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# Remarks on string solitons

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We consider generalized self-duality equations for $U(2r)$ Yang-Mills theory on $\mathbb{R}^8$ with quaternionic structure. We employ the extended ADHM method in eight dimensions to construct exact soliton solutions of the low-energy effective theory of the heterotic string.

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## I. INTRODUCTION

In [1], an exact multi–five-brane soliton solution of the heterotic string theory was presented. This solution represented an exact extension of the three-level supersymmetric five-brane solutions of [2]. Exactness is shown for the heterotic solution based on algebraic effective action arguments and (4,4) worldsheet supersymmetry. The gauge sector of the heterotic solution possesses $SU(2)$ instanton structure in the four-dimensional space transverse to the five-brane. An exact solution with $SU(2)\times SU(2)$ instanton structure was found in [3]. This soliton preserves four of the 16 supersymmetries. In [4] a one-brane solution of heterotic theory was found, which is an everywhere smooth solution of the equations of motion. The construction of this solution crucially involves the properties of octonions. One of the many bizarre features of this soliton is that it preserves only one of the 16 space-time supersymmetries, in contrast to previously known examples of supersymmetric solitons which all preserve half of the supersymmetries. A two-brane solution of heterotic theory was found in [5,6]. This soliton preserves two of the 16 supersymmetries and hence corresponds to $N=1$ space-time supersymmetry in $(2+1)$ dimensions transverse to the seven dimensions where the Yang-Mills instanton is defined. Some generalization of one- and two-brane solutions was found in [7,8]. All these solutions are conformal to a flat space. In dimension six, the possibility of the existence of a non-conformally flat solution on the complex Iwasawa manifold was discussed in [9–11].

In all the above-named papers instanton solutions in various dimensions are extended to heterotic string solitons. In this paper we employ the extended ADHM method in eight dimensions to construct exact soliton solutions of the low-energy effective theory of the heterotic string.

## II. GENERALIZED SELF-DUALITY ON $\mathbb{R}^8$

In the first place, we define a basis $V_\mu$ with $(\mu)=(\mu_0,\mu_1)$ on $\mathbb{R}^8\simeq\mathbb{H}\oplus\mathbb{H}$ as a collection of two quaternionic column vectors realized as $4\times 2$ matrices,

$$V_{\mu_0}=\begin{pmatrix} e^\dagger_{\mu_0}\\ 0_2\end{pmatrix}\quad\text{and}\quad V_{\mu_1}=\begin{pmatrix} 0_2\\ e^\dagger_{\mu_1}\end{pmatrix},\tag{1}$$

where $\mu_k$ is a four-valued index and the matrices $(e^\dagger_{\mu_k})=(i\sigma_1,i\sigma_2,i\sigma_3,1)$. As in [12] we introduce the anti-Hermitian matrices

$$N_{\mu\nu}=\tfrac{1}{2}(V_\mu V_\nu^\dagger - V_\nu V_\mu^\dagger).\tag{2}$$

Notice that for any $\mu,\nu=1,\ldots,8$, we have $N_{\mu\nu}\in sp(2)$. To introduce generalized self-duality equations on $\mathbb{R}^8$, we define the total antisymmetric tensor

$$T_{\mu\nu\rho\sigma}=\frac{1}{12}\mathrm{tr}(V^\dagger_\mu V_{[\nu}V^\dagger_\rho V_{\sigma]}).\tag{3}$$

Then by direct calculation one finds that the matrix-valued tensor $N_{\mu\nu}$ is self-dual in the sense of [13] (see also [14]); i. e. it satisfies the eigenvalue equations

$$\tfrac{1}{2}T_{\mu\nu\rho\sigma}N_{\rho\sigma}=N_{\mu\nu}.\tag{4}$$

It is well known that the subgroup of $SO(8)$ which preserves the quaternionic structure and therefore (4) is isomorphic to $Sp(1)\times Sp(2)/\mathbb{Z}_2$.

With the help of the tensor (3) one may introduce an analog of the self-dual Yang-Mills equations for $U(2r)$ gauge fields on $\mathbb{R}^8$. Indeed, if $F_{\mu\nu}$ is the $su(2r)$-valued Yang-Mills field, then the generalized self-dual Yang-Mills equation in eight dimensions is

$$\tfrac{1}{2}T_{\mu\nu\rho\sigma}F_{\rho\sigma}=F_{\mu\nu}.\tag{5}$$

Obviously, Eq. (5) is invariant under $Sp(1)\times Sp(2)/\mathbb{Z}_2\subset SO(8)$ and any gauge field fulfilling (5) satisfies the second-order Yang-Mills equations due to the Bianchi identities. In four dimensions $T_{\mu\nu\rho\sigma}$ reduces to $\varepsilon_{\mu\nu\rho\sigma}$ and, hence, (5) coincides with the standard self-dual Yang-Mills equations.

## III. 'T HOOFT-TYPE SOLUTIONS IN EIGHT DIMENSIONS

Now we construct a solution of Eq. (5) (cf. [15]). In the notations of the Appendix we choose $n=r=1$ and $k=2$. For the ADHM ingredients $a$, $b_i$, and $\Psi$, we propose the ansatz

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$$a = \begin{pmatrix} \Lambda 1_2 \\ 0_2 \end{pmatrix}, \qquad b_i = \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \quad \text{and} \quad \Psi = \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix}, \tag{6}$$

where $\Lambda$ is a real constant and $i = 0, 1$. With this selection we obtain

$$\Delta^\dagger \Delta = (\Lambda^2 + x^\dagger x) \otimes 1_2, \tag{7}$$

where $x = x_1 + x_2$ and $x_i = x^{\mu_i} e^\dagger_{\mu_i}$. It is obvious that the conditions (A2) and (A3) are satisfied. Next, Eq. (A4) becomes

$$\Lambda \Psi_0 + x^\dagger \Psi_1 = 0_2, \tag{8}$$

which is solved by the solutions

$$\Psi_0 = \varphi^{-1/2} 1_2 \quad \text{and} \quad \Psi_1 = -x \frac{\Lambda}{x^\dagger x} \varphi^{-1/2}, \tag{9}$$

where the function $\varphi$ is fixed by the normalization condition (A5):

$$\varphi = 1 + \frac{\Lambda^2}{x^\dagger x}. \tag{10}$$

The relation (A6) is verified by direct calculation. Hence, our $(\Delta, \Psi)$ satisfies all conditions (A2)–(A6), and we can define a gauge potential via (A7) and obtain from (A8) a self-dual gauge field on $\mathbb{R}^8$.

Now we choose $r = k = n = 2$. For the ADHM ingredients we propose the constant $8 \times 4$ matrices

$$a = \begin{pmatrix} \Lambda_0 + \Lambda_1 \\ Q_0 + Q_1 \end{pmatrix}, \qquad b_i = \begin{pmatrix} 0 \\ -E_i \end{pmatrix}, \tag{11}$$

where $\Lambda_i$ is a real matrix, $E = E_1 + E_2$ is the identity matrix, and $i = 0, 1$. (Here and below, we use the symbols $S_0$ and $S_1$ for the $4 \times 4$ matrix of the form

$$\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}, \tag{12}$$

where $s = s^{\mu_i} e^\dagger_{\mu_i}$, respectively). It is obvious that the matrix

$$\Delta^\dagger \Delta = \Lambda_i \Lambda_i + (Q_i - x_i E_i)^\dagger (Q_i - x_i E_i) \tag{13}$$

is real and nondegenerate. Hence, the conditions (A2) and (A3) are true. In order to construct a solution of Eq. (5), we must find a matrix $\Psi = \Psi(x)$ satisfying the conditions (A4)–(A6). Suppose

$$\Psi = \sum_{i=0}^{1} \begin{pmatrix} -E_i \\ U_i \end{pmatrix} W_i, \tag{14}$$

where $W_0$ and $W_1$ are real $4 \times 4$ matrices. Then by direct calculation we get that the matrix (14) satisfies the conditions (A4) and (A5) if and only if the nonzero elements $\lambda_i$, $q_i$, $u_i$, and $w_i$ of the matrices $\Lambda_i$, $Q_i$, $U_i$, and $W_i$, respectively, are connected by the following relations:

$$u_i^\dagger = \lambda_i (q_i - x_i)^{-1}, \tag{15}$$

$$w_i^2 = (1 + u_i^\dagger u_i)^{-1}, \tag{16}$$

where we do not sum on the recurring indices and the difference $q_i - x_i \neq 0$. Using (15) and (16) we easily prove the completeness relations (A6). Hence, our $(\Delta, \Psi)$ satisfies all conditions (A2)–(A6), and we can obtain from (A8) a self-dual gauge field on $\mathbb{R}^8$. Note that one may restrict our solutions to a subspace $\mathbb{R}^4 \subset \mathbb{R}^8$. In this case we get the 't Hooft-type instanton solutions in four dimensions.

Note that generalizations of the solution (9) have been described in the papers [16,17]. The construction of a solution which generalizes (14) can be found in [18]. However for our purposes this will not be necessary.

## IV. HETEROTIC STRING SOLITONS

As in the Refs. [1–6] we search for a solution to lowest nontrivial order in $\alpha'$ of the equations of motion that follow from the bosonic action

$$S = \frac{1}{2k^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \Big( R + 4(\nabla\phi)^2 - \frac{1}{3} H^2 - \frac{\alpha'}{30} \mathrm{Tr} F^2 \Big), \tag{17}$$

where the three-form antisymmetric field strength is related to the two-form potential by the familiar anomaly equation

$$H = dB + \alpha' (\omega_3^L(\Omega) - \frac{1}{30} \omega_3^{\mathrm{YM}}(A)) + \ldots, \tag{18}$$

where $\omega_3$ is the Chern-Simons three-form and the connection $\Omega_M$ is a non-Riemannian connection related to the usual spin connection $\omega$ by

$$\Omega_M^{AB} = \omega_M^{AB} - H_M^{AB}. \tag{19}$$

We are interested in solutions that preserve at least one supersymmetry. This requires that in ten dimensions there exist at least one Majorana-Weyl spinor $\epsilon$ such that the supersymmetry variations of the fermionic fields vanish for such solutions,

$$\delta\chi = F_{MN} \Gamma^{MN} \epsilon, \tag{20}$$

$$\delta\lambda = (\Gamma^M \partial_M \phi - \tfrac{1}{6} H_{MNP} \Gamma^{MNP}) \epsilon, \tag{21}$$

$$\delta\psi_M = (\partial_M + \tfrac{1}{4} \Omega_M^{AB} \Gamma_{AB}) \epsilon, \tag{22}$$

where $\chi$, $\lambda$, and $\psi_M$ are the gaugino, dilatino, and gravitino fields, respectively.

Let us now show that our instanton solutions can be extended to a solitonic solution of the heterotic string. Consider the action of the ten dimensional low-energy effective theory of the heterotic string. The bosonic part of this action is (17). If we have the solution (9), then we can construct a five-brane solution. Indeed, the supersymmetry variations are determined by a positive chirality the Majorana-Weyl $SO(9,1)$ spinor $\epsilon$. Because of the five-brane structure, it decomposes under $SO(9,1) \supset SO(5,1) \times SO(4)$ as

$$16 \rightarrow (4_+, 2_+) \oplus (4_-, 2_-), \tag{23}$$

where $\pm$ subscripts denote the chirality of the representation. Then the ansatz

$$g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu}, \tag{24}$$

$$H_{\mu\nu\lambda} = -\epsilon^{\sigma}_{\mu\nu\lambda} \partial_\sigma \phi, \tag{25}$$

with the constant chiral spinor $\epsilon$ solves the supersymmetry equations with zero background Fermi fields provided the Yang-Mills gauge fields satisfy the instanton self-dual condition (5). Substituting the explicit gauge field strength (A8) for the instanton (9) into the anomalous Bianchi identity

$$dH = \alpha' \left( \mathrm{tr} R \wedge R - \frac{1}{30} \mathrm{Tr} F \wedge F \right), \tag{26}$$

one obtains the following dilaton solution (cf. [1]):

$$e^{-2\phi} = e^{-2\phi_0} + 8\alpha' \frac{(x^\dagger x + 2\Lambda^2)}{(x^\dagger x + \Lambda^2)^2} + O(\alpha'^2). \tag{27}$$

Note that the obtained string solution is not identical to [2]. Indeed, the translation $x_{\mu_i} \rightarrow x_{\mu_i} + q_{\mu_i}$ introduces eight location parameters in our solution. Four parameters localize the instanton in the subspace $\mathbb{R}^4 \subset \mathbb{R}^8$. The other four parameters restrict the choice of $\mathbb{R}^4$ in $\mathbb{R}^8$. Since the five-brane is transverse to $\mathbb{R}^4$, it follows that its selection in $M_{9,1}$ is not arbitrary. The solution in [2] does not have these restrictions.

If we have the soliton solution (14), then we can construct a double-instanton string solution analogue of (27). In this case the Majorana-Weyl $SO(9,1)$ spinor $\epsilon$ decomposes under $SO(9,1) \supset SO(1,1) \times SO(4) \times SO(4)$ for the $M^{9,1} \rightarrow M^{1,1} \times M^4 \times M^4$ decomposition. The ansatz

$$g_{\mu_i\nu_i} = e^{2\phi} \delta_{\mu_i\nu_i}, \tag{28}$$

$$H_{m_i n_i p_i} = -\varepsilon_{m_i n_i p_i}{}^{s_i} \partial_{s_i} \phi, \tag{29}$$

where $i = 0$ or $1$, solves the supersymmetry equations with zero background Fermi fields. Substituting the gauge field strength (A8) for the ansatz (14) into (26), we get the following dilaton solution:

$$e^{-2\phi} = e^{-2\phi_0} + 8\alpha' \frac{(x_i^2 + 2\lambda_i^2)}{(x_i^2 + \lambda_i^2)^2} + O(\alpha'^2), \qquad i = 0, 1. \tag{30}$$

If we restrict the solutions (27) and (30) to a subspace $\mathbb{R}^4 \subset \mathbb{R}^8$, then we recover the heterotic string solitons as derived in [2].

Note also that there are different solutions with more worldsheet supersymmetry (cf. [1,3]). These symmetric solutions are characterized, from the spacetime point of view, by $dH = 0$. This condition requires, according to (26), that the curvature $R(\Omega)$ should cancel against the instanton Yang-Mills field $F$. Both the algebraic effective action arguments and the (4, 4) worldsheet supersymmetry arguments of [1] can be used in essentially the same manner to demonstrate exactness of the string solutions.

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## APPENDIX

Here, we give an extended ADHM construction of an $n$-instanton solution for $u(2r)$-valued gauge fields in $4k$ dimensions (see [7]). This construction is based on a complex $(2n+2r) \times 2r$ matrix $\Psi$ and a complex $(2n+2r) \times 2n$ matrix

$$\Delta = a + \sum_{i=0}^{k-1} b_i (x_i \otimes 1_n), \tag{A1}$$

where $a$ and $b_i$ are constant $(2n+2r) \times 2n$ matrices and $x_i = x^{\mu_i} e^\dagger_{\mu_i}$ is a $2 \times 2$ matrix. These matrices must satisfy the following conditions:

$$\Delta^\dagger \Delta = f^{-1}, \tag{A2}$$

$$[\Delta^\dagger \Delta, V_\mu \otimes 1_n] = 0, \tag{A3}$$

$$\Delta^\dagger \Psi = 0, \tag{A4}$$

$$\Psi^\dagger \Psi = 1_{2r}, \tag{A5}$$

$$\Psi \Psi^\dagger + \Delta f \Delta^\dagger = 1_{2n+2r}. \tag{A6}$$

The relation (A6) means that $\Psi\Psi^\dagger$ and $\Delta f \Delta^\dagger$ are projectors onto orthogonal complementing subspaces of $\mathbb{C}^{2n+2r}$. For $(\Delta, \Psi)$ satisfying (A2)–(A6) the gauge potential is chosen in the form

$$A = \Psi^\dagger d\Psi. \tag{A7}$$

Indeed, after straightforward calculation the components of the gauge field $F$ then take the form

$$F_{\mu\nu} = 2\Psi^\dagger b N_{\mu\nu} f b^\dagger \Psi, \tag{A8}$$

where the $(2n+2r) \times 2nk$ matrix $b = (b_0 \ldots b_{k-1})$ and $\mu, \nu = 0, \ldots, k-1$. It is obvious that for $k = 2$ the field strength (A8) satisfies the self-dual Yang-Mills equations (5).

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