 BPS correlators) was shown in [25]. For completeness in the rest of this section we discuss the comparison of (17) with the SUGRA vertex obtained by studying three-point functions of chiral primaries on $AdS_5 \times S^5$. Next, we outline the main steps of such a comparison, and leave a detailed discussion to Section below. The typical 3-point (cubic) SUGRA interaction on backgrounds of the form $AdS_n \times S^m$ is given by the overlap of bulk wave functions,

$$\begin{aligned} H_3 = (\Delta_3 - \Delta_1 - \Delta_2)\int_{\mathrm{AdS}} d^{n-1}x\sqrt{g_{\mathrm{AdS}}}g^{tt}_{\mathrm{AdS}}f_{I_1}f_{I_2}\bar{f}_{I_3} \\ \times\int_S d^m y\sqrt{g_S}Y^{I_1}Y^{I_2}\bar{Y}^{I_3}, \end{aligned} \tag{18}$$

with $f_I(x)$ and $Y^I(y)$ denoting eigenfunctions on $AdS_n$ and $S^m$ respectively. The total wave functions $\psi(x, y) = \sum_I f_I(x)Y^I(y)$ obey the linearized equation

$$(\Box_{\mathrm{AdS}} + \Box_S)\psi = 0.$$

Understanding the cubic interaction then relies on understanding bulk properties of AdS. From the GKP-W map one has the "holographic" formula

$$H_3 \sim (\Delta_3 - \Delta_1 - \Delta_2)C(I_1, I_2, I_3), \tag{19}$$

where $C(I_1, I_2, I_3)$ are coefficients in the 3-point correlator. For the highest-weight states one has that their energy $\Delta$ is given by the angular momenta $\Delta = j$. One also has[1]

$$C(j_1, j_2, j_3) \propto \delta_{j_1+j_2,j_3}, \tag{20}$$

which is the R-charge conservation condition. We find that the $\delta$-function forces the (highest-weight) vertex to vanish, $V^{\mathrm{h.w.}} = 0$. We emphasize that this implies that the holographic vertex is equal to 0 both on and off- shell. On the other hand, the collective vertex is seen to be *nonvanishing* off-shell and can therefore serve as the starting point for a raising-lowering procedure that one can apply to highest-weight states.

### Chiral primary interactions in $AdS_5 \times S^5$

Let us now examine the full interacting theory of chiral primaries, with the ultimate goal of showing agreement with the collective field calculation. We consider the case of $AdS_5 \times S^5$, which has been studied in [5]. The equation of motion for the chiral primary field $s$, of mass $m^2 = j(j-4)$, was found to be of the form

$$\begin{aligned} (\nabla_\mu\nabla^\mu - m_I^2)s^I = \sum_{J,K}(D_{IJK}s^J s^K + E_{IJK}\nabla_\mu s^J\nabla^\mu s^K \\ + F_{IJK}\nabla^{(\mu}\nabla^{\nu)}s^J\nabla_{(\mu}\nabla_{\nu)}s^K), \end{aligned}$$

where $\mu$ denotes $AdS_5$ coordinates, and the sphere dependence has already been integrated out. For the explicit form of the coefficients $D$, $E$ and $F$ we refer the reader to [5]. The derivative terms can be removed by the following field redefinition

$$s^I = s'^I + \sum_{J,K}(J_{IJK}s'^J s'^K + L_{IJK}\nabla^\mu s'^J\nabla_\mu s'^K), \tag{21}$$

where $L_{IJK} = \frac{1}{2}F_{IJK}$ and $J_{IJK} = \frac{1}{2}E_{IJK} + \frac{1}{4}F_{IJK}(m_I^2 - m_J^2 - m_K^2 + 8)$. The field redefinition dramatically simplifies the equation of motion, which becomes

$$(\nabla_\mu\nabla^\mu - m_I^2)s^I = \sum_{J,K}\lambda_{IJK}s^J s^K, \tag{22}$$

where $\lambda_{IJK} = D_{IJK} - (m_J^2 + m_K^2 - m_I^2)J_{IJK} - \frac{2}{5}L_{IJK}m_J^2 m_K^2$. Finally, after plugging in the coefficients $D$, $E$ and $F$, the action for the chiral primary $s$ becomes

[1]We note that in the appendix we will show in more detail the origin of the energy-conserving $\delta$-function in the 3-point function.

$$S = \int d^5x\sqrt{-g}\Big[-\nabla s^I \nabla \bar{s}^I - m_I^2|s^I|^2 - \frac{1}{2}\lambda_{IJK}(s^I s^J \bar{s}^K + \text{c.c})\Big], \tag{23}$$

where $m^2 = j(j-4)$, and the coupling constant [5] is

$$\lambda_{123} = (j_3 - j_1 - j_2)2\kappa \times \frac{\sqrt{j_1 j_2 j_3 (j_3^2 - 1)(j_3+2)(j_3-2)}}{\sqrt{(j_1^2-1)(j_2^2-1)(j_1+2)(j_2+2)}} \times f_{123},$$

$$f_{123} = \frac{1}{\sqrt{2\pi^3}} \frac{\sqrt{(j_1+1)(j_1+2)(j_2+1)(j_2+2)}}{\sqrt{(j_3+1)(j_3+2)}}. \tag{24}$$

The coefficient $f_{123}$ comes from the overlap integral over spherical harmonics on $S^5$. In global coordinates, the highest-weight state on $\text{AdS}_5 \times S^5$ takes the form

$$s = \frac{\sqrt{\Delta(\Delta-1)}}{\pi(\cosh\mu)^\Delta}. \tag{25}$$

The matrix element of the cubic Hamiltonian for the action (23) is then given by

$$\langle 3|H_3|12\rangle = \frac{1}{2^{3/2}\pi}\frac{(\Delta_1-1)(\Delta_2-1)(\Delta_3-1)}{(\Delta_3-1)(\Delta_3-2)} \times G_{123}\delta(j_1+j_2-j_3) = (\Delta_3-\Delta_1-\Delta_2)\frac{\sqrt{\Delta_1\Delta_2\Delta_3}}{N}\delta(j_1+j_2-j_3). \tag{26}$$

Since $\Delta = j$, this agrees with the collective field theory vertex (17)

$$H^{(3)} \sim -4\pi\sqrt{n_1 n_2 n_3}(n_1+n_2-n_3)a_1 a_2 a_3^\dagger, \tag{27}$$

apart from the absence of the $\delta$-function coming from conservation of angular momentum. Thus, we have shown that the collective field theory vertex is contained in the gravity description. We will discuss at a later stage the origin of two different pictures (for 1/2 BPS states) related to the appearance of a delta function term in the vertex.

## III. MATRIX MODEL MAPS

Our goal is to extend the Hamiltonian formulation from the highest-weight states of the bubbling 1/2 BPS configuration. In global coordinates $\text{AdS}_5 \times S^5$ can be written as

$$ds^2 = (-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2) + (\sin^2\theta d\theta^2 + d\phi^2 + \cos^2\theta d\tilde{\Omega}_3^2), \tag{28}$$

and, as we will show in more detail later, the chiral primary fields fluctuations read

$$\delta g \sim \left(\frac{\cos\theta}{\cosh\rho}e^{i\phi}\right)^j, \tag{29}$$

a highest-weight state of the isometry algebra. The collective droplet vertex represents an off-shell interaction of such fluctuations. The basic strategy that we will employ is then to use the resulting nonvanishing three-point interaction as a starting point for reconstructing the full 3-point vertex, i.e. involving more general states. The first ingredient in this program is the reconstruction of linearized wave functions:

$$\psi_{jnm}(t,\rho,\theta,\phi) \sim L_+^n J_-^{j-m}\psi_j\left(\frac{\cos\theta}{\cosh\rho}e^{i\phi}\right). \tag{30}$$

This was done in [23]. From the interactions of chiral primaries, we will develop an analogous raising-lowering Ward identity which relates $V_{j_1 n_1 j_2 n_2 j_3 n_3}$, the vertex for two-matrix states, to $V_{j_1 j_2 j_3}$, the one-matrix vertex. Toward this end it is important to describe the inclusion of the 1/2 BPS "bubbling" configurations of $\text{AdS}_5 \times S^5$ in the two-matrix (coordinate) picture. In the 2D coordinate space $(X_1, X_2)$, where the Hamiltonian is given by $H = \frac{1}{2}(X_1^2 + X_2^2 + P_1^2 + P_2^2)$ and the angular momentum by $J = X_1P_2 - X_2P_1$, one can introduce complex coordinates

$$Z = \frac{X_1 + iX_2}{\sqrt{2}}, \qquad \bar{Z} = \frac{X_1 - iX_2}{\sqrt{2}}, \tag{31}$$

with corresponding conjugate momenta

$$\Pi = \frac{P_1 + iP_2}{\sqrt{2}}, \qquad \bar{\Pi} = \frac{P_1 - iP_2}{\sqrt{2}}. \tag{32}$$

Switching to creation and annihilation operators,

$$Z = \frac{1}{\sqrt{2}}(A^\dagger + B), \qquad \bar{Z} = \frac{1}{\sqrt{2}}(A + B^\dagger), \tag{33}$$

$$\Pi = \frac{-i}{\sqrt{2}}(A^\dagger - B), \qquad \bar{\Pi} = \frac{i}{\sqrt{2}}(A - B^\dagger), \tag{34}$$

the Hamiltonian and angular momentum generators become

$$H = \text{Tr}(A^\dagger A + B^\dagger B), \qquad J = \text{Tr}(A^\dagger A - B^\dagger B). \tag{35}$$

So 1/2 BPS states having $H = J$ are described by a truncation to the sector where $B = 0$, and only $A$ oscillators remain. This condition can be translated into having a single matrix $X = (A + A^\dagger)$, with conjugate momentum $P = i(A - A^\dagger)$. In the phase space (matrix model) one has the corresponding canonical transformation

$$X = \frac{X_1 + P_2}{\sqrt{2}}, \qquad Y = \frac{X_1 - P_2}{\sqrt{2}}, \qquad P_X = \frac{P_1 - X_2}{\sqrt{2}}, \qquad P_Y = \frac{P_1 + X_2}{\sqrt{2}}, \tag{36}$$

with the fact that

$$J = X_1P_2 - X_2P_1 = \frac{1}{2}(X^2 + P_X^2) - \frac{1}{2}(Y^2 + P_Y^2) = \tilde{J}. \tag{37}$$

In the matrix model picture (which from now on we denote by a *tilde*), the R-charge transformation is not a coordinate transformation, but rather a canonical transformation (dynamical symmetry). This gives an explanation of the origin of the nonconservation in the 3-vertex of the cubic collective field theory that we have noted earlier: in the matrix model formulation we have two representations that are related by a canonical transformation. Next, we describe the matrix model map associated with LLM (*one-matrix* model), followed by the construction of [23] which extends it to *two matrices*. The LLM map is given by one central formula

$$Z(x_1, x_2, y) = \frac{y^2}{\pi}\int_D dx_1' dx_2' \frac{u(x_1', x_2', 0)}{[(\vec{x} - \vec{x}')^2 + y^2]^2}, \tag{38}$$

where the integral is defined over a domain $D$ and $u(x_1, x_2, 0) = \pm\frac{1}{2}$. It is a nonlinear map since the dynamical degree of freedom on the right-hand side is not $u(x_1, x_2, 0)$, but the boundary of the domain. Linearization leads to the following (linear) relationship (for a detailed derivation see [23]):

$$\delta g = \frac{1}{2\pi}\int_0^{2\pi} d\tau \frac{1 - 4a^2 - a^4 + 4a^3\cos(\tau - \phi)}{[1 + a^2 - 2a\cos(\tau - \phi)]^2}\delta\alpha(\tau), \tag{39}$$

where $a = \frac{\cos\theta}{\cosh\rho}$. On the right-hand side of the equation we have the small fluctuation $\delta\alpha(\tau)$ of the one-matrix collective field described by

$$H_2 = \frac{1}{2}\int dx\pi\phi_0(x)(\Pi^2 + (\partial_x\eta)^2) = \int d\tau(\delta\alpha(\tau))^2, \tag{40}$$

with

$$\phi_0(x) = \frac{1}{\pi}\sqrt{\mu - x^2}, \tag{41}$$

and $d\tau = \frac{dx}{\pi\phi_0(x)}$. On the left-hand side of (39) $\delta g$ denotes the fluctuation of a SUGRA chiral primary. In the notation of [23], $t$ and $\rho$ denote AdS coordinates, while $\theta$ and $\phi$ are sphere angles. For $\delta\alpha(\tau) \sim e^{ij\tau}$ one gets

$$\delta g \sim \left(\frac{\cos\theta}{\cosh\rho}e^{i\phi}\right)^j, \tag{42}$$

the correct chiral primary fields fluctuations. More precisely, denoting the Kernel by $K_{LLM}(\rho, \theta, \phi; \tau)$, one finds (see [23] for more details):

$$\delta g(t, \rho, \theta, \phi) = \frac{e^{ijt}}{2\pi}\int_0^{2\pi} d\tau K_{LLM}(\rho, \theta, \phi; \tau)\delta\alpha(\tau). \tag{43}$$

This is a *one-dimensional* map from the space $\tau = \int \frac{dx}{\pi\phi_0(x)}$ of a matrix model to the subspace of $AdS_5 \times S^5$ given by $\frac{\cos\theta}{\cosh\rho}e^{i\phi}$. The extension of the linearized LLM map to the two-matrix case was given in [23] and starts with the matrix observable

$$\psi(x, n) = \mathrm{Tr}((\delta(x - (A + A^\dagger))B^n)_{\mathrm{SYM}}). \tag{44}$$

This then leads to an eigenvalue problem

$$\hat{K}\psi = \omega\psi, \tag{45}$$

with solution

$$\tilde{\psi}_{jn}(\tau, \sigma) = \sin((j + 2n)\tau)e^{in\sigma}, \qquad \omega_{jn} = j + 2n. \tag{46}$$

Through a kernel constructed in [23], this maps into a nontrivial eigenfunction on AdS space:

$$\psi_{jn}(t, \rho, \phi, \theta) = e^{i\omega_{jn}t}\cos^j\theta\frac{1}{4\pi^2} \times \int_0^{2\pi} d\tau\int_0^{2\pi} d\sigma K_2(\rho, \phi; \tau, \sigma)\tilde{\psi}_{jn}. \tag{47}$$

Notice that the map is $2 \leftrightarrow 2$, mapping the two coordinates $\tau$, $\sigma$ of the matrix model into the space-time coordinates $\rho$, $\phi$. Furthermore, we have the following two remarks about the kernel $K_2$. First, when applied to the states with $n = 0$, it reduces to the kernel associated with the LLM map. Second, the map is essentially a reduction to action-angle variables associated with the nontrivial AdS Laplacian.

## IV. NONLINEAR ANALYSIS

We now come to the main consideration of this work and address the question of a nonlinear extension. In this section we will also address the issue concerning the presence of delta-function constraint in the 1/2 BPS interaction vertex. To simplify the discussion we start by considering what we refer to as the *nonrelativistic model*, which will allow us to present the main steps of our argument in explicit terms. Recall that in Sec. III we distinguished between the matrix model picture (i.e. the tilde representation with matrices $X$, $Y$ and conjugate momenta $P_X$, $P_Y$) and the coordinate space $(X_1, X_2)$. In the nonrelativistic approximation one directly replace the matrices with the corresponding coordinates, a procedure that is simple to implement based on density fields. In Sec. III we described (at the matrix level) the canonical transformation relating the two pictures in question, with

$$\tilde{H} = \tfrac{1}{2}(x^2 + y^2 + p_x^2 + p_y^2), \tag{48}$$

and similarly for $H$. The linear map (in the nonrelativistic approximation) which relates the two representations reads

$$\tilde{\psi}(x, y) = \int dx_1\int dx_2 K(\tilde{x}, \vec{x})\psi(x_1, x_2), \tag{49}$$

where the kernel is given by

$$K(x, y; x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{-ix_2(x-y)/\sqrt{2}} \delta\left(x_1 - \frac{x+y}{\sqrt{2}}\right). \quad (50)$$

It corresponds to a canonical transformation such that

$$J = x_1 p_2 - x_2 p_1 = \tfrac{1}{2}(p_x^2 + x^2) - \tfrac{1}{2}(p_y^2 + y^2) = \tilde{J}, \quad (51)$$

related to the change to the matrix model picture discussed in Sec. III. It maps (matrix model) eigenfunctions

$$\tilde{\psi}_{jn}(x, y) = e^{-(x^2+y^2)} H_{j+n}(x) H_n(y) \quad (52)$$

into (space-time) eigenfunctions

$$\psi_{jn}(r, \phi) = \frac{e^{ij\phi}}{\sqrt{2\pi}} L_n^j(r), \quad (53)$$

where in the space-time picture $x_1 + ix_2 = re^{i\phi}$. The vertices in the two pictures are denoted by $V$ and $\tilde{V}$ and are given by the overlap integral of three eigenfunctions:

$$V_{j_1n_1j_2n_2j_3n_3} = (\Delta_1 - \Delta_2 - \Delta_3) \int \frac{d^2\vec{x}}{\sqrt{\phi_0(x)}} \times \psi^*_{j_1n_1}(r, \phi)\psi_{j_2n_2}(r, \phi)\psi_{j_3n_3}(r, \phi), \quad (54)$$

with $\psi_{jn}$ given in (53), and similarly for $\tilde{V}$. The 3-point overlap $V_{j_1n_1j_2n_2j_3n_3}$ will then be roughly of the form

$$\begin{aligned} V_3 &\sim \int d\phi e^{i(-j_1+j_2+j_3)\phi} \int dr L_{n_1}^{j_1} L_{n_2}^{j_2} L_{n_3}^{j_3} \\ &\sim \delta(-j_1 + j_2 + j_3) \int dr L_{n_1}^{j_1} L_{n_2}^{j_2} L_{n_3}^{j_3} \\ &\equiv \delta(-j_1 + j_2 + j_3)\mathcal{V}, \end{aligned} \quad (55)$$

and (still) yield a conserving delta function. Let us briefly sketch what happens in the case of the *tilde* representation, with eigenfunctions now given by (52). The overlap integral takes the form

$$\tilde{V}_3 \sim \int dx e^{-3x^2} H_{j_1+n_1}(x) H_{j_2+n_2}(x) H_{j_3+n_3}(x) \times \int dy e^{-3y^2} H_{n_1}(y) H_{n_2}(y) H_{n_3}(y). \quad (56)$$

As one can verify, written in this basis the vertex no longer has a conserving $\delta$-function. Thus, as we commented earlier, the vertex $V$ has R-charge conservation

$$V_{j_1n_1j_2n_2j_3n_3} = \delta_{j_1,j_2+j_3}\mathcal{V}, \quad (57)$$

while $\tilde{V}$ does not. This is explained by the different action of the R-charge operator $\hat{J}$ in the two pictures. While in the present case one can easily show that for 2-point overlaps

$$\int dx_1 dx_2 \psi_{jn}\psi_{j'n'} = \int dx dy \tilde{\psi}_{jn}\tilde{\psi}_{j'n'}, \quad (58)$$

one cannot do that for the 3-point function overlap. In fact one can show explicitly that

$$V_{j_1n_1j_2n_2j_3n_3} \neq \tilde{V}_{j_1n_1j_2n_2j_3n_3}. \quad (59)$$

The basic theorem that we will establish in what follows is that the two Hamiltonians

$$H = \omega_{jn} A^\dagger_{jn} A_{jn} + (V_{j_1n_1j_2n_2j_3n_3} A^\dagger_{j_1n_1} A_{j_2n_2} A_{j_3n_3} + \text{H.c.}) \quad (60)$$

and

$$\tilde{H} = \sum_{jn} \omega_{jn} \tilde{A}^\dagger_{jn} \tilde{A}_{jn} + \sum_{\{j's,n's\}} (\tilde{V}^{(1)}_{j_1n_1j_2n_2j_3n_3} \tilde{A}^\dagger_{j_1n_1} \tilde{A}_{j_2n_2} \tilde{A}_{j_3n_3} + \tilde{V}^{(2)}_{j_1n_1j_2n_2j_3n_3} \tilde{A}^\dagger_{j_1n_1} \tilde{A}^\dagger_{j_2n_2} \tilde{A}^\dagger_{j_3n_3} + \text{H.c.}) \quad (61)$$

are in fact equivalent, with a nonlinear canonical transformation relating them. To demonstrate this statement, namely, the fact that (60) and (61) match, we would like to perform the following field redefinition:

$$\tilde{A}_N = A_N + c_{NMP} A_M A_P + d_{NMP} A^\dagger_M A_P + e_{NMP} A^\dagger_M A^\dagger_P. \quad (62)$$

We have simplified the notation by using the index $N$ to denote all quantum numbers $(j, n)$. The Hamiltonian in the *tilde* representation with this more compact notation takes the form

$$\begin{aligned} \tilde{H} &= \tilde{H}_2 + \tilde{H}_3 \\ &= \sum_N \omega_N \tilde{A}^\dagger_N \tilde{A}_N + \sum_{\{N,M,P\}} (\tilde{V}^{(1)}_{NMP} \tilde{A}^\dagger_N \tilde{A}_M \tilde{A}_P \\ &\quad + \tilde{V}^{(2)}_{NMP} \tilde{A}^\dagger_N \tilde{A}^\dagger_M \tilde{A}^\dagger_P + \text{H.c.}). \end{aligned} \quad (63)$$

Under the field redefinition (62) the quadratic part $\tilde{H}_2$ yields additional cubic terms, and the total Hamiltonian becomes

$$\begin{aligned} \tilde{H} = \sum_N \omega_N A^\dagger_N A_N + \sum_{N,M,P} [(&\tilde{V}^{(1)}_{NMP} + \omega_N c_{NMP} \\ &+ \omega_P \bar{d}_{PMN}) A^\dagger_N A_M A_P \\ &+ (\tilde{V}^{(2)}_{NMP} + \omega_N e_{NMP}) A^\dagger_N A^\dagger_M A^\dagger_P + \text{H.c.}] \end{aligned} \quad (64)$$

If we want this to match (60), we need the following conditions on the coefficients of the field redefinition:

$$\begin{aligned} \omega_N e_{NMP} &= -\tilde{V}^{(2)}_{NMP}, \\ \tilde{V}^{(1)}_{NMP} + \omega_N c_{NMP} + \omega_P \bar{d}_{PMN} &= V_{NMP}. \end{aligned} \quad (65)$$

Furthermore, we can obtain additional constraints on $c_{NMP}$, $d_{NMP}$ and $e_{NMP}$ by imposing appropriate commutation relations:

$$[\tilde{A}_N, \tilde{A}_{N'}] = 0, \quad (66)$$

$$[\tilde{A}_N, \tilde{A}^\dagger_{N'}] = \delta_{N,N'}. \quad (67)$$

Requiring (66) yields

$$d_{N'NM} = d_{NN'M},$$
$$e_{N'MN} - e_{NMN'} + e_{N'NM} - e_{NN'M} = 0. \tag{68}$$

The remaining commutation relation (67) gives

$$d_{NMN'} + \bar{c}_{N'MN} + \bar{c}_{N'NM} = 0,$$
$$\bar{d}_{N'MN} + c_{NMN'} + c_{NN'M} = 0. \tag{69}$$

This entirely fixes the nonlinear redefinition (62), showing that one can in fact connect the two Hamiltonians.

To summarize, we have described how the matrix level canonical transformation induces changes at the nonlinear level. One has two related pictures, one in which the R-symmetry is implemented as a coordinate symmetry (with the corresponding delta function) and another where the R-symmetry is dynamical, given by a canonical transformation. We have shown the equivalence of these two pictures through a nonlinear field transformation. Related field transformations have been identified previously at the Lagrangian level in [8].

## V. WARD IDENTITIES AND VERTEX RECONSTRUCTION

Our main goal is to establish that, starting from the vertex of highest-weight states, it is possible to build the vertex for more general states that are reachable by (in this case) $SL(2)$ raising/lowering procedure. Specifically, we will develop an identity that will allow us to generate such nontrivial vertices, by making use of the available Ward identities. We will again start from the simplified (nonrelativistic) model discussed in Sec. IV. This will then be followed by a discussion on the form of Ward identities in the AdS case.

### A. Nonrelativistic model

Recall that the Hamiltonian of the nonrelativistic model is given by

$$H = \frac{x^2 + y^2}{2} + \frac{p_x^2 + p_y^2}{2}, \tag{70}$$

or, in terms of creation and annihilation operators,

$$H = a^\dagger a + b^\dagger b. \tag{71}$$

Let us introduce complex variables

$$z = \frac{a^\dagger + b}{\sqrt{2}}, \qquad \bar{z} = \frac{a + b^\dagger}{\sqrt{2}}, \tag{72}$$

and corresponding conjugate momenta

$$\Pi = -i\partial_{\bar{z}} = i\frac{a^\dagger - b}{\sqrt{2}}, \qquad \bar{\Pi} = -i\partial_z = -i\frac{a - b^\dagger}{\sqrt{2}}. \tag{73}$$

These expressions can be combined to obtain

$$a = \frac{1}{\sqrt{2}}(\bar{z} + \partial_z), \qquad b = \frac{1}{\sqrt{2}}(z + \partial_{\bar{z}}). \tag{74}$$

The wave functions are then given by

$$|J, n\rangle \equiv \frac{(a^\dagger)^{J+n}(b^\dagger)^n}{\sqrt{(J+n)!n!}}|0\rangle, \tag{75}$$

and the generators $l_+$, $l_-$, $l_0$ by

$$l_+ = \frac{1}{2}(\partial_z\partial_{\bar{z}} + z\bar{z} - 1) - \frac{z}{2}\partial_z - \frac{\bar{z}}{2}\partial_{\bar{z}}, \tag{76}$$

$$l_- = \frac{1}{2}(\partial_z\partial_{\bar{z}} + z\bar{z} + 1) + \frac{z}{2}\partial_z + \frac{\bar{z}}{2}\partial_{\bar{z}}, \tag{77}$$

$$l_0 = -\partial_z\partial_{\bar{z}} + z\bar{z} - 1. \tag{78}$$

As we mentioned earlier, our goal is to use these generators to derive an identity for the cubic vertex, which is given by

$$\int d^2x \frac{1}{\sqrt{\phi_0(\vec{x})}}\eta\eta l_0\eta, \tag{79}$$

with $\vec{x} = (x, y)$. If we plug the standard mode expansion

$$\eta = \sqrt{2}\sum_{J=1}^{\infty}\sum_{n=0}^{\infty}(\bar{c}_{J,n}\psi_{J,n} + c_{J,n}\bar{\psi}_{J,n}) \tag{80}$$

into the vertex we find

$$\sum_{J=1}\sum_{n=0}\bigg[\bar{c}_{J_1,n_1}\bar{c}_{J_2,n_2}c_{J_3,n_3}\int dx\int dy \times\frac{1}{\sqrt{\phi_0(\vec{x})}}\psi_{J_1,n_1}\psi_{J_2,n_2}\bar{\psi}_{J_3,n_3} + \dots\bigg]P, \tag{81}$$

where we denote by $P$ the prefactor coming from the action of $l_0$ on the wave functions. Next, we would like to use the fact that

$$\psi_{J,n} = \frac{l_+}{\sqrt{(J+n)n}}\psi_{J,n-1}, \tag{82}$$

and focus on

$$V_1 = \int d^2x \frac{1}{\sqrt{\phi_0(\vec{x})}}\bar{\psi}_{J_1,n_1}\psi_{J_2,n_2}\psi_{J_3,n_3}. \tag{83}$$

Using (76), the vertex term above becomes

$$V_1 = \int d^2x \frac{1}{2\sqrt{\phi_0(\vec{x})}\sqrt{n_3(J_3+n_3)}}\bar{\psi}_{J_1,n_1} \times\psi_{J_2,n_2}(\partial_z\partial_{\bar{z}} + z\bar{z} - 1 - z\partial_z - \bar{z}\partial_{\bar{z}})\psi_{J_3,n_{3-1}}. \tag{84}$$

Let us treat each term in $V_1$ separately. We start from

$$
\begin{aligned}
T_1 &\equiv \int d^2x \frac{1}{2\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}} \bar{\psi}_{J_1,n_1}\psi_{J_2,n_2}\partial_z\partial_{\bar{z}}\psi_{J_3,n_3-1} \\
&= \int d^2x \frac{1}{2\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}} \bar{\psi}_{J_1,n_1}\psi_{J_2,n_2} \\
&\quad \times [-l_0 + z\bar{z} - 1]\psi_{J_3,n_3-1} \\
&= \int d^2x \frac{1}{2\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}} \bar{\psi}_{J_1,n_1}\psi_{J_2,n_2} \\
&\quad \times [-J_3 - 2n_3 + z\bar{z} - 1]\psi_{J_3,n_3-1}. \qquad (85)
\end{aligned}
$$

We then look at the term

$$
\begin{aligned}
T_2 &\equiv -\int d^2x \frac{1}{2\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}} \bar{\psi}_{J_1,n_1}\psi_{J_2,n_2} z\partial_z\psi_{J_3,n_3-1} \\
&= \int d^2x \frac{1}{2\sqrt{n_3(J_3+n_3)}} \psi_{J_3,n_3-1}\psi_{J_2,n_2} \\
&\quad \times \left[\frac{1}{2\sqrt{\phi_0}} + \frac{z}{\sqrt{\phi_0}}\partial_z + \frac{z}{2}\partial_z\phi_0^{-1/2}\right]\bar{\psi}_{J_1,n_1} \\
&\quad + \int d^2x \frac{1}{2\sqrt{n_3(J_3+n_3)}} \psi_{J_3,n_3-1}\bar{\psi}_{J_1,n_1} \\
&\quad \times \left[\frac{1}{2\sqrt{\phi_0}} + \frac{z}{\sqrt{\phi_0}}\partial_z + \frac{z}{2}\partial_z\phi_0^{-1/2}\right]\psi_{J_2,n_2}. \qquad (86)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
T_3 &\equiv -\int d^2x \frac{1}{2\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}} \bar{\psi}_{J_1,n_1}\psi_{J_2,n_2} \bar{z}\partial_{\bar{z}}\psi_{J_3,n_3-1} \\
&= \int d^2x \frac{1}{2\sqrt{n_3(J_3+n_3)}} \psi_{J_3,n_3-1}\psi_{J_2,n_2} \\
&\quad \times \left[\frac{1}{2\sqrt{\phi_0}} + \frac{\bar{z}}{\sqrt{\phi_0}}\partial_{\bar{z}} + \frac{\bar{z}}{2}\partial_{\bar{z}}\phi_0^{-1/2}\right]\bar{\psi}_{J_1,n_1} \\
&\quad + \int d^2x \frac{1}{2\sqrt{n_3(J_3+n_3)}} \psi_{J_3,n_3-1}\bar{\psi}_{J_1,n_1} \\
&\quad \times \left[\frac{1}{2\sqrt{\phi_0}} + \frac{\bar{z}}{\sqrt{\phi_0}}\partial_{\bar{z}} + \frac{\bar{z}}{2}\partial_{\bar{z}}\phi_0^{-1/2}\right]\psi_{J_2,n_2}. \qquad (87)
\end{aligned}
$$

We now collect all terms and, using the definitions of $l_0$ and $l_-$, find

$$
\begin{aligned}
V_1 = \int d^2x \frac{\psi_{J_3,n_3-1}}{\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}} \Big[ &\psi_{J_2,n_2}\Big[ l_- + \frac{l_0}{2} - z\bar{z} \\
&+ \frac{1}{2} + \frac{z\sqrt{\phi_0}}{4}\partial_z\phi_0^{-1/2} + \frac{\bar{z}\sqrt{\phi_0}}{4}\partial_{\bar{z}}\phi_0^{-1/2}\Big]\bar{\psi}_{J_1,n_1} \\
&+ \bar{\psi}_{J_1,n_1}\Big[ l_- + \frac{l_0}{2} - z\bar{z} + \frac{1}{2} + \frac{z\sqrt{\phi_0}}{4}\partial_z\phi_0^{-1/2} \\
&+ \frac{\bar{z}\sqrt{\phi_0}}{4}\partial_{\bar{z}}\phi_0^{-1/2}\Big]\psi_{J_2,n_2} \\
&+ \bar{\psi}_{J_1,n_1}\psi_{J_2,n_2}\Big[ z\bar{z} - 1 - \frac{2n_3+J_3}{2}\Big]\Big]. \qquad (88)
\end{aligned}
$$

We can rewrite the vertex above in the following way:

$$
\begin{aligned}
&\int d^2x \frac{\bar{\psi}_{J_1,n_1}\psi_{J_2,n_2}}{\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}}\left(l_+ + \frac{l_0}{2}\right)\psi_{J_3,n_3-1} \\
&\quad = \int d^2x \frac{\psi_{J_3,n_3-1}\psi_{J_2,n_2}}{\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}}\left(l_- + \frac{l_0}{2}\right)\bar{\psi}_{J_1,n_1} \\
&\qquad + \int d^2x \frac{\psi_{J_3,n_3-1}\bar{\psi}_{J_1,n_1}}{\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}}\left(l_- + \frac{l_0}{2}\right)\psi_{J_2,n_2} \\
&\qquad + \int d^2x \frac{\bar{\psi}_{J_1,n_1}\psi_{J_2,n_2}\psi_{J_3,n_3-1}}{\sqrt{\phi_0}\sqrt{n_3(J_3+n_3)}} \\
&\qquad \times \left[-z\bar{z} + \frac{z\sqrt{\phi_0}}{2}\partial_z\phi_0^{-1/2} + \frac{\bar{z}\sqrt{\phi_0}}{2}\partial_{\bar{z}}\phi_0^{-1/2}\right]. \qquad (89)
\end{aligned}
$$

The last line vanishes, since $\phi_0 = e^{-2z\bar{z}}$. Thus, we find the following identity:

$$
\int d^2x \frac{1}{\sqrt{\phi_0}}\left[ l_+^{(3)} - l_-^{(1)} - l_-^{(2)} + \frac{l_0^{(3)} - l_0^{(1)} - l_0^{(2)}}{2}\right] \times \bar{\psi}_{J_1,n_1}\psi_{J_2,n_2}\psi_{J_3,n_3-1} = 0. \qquad (90)
$$

Notice that this identity can be used to relate the vertex for single-matrix states (highest-weight states with $n_i = 0$) to vertices of multimatrix states. In this sense, it provides a generating mechanism for constructing nontrivial interactions starting from the (simpler) 1/2 BPS sector of the theory.

### B. Interactions in AdS

We now move on to the case of real interest, interactions in AdS $\times$ $S$. For simplicity we consider $\mathrm{AdS}_2 \times S^2$. The generators are given by

$$
l_\pm = i[\cos\rho\,\partial_\rho \mp i\sin\rho\,\partial_t], \qquad l_0 = i\partial_t, \qquad (91)
$$

and the eigenfunctions (denoted by $\phi_n^\lambda$ to distinguish them from those of the nonrelativistic model) by

$$
\phi_n^\lambda(t,\rho) = c(\lambda) \times \sqrt{\frac{n!}{\Gamma(n+2\lambda)}} e^{-i(n+\lambda)(t+(\pi/2))}(\cos\rho)^\lambda C_n^\lambda(\sin\rho), \qquad (92)
$$

with

$$
c(\lambda) = \frac{\Gamma(\lambda)2^{\lambda-1}}{\sqrt{\pi}}. \qquad (93)
$$

Starting from $\int d\rho[l_+^{(1)}\bar{\phi}_{n_1}^{\lambda_1}]\phi_{n_2}^{\lambda_2}\phi_{n_3}^{\lambda_3}$ at $t = 0$ and integrating by parts with respect to $\rho$ we find

$$i\int d\rho \sin\rho \bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3} + i\int d\rho \sin\rho[l^{(2)}_0 + l^{(3)}_0 - l^{(1)}_0]\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3} + \int d\rho[l^{(2)}_- + l^{(3)}_- - l^{(1)}_+]\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3} = 0 \quad (94)$$

We can eliminate the $\sin\rho$ terms from the recursion relation by using

$$\sin\rho = \frac{l_- - l_+}{2il_0} \quad (95)$$

on the "1" leg to obtain:

$$\int d\rho\left[\frac{l^{(1)}_- - l^{(1)}_+}{2(n_1+\lambda_1)}\right]\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3} - \int d\rho[l^{(2)}_- + l^{(3)}_- - l^{(1)}_+]\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3} + \int d\rho\left[\frac{l^{(1)}_- - l^{(1)}_+}{2(n_1+\lambda_1)}\right] \times [n_1+\lambda_1+n_2+\lambda_2+n_3+\lambda_3]\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3} = 0. \quad (96)$$

### *1. Use of the Ward identity*

We can use this recursion relation to evaluate the overlap integral of a product of *any* three eigenfunctions given the overlap of highest-weight eigenfunctions. Inserting $n_1 = n_1, n_2 = n_3 = 0$ into the Ward identity and using ($C^\lambda_0 = 1$, $C^\lambda_1(x) = 2\lambda x$)

$$\begin{aligned} l_-\phi^\lambda_m &= e^{-it}\sqrt{m(m-1+2\lambda)}\phi^\lambda_{m-1},\\ l_-\bar{\phi}^\lambda_n &= -e^{-it}\sqrt{(n+1)(n+2\lambda)}\bar{\phi}^\lambda_{n+1},\\ l_+\bar{\phi}^\lambda_n &= -e^{+it}\sqrt{n(n-1+2\lambda)}\bar{\phi}^\lambda_{n-1}, \end{aligned} \quad (97)$$

we obtain

$$\int d\rho\bar{\phi}^{\lambda_1}_{n_1+1}\phi^{\lambda_2}_0\phi^{\lambda_3}_0 = \frac{1-n_1-\lambda_1+\lambda_2+\lambda_3}{1+n_1+\lambda_1+\lambda_2+\lambda_3} \times \frac{\sqrt{n_1(n_1-1+2\lambda_1)}}{\sqrt{(n_1+1)(n_1+2\lambda_1)}} \times e^{2it}\int d\rho\bar{\phi}^{\lambda_1}_{n_1-1}\phi^{\lambda_2}_0\phi^{\lambda_3}_0. \quad (98)$$

This relation allows us to determine $\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_0\phi^{\lambda_3}_0$ for any $n_1$, once we know its value for $n_1 = 0, 1$. To obtain the value when $n_1 = 1$, insert $n_1 = n_2 = n_3 = 0$ into the Ward identity. The resulting identity implies that $\int d\rho\bar{\phi}^{\lambda_1}_1\phi^{\lambda_2}_0\phi^{\lambda_3}_0 = 0$. Next, set $n_1 = n_1$, $n_2 = n_2$ and $n_3 = 0$. In this case, we find

$$\alpha_1\int d\rho\bar{\phi}^{\lambda_1}_{n_1+1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_0 + \alpha_2\int d\rho\bar{\phi}^{\lambda_1}_{n_1-1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_0 + \alpha_3\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2-1}\phi^{\lambda_3}_0 = 0, \quad (99)$$

where

$$\alpha_1 = -e^{-it}\sqrt{(n_1+1)(n_1+2\lambda_1)}\frac{1+n_1+\lambda_1+n_2+\lambda_2+\lambda_3}{2(n_1+\lambda_1)}, \quad (100)$$

$$\alpha_2 = e^{it}\sqrt{n_1(n_1-1+2\lambda_1)}\frac{1-n_1-\lambda_1+n_2+\lambda_2+\lambda_3}{2(n_1+\lambda_1)}, \quad (101)$$

$$\alpha_3 = -e^{-it}\sqrt{n_2(n_2-1+2\lambda_2)}. \quad (102)$$

If we set $n_2 = 1$ we have

$$\alpha_1\int d\rho\bar{\phi}^{\lambda_1}_{n_1+1}\phi^{\lambda_2}_1\phi^{\lambda_3}_0 + \alpha_2\int d\rho\bar{\phi}^{\lambda_1}_{n_1-1}\phi^{\lambda_2}_1\phi^{\lambda_3}_0 + \alpha_3\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_0\phi^{\lambda_3}_0 = 0, \quad (103)$$

which (starting from $n_1 = 0$) determines $\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_1\phi^{\lambda_3}_0$ for all $n_1$. Next, set $n_2 = 2$ to obtain

$$\alpha_1\int d\rho\bar{\phi}^{\lambda_1}_{n_1+1}\phi^{\lambda_2}_2\phi^{\lambda_3}_0 + \alpha_2\int d\rho\bar{\phi}^{\lambda_1}_{n_1-1}\phi^{\lambda_2}_2\phi^{\lambda_3}_0 + \alpha_3\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_1\phi^{\lambda_3}_0 = 0, \quad (104)$$

which (starting from $n_1 = 0$) fixes $\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_2\phi^{\lambda_3}_0$ for all $n_1$. Continuing in this way, it is clear that we can determine $\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_0$, for all $n_1$, $n_2$. Finally, set $n_1 = n_1$, $n_2 = n_2$ and $n_3 = n_3$. In this case, we find

$$\alpha_1\int d\rho\bar{\phi}^{\lambda_1}_{n_1+1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3} + \alpha_2\int d\rho\bar{\phi}^{\lambda_1}_{n_1-1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3} + \alpha_3\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2-1}\phi^{\lambda_3}_{n_3} + \alpha_4\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_{n_2}\phi^{\lambda_3}_{n_3-1} = 0, \quad (105)$$

where

$$\alpha_1 = -e^{-it}\sqrt{(n_1+1)(n_1+2\lambda_1)} \times \frac{1+n_1+\lambda_1+n_2+\lambda_2+n_3+\lambda_3}{2(n_1+\lambda_1)}, \quad (106)$$

$$\alpha_2 = e^{it}\sqrt{n_1(n_1-1+2\lambda_1)} \times \frac{1-n_1-\lambda_1+n_2+\lambda_2+n_3+\lambda_3}{2(n_1+\lambda_1)}, \quad (107)$$

$$\alpha_3 = -e^{-it}\sqrt{n_2(n_2-1+2\lambda_2)}, \quad (108)$$

$$\alpha_4 = -e^{-it}\sqrt{n_3(n_3-1+2\lambda_3)}. \quad (109)$$

If we take $n_2 = 0$ and $n_3 = 1$ we can determine $\int d\rho\bar{\phi}^{\lambda_1}_{n_1}\phi^{\lambda_2}_0\phi^{\lambda_3}_1$ for all $n_1$. Setting $n_2 = 1$ and $n_3 = 1$, we can find $\int d\rho \bar{\phi}_{n_1}^{\lambda_1} \phi_1^{\lambda_2} \phi_1^{\lambda_3}$ for all $n_1$. Next, set $n_2 = 2$ and $n_3 = 1$ to get $\int d\rho \bar{\phi}_{n_1}^{\lambda_1} \phi_2^{\lambda_2} \phi_1^{\lambda_3}$ for all $n_1$. Inching one step at a time we can determine the full vertex.

### *2. Check of the Ward identity*

To check the action of the generators we checked:

$$l_- \phi_0^\lambda = i[\cos\rho\,\partial_\rho + i\sin\rho\,\partial_t]\Big(c(\lambda)\sqrt{\frac{1}{\Gamma(2\lambda)}} \times e^{-i\lambda(t+(\pi/2))}(\cos\rho)^\lambda\Big) = 0, \tag{110}$$

$$\begin{aligned} l_- \bar{\phi}_1^\lambda &= i[\cos\rho\,\partial_\rho + i\sin\rho\,\partial_t]\Big(c(\lambda)\sqrt{\frac{1}{\Gamma(1+2\lambda)}} \times e^{i(1+\lambda)(t+(\pi/2))}(\cos\rho)^\lambda 2\lambda \sin\rho\Big) \\ &= -\sqrt{2(1+2\lambda)}\bar{\phi}_2^\lambda e^{-it}, \end{aligned} \tag{111}$$

$$\begin{aligned} l_+ \bar{\phi}_1^\lambda &= i[\cos\rho\,\partial_\rho - i\sin\rho\,\partial_t]\Big(c(\lambda)\sqrt{\frac{1}{\Gamma(1+2\lambda)}} \times e^{i(1+\lambda)(t+(\pi/2))}(\cos\rho)^\lambda 2\lambda \sin\rho)\Big) \\ &= -\sqrt{2\lambda}\bar{\phi}_0^\lambda e^{it}. \end{aligned} \tag{112}$$

As a partial check of the results of the previous section, we will evaluate (98) for $n_1 = 1$ and explicitly verify that it is correct. After setting $n_1 = 1$ we have

$$\int d\rho \bar{\phi}_2^{\lambda_1} \phi_0^{\lambda_2} \phi_0^{\lambda_3} = \frac{\lambda_2 + \lambda_3 - \lambda_1}{2 + \lambda_1 + \lambda_2 + \lambda_3} \times \frac{\sqrt{2\lambda_1}}{\sqrt{2(1+2\lambda_1)}} e^{2it} \int d\rho \bar{\phi}_0^{\lambda_1} \phi_0^{\lambda_2} \phi_0^{\lambda_3}. \tag{113}$$

Now,

$$\begin{aligned} \int d\rho \bar{\phi}_2^{\lambda_1} \phi_0^{\lambda_2} \phi_0^{\lambda_3} &= c(\lambda_1)c(\lambda_2)c(\lambda_3) \\ &\times \sqrt{\frac{2!}{\Gamma(2+2\lambda_1)\Gamma(2\lambda_2)\Gamma(2\lambda_3)}} \\ &\times e^{-i(\lambda_2-\lambda_3-\lambda_1)(t+(1/2)\pi)} e^{2i(t+(1/2)\pi)} \\ &\times \int d\rho (\cos\rho)^{\lambda_1+\lambda_2+\lambda_3} C_2^{\lambda_1}(\sin\rho). \end{aligned} \tag{114}$$

Using

$$\begin{aligned} \sqrt{\frac{2!}{\Gamma(2+2\lambda_1)\Gamma(2\lambda_2)\Gamma(2\lambda_3)}} &= \sqrt{\frac{2!}{(1+2\lambda_1)2\lambda_1}}\sqrt{\frac{1}{\Gamma(2\lambda_1)\Gamma(2\lambda_2)\Gamma(2\lambda_3)}}, \\ e^{2i(t+(1/2)\pi)} &= -e^{2it}, \\ \int d\rho (\cos\rho)^{\lambda_1+\lambda_2+\lambda_3} C_2^{\lambda_1}(\sin\rho) &= \frac{\lambda_1(\lambda_1 - \lambda_2 - \lambda_3)}{2+\lambda_1+\lambda_2+\lambda_3} \int d\rho (\cos\rho)^{\lambda_1+\lambda_2+\lambda_3}, \end{aligned} \tag{115}$$

it is trivial to verify the identity.

In conclusion, in this section we have demonstrated the existence of $[SL(2)]$ Ward identities. We have shown that these identities contain the necessary information to specify the cubic interaction vertex for general states from the knowledge of the vertex for the highest-weight states. Since we have shown that the one-matrix collective field theory correctly describes the latter case, we therefore have a scheme of reconstruction of the full vertex. This discussion was presented in the simplest $AdS_2$ framework; it is clear, however, that this procedure is valid in general. Nevertheless it will be important to develop the details in the higher dimensional case. In particular, there should be significant information on interactions in the 1/4 BPS sector where progress has recently been accomplished at the SUGRA level [20,26].

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## APPENDIX: THE VERTEX IN THE TWO REPRESENTATIONS

In this appendix we show in some detail the origin of the energy-conserving $\delta$-function in the 3-point function in the original, *nontilde* representation. Recall that a typical cubic term in the Hamiltonian takes the form

$$H_3^{(1)} = \frac{\omega_3(\omega_3 - \omega_1 - \omega_2)}{\sqrt{\omega_1\omega_2\omega_3}} A_1 A_2 A_3^\dagger \int d^D X \sqrt{-g} g^{tt} \psi_1 \psi_2 \bar{\psi}_3, \tag{A1}$$

where the integral is over the spatial coordinates only, and the $\mathrm{AdS}_{d+1} \times S^{d+1}$ wave function is given by $\psi(x, y) = \sum_I f_I(x) Y^I(y)$. Here $x$ and $y$ denote $\mathrm{AdS}_{d+1}$ and $S^{d+1}$ coordinates, respectively. Furthermore, in general we have $f(x) = f(t, \rho, \Omega_{d-1}) = e^{-i\omega t}\Psi(\rho, \Omega_{d-1})$. Incorporating the (trivial) time dependence $e^{-i\omega t}$ into the creation/annihilation operators of (A1), we see that the 3-vertex

$$V_3 = \int d^D X \sqrt{-g} g^{tt} \psi_1 \psi_2 \bar{\psi}_3 \tag{A2}$$

can be written as

$$V_3 = \int d^d x \sqrt{-g_{AdS}} g^{tt} \Psi_1 \Psi_2 \bar{\Psi}_3 \int d^{d+1} y \sqrt{g_S} Y_1 Y_2 \bar{Y}_3, \equiv \mathcal{F}_{\mathbf{123}} \mathcal{G}_{\mathbf{123}}, \tag{A3}$$

where we defined

$$\mathcal{F}_{\mathbf{123}} \equiv \int d^d x \sqrt{-g_{AdS}} g^{tt} \Psi_1 \Psi_2 \bar{\Psi}_3, \qquad \mathcal{G}_{\mathbf{123}} \equiv \int d^{d+1} y \sqrt{g_S} Y_1 Y_2 \bar{Y}_3. \tag{A4}$$

For simplicity, we now restrict ourselves to $\mathrm{AdS}_2 \times S^2$, with (global coordinates) metric

$$ds^2 = -\sec\rho^2 dt^2 + \sec\rho^2 d\rho^2 + \sin\theta^2 d\phi^2 + d\theta^2. \tag{A5}$$

On the sphere one has

$$\Box_{S^2} Y_j^{\bar{m}} = -j(j+1) Y_j^{\bar{m}}, \tag{A6}$$

where $Y_j^{\bar{m}}(\phi, \theta) = \tilde{N}_j^{\bar{m}} e^{i\bar{m}\phi} P_j^{\bar{m}}(\cos\theta)$, $\bar{m} = -j, -j+1, \ldots, j$ and $\tilde{N}_j^{\bar{m}}$ is the proper normalization. The AdS wave functions which satisfy the wave equation

$$\Box \mathrm{AdS}_2 f = \cos\rho^2(-\partial_t^2 + \partial_\rho^2) f = m^2 f \tag{A7}$$

are given by

$$f_{\omega,\lambda}(t, \rho) = N_{\omega-\lambda}^\lambda e^{-i\omega t} (\cos\rho)^\lambda C_{\omega-\lambda}^\lambda(\sin\rho), \qquad \omega = \lambda + n, \qquad n = 0, 1, 2, \ldots. \tag{A8}$$

Here $C_{\omega-\lambda}^\lambda(\sin\rho)$ are Gegenbauer polynomials, $N_{\omega-\lambda}^\lambda$ is a normalization factor and $\lambda$ is related to the mass of the field (also note $0 \le \rho < \frac{\pi}{2}$). For chiral primaries the mass turns out to be $m^2 = j(j-1)$, and the highest-weight state is given by $\lambda = j$ and $n = 0$:

$$\psi_{\mathrm{h.w.}}(t, \rho, \phi, \theta) = N_0^j \tilde{N}_j^j e^{-ijt} (\cos\rho)^j e^{ij\phi} P_j^j(\cos\theta). \tag{A9}$$

In order for the spherical harmonics and the Gegenbauer polynomials to be $\delta$-function normalized we must take

$$\tilde{N}_j^{\bar{m}} = \sqrt{\frac{(2j+1)(j-\bar{m})!}{4\pi(j+\bar{m})!}}, \qquad N_n^\lambda = \frac{\Gamma(\lambda) 2^{\lambda-1/2}}{\sqrt{\pi}} \sqrt{\frac{n!(n+\lambda)}{\Gamma(n+2\lambda)}}. \tag{A10}$$

For highest-weight states on $\mathrm{AdS}_2 \times S^2$ the overlap integrals are (defining $j \equiv j_1 + j_2 + j_3$):

$$\begin{aligned} \mathcal{F}_{j_1 j_2 j_3} &= \int_0^{\pi/2} d\rho \sqrt{-g_{\mathrm{AdS}_2}} g^{tt} \Psi_{j_1} \Psi_{j_2} \bar{\Psi}_{j_3} \\ &= N_0^{j_1} N_0^{j_2} N_0^{j_3} \int d\rho (\cos\rho)^j C_0^{j_1} C_0^{j_2} C_0^{j_3} \\ &= N_0^{j_1} N_0^{j_2} N_0^{j_3} \left( \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{1}{2} + \frac{j}{2})}{\Gamma(1 + \frac{j}{2})} \right), \end{aligned} \tag{A11}$$

$$\begin{aligned} \mathcal{G}_{j_1 j_2 j_3} &= \int d^2 y \sqrt{g_{S^2}} Y_{j_1}^{j_1} Y_{j_2}^{j_2} \bar{Y}_{j_3}^{j_3} \\ &= \delta(j_1 + j_2 - j_3) \frac{\tilde{N}_{j_1}^{j_1} \tilde{N}_{j_2}^{j_2} \tilde{N}_{j_3}^{j_3}}{(2\pi)^{3/2}} \\ &\quad \times \int_0^\pi d\theta \sin\theta P_{j_1}^{j_1} P_{j_2}^{j_2} P_{j_3}^{j_3} \\ &= \delta(j_1 + j_2 - j_3) \frac{(-1)^j \sqrt{\pi}}{(2\pi)^{3/2}} \prod_{i=1}^3 ((2j_i - 1)!! \tilde{N}_{j_i}^{j_i}) \\ &\quad \times \frac{\Gamma(1 + \frac{j}{2})}{\Gamma(\frac{3}{2} + \frac{j}{2})}, \end{aligned} \tag{A12}$$

where we used $C_0^j = 1$ and $P_j^j(x) = (-1)^j (2j-1)!!(1-x^2)^{j/2}$. Note the appearance of the $\delta(j_1 + j_2 - j_3)$ term in $\mathcal{G}_{j_1 j_2 j_3}$, coming from the $\int d\phi e^{ij(\phi_1 + \phi_2 - \phi_3)}$ integral. After some Gamma function cancellations, we are left with the following 3-vertex:

$$\mathcal{F}_{j_1 j_2 j_3} \mathcal{G}_{j_1 j_2 j_3} = \delta(j_1 + j_2 - j_3) \frac{(-1)^j \pi}{(2\pi)^{3/2}} \times \prod_{i=1}^3 ((2j_i - 1)!! N_0^{j_i} \tilde{N}_{j_i}^{j_i}) \frac{1}{j+1}, \tag{A13}$$

where we used $\Gamma(n/2) = \sqrt{2\pi}(n-2)!! 2^{-n/2}$. Finally, using

$$N_0^{j_i} = \frac{\Gamma(j_i) 2^{j_i} \sqrt{j_i}}{\sqrt{2\pi} \Gamma(2j_i)}, \qquad \tilde{N}_{j_i}^{j_i} = \sqrt{\frac{2j_i + 1}{2(2j_i)!}}, \quad \text{and} \quad (2j_i - 1)!! = \frac{(2j_i)!}{2^{j_i}(j_i)!}, \tag{A14}$$

we find

$$(2j_i - 1)!!N_0^{j_i}\tilde{N}_{j_i}^{j_i} = \sqrt{\frac{2j_i + 1}{2\pi}}. \tag{A15}$$

The final expression for the 3-point function overlap is

$$\begin{aligned} V_3 &\equiv \mathcal{F}_{j_1j_2j_3}\mathcal{G}_{j_1j_2j_3} \\ &= \delta(j_1 + j_2 - j_3)\left[\frac{(-1)^j}{(8\pi^2)(j+1)} \right. \\ &\quad \left. \times \sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}\right]. \end{aligned} \tag{A16}$$

Thus, (for the simple case of $\mathrm{AdS}_2 \times S^2$) we have explicitly shown the origin of the delta-function term, and presented the final expression for the 3-point overlap. This calculation can be repeated for the more general wave functions given in (53). The 3-point overlap will then be roughly of the form

$$\begin{aligned} V_3 &\sim \int d\phi e^{i(j_1+j_2-j_3)\phi} \int drL_{n_1}^{j^1}L_{n_2}^{j^2}L_{n_3}^{j^3} \\ &\sim \delta(j_1 + j_2 - j_3) \int drL_{n_1}^{j^1}L_{n_2}^{j^2}L_{n_3}^{j^3}, \end{aligned} \tag{A17}$$

and still yield a conserving delta function. On the other hand, in the case of the *tilde* representation the wave functions are

$$\tilde{\psi}(x, y) = e^{-(x^2+y^2)}H_{j+n}(x)H_n(y), \tag{A18}$$

and the overlap integral takes the form

$$\begin{aligned} V_3 &\sim \int dxe^{-3x^2}H_{j_1+n_1}(x)H_{j_2+n_2}(x)H_{j_3+n_3}(x) \\ &\quad \times \int dye^{-3y^2}H_{n_1}(y)H_{n_2}(y)H_{n_3}(y). \end{aligned} \tag{A19}$$

As one can verify, written in this basis the vertex no longer has a conserving $\delta$-function.


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