Two-dimensional topological field theories coupled to four-dimensional BF theory

Merced Montesinos*

Departamento de Física, Cinvestav, Avenida Instituto Politécnico Nacional 2508, San Pedro Zacatenco, 07360, Gustavo A. Madero, Ciudad de México, México

Alejandro Perez+

Centre de Physique Théorique‡, Campus de Luminy, 13288 Marseille, France and Universidade Federal do Espiritu Santo, Campus Goiabeiras, 29060-900 Vitoria, Brasil.

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Four-dimensional BF theory admits a natural coupling to extended sources supported on two-dimensional surfaces or string world sheets. Solutions of the theory are in one to one correspondence with solutions of Einstein equations with distributional matter (cosmic strings). We study new (topological field) theories that can be constructed by adding extra degrees of freedom to the two-dimensional world sheet. We show how two-dimensional Yang-Mills degrees of freedom can be added on the world sheet, producing in this way, an interactive (topological) theory of Yang-Mills fields with BF fields in four dimensions. We also show how a world sheet tetrad can be naturally added. As in the previous case the set of solutions of these theories are contained in the set of solutions of Einstein's equations if one allows distributional matter supported on two-dimensional surfaces. These theories are argued to be exactly quantizable. In the context of quantum gravity, one important motivation to study these models is to explore the possibility of constructing a background-independent quantum field theory where local degrees of freedom at low energies arise from global topological (world sheet) degrees of freedom at the fundamental level.

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I. INTRODUCTION

Topological field theories are simple examples of background-independent field theories for which quantization can be completely worked out. These theories are a natural playground where conceptual as well as technical issues in background-independent quantum theory can be addressed in detail. Three-dimensional vacuum general relativity is an important example of a topological field theory. Interestingly, the topological nature of the theory can be maintained if matter is added in the form of topological defects representing massive and spinning point particles [1]. Interest in the quantization of $2 + 1$ gravity coupled to point particles has been revived in the context of the spin foam [2] and loop quantum gravity [3] approaches to the nonperturbative and background-independent quantization of gravity. On the one hand, this simple system provides a nontrivial example where the strict equivalence between the covariant and canonical approaches can be demonstrated [4]. On the other hand, intriguing relationships with field theories with infinitely many degrees of freedom have been obtained [5,6]. The generalization of these models to higher dimensions has been studied in [7]. As it is shown there, membranelike defects of dimension $d - 3$ are a natural form of matter that couples to d-dimensional BF theory [8]. The resulting theory is in turn also a topological theory and can be completely quantized using the techniques of loop quantum gravity. Among these higher dimensional models the four-dimensional one (which couples to stringlike defects) is of singular interest due to the special role played by 4-dimensional BF theory in the construction of spin foam models of four-dimensional quantum gravity.

At first look these strings are a rather dull form of matter: at their location there are conical singularities of the curvature tensor and the equations of motion imply that the string world sheet is locally flat [9] (vibrational modes of the strings are pure gauge). Nevertheless, as we will argue in this paper, the feature that makes these strings interesting is the fact that they are extended objects (this is also behind their exotic statistical properties [10]). This will allow us to couple four-dimensional BF theory with more physically appealing degrees of freedom. As the set of possibilities is quite vast, we will restrict our attention to certain world sheet theories that satisfy the following two properties: (a) they can be naturally (or minimally) coupled to BF theory in 4d, and (b) the coupled system defines a (topological) theory with no local degrees of freedom. Because of the close relationship between four-dimensional BF theory and gravity, requirement (a) is expected to produce physically interesting models, as they might provide natural candidates for the coupling of spin foam models of gravity with natural forms of matter.

*merced@fis.cinvestav.mx
+perez@cpt.univ-mrs.fr
‡Unité Mixte de Recherche (UMR 6207) du CNRS et des Universités Aix-Marseille I, Aix-Marseille II, et du Sud Toulon-Var; laboratoire afilié à la FRUMAM (FR 2291)

Requirement (b) implies that the models studied here are expected to be nonperturbatively quantizable.

We believe that the study of these simple topological models can be of more relevance than a simple exercise in the application of nonperturbative quantization techniques. We would like to explore the possibility that topological theories, containing low dimensional objects, could be used to construct a background-independent quantum field theory with infinitely many (“quasilocal”) degrees of freedom. This is in fact our motivation for imposing requirement (b) above.

The article is organized as follows: In Sec. II we briefly review the coupling of strings to four-dimensional BF theory. In Sec. III we show how Yang-Mills degrees of freedom can be added to the strings. We analyze the equations of motion of the coupled system and perform the canonical analysis to prove that the theory is topological. In Sec. IV we add a tetrad field on the world sheet and obtain an interesting model whose equations of motion resemble those of general relativity in a curious way. In Sec. V we study a purely two-dimensional model of background-independent Yang-Mills theory which naturally follows from the results of the previous sections. In Sec. VII we present a speculative discussion about the possibility of using topological theories of the type introduced in this paper in order to define a background-independent quantum field theory with infinitely many degrees of freedom.

II. STRINGS COUPLED TO FOUR-DIMENSIONAL BF THEORY

The coupling of $(d-3)$-dimensional membranes to d-dimensional BF theory (defined for a large class of structure groups) was introduced in [7]. Here we concentrate on the case of strings coupled to four-dimensional BF theory with structure group $SO(3,1)$ (see Refs. [11] for its canonical analysis and Refs. [12,13] for alternative action principles). If we denote $\mathcal{M}$ the four-dimensional space-time manifold and $\mathcal{W}\subset\mathcal{M}$ the two-dimensional world sheet of the string, the action defining the coupling is given by

$$S_{ST-\mathrm{BF}} = \int_{\mathcal{M}} B_{IJ}\wedge F^{IJ}(A) + \tau\int_{\mathcal{W}}(B+d_A q)^{IJ}p_{IJ}, \quad (1)$$

where $I, J = 1, \ldots, 4$, and if we denote $T_{IJ}\in so(3,1)$ the generators of the Lie algebra then $q = q^{IJ}T_{IJ}$ is a $so(3,1)$-valued 1-form on $\mathcal{W}$ and $p = p^{IJ}T_{IJ}$ is a $so(3,1)$-valued function on $\mathcal{W}$. This action is invariant under the gauge transformations:

$$\begin{aligned} &B\mapsto gBg^{-1}, \qquad B\mapsto B+d_A\eta,\\ &A\mapsto gAg^{-1}+gdg^{-1}, \qquad q\mapsto q-\eta, \\ &q\mapsto gqg^{-1}, \qquad p\mapsto gpg^{-1}, \end{aligned} \quad (2)$$

where $g\in C^\infty(\mathcal{M},G)$ and η is any $\mathfrak{g}$-valued $(d-3)$-form. Varying the action with respect to the B field implies that the connection A is flat except at $\mathcal{W}$:

$$F = -\tau p\,\delta_{\mathcal{W}}, \quad (3)$$

where $\delta_{\mathcal{W}}$ is the distributional 2-form (current) associated to the string world sheet. So, the string causes a conical singularity in the otherwise flat connection A. The strength of this singularity is determined by the field p, which plays the role of a “momentum density” for the string. Note that while the connection A is singular in the directions transverse to $\mathcal{W}$, it is smooth and indeed flat when restricted to $\mathcal{W}$. Thus, the equation of motion obtained from varying q makes sense:

$$d_A p = 0. \quad (4)$$

This expresses conservation of momentum density and in fact implies that the field p remains in the same conjugacy class, hence it can be written as $p=\tau\lambda v\lambda^{-1}$ for $v\in so(3,1)$ a normalized vector and $\lambda\in SO(3,1)$. The constant τ defines the string tension. Conjugacy classes of $so(3,1)$ are labeled by the two Lorentz Casimirs. So far, we have fixed only one by choosing the string tension $\tau^2 = p_{IJ}p^{IJ}$. The other Casimir defines an extra parameter $s = p_{IJ}p_{KL}\epsilon^{IJKL}$ (the geometric meaning of s will be discussed below). Notice that the strength of the conical singularity at the location of the strings is, in this sense, nondynamical. This will change in the model of Sec. III.

Assuming that the space-time manifold is of the form $\mathcal{M}=\Sigma\times\mathbb{R}$, we choose local coordinates (t, x^a) for which Σ is given as the hypersurface $\{t=0\}$. By definition, x^a with $a=1,2,3$ are local coordinates on Σ. We also choose local coordinates (t, s) on the 2-dimensional world sheet $\mathcal{W}$, where $s\in[0,2\pi]$ is a coordinate along the one-dimensional string formed by the intersection of $\mathcal{W}$ with Σ. Performing the standard Legendre transformation one obtains $E^a_i = \epsilon^{abc}B_{ibc}$ as the momentum canonically conjugate to A^i_b. Similarly, p_{IJ} is the momentum canonically conjugate to $q^{IJ}_1 = q^{IJ}_a(\partial_\sigma)^a$. The phase-space variables satisfy the following constraints:

$$L_{IJ} := D_a E^a_{IJ} - 2\delta_{\mathcal{S}}[q_{1[I|M|}p^M_{J]}]\approx 0, \quad (5)$$

$$K^a_{IK} := \epsilon^{abc}F^{IJ}_{bc}(x) + \delta_{\mathcal{S}}[p^{IJ}(\partial_\sigma)^a]\approx 0. \quad (6)$$

Here $\mathcal{S}\subset\Sigma$ denotes the one-dimensional curve representing the string, parametrized by $x_{\mathcal{S}}(s)$, and for any field ϕ on $\mathcal{S}$, we define

$$\delta_{\mathcal{S}}[\phi] := \int_{\mathcal{S}}\phi\,\delta^{(3)}(x-x_{\mathcal{S}}(s)).$$

The constraint (5) is the modified Gauss law of BF theory due to the presence of the string. The constraint (6) is the modified curvature constraint containing the dynamical information of the theory. This constraint implies that the connection A is flat away from the string $\mathcal{S}$. Some care must be taken to correctly interpret the constraint for points on $\mathcal{S}$. By analogy with the case of 3d gravity, the correct interpretation is that the holonomy of an infinitesimal loop circling the string at some point $x \in \mathcal{S}$ is $\exp(-p(x)) \in G$, where $p = \tau\lambda v\lambda^{-1}$ as before. This describes the conical singularity of the connection at the string world sheet.

The BF phase-space variables satisfy the standard commutation relations:

$$\{E^a_i(x), A^j_b(y)\} = \delta^a_b\delta^j_i\delta^{(3)}(x-y), \qquad \{E^a_i(x), E^b_j(y)\} = \{A^i_a(x), A^j_b(y)\} = 0. \tag{7}$$

The phase space of the string is parametrized in terms of the momentum p^{IJ} and the "total angular momentum" $J_{IJ} = 2q_{1[I|M|}p^M{}_{J]}$. The Poisson brackets of these variables are given by

$$\{p_{IJ}(s), J_{KL}(s')\} = c^{ST}_{IJKL}p_{ST}(s)\delta^{(1)}(s-s'), \qquad \{J_{IJ}(s), J_{KL}(s')\} = c^{ST}_{IJKL}J_{ST}(s)\delta^{(1)}(s-s'), \tag{8}$$

where c^{ST}_{IJKL} are the structure constants of $so(3,1)$, and

$$\{J_{IJ}(s), \lambda(s')\} = -T_{IJ}\lambda(s)\delta^{(1)}(s-s'). \tag{9}$$

The string variables are still subject to the following first class constraints:

$$\mathrm{tr}[T_{IJ}\lambda z\lambda^{-1}]J^{IJ} = 0, \qquad \mathrm{tr}[p\lambda z\lambda^{-1}] = \tau\mathrm{tr}[vz], \tag{10}$$

where $z \in \mathfrak{g}$ is such that $[z, v] = 0$. The last constraint is the generalization of the mass shell condition for point particles in 3d gravity. The Poisson bracket of the string variables with the BF variables is trivial, as well as the Poisson brackets among the p_{IJ}.

Geometrical interpretation

Here we present a brief account of the analysis carried out in [9]. The full set of equations of motion of the theory is

$$F(A) = -p\delta_{\mathcal{W}}, \qquad d_AB = -[q,p]\delta_{\mathcal{W}}, \qquad d_Ap|_{\mathcal{W}} = 0, \qquad \phi^*_{\mathcal{W}}(B + d_Aq) = 0, \tag{11}$$

where $\phi^*_{\mathcal{W}}$ denotes the pullback of the corresponding 2-forms to $\mathcal{W}$. Therefore, the field configuration $A = 0$, $B = 0$, $q = 0$, $p =$ constant gives a solution to the equations of motion in an open region $U \subset M$ such that any open set containing points of $\mathcal{W}$ has points outside U. Since the theory is topological, all the solutions are equivalent to this one in U through a gauge transformation. Let us assume that we have a coordinate system in U with coordinate functions X^I, (for $I = 1, \cdots, 4$). In order to recover an interpretation of fields on a flat background we can make a gauge transformation of the type (2) with gauge parameter $\eta^{IJ} = X^{[I}dX^{J]}$. In this gauge the solution is

$$B^{IJ}_{ab} = e^I_{[a}e^J_{b]} = \delta^I_{[a}\delta^J_{b]}, \qquad q^{IJ}_a = X^{[I}d_aX^{J]}. \tag{12}$$

We see that in this gauge the B field defines a flat background geometry. There is still the residual gauge freedom that maintains this property of the B field given by gauge transformations of the form $\eta_0 = df$ for some arbitrary f. We call this family of gauges *flat gauges*. The integrability conditions that follow from the equation $dB = [q,p]\delta_{\mathcal{W}}$ imply that $d[p,q] = 0$ or equivalently that $[p,q] = d\alpha$ for some potential α. If $\alpha = 0$, it can be shown that $[p,q] = 0$ has nontrivial solutions if $s = p_{IJ}p_{KL}\epsilon^{IJKL} = 0$.[1] In that case the string world sheet $X^I(\sigma, t)$ is given by a plane in Minkowski space-time passing through the origin defined by either the equation $p^{IJ}X_I = 0$ or $\star p^{IJ}X_I = 0$. We can translate the plane off the origin by choosing $\alpha^{IJ} = C^{[I}X^{J]}$ (this choice sends X^I to $X^I + C^I$). If $s \neq 0$ then equation $[p,q] = 0$ implies $X^I = 0$.

One can establish a strict connection between these solutions and solutions of general relativity representing a cosmic string. In cylindrical coordinates $\{\partial_t, \partial_r, \partial_\varphi, \partial_z\}$, such that the string is lying along the z axis and goes through the origin, the metric of a cosmic string solution of tension τ is:

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu = -dt^2 + dr^2 + (1-a)^2r^2d\varphi^2 + dz^2, \tag{13}$$

where $a = (1 - 4G\tau)$, G is the Newton constant. The dual coframe for the above metric is written

$$e^0 = dt, \qquad e^1 = \cos\varphi dr - ar\sin\varphi d\varphi, \qquad e^2 = \sin\varphi dr + ar\cos\varphi d\varphi, \qquad e^3 = dz, \tag{14}$$

such that $ds^2 = e^I \otimes e^J\eta_{IJ}$. The spin connection (s.t. $d_Ae = 0$) is

$$A = A^{IJ}_\mu J_{IJ}dx^\mu = 4G\tau J_{12}d\varphi, \tag{15}$$

where J_{IJ} are the $so(3,1)$ generators. We can identify now the string momentum p above, namely $p^{IJ}J_{IJ} = \tau J_{12}$. From the distributional identity $dd\varphi = 2\pi\delta^2(r)dxdy$ ($x = r\cos\varphi$, $y = r\sin\varphi$), it is immediate to compute the torsion $T = T^0e_0$ and curvature $F = F^{12}\sigma_{12}$ of the cosmic string induced metric:

$$T^0 = 0, \qquad F^{12} = 8\pi G\tau\delta^2(r)dxdy. \tag{16}$$

The above fields are clearly a solution of Einstein's equations with distributional matter

$$\epsilon_{IJKL}e^J \wedge F^{KL} = 8\pi G\tau\epsilon_{IJKL}e^JJ^{KL}_{12}\delta_{\mathcal{W}}. \tag{17}$$

The previous solution is in one to one correspondence with the solution of (11)

$$B = *(e \wedge e), \qquad A = 4G\tau J_{12}d\varphi, \qquad p = \tau J_{12}, \qquad q^{IJ} = (zdt - tdz)\delta^{[I}_0\delta^{J]}_3 = (zdt - tdz)J^{IJ}_{21}. \tag{18}$$

[1] If we allow for complex p^{IJ}, then solutions also exist if p^{IJ} is self-dual (or anti self-dual).

One can construct a two strings solution by "superimposing" two solutions of the previous kind at different locations (notice that the equations are nonlinear so the new solution is not the sum of two solutions). It can be shown that the torsion $d_A B$ is proportional to the distance separating the world sheets in the flat-gauge where $B^{IJ}_{ab} = \delta^{[I}_a \delta^{J]}_b$. More strings can be added in a similar fashion.

III. MINIMAL COUPLING OF WORLD-SHEET YANG-MILLS WITH 4D BF THEORY

Yang-Mills theory in two dimensions can be written in a way that resembles BF theory if one is given a 2-form field ρ, namely

$$S_{\rm YM} = \int_{\mathcal{W}} [\mathcal{E}_a F^a(A) + \rho \mathcal{E}_a \mathcal{E}^a], \tag{19}$$

where $a = 1, \ldots, \dim(\mathfrak{g})$ are internal indices labeling the elements of a basis of the Lie algebra $\mathfrak{g}$ of the gauge group of our choice G (we require G to be compact and $\mathfrak{g}$ to have an invariant metric with which we raise and lower internal indices). The field $A = (A^a_\mu dx^\mu) \otimes J_a$ is the $\mathfrak{g}$-valued connection 1-form, $[J_a, J_b] = f^c{}_{ab} J_c$ where $f^c{}_{ab}$ are the structure constants with respect to the basis $\{J_a\}$. Under these assumptions the internal metric can be taken as $k_{ab} = c{\rm Tr} J_a J_b$ (assuming a matrix form for the generators J_a and c is a constant that depends on the dimension of the representation of the generators J_a). The field $\mathcal{E}_a$ is a collection of $\dim(\mathfrak{g})$ many 0-forms. One can show that if ρ is nondegenerate (i.e., a volume form) then the previous action is equivalent to the standard Yang-Mills action

$$S_{\rm YM} = \int_{\mathcal{W}} \sqrt{-g} g^{\mu\nu} g^{\rho\sigma} F^a_{\mu\rho} F_{\nu\sigma a},$$

where the 2d metric $g_{\mu\nu}$ is such that $\rho = \sqrt{-g} dx^1 \wedge dx^2$. If one makes the canonical analysis of the BF-like action above one finds that the total Hamiltonian is not weakly vanishing due to the presence of the background structure given by the (nondynamical) ρ. It is also easy to check through the canonical analysis that the theory has no local degrees of freedom. Sometimes it is said that 2d YM is topological; however, this is not strictly the case because, even though the degrees of freedom are global (and certainly tied to the topology of $\mathcal{W}$), they are also related to the background structure ρ.

The simplest way of coupling two-dimensional Yang-Mills theory with four-dimensional BF theory to produce a background-independent field theory is to combine the B field and the world sheet variable p to build a volume 2-form $\rho = B^{IJ} p_{IJ}$ on the world sheet. The result is given by the following action:

$$S_{\rm BFYM} = \int_{\mathcal{M}} B_{IJ} \wedge F^{IJ}(\omega) + \int_{\mathcal{W}} ([B^{IJ} \mathcal{E}_a \mathcal{E}^a - d_\omega q^{IJ}] p_{IJ} + \mathcal{E}_a F^a(A)). \tag{20}$$

The equations of motion of the new model are

$$F(\omega) + \delta_{\mathcal{W}}[\mathcal{E}_a \mathcal{E}^a p] = 0, \qquad d_\omega B + \delta_{\mathcal{W}}[qp] = 0, \qquad \phi^*_{\mathcal{W}}(\mathcal{E}_a \mathcal{E}^a B - d_\omega q) = 0, \tag{21}$$

and

$$2B \cdot p \mathcal{E}^a + F^a(A) = 0. \tag{22}$$

We have not explicitly written the equations $d_\omega p = 0$, and $d_A \mathcal{E}^a = 0$, as they are implied by the integrability conditions arising from the Bianchi identities for the curvature of ω and A, respectively.

Now we show that the new model is in fact a topological field theory (i.e. background-independent theory with no local degrees of freedom). In order to do this, we perform the 3 + 1 decomposition of the previous action and analyze its phase-space structure. The unconstrained phase space is parametrized by the canonical variables $(E^\mu_{IJ}, \omega^{KL}_\nu)$ and (p^{IJ}, q^{KL}_1) (given in the previous section) plus the Yang-Mills canonical pairs $(\mathcal{E}_a, \mathcal{A}^b_1)$. The constraints relating the bulk degrees of freedom with the ones on the world sheet are

$$L_{IJ} := d_{\omega_\mu} E^\mu_{IJ} + 2\delta_{\mathcal{S}}[q_{[I|M|} p^M{}_{J]}] \approx 0, \tag{23}$$

$$K^{\mu IJ} := \epsilon^{\mu\nu\rho} F^{IJ}_{\nu\rho}(x) + \delta_{\mathcal{S}}[\mathcal{E}_a \mathcal{E}^a p^{IJ} \partial^\mu_\sigma] \approx 0. \tag{24}$$

Notice that L_{IJ} is precisely the same as (5), while K^{IJ} is a simple modification of (6). In fact there are new constraints

$$G_a := d_{\mathcal{A}} \mathcal{E}_a \approx 0, \tag{25}$$

which is the standard Gauss law of Yang-Mills. These equations (together with Hamilton's equations of motion) imply that $\mathcal{E}_a \mathcal{E}^a =$ constant. It is easy to see that the constraint algebra closes forming a first class system of $6 + 18 + \dim(\mathfrak{g})$ local constraints for the same number of configuration variables $\{q^{IJ}_1, \omega^{IJ}_\mu, \mathcal{A}^a_1\}$. The model has no local degrees of freedom.[2] The curvature constraint implies that the space-time connection is flat in the bulk and there is a conical singularity at the string. The strings on Σ can be viewed as flux lines of Yang-Mills electric field which back react with the environment producing a conical singularity whose strength is modulated by the Yang-Mills "energy density" $\rho_{\mathcal{E}} = \delta_{\mathcal{S}}[\mathcal{E}_a \mathcal{E}^a p_{IJ}]$. As mentioned in the introduction the strength of the curvature singularity is now dynamical.

IV. ADDING A WORLD SHEET "FRAME" FIELDS

The idea follows from the observation that the two-dimensional field theory defined by the following action

[2]There is a subtlety concerning the constraints K_{IJ}. In fact when we are away from the string the source term vanishes and the Bianchi identity implies that only 3 out of the 6 ones are independent. On the string, the Bianchi identity implies $d_\omega p = 0$ which is indeed an independent condition.

has no local degrees of freedom

$$S = \int_{\mathcal{W}} ([dq^{IJ} + *(e^I \wedge e^J)]p_{IJ} + \pi_I de^I), \qquad (26)$$

where $*(e^I \wedge e^J) = \frac{1}{2}\varepsilon^{IJ}_{KL} e^K \wedge e^L$, $\mathcal{W}$ is a two-dimensional surface, $q^{IJ} = -q^{JI}$ is a set of six 1-forms on $\mathcal{W}$, e^I is a set of four 1-forms on $\mathcal{W}$, $p_{IJ} = -p_{JI}$ is a set of six 0-forms (functions) on $\mathcal{W}$, π_I is a set of four 0-forms (functions) on $\mathcal{W}$. In principle, there are other terms that can also be added to the action, for instance, $(d * q^{IJ})p_{IJ} = dq^{IJ} * p_{IJ}$ and $(e^I \wedge e^J)p_{IJ}$.

In order to count the number of degrees of freedom, let us perform the canonical analysis of this model. Let $(y^a) = (y^0, y^1) = (\tau, \sigma)$ be local coordinates on $\mathcal{W}$ which is assumed to have the form $\mathcal{W} = S \times \mathbb{R}$; the coordinate time τ labels the points along $\mathbb{R}$ and the space coordinate σ labels the points on $\mathcal{S}$ which is assumed to have the topology of S^1. Therefore, using

$$q^{IJ} = q^{IJ}_a dy^a = q^{IJ}_0 d\tau + q^{IJ}_1 d\sigma, \qquad e^I = e^I_0 d\tau + e^I_1 d\sigma, \qquad (27)$$

the action (26) becomes

$$S = \int_{\mathbb{R}} d\tau \int_{\mathcal{S}} d\sigma (\dot{q}^{IJ}_1 p_{IJ} + \dot{e}^I_1 \pi_I - \lambda^{IJ}\mathcal{D}_{IJ} - \lambda^I \mathcal{G}_I), \qquad (28)$$

where $\lambda^{IJ} := -q^{IJ}_0$ and $\lambda^I := -e^I_0$ are Lagrange multipliers imposing the constraints

$$\mathcal{D}_{IJ} = \partial_\sigma p_{IJ} \approx 0, \qquad (29)$$

$$\mathcal{C}_I = \partial_\sigma \pi_I + \varepsilon^{KL}{}_{IJ} e^J_1 p_{KL} \approx 0. \qquad (30)$$

There are no more constraints. Smearing the constraints with test fields

$$D(N) = \int_{\mathcal{S}} d\sigma N^{IJ} \mathcal{G}_{IJ}, \qquad C(a) = \int_{\mathcal{S}} d\sigma a^I \mathcal{C}_I, \qquad (31)$$

to compute their Poisson brackets leads to

$$\{D(N), D(M)\} = 0, \qquad \{D(N), C(a)\} = 0, \qquad \{C(a), C(b)\} = D(*[a, b]), \qquad (32)$$

with $[a, b]^{IJ} := a^I b^J - a^J b^I$. Thus, all the 10 constraints are first-class for the 10 configuration variables (q^{IJ}_1, e^I_1). Therefore, the system has no local degrees of freedom, it is a topological field theory.

In the spirit of what was done in the previous section now we couple this world sheet action to the four-dimensional BF theory in such a way to maintain the topological character of the model. There is a natural choice for the coupling leading to the new model introduced in this section, namely:

$$S_{\text{BFYMGR}} = \int_{\mathcal{M}} B_{IJ} \wedge F^{IJ}(\omega) + \int_{\mathcal{W}} ([B^{IJ}\mathcal{E}_a\mathcal{E}^a - d_\omega q^{IJ} + *(e^I \wedge e^J)]p_{IJ} + \pi_I d_\omega e^I + \mathcal{E}_a F^a(A)). \qquad (33)$$

We call this model BFYMGR (where GR stands for general relativity) due to the suggestive similarity of the equations of motion with those of general relativity in the first order formalism. In order to make this statement more explicit let us analyze the equations of motion of the model. The observation is that on the world sheet variations with respect to p imply that $B = \mathcal{E}^{-2}(d_\omega q - *(e \wedge e))$, hence the B field is simple up to a gauge transformation. Therefore, the simplicity constraints that reduce BF theory to general relativity are satisfied on the world sheet. The conclusion is more transparent if we study the remaining equations of motion. For instance we have

$$F^{IJ} = -p^{IJ}\mathcal{E}^2\delta_{\mathcal{W}} \rightarrow \bar{F}^{IJ}_{\mu\nu} = -p^{IJ}\mathcal{E}^2 \quad \text{and} \quad \varepsilon_{IJKL} e^J p^{KL} = d_\omega \pi_I, \qquad (34)$$

where $\bar{F}^{IJ}_{\mu\nu}$ is the smearing of the curvature tensor on a two-dimensional surface dual to the world sheet along the coordinates $\mu - \nu$, more precisely

$$\bar{F}^{\ IJ}_{\mu\nu} := \int_{\mu-\nu} F^{IJ}.$$

Now we can appropriately combine the previous equations and obtain

$$\epsilon^{\mu\nu\rho\tau}\varepsilon_{IJKL} e^J_\nu \bar{F}^{KL}_{\rho\tau} = \epsilon^{\mu\nu}(d_\omega \pi_I)_\nu \mathcal{E}^2, \qquad (35)$$

where $\epsilon^{\mu\nu} := \epsilon^{\mu\nu\rho\tau}(dt)_\rho(d\sigma)_\tau$, and we have assumed that $\mathcal{E}^2$ is nonvanishing in order to bring it to the right-hand side. The previous equation has a suggestive similarity to Einstein's equation with source $T_{\mu\nu} = t_{I(\mu}e^I_{\nu)}$ where $t_{I\mu} = (d_\omega \pi_I)_\mu \mathcal{E}^2$. This is why we call this topological model BFYMGR.

We have emphasized the similarity of this model with Einstein's theory of gravity in order to motivate the introduction of this model. Now let us stress why this is quite different in fact. The main reason is that, in contrast with general relativity, this model is a topological theory with no local excitations. This conclusion becomes transparent in the Hamiltonian analysis which yields the following set of constraints for the canonical variables $(E^\mu_{IJ}, \omega^{KL}_\nu)$, (p^{IJ}, q^{KL}_1), $(\mathcal{E}_a, \mathcal{A}^b_1)$, and (π_I, e^J_1)

$$L_{IJ} := d_{\omega_\mu} E^\mu_{IJ} + 2\delta_{\mathcal{S}}[q_{[I|M|}p^M{}_{J]}] \approx 0,$$
$$K^{\mu IJ} := \epsilon^{\mu\nu\rho} F^{IJ}_{\nu\rho}(x) + \delta_{\mathcal{S}}[\mathcal{E}_a\mathcal{E}^a p^{IJ}\partial^\mu_\sigma] \approx 0,$$
$$G_a := d_{\mathcal{A}}\mathcal{E}_a \approx 0,$$

which are just the same as (6), (23), and (25) in addition to the new world sheet constraints

$$C_I := d_\omega \pi_I + 2e^J * p_{IJ} \approx 0. \qquad (36)$$

It is easy to see using the results of the previous sections that the constraints form a first-class set of $24 + \dim(\mathfrak{g})$ local constraints for the same number of configuration variables. The degrees of freedom are global.

V. A TWO-DIMENSIONAL BACKGROUND-INDEPENDENT YANG-MILLS THEORY

Using what we have learned, we can also define a two-dimensional background-independent Yang-Mills theory by making the 2-form ρ appearing in Eq. (19) dynamical in an world sheet intrinsic way: namely $\rho = (e^I \wedge e^J)p_{IJ}$. The resulting action is

$$S_{\rm TYM} = \int_{\mathcal{W}} ([dq^{IJ} + *(e^I \wedge e^J) + e^I \wedge e^J \mathcal{E}_a \mathcal{E}^a] p_{IJ} + \mathcal{E}_a F^a(A) + \pi_I de^I). \tag{37}$$

The canonical analysis performed along the lines of the one corresponding to the previous model leads to the following constraints

$$\mathcal{G}_a = d_A \mathcal{E}_a \approx 0, \tag{38}$$

$$\mathcal{D}_{IJ} = \partial_\sigma p_{IJ} \approx 0, \tag{39}$$

$$\mathcal{C}_I = \partial_\sigma \pi_I + 2 e^J_1 p_{IJ} \mathcal{E}_a \mathcal{E}^a + 2 e^J_1 * p_{IJ} \approx 0. \tag{40}$$

The first one is the familiar Gauss law of Yang-Mills theory while the remaining ones correspond to the appropriate modification of the ones obtained above. The constraint algebra gives

$$\begin{aligned} &\{G(\alpha), G(\beta)\} = G([\alpha,\beta]_{\mathfrak{g}}),\\ &\{D(N), D(M)\} = 0,\\ &\{D(N), C(a)\} = 0,\\ &\{C(a), C(b)\} = D(*[a,b] + \mathcal{E}^2[a,b]) + G(2[a,b]^{IJ} p_{IJ}\mathcal{E}), \end{aligned} \tag{41}$$

with $[a,b]^{IJ} := a^I b^J - a^J b^I$ and $[\alpha,\beta]_{\mathfrak{g}}$ is the commutator in the Lie algebra $\mathfrak{g}$. The constraint algebra closes and gives a first-class system. As before, we have $10 + \dim(\mathfrak{g})$ local constraints for the same number of configuration variables; hence the system is a topological field theory.

We end this section with a remark. Notice that the constraint algebra has field-dependent structure constants. This is characteristic of the constraint algebra of general relativity, although here the field dependence is much simpler since the quantity $\mathcal{E}^2$ is constant on the world sheet due to the Gauss constraint. These are genuine field-dependent structure constants.

VI. QUANTIZATION

We have shown how the coupling of four-dimensional BF theory to strings introduced in [7] allows for the definition of a large class of topological field theories with physically interesting kinematical degrees of freedom. The set of possibilities is indeed very large so we have concentrated here on two cases of special interest: world sheet Yang-Mills theories defined in terms of structure groups G possessing an ad_G invariant metric in their Lie algebra $\mathfrak{g}$, and a world sheet tetrad (with intriguing resemblance with general relativity).

The fact that these models are topological indicates that their nonperturbative quantization should be well-defined. Indeed the quantization of the model of Sec. III follows straightforwardly from the results of [7,9]. This should be clear from the fact that the phase-space structure presented in Sec. III is quite similar to the one of the theory briefly reviewed in Sec. II whose loop quantization is set up in [7] and completely worked out in [9]. The only new ingredients are the Yang-Mills unconstrained degrees of freedom which are especially well suited for the application of loop variables techniques.

More precisely, a basis of the kinematical Hilbert space—space of solutions of all quantum constraints with the sole exception of the curvature constraint (24)—of the model (20) is given by: (1) A bulk $SO(3,1)$ spin network functional of the $SO(3,1)$-connection A based on a graph $\gamma \in \Sigma$ with open ends at n points on the string $\mathcal{S}$, (2) an n-point spin functional of λ (recall that the variable $p = \lambda v \lambda^{-1}$ for $v\mathfrak{g}$ normalized and $\lambda \in G$), (3) a functional of the G-connection $\mathcal{A}$ given by the trace of the Wilson loop of $\mathcal{A}$ around the string $\mathcal{S}$ in an unitary irreducible representation of G (Fig. 1). If G is compact we can always think of the latter quantum number as $n \in \mathbb{N}$, where n labels the nth eigenvalue ϵ_n of the square of the electric field $\widehat{\mathcal{E}^a\mathcal{E}_a}$. The physical Hilbert space is obtained by imposing the

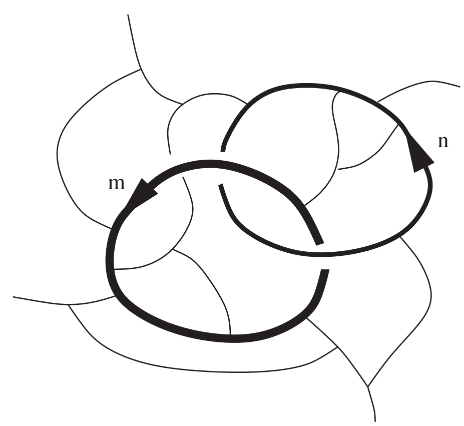

FIG. 1. The elements of a natural basis of the kinematical Hilbert can be written as the product of: (1) A functional of the Lorentz connection labeled by a graph in space and the assignment of unitary irreducible representations of the Lorentz group, i.e., a $SO(3,1)$ *spin-network state* (represented by the thin-lines graph), (2) An n-point spin function (represented here by the endpoints of the thin-lines-graph on the strings; see [7] for the precise definition), (3) A functional of the Yang-Mills connection given by the product of Wilson loops on a unitary representation of the structure group G along the each string component.

quantum version of the constraint (24). This amounts for requiring the holonomy of loops around the string carrying Yang-Mills quantum flux number $n \in \mathbb{N}$ to be in the conjugacy class of $\exp(-\epsilon_n v)$. The techniques developed in [9] can be simply extended to treat this case.

Another important remark concerns the relationship of this model with 4-dimensional Yang-Mills fields coupled to general relativity. There is a close relationship between $SO(3,1)$ BF theory and general relativity [14–17]. More precisely one can obtain the action of general relativity in the first order formulation by constraining the B field to be of the form $B = *(e \wedge e)$ for a tetrad field e. This idea is in fact at the core of the definition of many spin foam models for four-dimensional quantum gravity [18]. Here we would like to point out that if such constraint is imposed on the B field appearing in the action (20) then the naive quantum amplitude for a world sheet configuration with quantized Yang-Mills electric field squared ϵ_n is proportional to $\exp(iA_p[\mathcal{W}]\epsilon_n)$ where $A_p[\mathcal{W}]$ is the area of the world sheet computed with the area form $(e \wedge e)^{*IJ} p_{IJ}$. This is precisely the functional dependence of the Yang-Mills amplitude in any dimension [19]. We think that the model presented here might present a new perspective for the definition of a natural coupling of Yang-Mills fields with gravity in the context of spin foam models of quantum gravity.

It would be interesting to undertake the quantization of the model of Sec. IV. This would require the nonperturbative quantization of the tetrad field e^I_1 and its conjugate momentum π_I. We would like to study this question in detail in the future. Nevertheless, it seems clear that topological invariance should considerable simplify matters. It seems that if this question can be resolved then one should be able to quantize the model of Sec. V. An interesting feature of these models (from the loop quantum gravity perspective) is that their constraint algebras represent simpler models of that of general relativity, since as in the latter, they possess field-dependent structure constants. Perhaps some technical issues concerning the quantization of such theories can be clarified in this simpler context. The model of Sec. IV is in addition interesting because of its additional resemblance to general relativity.

VII. SOME SPECULATIVE REMARKS

Let us finish with more speculative considerations which are however an important additional motivation for the study presented here. The most fundamental question of loop quantum gravity is whether one can construct a quantum field theory in the absence of a nondynamical background metric. Several known results such as the quantization of Chern-Simons theory, 2 + 1 gravity, BF theory, etc., show that this is possible at least when dealing with topological field theories. The difficult question is whether one can construct an explicit nontrivial example of background-independent quantum field theory (with infinitely many degrees of freedom, i.e., infinitely many physical observables). One can argue that the entire framework of standard quantum field theory is based on the notion of *particle*, where Fourier modes are the basic building block in the construction of standard quantum field theories. Similarly, we would like to explore the possibility that the finitely many degrees of freedom encoded in topological models, of the kind presented here, might be put together (be "second quantized") in order to define a QFT with infinitely many degrees of freedom. Our ideas are at this stage rather heuristic with some aspects based in unproven assumptions motivated by properties of very simple models [20]. The degree to which these assumptions can be made into factual statements will be explored elsewhere.

The basic idea goes as follows: In the model of [7] as well as those presented here, the topology of the spacetime manifold $\mathcal{M}$ and the embedded world sheet $\mathcal{W}$ are held fixed. Under these conditions the transition amplitudes between kinematical states can be computed. When the topology of the world sheet is trivial (e.g. a cylinder $\mathcal{W} = S^1 \times \mathbb{R}$ or an ensemble of any arbitrary number of disconnected cylinders) these amplitudes can be used to define the so called physical inner product of the (canonically defined) quantum theory. Let us call $\mathcal{H}_n$ with $n \in \mathbb{N}$, the physical Hilbert space so defined for the quantum theory associated with the classical configuration space containing n disconnected strings. One can construct a theory with infinitely many degrees of freedom defining the "Fock" space $\mathcal{F} = \oplus_{n=0}^{\infty}\mathcal{H}_n$ with the infinite set of quantum observables associated with the multistring states (for the explicit construction in the particle case see [6]). However, from our perspective[3] such a theory seems rather trivial because there is no interaction between the $\mathcal{H}_n$'s for different values of n.

When the world sheet topology is nontrivial (e.g. it has branching components as in Fig. 2 and/or nonvanishing genus) the quantum amplitudes are still well-defined (in the spin foam representation) but have no clear-cut physical interpretation.[4] It is tempting to interpret these amplitudes as providing the definition of physical interacting transition amplitudes in a theory where the kinematical Hilbert space is the Hilbert space $\mathcal{F}$ defined above. This interpretation would be consistent if: (1) the sum over world sheet topologies would be convergent, and (2) the transition amplitudes define a positive semidefinite inner product in $\mathcal{F}$. This last requirement is highly nontrivial—it is the counterpart of unitarity in background-dependent quantum field theory. If these conditions hold, this would provide a consistent way of rendering the world sheet

[3]In [6] the context in which $\mathcal{F}$ is introduce is quite different. There one uses it to setup a perturbation theory.

[4]A field theoretic interpretation as Feynman diagrams in the context of perturbation theory of an associated effective field theory is proposed in [5].

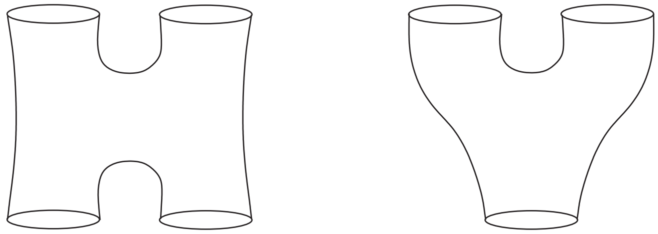

FIG. 2. Interacting string world sheets.

topology dynamical achieving the goal of defining a nontrivial (i.e. interacting) quantum field theory with infinitely many degrees of freedom: the latter given by the ensemble global degrees of freedom of all world sheet topologies.

Because of the fact that topology of two-dimensional orientable manifolds is characterized by a single integer (the genus g) condition (1) above can be satisfied if the amplitudes are suitably damped for high g. In fact the sum over two-dimensional topologies does converge in simple models such as 2d BF theory (see for instance [20]). Some positive indication that property (2) could be realized for models of the kind presented here also comes from the study of this simple case. However, the model in [20] is too simple and the sum over world sheets does not lead to a theory with infinitely many degrees of freedom. If the sum over world sheet topologies could be achieved in the models presented here, due to the nontrivial character of the degrees of freedom involved, we believe that they might lead to nontrivial examples of background-independent field theories with infinitely many degrees of freedom. We would like to explore this possibility in the future.

VIII. DISCUSSION

We have shown how the extended nature of the conical defects that naturally couple to four-dimensional BF theory allow for the introduction of physically interesting world sheet fields while keeping the topological character of the theory. These models are expected to be nonperturbatively quantizable. In particular, the coupling of Yang-Mills theory with BF theory described in Sec. III can be quantized in a rather direct way by using the techniques of Refs. [7,9]. For this theory we get at a remarkably simple description of states in the kinematical Hilbert space where bulk-geometry spin network states are dual to Yang-Mills electric field flux lines (see Fig. 1). The strength of the conical singularities at the location of flux lines is proportional to the electric field square.

The models are in close relationship with gravity in at least two independent ways. On the one hand, as we argued in the subsection of Sec. II, solutions of the topological models are in one to one correspondence with solutions of Einstein's equations. This correspondence between solutions has to be interpreted with due care as the gauge symmetry of our models is much larger than the one of general relativity. In particular local excitations such as gravitons are pure gauge in our models. Nevertheless the correspondence among solutions might be of relevance if some of the hopes described in the previous section could be realized. On the other hand, our models are linked to gravity along the well-known relationship between four-dimensional BF theory and general relativity explicitly exhibit in the Plebansky formulation of gravity. In particular, it would be interesting to compare our models with the coupling to Yang-Mills theories proposed in [21].

These models are simple but nontrivial. In particular, the presence of geometric degrees of freedom as well as matterlike degrees of freedom make them potentially useful for the study of various conceptual difficulties in nonperturbative quantum gravity.

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APPENDIX

Here we construct a truly background-independent model which will lead to a genuinely topological theory. The discussion of the first part of this paper gives a clear way to defining a background-independent analog. The action is

$$S = \int_{\mathcal{W}} [\mathcal{E}_a F^a(A) + (\beta(e^I \wedge e^J)^* + \gamma e^I \wedge e^J) p_{IJ} \mathcal{E}_a \mathcal{E}^a + p_{IJ} F^{IJ}(\omega) + \pi_I d_\omega e^I], \qquad \text{(A1)}$$

where $F = (\frac{1}{2} F^a{}_{\mu\nu} dx^\mu \wedge dx^\nu) \otimes J_a$ with $F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} A^b_\mu A^c_\nu$ and—in order to make the e^I and p_{IJ} fields dynamical—added the natural term $p_{IJ} F^{IJ}(\omega) + \pi_I d_\omega e^I$ which also requires the introduction of the connection ω^{IJ}. Of course there are other additional fields which can be added to the action (A1) but, for the moment, let us look just at this action. The parameters β, and γ are coupling constants.

After the $1+1$ decomposition, $(x^\mu) = (x^1, x^2) = (\tau, \sigma)$, each of the terms become: The action becomes (neglecting space boundary terms)

$$S = \int d\tau \wedge d\sigma [\mathcal{E}_a \dot{A}^a{}_1 + \pi_{IJ} \dot{\omega}^{IJ}{}_1 + p_I \dot{e}^I{}_1 - \lambda^a \mathcal{G}_a - \lambda^I \mathcal{C}_I - \lambda^{IJ} \mathcal{D}_{IJ}], \qquad \text{(A2)}$$

with

$$\mathcal{G}_a := d_A \mathcal{E}_a,$$
$$\mathcal{C}_I := d_\omega \pi_I + \beta p^*_{IJ} e^J{}_1 \mathcal{E}_a \mathcal{E}^a + \gamma p_{IJ} e^J{}_1 \mathcal{E}_a \mathcal{E}^a, \qquad \text{(A3)}$$
$$\mathcal{D}_{IJ} := d_\omega p_{IJ} + \frac{1}{2}(\pi_I e_{J1} - \pi_J e_{I1}).$$

Smearing the constraints with test fields

$$G(\alpha) := \int_{\mathcal{S}} d\sigma \alpha^a \mathcal{G}_a, \qquad C(\lambda) := \int_{\mathcal{S}} d\sigma \lambda^I \mathcal{C}_I, \qquad D(N) := \int_{\mathcal{S}} d\sigma N^{IJ} \mathcal{D}_{IJ}, \tag{A4}$$

the constraint algebra gives

$$\{C(\lambda), C(\Lambda)\} = \int_{\mathcal{S}} \Big([\lambda, \Lambda]^{IJ} \Big(\frac{2\gamma}{\phi^2} \mathcal{E}_a \mathcal{E}^a \Big) D_{IJ} + \frac{4\gamma}{\phi^2} \mathcal{E}^a [\lambda, \Lambda]^{IJ} p_{IJ} G_a \Big),$$
$$[\lambda, \Lambda]^{IJ} := \frac{1}{2} (\lambda^I \Lambda^J - \lambda^J \Lambda^I), \tag{A5}$$

$$\begin{aligned} &\{G(\alpha), G(\beta)\} = G([\alpha, \beta]), \\ &\{G(\alpha), C(M)\} = \{G(\alpha), D(N)\} = 0, \\ &\{D(N), D(M)\} = D([N, M]), \\ &\{D(N), C(\beta)\} = C(N \cdot \beta), \\ &\{C(M), C(N)\} = \mathcal{E}^2 D([N, M]) + 2G([N, M] \cdot p\mathcal{E}), \end{aligned} \tag{A6}$$

where $[N, M]^I$ and $(N \cdot \beta)^I := N^{IJ} \beta_J$ is the commutator in the Lie algebra $so(4)$. The constraints are all first class which leads to the conclusion that there are zero local degrees of freedom.

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