Stability of polytropes

Christian Frønsdal
Physics Department, University of California, Los Angeles, California 90095-1547 USA
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This paper is an investigation of the stability of some ideal stars. It is intended as a study in general relativity, with emphasis on the coupling to matter, aimed at a better understanding of strong gravitational fields and "black holes." This contrasts with the usual attitude in astrophysics, where Einstein's equations are invoked as a refinement of classical thermodynamics and Newtonian gravity. Our work is based on action principles for systems of metric and matter fields, well-defined relativistic field models that we hope may represent plausible types of matter. The thermodynamic content must be extracted from the theory itself. When the flow of matter is irrotational, and described by a scalar density, we are led to differential equations that differ little from those of Tolman, but they admit a conserved current, and stronger boundary conditions that affect the matching of the interior solution to an external metric and imply a relation of mass and radius. We propose a complete revision of the treatment of boundary conditions. An ideal star in our terminology has spherical symmetry and an isentropic equation of state, $p = a\rho^\gamma$, a and γ piecewise constant. In our first work it was assumed that the density vanished beyond a finite distance from the origin and that the metric is to be matched at the boundary to an exterior Schwartzchild metric. But it is difficult to decide what the boundary conditions should be and we are consequently skeptical of the concept of a fixed boundary. We investigate the double polytrope, characterized by a polytropic index $n \leq 3$, in the bulk of the star and a value larger than five in an outer atmosphere that extends to infinity. It has no fixed boundary but a region of critical density where the polytropic index changes from a value that is appropriate for the bulk of the star to a value that provides a crude model for the atmosphere. The boundary conditions are now natural and unambiguous. The existence of a relation between mass and radius is confirmed, as well as an upper limit on the mass. The principal conclusion is that all the static configurations are stable. There is a solution that fits the Sun. The masses of white dwarfs respect the Chandrasekhar limit. The application to neutron stars has surprising aspects.

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I. INTRODUCTION

The problem on which we hope to throw some light in this paper is the application of general relativity to starlike systems that can be described by mass, radius, density, and pressure; the flow is assumed to be irrotational and to be controlled by a velocity potential.

In contrast with the traditional treatment we introduce the matter component into Einstein's equation for the metric by adding an appropriate matter contribution to the Einstein-Hilbert action. The difference, at first sight, seems minor, for the equations associated with either approach are nearly identical, especially in the static case. An important consequence for the study of equilibrium configurations is that the action principle fixes an integration constant that is usually left free. This results in a strengthening of the conditions for matching the interior metric to an exterior Schwartzchild metric at the boundary of the star [see Eq. (4)], or asymptotically at infinity. Consequently, the action principle has some additional predictive power, likely to bring it down, perhaps, but worth investigating.

We do not take it for granted that the matter distribution conforms to the precepts of classical thermodynamics; instead it is expected that the interpretation is implied by the theory itself. A simple choice of interaction does indeed lead to an equation of state of the familiar type. The difference in attitude therefore does not, by itself, lead to any dramatic contrast with the traditional approach. The future inclusion of radiative effects may change that.

The simplest choice of action leads to an equation of state of the form

$$p = \frac{a}{n}\rho^\gamma, \qquad \gamma = 1 + \frac{1}{n}, \tag{1}$$

where a, γ are constants, eventually piecewise constant.

The main result of an earlier investigation [1] was a relation between the mass and the radius of any equilibrium configuration. The original purpose of the present paper was to study the stability of those configurations. In the course of this work we have become somewhat skeptical about the appropriateness of naive boundary conditions; that is, the assumed continuity of the metric, regardless of the behavior of density and pressure that it implies, at a fixed boundary. A large part of this paper is directed to a reevaluation of these questions.

A. Double polytrope

Consider the process that leads to the formation of a star, assuming that the primordial matter is homogeneous, as is reasonable if stars are a late result of a long process of condensation of a hydrogen cloud. Condensation is a result of gravitational attraction and the first effect produced by the attraction is an increase in density. All subsequent development is ultimately attributable to this primordial increase in the density. If, as is always taken for granted in studies of stellar structure, the equation of state is nearly polytropic, then the basic, underlying reason for a change in the index n must be the variation of density. A popular model is a polytrope with $n \approx 3$ in a region of moderately high density and $n > 5$ in the outer atmosphere. In the early stages with low density the index may be larger than five almost everywhere, characteristic of a distribution that extends to infinity, but as this would imply a singularity at the origin a change must take place near the center. Whatever happens, the primal cause is the variation of density. That is, the position of the boundary must be determined by the density, rather than the other way around. It follows that, if the index changes abruptly, then it is the result of a rapidly changing density, as in

$$n[\rho] = \frac{n_1(\rho/\rho_{\rm cr})^K + n_2}{1 + (\rho/\rho_{\rm cr})^K}, \tag{2}$$

where K is a suitable large number and $\rho_{\rm cr}$ is a critical density. (Another representation for the approximately piecewise function will be explored at the end, with interesting results.)

Indeed, in an approximation where the only variables to be taken into account, besides the components of the metric field, are density and pressure, this would appear to be the only possible approach: the boundary is defined to be the region of critical density. Note that this "boundary" need not coincide with the visual boundary of the star.

In this paper, after attempting a more traditional approach to localizing the surface of the star, and remaining unconvinced of the aptness of it, we shall concentrate on trying to understand the double polytrope with this type of equation of state.

The main conclusion is that all the static solutions, with natural boundary conditions applied at the center and at infinity, are stable. The white dwarfs respect the Chandrasekhar limit on the mass, not because heavier stars are unstable, but because they do not exist. A model for the Sun is included; the application to neutron stars offers some new dimensions.

B. Summary

An unfamiliar aspect of this work is the use of an action principle for the complete system of metric and matter fields. Matter is assumed to be irrotational and polytropic, thus fully described by fields of density and pressure, with the action

$$A_{\rm matter} = \int d^4x\sqrt{-g}\mathcal{L}, \qquad \mathcal{L} = \frac{\rho}{2}(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - c^2) - V[\rho]. \tag{3}$$

The eventual presence of (electromagnetic) radiation will require additional terms to be added. The associated energy-momentum tensor provides the right-hand side of Einstein's equation $R_{\mu\nu} - g_{\mu\nu}\frac{R}{2} = 8\pi G T_{\mu\nu}$. We look for solutions that are spherically symmetric, such that the metric, in terms of coordinates t, r, θ, and ϕ, takes the form

$$(ds)^2 = e^\nu (dt)^2 - e^\lambda (dr)^2 - r^2 d\Omega^2,$$

with ν and λ depending on r and t only.

Homogeneous polytropes are characterized by a potential $V[\rho]$ of the form $V[\rho] = a\rho^\gamma$, with γ and a constant, which leads to the equation of state $p = a(\gamma - 1)\rho^\gamma$. When γ is not constant, the polytropic equation of state is slightly modified in the region of critical density. As is usual, we study the time development of the system under the assumption that it is initially in an equilibrium configuration.

Our first calculations [1] posited a fixed boundary beyond which the density is zero and the metric is that of Schwartzchild's exterior solution. The static configurations of this model are essentially the same as in the traditional approach, except for an important difference with respect to the boundary conditions. In contrast with the situation in the usual approach, we must match both of the metric functions ν, λ of the interior solutions to an external Schwartzchild metric. Integration proceeds from the center and the boundary is at a point $r = R$ where

$$\nu(R) + \lambda(R) = 0. \tag{4}$$

Within the traditional approach this condition is ineffective since the boundary value of ν is just an integration constant.

We wish to calculate the time development to first order in the deviation from equilibrium. This was first done by Chandrasekhar in 1931 [2], and as far as we know the same method has been followed by all later investigators. All these studies are characterized by what we think are insufficiently motivated boundary conditions. In the first place it is not sure that one knows which of the fields, metric components, density, pressure, should be required to be continuous at the boundary. The boundary is not at a fixed point but varies from one static configuration to another and with time. Consequently it is unnatural to restrict the fluctuations by the condition that the radius remain fixed. Another question that imposes itself is that of the mass. We define the mass in terms of the asymptotic metric; does it echo the oscillations or does it remain fixed?

In view of the fact that we have come to view the Schwartzchild solution as the metric of a singular, limiting mass distribution [1,3,4], we felt that one way to clarify these questions would be to replace the outer

Schwartzchild metric with another polytrope, with the index $n > 5$ as is appropriate for a mass distribution that extends to infinity. This exterior metric rapidly approaches the Schwartzchild metric at moderate distances. But the boundary conditions continue to present the same, difficult problems.

We have come to believe that the very idea of a fixed boundary is unnatural, and an obstruction to understanding what is going on. For this reason we make a new start, with another version of the double polytrope, an ideal star in which the equation of state is essentially polytropic near the center, with an index that is nearly constant, but changes to a larger value in a "boundary" region of critical density, and essentially constant outside this region. The "surface" of this star is a region in which the index makes a sudden or gradual change from one value to the other, as in

$$p = \frac{a}{n} f^{n+1}, \qquad \rho = f^n, \qquad n = 3 + \frac{3}{1 + (f/f_{\rm cr})^K}. \quad (5)$$

This defines a 2-parameter family of equations of state. For reasons that will be explained, the most plausible models have $f_{\rm cr} = \rho_{\rm cr} = 1$.

C. Results

Our earlier calculations were made with a fixed polytropic index for the interior, and a matching Schwartzchild metric for the empty exterior. In this case the parameter a can be varied by simple rescaling. We investigated these solutions but, having great difficulties in selecting the proper boundary conditions, we abandoned that approach.

Using the new equation of state, in Eq. (5), and the natural boundary conditions (regularity at the center and falloff at infinity) we recalculated the static configurations. With $n = 3$ (inside) and $n = 6$ (outside), solutions were found for values of the parameter a ranging from 10^{-6} to $1/5.765$ and no solutions were found for larger values of a. A relation between radius and mass emerges in all cases considered, including:

(1) Polytropic index $1 \le n \le 3$ (near the center) and 6 (at large distances).

(2) Index $n = 3$ and 15.

All these static solutions appear to be stable to radial perturbations. It may be objected that the equation of state used here is somewhat special, not to say *ad hoc*. We are nevertheless justified in concluding that instabilities of polytropes found previously are characteristic of a restricted class of boundary conditions; they are not generic.

An important consequence of the fact that the dynamics is formulated as an action principle is the existence of a conserved current. With the usual boundary conditions at the center, and natural boundary conditions at infinity, we find that the asymptotic mass is a constant of the motion.

An application to the Sun predicts the central density and pressure, close to the values obtained within the traditional approach. In applications to white dwarfs the constant a, a free parameter in other cases, is known. In the case of complete degeneracy, the polytropic index is equal to 3, and in this case a unique mass is predicted, very close to the mass of the Sun. This limit, here derived from a theory in which all the stars are stable, is close to the limiting value obtained by Chandrasekhar from stability considerations [2].

The instabilities of white dwarfs discovered by Chandrasekhar are difficult to interpret, as witness the reservations expressed by Eddington [5]. The first question that comes to mind is the future development of a star that starts from static but unstable initial conditions. This could not be answered within the context in which the instabilities appeared, because that context was not a mathematically well-defined model. Instead, Chandrasekhar's results have been taken to mean that there are limitations to the range of physical parameters (mass and size) that are possible, given the assumed thermodynamic properties of the star. With this conclusion, our results are in perfect agreement: There is a largest mass, beyond which the problem is not that the static solutions are unstable, but that they do not exist.

The maximal mass of a neutron star can be obtained once the critical density is known. Using commonly accepted values, we again recover the traditional limit. We explore some variations of the representation used for the piecewise almost constant function $n[\rho]$ and discover an unexpected and interesting density profile.

II. MATTER MODEL AND EQUATIONS OF MOTION

The dynamical variables of a simple hydrodynamical system are ρ, a scalar density that will be interpreted as a mass density, a velocity field $\vec{v}$, and the pressure p. The fundamental equations are the equation of continuity, $\dot{\rho} + \mathrm{div}(\rho\vec{v}) = 0$ and the hydrostatic equation

$$-\rho \frac{D}{Dt}\vec{v} = \mathrm{grad}\, p.$$

We limit ourselves to laminar flow, setting $\vec{v} = -\mathrm{grad}\Phi$, in terms of a scalar velocity potential. If we further define a functional $V[\rho]$ by $p = \rho(dV/d\rho) - V$, then V is defined up to a term linear in ρ and we can express the hydrodynamical equation by

$$-\frac{D}{Dt}\vec{v} = \mathrm{grad}\frac{dV}{d\rho}.$$

It is well known that this equation, with the continuity equation, are the Euler-Lagrange equations of the action

$$A = \int dt \int d^3x(\rho(\dot{\Phi} - \vec{v}^2/2 - \phi) - V[\rho]).$$

We have included the gravitational potential ϕ. Variation of ρ now gives

$$-\rho\frac{D}{Dt}\vec{v}-\text{grad}\phi=\text{grad}p.$$

A straightforward relativistic version of this theory is obtained by taking the matter action to be

$$A_{\text{matter}}=\int d^4x\sqrt{-g}\Big(\frac{\rho}{2}(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}-c^2)-V[\rho]\Big)$$
$$=:\int d^4x\sqrt{-g}\mathcal{L}.$$

The nonrelativistic limit is recovered by setting $\psi=c^2t+\Phi$.

We add this matter action to the Einstein-Hilbert action and restrict the metric to the spherically symmetric form

$$(ds)^2=e^{\nu}(dt)^2-e^{\lambda}(dr)^2-r^2d\Omega^2.$$

Einstein's equations then reduce to

$$G_t^t=-e^{-\lambda}\Big(\frac{-\lambda'}{r}+\frac{1}{r^2}\Big)+\frac{1}{r^2}=8\pi G(e^{-\nu}\rho\dot{\psi}^2-\mathcal{L}),\quad(6)$$

$$G_r{}^r=-e^{-\lambda}\Big(\frac{\nu'}{r}+\frac{1}{r^2}\Big)+\frac{1}{r^2}=8\pi G(-e^{-\lambda}\rho(\psi')^2-\mathcal{L}),\quad(7)$$

$$G_t^r=e^{-\lambda}\frac{\dot{\lambda}}{r}=-8\pi Ge^{-\lambda}\rho\psi'\dot{\psi},\quad(8)$$

and are supplemented by the wave equations, from variation of the fields ρ and ψ,

$$\frac{1}{2}(e^{-\nu}\dot{\psi}^2-e^{-\lambda}(\psi')^2-1)=\frac{dV}{d\rho},\quad(9)$$

$$\partial_t(e^{(-\nu+\lambda)/2}r^2\rho\dot{\psi})-(e^{(\nu-\lambda)/2}r^2\rho\psi')'=0.\quad(10)$$

The Lagrangian density $\mathcal{L}$, evaluated on shell, is interpreted as the pressure, subsequently denoted p. This is not only because of its appearance in the expression for the energy-momentum tensor [in Eqs. (6)–(8)] but also because of the fact that $\mathcal{L}=\rho(dV/d\rho)-V$, which is a familiar expression for pressure in terms of internal energy (see Ref. [6] , page 304). (Extending this to an off shell identification of the pressure with the Lagrangian density would be a mistake.)

The function λ is often replaced by the function M defined by

$$M=:\frac{r}{2G}(1-e^{-\lambda}),\qquad e^{-\lambda}=1-\frac{2MG}{r};\quad(11)$$

then Eqs. (6) and (7) can be written as follows,

$$M'=4\pi r^2(e^{-\nu}\rho\dot{\psi}^2-\mathcal{L}),\quad(12)$$

$$re^{-\lambda}\nu'=1-e^{-\lambda}+8\pi Gr^2(e^{-\lambda}\rho\psi'^2+\mathcal{L}).\quad(13)$$

The two equations can be combined to yield

$$(\nu+\lambda)'=8\pi Gre^{\lambda}\rho(e^{-\nu}\dot{\psi}^2+e^{-\lambda}\psi'^2).\quad(14)$$

The differences between this theory and the usual one are mainly as follows:

(i) Equation (9) looks unfamiliar, but taking the derivative with respect to r one recovers the usual force equation with only minor changes. In the static case, when $\dot{\psi}=1$, $\psi'=0$ and ν, λ are time independent, this becomes the hydrostatic condition,

$$p'/\rho=-\tfrac{1}{2}\nu'e^{-\nu}$$

and this has exactly the same form in both theories. But Eq. (9) is stronger than its derivative; it furnishes an additional constraint at the boundary, and it is this equation that provides a relation between the radius and the mass. In the traditional approach, the boundary of a polytrope is often chosen to be at the point where the pressure becomes zero, that always exists if the polytropic index $n=(\gamma-1)^{-1}$ is less than five. There is no effective or meaningful matching of the field ν to an external Schwartzchild metric, a fact that, in our opinion, makes the whole proceeding unsatisfactory. In our model this matching is expressed by

$$\nu(R)+\lambda(R)=0,\qquad 1-e^{\nu(R)}=\frac{2mG}{R}.\quad(15)$$

The first equation determines R and the second gives the value of the asymptotic, Schwartzchild mass m.

(ii) Another significant difference is that the old approach has no intrinsic conserved current (a conserved baryonic current is often introduced by hand) [7], while the new theory does, namely

$$J^{\mu}=\sqrt{-g}\rho g^{\mu\nu}\partial_{\nu}\psi.\quad(16)$$

The existence of this conserved current does not by itself assure us that the quantity

$$\bar{m}=\int d^3x\sqrt{-g}J^0$$

is a constant of the motion. However, with the new equation of state introduced later, and the natural boundary conditions that it entails, it does indeed turn out that $\bar{m}$ is conserved (see, Sec. III).

(iii) In the static case, the pressure of the model corresponds exactly to the pressure as defined by Tolman's formula,

$$T_{\mu\nu}=(\hat{\rho}+p)U_{\mu}U_{\nu}-pg_{\mu\nu}$$

while Tolman's density $\hat{\rho}$ is replaced by $\rho+(\rho V)'$. We do not try to guess the precise physical interpretation of ρ and p but try instead to obtain results in terms of quantities that we feel sure are physical, such as the gravitational, Schwartzchild mass m, uniquely defined by the asymptotic gravitational potential. The choice of $V\propto\rho^{\gamma}$ leads to $p\propto\rho^{\gamma}$, but if an equation of the form $p\propto\hat{\rho}^{\gamma}$ is more successful, then that too can be accommodated. Both types are natural extensions of the nonrelativistic, polytropic equation of state.

In the static case, when $\dot{\psi} = 1$ and $\psi' = 0$, we find a curious, special solution, with f constant, namely

$$e^{-\nu} = 1 + 2a\gamma f, \qquad e^{-\lambda} = 1 + 8\pi G p r^2. \tag{17}$$

As a global solution it is of no interest, but we shall find a solution for which f remains nearly constant over a finite interval.

In the paper [1], we examined static solutions with boundary conditions determined by matching the metric of the interior polytrope to an exterior Schwartzchild metric. As explained above, the radius and the mass were determined by the two conditions in Eq. (15). Solutions were given for $n = 1, 2, 3, 4, 6, 10, \ldots$, but only the case $n = 3$ will be invoked here; Table I reproduces the data for this case.

There appears to be an upper limit to the mass of about 1.02, and a lower limit on the radius of about 1.96, in units where $c = G = a = 1$. The number in the third row is the position of the first zero of the pressure. Since the scale is not fixed, the most significant data are the range of the dimensionless ratio $2mG/R$ and the correlation of this number with $\nu(0)$. This is shown in Fig. 1, lower curve.

A. Oscillations around the static solutions

In this section we use units such that $8\pi G = 1$ and fix the scale by setting $a = 1$. We linearize the equations at the static solution. The equation for G_r^r gives δp,

$$r e^{\lambda} \delta p = \delta\nu' - \left(\nu' + \frac{1}{r}\right)\delta\lambda, \tag{18}$$

and so does "Newton's equation," Eq. (9)

$$(e^{-\nu/2})\delta(e^{-\nu/2}\dot{\psi}) = \delta\frac{dV}{d\rho} = \frac{1}{\rho}\delta p,$$

or

$$\delta p = e^{-\nu}\rho(\delta\dot{\psi} - \delta\nu/2). \tag{19}$$

Eliminating δp from these two gives

$$r\rho e^{\lambda-\nu}(\delta\dot{\psi} - \delta\nu/2) = \delta\nu' - \left(\nu' + \frac{1}{r}\right)\delta\lambda. \tag{20}$$

Equation (14) for $G_t^t - G_r^r$ gives $\delta\rho$,

$$r e^{\lambda-\nu}(\delta\rho + \rho\delta\lambda + 2\rho(\delta\dot{\psi} - \delta\nu/2)) = \delta\nu' + \delta\lambda',$$

or

$$r\delta\rho = -2r\rho(\delta\dot{\psi} - \delta\nu/2) - r\rho\delta\lambda + e^{\nu-\lambda}(\delta\nu' + \delta\lambda'). \tag{21}$$

Using Eq. (20) we eliminate the time derivative and obtain a constraint,

$$r\delta\rho = e^{\nu-\lambda}\left(\delta\lambda' - \delta\nu' + \left(\nu' - \lambda' + \frac{2}{r}\right)\delta\lambda\right). \tag{22}$$

TABLE I. Mass/radius relation for $n = 3$/exterior Schwartzchild.

$-\nu(0)$	R		$-\nu(R)$	$2mG$	$2mG/R$
.00001	18 750	60 000	0.000 054	1.017	0.000 054
0.001	18 777	6000	0.000 542	1.015	0.000 54
0.005	374	1200	0.002 70	1.001	0.0027
0.02	92.4	300	0.010 73	0.991	0.011
0.1	17.4	57	0.0516	0.895	0.051
0.3	5.03	18.6	0.1388	0.680	0.135
0.4	3.59	14.4	0.1721	0.618	0.172
0.45	3.12	13.1	0.1863	0.582	0.187
0.475	2.90	12.7	0.2026	0.567	0.196
0.49	2.84	12.5	0.1955	0.555	0.195
0.5	2.77	12.3	0.1927	0.547	0.197
0.6	2.30	11.4	0.217	0.489	0.213
0.7	2.03	11.5	0.232	0.441	0.217
0.8	1.96	13.6	0.2012	0.396	0.202
0.85	2.022	14.4	0.1875	0.379	0.197
0.9	2.17	16.8	0.1688	0.365	0.168
0.95	2.43	20.5	0.1458	0.354	0.146
1	2.91	26	0.120	0.350	0.120
1.2	7.86	48	0.0524	0.404	0.051
1.3	8.99	49	0.0514	0.450	0.050
1.5	7.64	47	0.0650	0.497	0.065
2.0	4.82	29	0.1000	0.482	0.100
2.065	4.66	28.5	0.1021	476	0.102

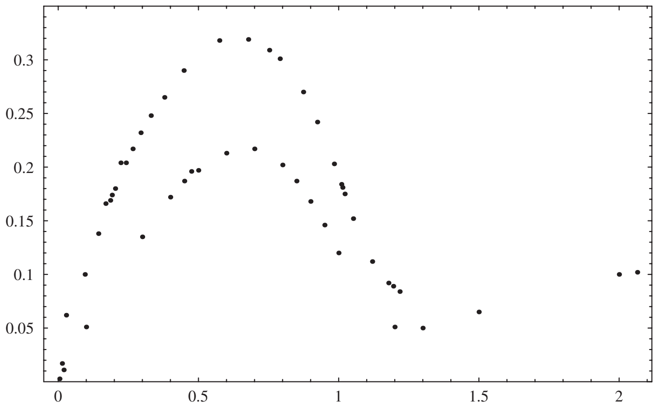

FIG. 1. The lower curve shows the essential information from Table I. The abscissa is $\nu(0)$ and the ordinate is the dimensionless ratio $2mG/R$, where R is the boundary defined by Eq. (15). The upper curve shows the same information from Table II, the ordinate here is $2mG/R_{\rm cr}$.

The original set of equations is equivalent to Eqs. (19) and (21) that determine δp and $\delta\rho$, and the two equations

$$r\rho e^{\lambda-\nu}(\delta\dot\psi - \delta\nu/2) = \delta\nu' - \left(\nu' + \frac{1}{r}\right)\delta\lambda \tag{23}$$

and

$$\delta\dot\lambda = r\rho\psi' \tag{24}$$

for the functions λ and ψ. The only difficulty is that there is no determination of $\delta\nu$; to fix it we have to use the equation of state, the choice may affect stability.

B. Equation of state

We express V and ρ parametrically, in terms of the Emden function f

$$V = af\rho, \qquad \rho = f^n, \qquad p = \rho\frac{dV}{d\rho}V,$$

with

$$n = n_1 + \frac{n_2 - n_1}{1 + f^K}.$$

This function takes the constant value n_1 (in most of our calculations $n_1 = 3$) in the inner regions where $f > 1$, and the constant value n_2 (the values 6 and 15 were explored) in the atmosphere. The critical region is thus at the place where f takes the value unity; f and ρ are entered as multiples of their critical values. This choice, with a constant, gives the best approximation to an polytropic equation of state, as we shall see.

In regions where n is constant we have $p = (a/n)f^{n+1}$. In the boundary region there will of course be deviations from this equation of state. We calculate

$$\frac{dV}{d\rho} = af\frac{1 + n - N\ln f}{n - N\ln f}, \qquad N = \frac{(n_1 - n)(n - n_2)}{n_1 - n_2}K.$$

Because of the high value of K that was used ($K = 50$), the function N vanishes except in a narrow interval of r where $n_1 < n < n_2$. Of course, because K is large, N is not small in this region. However, for the same reason, the function f varies little from its critical value in this interval. Therefore, taking this critical value to be unity, making $f \approx 1$ in the interval, is highly beneficial. The two functions $\rho\frac{d}{d\rho}V$, γV are nearly indistinguishable over the entire interval $0 < r < \infty$. The deviation is small, in a small region, and it will be ignored in the calculations. (Most results show a remarkable insensitivity to details of the equation of state near the critical point.) Thus we set $p = a\rho^\gamma$ and

$$\rho\delta p = \gamma p\delta\rho. \tag{25}$$

For a stationary solution, when $\delta\dot\lambda = s\delta\lambda$ for some number s that we hope will have to be pure imaginary, from Eqs. (20), (24), and (25),

$$\delta\nu' = re^\lambda\delta p + \left(\nu' + \frac{1}{r}\right)\delta\lambda, \qquad r\rho\psi' = s\delta\lambda,$$

$$\delta\lambda' = rKe^\lambda\delta p + \left(\lambda' - \frac{1}{r}\right)\delta\lambda, \qquad K = 1 + e^{-\nu}\rho/\gamma p.$$

Elimination of δp leads to

$$\delta\nu' = re^{\lambda-\nu}\rho(s\delta\psi - \delta\nu/2) + \left(\nu' + \frac{1}{r}\right)\delta\lambda,$$

$$r\rho\psi' = s\delta\lambda,$$

$$\delta\lambda' = sr\rho Ke^{\lambda-\nu}\delta\psi - (r\rho K/2)e^{\lambda-\nu}\delta\nu + \left(\lambda' - \frac{1}{r}\right)\delta\lambda,$$

or

$$(d/dr)\begin{pmatrix}\delta\nu\\ \delta\psi\\ \delta\lambda\end{pmatrix} = \begin{pmatrix} -(r\rho/2)e^{\lambda-\nu} & s(r\rho)e^{\lambda-\nu} & \nu' + \frac{1}{r}\\ 0 & 0 & (s/r\rho)\\ (r\rho K/2)e^{\lambda-\nu} & s(r\rho)Ke^{\lambda-\nu} & \lambda' - \frac{1}{r}\end{pmatrix}\begin{pmatrix}\delta\nu\\ \delta\psi\\ \delta\lambda\end{pmatrix}. \tag{26}$$

From this it is easy to see that the integration from $r = 0$ can proceed. We start at $r = 10^{-10}$ with $\delta\psi = \delta\lambda = 0$ and $\delta\nu \neq 0$. This will make $\delta\nu'$ and $\delta\lambda'$ of order r, $\delta\lambda$ of order r^2, and $\delta\nu - \delta\nu(0)$ of order r^2.

The solutions include a simple gauge transformation; when it is ignored the system can be reduced to a single, second order differential equation for the function $L = re^{-\lambda}\delta\lambda$

$$\ddot L = \frac{r\rho}{2}\left(\nu' + \frac{1}{r}\right)L + r^2\rho e^{-(\nu+3\lambda)/2}\left(\frac{e^{(\lambda+3\nu)/2}L'}{r^2\rho K}\right)'. \tag{27}$$

This equation shows that the stationary solutions have frequencies determined by a self-adjoint Sturm-Liouville operator, with a domain of functions L that satisfy a condition that fixes the value of L'/L at the boundary and such that L/r^3 is regular at the origin.

C. Boundary conditions, difficulties

Round 1—Here we report the result of studying the stability of the static solutions in the case that the star has a fixed polytropic index in the interior and the metric is matched to an exterior Schwartzchild metric at the value of r determined by Eq. (15). (These are the solutions listed in Table I.) We adopt Eddington's boundary conditions at the center [5], so that the solutions of the static equations of motion for fixed values of n are indexed by the value $\nu(0)$ of the function ν at the center. Matching of the solution to the exterior metric [Eq. (15)] determines both the radius and the gravitational mass. The result is a relation between mass and radius, for each value of n, reproduced for the case $n = 3$ in Table I and in Fig. 1, lower curve.

The stability of a star is to be determined by solving Eqs. (26) or (27) with some boundary conditions. It is difficult, however, to understand what boundary conditions are appropriate.

To deal with Eq. (27) from the point of view of the Sturm-Liouville theory, one looks for solutions with harmonic time dependence, $L(x, t) = e^{i\omega t}L(x, 0)$. One identifies the range of the parameter ω with the spectrum of an operator in a Hilbert space constructed from a space of functions of r. An acceptable set of boundary conditions must make this operator self-adjoint.

It has always been assumed that the origin is a regular point. It is difficult to find a real justification for this, since the center of the star is a region about which one has very little information. Nevertheless, we follow this precedent since it helps to give a precise mathematical sense to our model. Thus ν has a well-defined value at $r = 0$, λ is of the order of r^2, and the function L is of the order of r^3. In this case the possible additional boundary conditions amount to fixing the value of $L'(r)/L(r)$ at the surface. If we fix the boundary and suppose that $L = 0$ there, then we find a discrete set of oscillating solutions and, in some cases, decaying solutions, in agreement with the findings of [2]. But the radius of the star is determined by the first of the conditions (15), and that means that the surface of the star is pulsating, as we have verified numerically. This seems not at all unnatural, the difficulty is that the asymptotic mass, as determined by the matching of the metrics, is also pulsating. We are very skeptical of these results.

In the traditional treatment the mass within a sphere of radius R is defined by the integral

$$M(R) = \int^R d^3x\hat{\rho}.$$

The equations of motion equate this quantity to $(R/2G) \times (1 - e^{-\lambda})$ and if the metric is asymptotically Schwartzchild then it tends, as R tends to infinity, to the Schwartzchild mass. But in the simplest models $\hat{\rho}$ vanishes beyond the surface of the star and there is no matching of the interior metric to an exterior Schwartzchild metric. Hence $M(\infty)$ is "the mass" of the star by definition only. Oscillations of this quantity do not propagate to infinity. This is the reason why the problem does not arise in connection with the work of Chandrasekhar.

Round 2—Fixing the mass would seem to be the more reasonable boundary condition, since it is determined asymptotically in a region where the density is zero. It does not seem possible that oscillations of a finite star propagate to infinity through an infinite region of empty space. To understand this better we should remember that the external Schwartzchild metric, according to our interpretation [1], is not the metric of empty space, but a singular limit of a family of metrics of spaces with non-vanishing density. Consequently, in our next attempt we replaced the empty Schwartzchild exterior by a crude approximation for the atmosphere, another polytrope. The density distribution now extends to infinity, though it falls off extremely rapidly and the metric soon becomes indistinguishable from that of Schwartzchild. The asymptotic mass is a property of the exterior polytrope, but we were unable to match the two polytropes at the boundary in such a way that this mass would remain constant. Still we cannot claim to have excluded this possibility completely.

It has been traditional since the beginning, to admit a discontinuous behavior of density and pressure at the surface of a star. This is reasonable if the star is cold, but perhaps less likely to be typical of a polytrope. Facing doubts of this kind, and the difficulties discussed in the preceding paragraph, we came to the realization that it may be better to give up the idea of a fixed boundary and introduce the equation of state described in the introduction, thus allowing all the fields to vary continuously throughout.

III. IMPROVED BOUNDARY CONDITIONS

Round 3—We suppose that there is a region of critical density ρ_{cr}, where the polytropic index changes more or less abruptly from a value $n_1 < 5$ (in our calculations $n_1 \leq 3$) that is appropriate for the bulk of the star, to a value $n_2 > 5$ (actually 6 or 15) that we hope may be appropriate for the atmosphere. Precisely,

$$V[\rho] = a\rho_{cr}\tilde{\rho}^{\gamma}, \qquad \gamma = 1 + \frac{1}{n}, \qquad n = \frac{n_1\tilde{f}^K + n_2}{1 + \tilde{f}^K}, \tag{28}$$

where K is a suitably large number (actually 50) and $\tilde{f} = f/f_{cr}$, $\tilde{\rho} = \rho/\rho_{cr}$ are all dimensionless. The critical density now appears as a common factor of the energy-momentum tensor, and on the right-hand sides of Eq. (6)–(8), together with G. We shall drop the tildes on $\tilde{f}$ and $\tilde{\rho}$, so that f and ρ are henceforth given as multiples of their critical values. Then Eqs. (6)–(10) remain valid if the factor G in (6)–(8) is replaced by $G\rho_{\rm cr}$. Finally, we choose our unit of length such that this factor is equal to unity,

$$G\rho_{\rm cr} = 1.$$

The pressure in these units is $(a/n)\tilde{\rho}^\gamma$ and the critical pressure is $(a/\bar{n})\rho_{\rm cr}$, $\bar{n} = (n_1 + n_2)/2$.

There is no longer any question of matching to an exterior Schwartzchild metric, instead we require that the metric approach the Schwartzchild form at large distances, to order $1/r$. The mass is determined by this asymptotic metric,

$$2mG := \lim r\lambda(r) = -\lim r\nu(r).$$

In the traditional approach the second condition is not effective, since only the derivative ν' of the function ν appears in Einstein's equation. The extra condition that comes from the action principle guarantees that the metric is asymptotically Schwartzchild (as $r^3\rho(r) \to 0$) so that the two limits always coincide. The indices n_1, n_2 were given the values 3, 6 in our initial calculations, Table II. A larger value of n_2 makes the metric approach more quickly to the Schwartzchild form. The exponent K determines the abruptness of the change of n.

Since the index is not constant, the value of a can no longer be reduced to unity by a change of scale. As before, we assume that all the fields are regular at the origin. A star is characterized by the parameters $f_{\rm cr}$ and a. We choose a value of a and determine allowed value(s) of $\nu(0)$ by demanding that $-r\nu(r)$ tend to a finite limit (twice the mass m) at infinity, and establish in this way a correlation between R and m. Results are given in Fig. 2 and in Table II. The upper curve in Fig. 2 shows the dimensionless number $2mG/R_{\rm cr}$ versus $\nu(0)$, for the case $n = 3, 6$ (3 inside and 6 outside). The lower curve was obtained by matching the interior solution ($n = 3$) to an exterior Schwartzchild metric. The difference between the two curves is easily explained since the upper curve refers to the critical radius $R_{\rm cr}$ while the lower curve is $2mG/R$, where R is the boundary.

A. All static solutions are stable

When there is no fixed boundary, and the star extends to infinity, the asymptotic behavior becomes important; for the function L that embodies the oscillations around a static solution we find

$$L \propto \sin(r^{3/2}b)/r^k$$

with k and b constant. Values of the exponent k determined by numerical calculations for $n = 6$ ($k = 5/2$) and $n = 10$ ($k = 9/2$) at large distances are such that the metric fluctuations $\delta\nu$ and $\delta\lambda$ fall of faster than $1/r$. It follows that the mass (defined by the asymptotic metric) is unaffected by the oscillations.

With the improved boundary conditions, the real spectrum of frequencies is continuous and apparently the entire real line. No unstable solutions were found. This could have been anticipated by inspection of Eq. (27). The factor ρ in the first term on the right-hand side makes this term fall off very fast at infinity, which suggests that this term cannot affect the spectrum of ω^2. The conclusion must be that the instabilities first discovered by Chandrasekhar [2] are imposed on the theory by the choice of boundary conditions.

B. Constants of the motion

The conservation law for the current (16) can be integrated to yield

$$\frac{d}{dt}\int_0^\infty \sqrt{{\rm e}^{(-\nu+\lambda)/2}}r^2\rho\dot{\psi}dr = [\sqrt{{\rm e}^{(\nu-\lambda)/2}}r^2\rho\psi']_0^\infty.$$

In view of the boundary conditions at the origin,

$$\frac{d}{dt}\int_0^\infty \sqrt{{\rm e}^{(-\nu+\lambda)/2}}\rho\dot{\psi}r^2dr = \lim_{r\to\infty}[\sqrt{{\rm e}^{(\nu-\lambda)/2}}r^2\rho\psi'].$$

The factor ρ on the right-hand side suggests that there is no flux at infinity, but in fact the flux $r\rho\psi'$ is equal to $-\delta\dot{\lambda}/8\pi$ by Eq. (8). For a static configuration both sides of this equation are zero; for a first order deviation from a static configuration we have

$$\frac{d}{dt}\int \sqrt{{\rm e}^{(-\nu+\lambda)/2}}r^2\rho\delta\dot{\psi}drd\Omega = \frac{1}{2}\lim_{r\to\infty}[\sqrt{{\rm e}^{(\nu-\lambda)/2}}r\delta\dot{\lambda}].$$

If the perturbed and unperturbed metrics both tend to Schwartzchild at infinity, then $r\delta\dot{\lambda} \to 2\dot{m}$ so that, finally,

$$\frac{d}{dt}\int \sqrt{{\rm e}^{(-\nu+\lambda)/2}}r^2\rho\delta\dot{\psi}drd\Omega = dm/dt. \qquad (29)$$

It is not *a priori* obvious that the asymptotic mass is a constant of the motion, but a result of our calculations is that $r\delta\lambda$ tends to zero at infinity so that in fact $\dot{m} = 0$. The asymptotic mass is a constant of the motion and so is the quantity

$$\int d^3x\sqrt{-g}g^{tt}\rho.$$

In our numerical study this number turns out to be bounded upwards by the asymptotic mass, the difference being greater in the case of strong gravitational fields.

IV. APPLICATIONS

A. Modelling the Sun

The results for $n_1 = 3$, $n_2 = 6$ (Table II) and for $n_1 = 3/2$, $n_2 = 6$ are shown in Fig. 2. The graph of allowed values of ($R_{\rm cr}$, $2mG$) has a lower branch with small internal pressure, a maximal value of $R_{\rm cr}$, and an upper branch with increasing central pressure and a maximum value of the mass.

TABLE II. Mass/radius relation for $n = 3$ and 6.

$-\nu(0)$	R	$1/a$	$-\nu(R)$	$2mG$	$2m\frac{G}{R}$
$0.346\,977 \times 10^{-5}$	(8.25×1^{5})	1.35×10^{-9}	$[3.22 \times 10^{-4}]$	1.079 18	1.011
$0.408\,938 \times 10^{-5}$	(7×10^{5})	1.72×10^{-9}	$[3.47 \times 10^{-4}]$	1.079 18	1.007
$0.477\,094 \times 10^{-5}$	(6×10^{5})	2.175×10^{-9}	$[3.77 \times 10^{-4}]$	1.079 18	1.011
$0.572\,512 \times 10^{-5}$	(5×10^{5})	2.83×10^{-9}	$[4.13 \times 10^{-4}]$	1.079 18	1.006
$0.715\,640 \times 10^{-5}$	(4×10^{5})	4.00×10^{-9}	$[4.62 \times 10^{-4}]$	1.079 18	1.012
$0.954\,190 \times 10^{-5}$	(3×10^{5})	6.155×10^{-9}	$[5.34 \times 10^{-4}]$	1.079 19	1.011
$0.143\,128 \times 10^{-4}$	(2×10^{5})	1.13×10^{-8}	$[6.52 \times 10^{-4}]$	1.079 19	1.011
$0.190\,837 \times 10^{-4}$	(1.5×10^{5})	1.74×10^{-8}	$[7.58 \times 10^{-4}]$	1.079 19	1.011
$0.286\,255 \times 10^{-4}$	(10^{5})	3.2×10^{-8}	$[9.23 \times 10^{-4}]$	1.079 19	1.012
$0.572\,512 \times 10^{-4}$	(50 000)	0.905×10^{-7}	[0.000 585]	1.079 21	1.012
0.000 572 505 63	(5000)	2.9×10^{-6}	[0.004 15]	1.079 42	1.025
0.002 385 341 6	(1200)	0.000 024 2	[0.008 44]	1.080 17	1.008
0.014 086 8	(200)	0.000 349	[0.020]	1.085 26	1.012
0.028 611 53	(100)	0.000 970	[0.029]	1.091 74	0.970
0.047 680 7	(60)	0.002 02	[0.038]	1.101 01	0.939
0.057 218 4	(50)	0.002 61	[0.042]	1.105 91	0.923
0.071 534 195	(40)	0.003 57	[0.0465]	1.113 59	0.903
0.095 436 8	(30)	0.005 29	[0.0535]	1.179 42	0.869
0.143 546 8	(20)	0.009 05	[0.0655]	1.1579	0.809
0.169 284 5	(17)	0.011 10	[0.0687]	1.176 01	0.778
0.186 029 25	(15.5)	0.012 44	[0.0736]	1.188 46	0.759
0.192 388 6	(15)	0.012 95	[0.0748]	1.193 34	0.752
0.203 544 5	(14.2)	0.013 82	[0.0766]	1.202 09	0.740
0.223 019 5	(13)	0.015 34	[0.0796]	1.218	0.719
0.242 466	(12)	0.016 80	[0.0825]	1.234 75	0.698
0.265 818 9	(11)	0.018 53	[0.0855]	1.256 06	0.676
0.294 494 4	(10)	0.020 53	[0.0884]	1.284 18	0.649
0.330 789	(9)	0.022 86	[0.0920]	1.3232	0.617
0.378 856 5	(8)	0.025 50	[0.0950]	1.3818	0.577
0.447 908	(7)	0.028 40	[0.0980]	1.4832	0.527
0.575 115	(6)	0.030 60	[0.0962]	1.749	0.450
0.677 60	(5.7586)	0.029 00	[0.0910]	2.092 84	0.401
0.752 918 5	(5.9)	0.025 96	[0.0840]	2.485 05	0.372
0.790 723	(6.1)	0.023 90	[0.0793]	2.7564	0.360
0.874	(7)	0.018 33	[0.0680]	3.6683	0.339
0.924 04	(8)	0.014 66	[0.0605]	4.5583	0.332
0.983 69	(10)	0.010 39	[0.0512]	6.2787	0.329
1.009 24	(11.245)	0.008 74	[0.0474]	7.3522	0.330
1.013 842	(11.5)	0.008 46	[0.0468]	7.5735	0.330
1.022 385	(12)	0.007 95	[0.0455]	8.0092	0.330
1.051 813	(14)	0.006 40	[0.042]	9.780	0.335
1.119 438	(20)	0.003 95	[0.0353]	14.473	0.353
1.177 98	(25)	0.003 05	[0.0332]	21.073	0.381
1.194 58	(26)	0.002 95	[0.0331]	22.446	0.391
1.218 345 1	(27)	0.002 80	[0.0332]	24.114	0.393

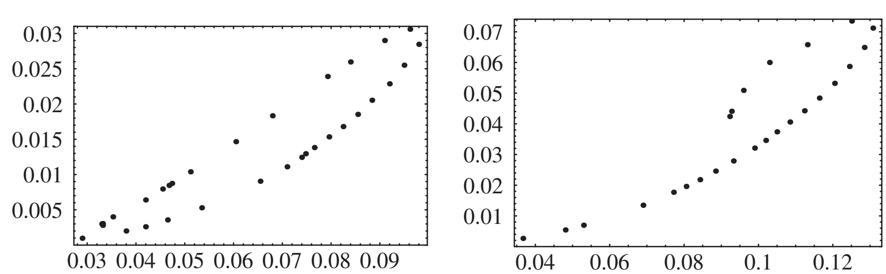

FIG. 2. The mass-radius relation obtained with different boundary equations of state. On the left $n = 3, 6$, on the right $n = 3/2, 6$.

The bulk of the sun is often modeled with a polytrope with index three. The parameter values are

$$(2MG)_{\rm Sun} = 2.95 \times 10^5 \text{ cm},$$
$$R_{\rm Sun} = 6.96 \times 10^{10} \text{ cm},$$

and the ratio is 0.424×10^{-5}. We must not equate $R_{\rm sun}$ with the value $R_{\rm cr}$ of the radius at which the density takes the critical value; this may occur deep in the interior.

If the critical density is $\rho_{\rm cr} = k^2 \text{ g/cm}^3$, then

$$\rho_{\rm cr} G = 0.7414 \times 10^{-28} k^2/\text{cm}^2$$

and the unit of length used in our calculations is therefore

$$\ell = \frac{1.16}{k} \times 10^{14} \text{ cm}.$$

Here are two examples from Table II.

Example 1. Take $1/a = 10^5$, the table gives $2mG = 0.32 \times 10^{-7}\ \ell = 37k^{-1} \times 10^5$. To agree with the Sun value we need $k = 37/2.95 = 12.54$. Thus $\rho_{\rm cr} = 158 \text{ g/cm}^3$, which may be a little high. The critical radius is $9.23 \times 10^{-4}\ell = 0.854 \times 10^{10}$ and the critical pressure is $p_{\rm cr} = (a/3)\rho_{\rm cr} = 0.527 \times 10^{-3}$ or $4.74 \times 10^{17} \text{ dyn/cm}^2$.

Example 2. Take $1/a = 2 \times 10^5$, the table gives $2mG = 0.113 \times 10^{-7}\ \ell = 0.131k^{-1} \times 10^7$. Here we need $k = 13.1/2.95 = 4.44$. Thus $\rho(0) = 1.257 \times \rho_{\rm cr} = 24.8 \text{ g/cm}^3$, which may be too low. The critical radius is

$$R_{\rm cr} = 6.52 \times 10^{-4}\ell, \qquad \ell = 1.7 \times 10^{10}.$$

This is about $R_{\rm Sun}/4$. The density profile, shown in Fig. 3, shows that this is quite reasonable. The pressure is

$$p(0) = \frac{a}{3}\rho_{\rm cr} f(0)^4 = 0.446 \times 10^{-4},$$

or $0.401 \times 10^{17} \text{ dyn/cm}^2$.

Given the known values of the mass and the radius of the Sun, the traditional polytrope model predicts the values $\rho(0) = 76.39 \text{ g/cm}^3$ and $p(0) = 1.24 \times 10^{17} \text{ dyn/cm}^3$ for the density and pressure at the center. Our dynamical model gives slightly different values for the central density and pressure. Various refinements, such as larger values of K and/or n_2, may affect these predictions to a limited extent.

Let us compare the effect of different boundary conditions for fixed values of $\nu(0)$ and a. In the case of Example 2, in units of 10^{10} cm:

(i) The Emden function vanishes at 12.3. This would be the prediction for the radius of the standard approach with this choice of parameters (not the best choice). The density predicted by the model at this point is about $1/5000$ of the critical value.

(ii) The actual value of the visual radius is 6.96.

(iii) Matching to an external Schwartzchild metric would fix the boundary at 3.9. The density at this point is about $1/8$ of the critical value, dropping abruptly to zero.

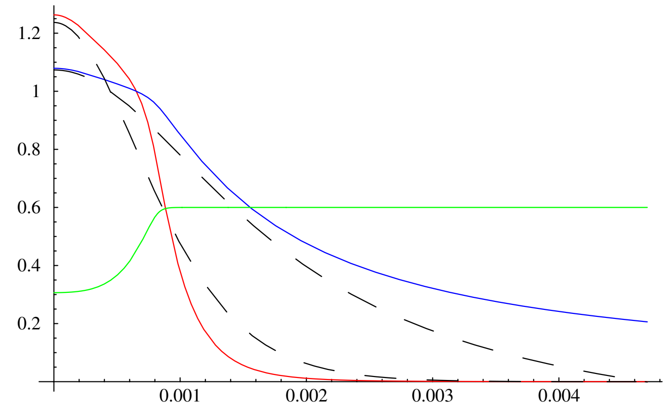

FIG. 3 (color online). Various functions plotted against the distance from the center. The function with the step is $n(r)/10$. The other solid curves show $f(r)$ and $\rho(r)$. The dashed curves show analogous results with a polytropic index fixed at the value of three. What cannot be seen is that the latter pass to negative values at $r = 0.0047$, while the solid curves never do.

(iv) The critical radius is at 1.7.

(v) The density profile is shown in Fig. 3, and the density predicted by the standard theory is shown for comparison.

All three models can be made to give the correct radius and mass of the Sun. They differ, but not very greatly, in the predictions for central density and pressure, and they differ considerably in the degree to which they seem to be physically reasonable.

Figure 4 shows profiles of the metric functions.

B. Strong gravitational fields

A good indicator of the strength of the gravitational field is the dimensionless ratio $2mG/R_{\rm cr}$. The maximum values are, for the case that $n_2 = 6$

$$n_1 = 3, 5/2, 2, 3/2, 1, 1/10$$
$$\frac{2mG}{R_{\rm cr}} = 0.33, 0.43, 0.50, 0.59, 0.67, 0.77$$

As the value of n_2 is increased the falling off of the density outside the critical radius becomes much more rapid. Nevertheless, there is always a region where the gravitational field is less strong and where the density is not yet negligible. Thus it is difficult to imagine a situation where no radiation escapes. A much simpler assumption to explain the astrophysical "black holes" would be for the star, or at least the atmosphere, to be nonradiating on account of the temperature being very low. But in the polytropic models the temperature falls off much more slowly than the density.

C. Chandrasekhar limit

Under conditions that are believed to prevail in a white dwarf, the value of the parameter a can be calculated.

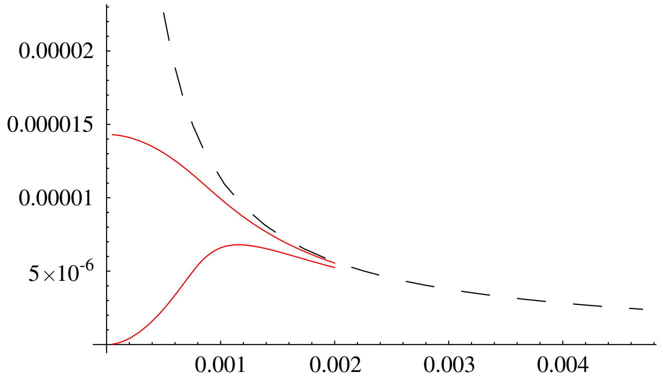

FIG. 4 (color online). The metric functions λ (lower curve) and $-\nu$ in the case that best fits the Sun. The dashed line is the function $-\nu$ for the Schwartzchild metric with the same mass.

Normally the equation of state is expressed as $p = K\rho^{\gamma}$, with pressure and density given in units of g/cm^3. In that case one obtains

$$K = 0.548 \times 10^{-6}, \qquad a = 3f_{cr}K.$$

Our unit of length is

$$\ell = \frac{1.16}{f_{cr}^{3/2}} \times 10^{14} \text{ cm}.$$

The numbers posted for $2mG$, to be expressed in centimeters, must be multiplied by ℓ, including the factor $f_{cr}^{-3/2} \propto a^{-3/2}$. To find the maximum value of the mass we have evaluated the product of $2mG$ in the table by this factor, with the result that it is essentially constant at 1.01 in the upper half of the table and eventually decreases to about 0.33 near the bottom of the table. The model is thus in agreement with the traditional treatment, in predicting a unique mass in the case of weak gravitational fields, and a maximum value of the mass that is reached roughly in the interval $0 < a < 0.01$. This maximum value is

$$2mG\ell = 1.16 \times 1.01 \times 10^{14}(3K)^{3/2} = 0.84(2mG)_{Sun}.$$

When the mass is close to this upper limit the ratio

$$2mG/R_{cr} < 0.334.$$

The maximum value is 788 times the value of $2mG/R_{Sun}$, so it would appear that a white dwarf of this mass may have a radius as little as 1/100th that of the Sun.

The model makes these predictions, basically the same as the theory of Chandrasekhar, but without invoking instabilities. All the stars described by the model are stable. Field profiles are shown in Figs. 3 and 4.

D. Neutron stars

According to Oppenheimer and Volkoff [8], a model of a neutron star must have a very rigid equation of state. We tried $n = 1/10$ in the bulk and $n = 6$ outside and in this way we reach the higher value of 0.77 for the ratio $2mG/R_{cr}$. The model does not furnish the scale; that is, in our case, the value of ρ_{cr}. It is usual to place the central density of a neutron star between 10^{14} and 10^{15} g/cm^2. Taking $\rho_{cr} = 10^{14}$ g/cm^2 gives us the scale $\ell = 1.16 \times 10^7$. Two sample solutions are

$$a = 1/2, \qquad 2mG = 9 \times 10^5 \text{ cm}, \qquad R_{cr} = 17 \text{ km},$$

which is three solar masses and 2.4×10^{-5} solar radii and

$$a = 4/3, \qquad 2mG = 10^6, \qquad R_{cr} = 13 \text{ km}.$$

If the density is 9 times greater the numbers will be 3 times smaller. All these numbers are consistent with current estimates.

It was found that the formula that was used for the variable polytropic index fails, in this case of a low internal value, to give a constant value in the interior; the value is close to n_1 near the center, but begins to increase at about half the critical radius. The natural remedy would be to increase the value of the exponent K, but in that case the power of MATHEMATICA to handle very large numbers is overtaxed. An alternative formula, namely

$$n[f] = \frac{n_1 + n_2}{2} + \frac{n_1 - n_2}{2} \frac{f - 1}{\sqrt{(f-1)^2 + \epsilon}} \tag{30}$$

approaches a step function very well if ϵ is very small, and for $3 \geq n_1 \geq 1$ this formula reproduces the same relation between mass and radius as the one used to construct the tables, to a very good accuracy. However, the density profile is affected in an interesting way as ϵ is decreased.

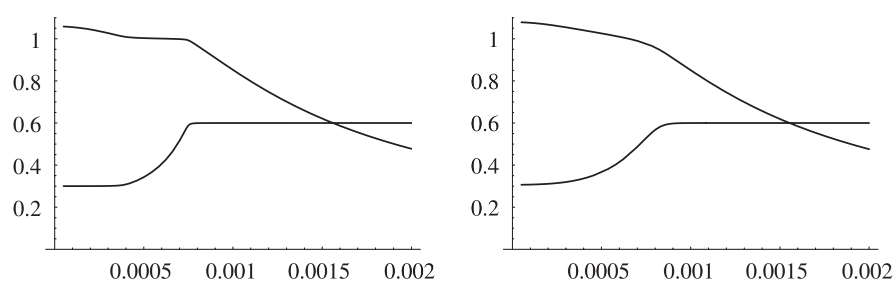

FIG. 5. Contrast between two ways of smoothing the transition at the critical density, on the left formula (28), on the right (30). In both cases, the parameters used are those that best fit the Sun. The lower curve shows the function $n(r)$.

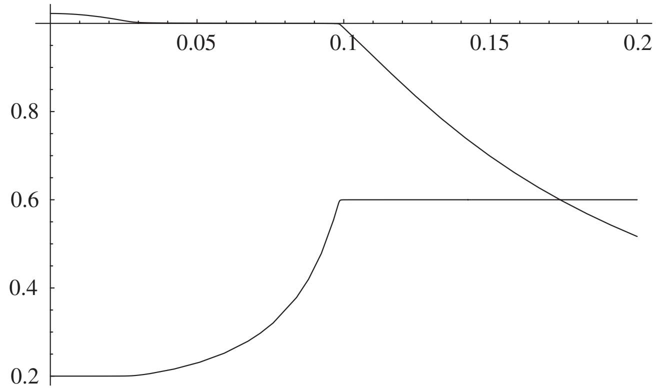

FIG. 6. Here formula (28) was used with $n = 2, 6$. The platform of constant density reaches almost to the center. The lower curve is the function $n(r)/10$.

In an intermediary range of the radial variable the density is constant. This phenomenon is specific to the model, to the use of a variational principle. Consequently, we succeed in fixing $n = n_1$ in the region interior to this platform only.

As shown in Fig. 5, the effect is not very important in the case $n_1 = 3$, $n_2 = 6$, but for smaller values of n_1 it is decisive. Figure 6 shows the density profile for the case $n_1 = 2$, $n_2 = 6$, and $\epsilon = 10^{-6}$.

Thus it seems that attempts to fix n at a very low value leads instead to a stratification of the star, with a middle region in which the density is constant. This has an uncanny similarity with the accepted picture of neutron stars. If the value of n_1 is decreased below 0.3 or 0.2 this middle region reaches the center, and the theoretically uncertain core shrinks to nothing. We do not have the temerity to suggest that this picture of a neutron star corresponds to reality, but we find it fascinating.

V. DISCUSSION OF THE RESULTS

(1) The success of the polytropic equation of state in accounting for a wide range of astronomical objects is well-known and almost miraculous. One may feel, nevertheless, that there is some question about the best choice of boundary conditions. There is also something a little unsatisfactory about the definition of mass and mass density. We have advocated the use of a variational principle, and we have found that there is a simple and natural matter Lagrangian that allows us to reproduce all of the results of the traditional approach. It also provides a conserved quantity, something that is often added as an additional ingredient to Tolman's theory. The new equation of state, that interpolates between an interior polytrope and an exterior polytrope that extends to infinity, is justified on physical grounds; see Sec. I. It wipes out all the uncertainties that are presented by the more traditional approach using a fixed boundary [5,2], and it one allows to establish the existence of a constant of the motion related to the conserved current. The idea of a double polytrope is not new, for Chandrasekhar has proposed a very similar equation of state [7,8]. With this equation of state all the examined double polytropes are stable, which is an extremely satisfactory resolution of the paradox presented by Chandrasekhar's limit. It is worth emphasizing that the only practical test of Chandrasekhar's prediction is the observation of an upper limit on the mass; this is a prediction of our model as well.

(2) In this paper it is taken for granted that the metric approaches the Schwartzchild form at infinity; in particular, that the function $r\nu(r)$ has a limit at infinity. What is called "the mass" or "asymptotic mass," and denoted m, is defined by

$$2mG = -\lim_{r\to\infty} r\nu(r).$$

The traditional approach defines "the mass" as the integral

$$\int d^3x\hat{\rho}.$$

Its value, with our treatment of the boundary conditions, is the asymptotic mass m. Of all the textbooks that we have consulted, only one (Ref. [9], pages 12–13) expresses any discomfiture with this formula; they say that it is "treacherous," because a factor $\sqrt{-g}$ is missing in the measure. Indeed, if in Tolman's formula the field with components U^μ is a vector field, and T is a tensor field, then manifestly ρ and p are scalar fields. The correct expression would be

$$\int d^3x\sqrt{-g}\hat{\rho}U^0,$$

but not being conserved there is nothing to recommend it.

The difficulty in the traditional theory arises because it does not have a conserved current, ultimately because it is not formulated as an action principle. Our model is, and we have shown that, with the boundary conditions that we have adopted, there is a conserved quantity,

$$\frac{1}{2}\int_0^\infty \sqrt{-g}g^{tt}\rho\dot{\psi}r^2drd\Omega.$$

The value is always less than m. The only meaningful concept of energy in general relativity is the Arnowit, Deser, and Misner energy; therefore we make no suggestion as to the proper name for the above integral. The important point is that both it and the asymptotic mass are constants of the motion. It is our opinion that this solves a difficulty in the applications of general relativity to mass distributions in general and to the structure of stars in particular.

(3) The paradox that is presented by the Chandrasekhar limit is that his theory fails to explain what happens to an unstable configuration. Our model reproduces the limits, but the unstable configurations of Chandrasekhar's theory do not have counterparts in the model. The problem of explaining or predicting what happens to them simply does not arise.

(4) Observation of most real stars depends on the fact that they are luminous. The role of radiation in determining the equation of state has been overlooked in this paper and this is an obstacle to further phenomenological applications. The question of how radiation is to be included in the model is interesting. Of course, matter has to interact with radiation, but since the interior of a star is neutral on a large scale this may not be the first item to take up. Radiation can be introduced by adding the Maxwell action to the gravitational and matter actions, but this is probably not the best way. Here we suggest, but offer no justification for it, to include a contribution to the action of the form

$$A_{\rm Radiation} = \int d^4x\sqrt{-g}\bigg(\frac{-\sigma F^2}{16\pi} - W[\sigma]\bigg)$$

where σ is a scalar field, the density of radiation, and the functional $W[\sigma]$ represents an internal energy of the photon gas. The modification of the Maxwell action by the factor σ and the internal energy W is inspired by analogy with the fact that we need the factor ρ and the internal energy V in the matter action. It is probable that the gravitational action should be similarly modified, to reflect the existence of a background of soft gravitons (compare Refs. [10,11]).

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