Stable cosmological models driven by a free quantum scalar field

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In the mathematically rigorous analysis of semiclassical Einstein equations, the renormalization of the stress-energy tensor plays a crucial role. We address such a topic in the case of a scalar field with both arbitrary mass and coupling with gravity in the hypothesis that the underlying algebraic quantum state is of the Hadamard type. Particularly, if we focus on highly symmetric solutions of the semiclassical Einstein equations, the envisaged method displays a de Sitter-type behavior even without an *a priori* introduced cosmological constant. As a further novel result, we shall show that these solutions turn out to be stable.

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I. INTRODUCTION

A landmark in present-day observational cosmology has been set by means of the measurement of the type IA supernovae redshift which, as a by-product, proved that the Universe is undergoing a phase of accelerated expansion. Such a result, also combined with the most recent data collected in several other experiments, suggests that, in order to explain the present state of our Universe, we must take into account the presence of a "dark energy" playing the role of an effective cosmological constant. From a theoretical point of view, we still lack a full-fledged satisfactory model for dark energy, and such a problem was tackled in the past in several ways, the most notable being by means either of a yet unobserved classical scalar field coupled to gravity [1,2] or of a modified theory of gravity itself (see [3] and references therein for a recent review).

In the present paper, our aim is to consider the backreaction of a massive quantum scalar field coupled to gravity in order to discuss the role played by quantum effects in the framework of cosmological models. The interest in backreaction effects of quantum fields in cosmology is not new since, already in the 1980s, Starobinsky [4] addressed the same topic taking into account a massless scalar field conformally coupled to gravity (see also [5]). The end point of Starobinsky's seminal paper was the construction of a graceful exit from a de Sitter phase of rapid expansion. By using the quantum property of the source fields, he observed that such a de Sitter spacetime is an unstable solution of the semiclassical Einstein equations (see also [6]). More recently, in [7], Shapiro and Sola also considered the massive case in a similar way. They obtained as well a smooth exit from an inflationary phase. Since this is a topic partly far away from our goals, we shall consider anew such a case, namely, we study the semiclassical Einstein equation

$$G_{ab} = 8\pi G\langle T_{ab}\rangle_\omega,$$

where the left-hand side is the standard Einstein tensor whereas the right-hand side is the expectation value for the stress-energy tensor in the state ω. It is a well-known problem that the latter gives origin to divergences. Hence, it is compulsory to invoke a renormalization procedure, and, within this perspective, we would like to carry on our analysis along the lines discussed by Wald, by using the point-splitting regularization.

In a series of papers [8,9], Wald sets out five axioms that need to be satisfied to have a renormalized stress-energy tensor that can be used in order to have possible meaningful semiclassical solutions of the Einstein equation. By sticking to such a perspective, we shall show that, in some physically motivated limits, we can find a stable solution to the semiclassical Einstein equation. This leads to a great difference from the original Starobinsky model, where, on the opposite, an unstable behavior is displayed. To this end, we must bear in mind the following message already conveyed to us in [10,11]: The renormalization of the stress-energy tensor suffers of some ambiguities encoded in a modification of the action by the addition of terms depending only on the curvature and on the parameters describing the fields such as, for example, the mass. This arbitrariness is then encoded in the renormalization parameters present in front of this arbitrary term. In the forthcoming discussion, we shall fix the renormalization parameters by requiring a physically meaningful theory and invoking the principle of general local covariance [12]. It will also turn out that the original result due to Starobinsky in the case of conformal invariant fields corresponds to another choice of the renormalization constants; hence, by employing a different criterion, the system under analysis displays a rather physically different behavior.

For a more mathematically oriented reader, a few more comments are in due course. Since we are interested in solutions of the semiclassical Einstein equation, where quantum matter acts as a source for the gravitational field,

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we need to employ a quantization scheme independent from the spacetime itself. Such a conceptual problem was recently addressed in a work by Brunetti, Fredenhagen, and Verch [12]. They showed that it is possible to simultaneously quantize on all spacetimes, and the quantization scheme in this framework corresponds to assigning a functor between the category of spacetimes (Man) and the category of local algebra (Loc) generated fields. Furthermore, such a functor transforms covariantly under any local transformation. Unfortunately, while fields also transform covariantly under isometries, a similar conclusion cannot be drawn for states. Therefore, since we are interested in expectation values of fields, we are forced to select a class of the mentioned states enjoying some suitable physical properties and in the framework of Friedmann-Robertson-Walker (FRW) spacetimes; this naturally leads to selection of the class of the so-called adiabatic states. Starting from these premises, we are now ready to use, within this abstract scheme of analysis, quantum matter as a source for the gravity, whereas the role of Einstein's equations will select a particular set of objects in Man, as a sort of consistency check. To rephrase, even if we can quantize in all of the spacetimes simultaneously, once a family of states is chosen, only in a few of those spacetimes do the semiclassical Einstein equations hold true.

After fixing some notation, in the next section we shall recall briefly the renormalization procedure that we shall employ. In the third section we shall perform a suitable choice for the quantum state, and then we will discuss the associated solutions of the semiclassical Einstein equations. In the fourth section we shall justify this hypothesis by means of physical motivations. Finally, some conclusions are drawn in the last section.

Einstein's equation and cosmological backgrounds

To set notations and conventions, let us clarify that our aim is to consider spacetimes whose metric is used in the description of the Universe. Hence, we stick to the standard convention of requiring the cosmological principle to hold true; this straightforwardly leads to the full class of Friedmann-Robertson-Walker metrics, and, particularly, here we shall consider only those with a spatial flat section. In a Cartesian reference frame, the metric reads

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \qquad i, j = 1, \ldots, 3, \quad (1)$$

where $a(t)$ can be interpreted as usual as the expansion factor and it is the only function to be determined out of (the semiclassical) Einstein's equations. A standard calculation shows that we can employ only the identity between traces, i.e.,

$$-R = 8\pi\langle T\rangle_\omega, \quad (2)$$

together with the conservation law for the stress-energy tensor, namely,

$$\nabla^a\langle T_{ab}\rangle_\omega = 0. \quad (3)$$

As already remarked in the introduction, $\langle T\rangle_\omega$ stands for the expectation value of the trace of the stress-energy tensor. We stress to a potential reader that the semiclassical Einstein equations are not fully equivalent to (2) and (3). As a matter of fact, in these last two equations there is a residual freedom to add to T_{ab} a conserved traceless stress-energy tensor T^0_{ab}. Such extra freedom amounts to the choice of an initial condition for $\dot{a}(t)$ in (2) at $t = t_0$. In other words, to fulfill the semiclassical Einstein equations, we need to ensure that the identity $G_{00} = 8\pi\langle T_{00}\rangle_\omega$ is satisfied at the initial time t_0.

II. MASSIVE SCALAR FIELD

As we already emphasized in the introduction, we shall employ a real scalar field ϕ as the prototype to discuss the quantum behavior of classical matter on a FRW background (1). Therefore, the classical dynamic of our system is governed by

$$P\phi = 0, \qquad P := -\Box + \xi R + m^2, \quad (4)$$

where $\xi \in \mathbb{R}$ and R is the scalar curvature, whereas m is the mass of the field. Bearing in mind that, unless stated otherwise, our convention for the metric signature is $(-, +, +, +)$, (1) entails the following identity $R = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})$, where each dot stands for derivation with respect to t. In what follows, we shall indicate $H = \dot{a}/a$. Setting $\xi = \frac{1}{6}$ corresponds to the so-called conformal coupling.

A. Quantization procedure: States and Hadamard condition

In this paragraph, we shall start dealing with the quantum behavior of the solutions of (4), and, to this avail, we shall stick to the realm of the algebraic formulation of quantum field theory. Since a detailed analysis of the main ingredients and results would require a review on its own just for the massive scalar field, we shall point an interested reader to [13,14]. Therefore, to cut a long story short, let us state that, for our purposes, it suffices to remember that, the FRW spacetime being globally hyperbolic, there exists a standard procedure to assign a $*$ algebra, say, $\mathcal{W}$, out of (4) [13,14]. Afterwards, we need to add a further ingredient, namely, a state ω: $\mathcal{W} \to \mathbb{C}$, which is the key tool out of which we can calculate the relevant objects, i.e., expectation values of the fields on that state, more commonly referred to as n-point functions, which we shall denote from now on as $\omega_n = \langle\phi(x_1)\ldots\phi(x_n)\rangle$. From a formal perspective, these objects must be thought of as distributions in $\mathcal{D}'(M^n)$, and the singular structure, proper in general of distributions, arises whenever we perform in ω_n a coincidence limit.

Therefore, in order either to formulate a mathematically meaningful field theory or to construct a theory which allows us to perform calculations going beyond the pure formal level, the selection of a suitable class of states is one of the main, if not the most important, tasks. To this avail, we shall impose some reasonable constraints, and the first requires us to restrict our attention to the so-called quasifree states. These are characterized by the following property: All of the odd n-point functions vanish, while all of the even ones can be reconstructed out of sums of products of the two-point function. In other words, quasifree states are fully determined once $\omega_2(x, y)$ is known. In the forthcoming sections, we shall display how the above requirement is relevant to our discussion. In particular, we shall show that also the stress-energy tensor can be fully determined only out of ω_2, and this is the key nongeometrical ingredient in the semiclassical Einstein equation.

Nonetheless, "quasifree" is not a sufficient requirement for our ω to satisfy, and, particularly, a second and most important hypothesis must be imposed, namely, the state shall be Hadamard. On a practical ground, from such a condition we can infer that the singular structure for the two-point function is fixed as

$$\omega_2(x, y) = \frac{1}{8\pi^2}\left(\frac{u(x, y)}{\sigma(x, y)} + v(x, y)\log\sigma(x, y) + w(x, y)\right), \tag{5}$$

where σ is half of the square of the geodesic distance in the FRW background. The functions u, v, and w, also known as Hadamard coefficients, are smooth, and u and v can be uniquely determined once the equation of motion and the metric of the underlying background are fixed. In the above expression, it turns out that u is the square root of the so-called van Vleck-Morette determinant, which depends only on g_{ab}; i.e., u can be reconstructed only out of the geometric properties of the manifold on which our fields live. On the opposite, w is the contribution to the Hadamard function which depends upon the state that we have selected. Therefore, all of the information of the singular part in (5) is encoded in

$$H(x, y) = \frac{1}{8\pi^2}\left(\frac{u(x, y)}{\sigma(x, y)} + v(x, y)\log\sigma(x, y)\right),$$

which has a universal structure in every Hadamard state. Hence, this is the contribution that we can subtract from the two-point function in order to get a smooth behavior; in other words, this amounts to regularizing the state. As a notational convention, from now on, we shall refer to $v(x, x)$ by means of the symbol $[v]$. Furthermore, $v(x, y)$ admits an asymptotic expansion in powers of the geodesic distance: $v(x, y) = \sum_{n=0}^{\infty} v_n(x, y)\sigma^n(x, y)$. In the forthcoming discussion, the coefficient v_1 will play a distinguished role.

B. Stress-energy tensor

The stress-energy tensor for a quantum real scalar field ϕ with mass m and coupling to curvature ξ can be written as

$$T_{ab} := \partial_a\phi\partial_b\phi - \frac{1}{6}g_{ab}(\partial_c\phi\partial^c\phi + m^2\phi^2) - \xi\nabla_a\partial_b\phi^2 + \xi\left(R_{ab} - \frac{R}{6}g_{ab}\right)\phi^2 + \left(\xi - \frac{1}{6}\right)g_{ab}\Box\phi^2.$$

Since the key ingredient to our analysis is the trace and the conservation equation for T_{ab}, let us switch from the previous formula to

$$T = -3(\tfrac{1}{6} - \xi)\Box\phi^2 - m^2\phi^2, \qquad \nabla_a T^a{}_b = 0.$$

We stress to the reader that here we employ a nonstandard form for T_{ab}; i.e., it differs from the more familiar one by a term proportional to $\frac{1}{3}((P\phi)\phi + \phi(P\phi))g_{ab}$ [15]. At a classical level, this contribution vanishes since, on shell, $P\phi = 0$, but nonetheless it represents an important feature in a full-fledged analysis of the underlying quantum theory, since, in this case, it is different from zero. Furthermore, encompassing such a term in the stress-energy tensor automatically accounts for the trace anomaly, which, on the opposite, was usually added by hand. As shown in [9–11,15,16], this automatically arises in the quantum theory once the point-splitting regularization is performed. We also exploit the latter to regularize the operator T_{ab} in order, subsequently, to calculate its expectation value on a quasifree Hadamard state. Such an expression would be quite cumbersome in the text and also of little avail; therefore, an interested reader can refer to the Appendix for more details.

Notice that the envisaged conservation equation for the quantum stress-energy tensor, namely, $\nabla_a\langle T^{ab}\rangle_\omega = 0$, holds true due to the following identities:

$$8\pi^2\langle\phi P\phi\rangle_\omega = 6[v_1], \qquad 8\pi^2\langle(\nabla_a\phi)(P\phi)\rangle_\omega = 2\nabla_a[v_1],$$

where $[v_1]$ is here explicitly given in the Appendix in Eq. (A1). The heritage of such a conservation law is the change of the expectation value for the trace of T_{ab} by means of a purely quantum term:

$$\langle T\rangle_\omega := \left(-3\left(\frac{1}{6} - \xi\right)\Box - m^2\right)\frac{[w]}{8\pi^2} + \frac{2[v_1]}{8\pi^2},$$

where the dependence upon the state is encoded in the term $[w]$.

To conclude, we point out to a potential reader that, due to $[v_1]$, the above trace is nonvanishing also in a conformal field theory [9].

C. Remaining freedom in the definition of T_{ab}

By means of point-splitting regularization, we have fixed the expectation value of $\langle T\rangle_\omega$ in the so-called minimal regularization prescription, namely, we have subtracted only the singular part from the two-point function. Nonetheless, as discussed by Wald [9], in the renormalization prescription, there is still a freedom of geometric nature. In detail, we can add a tensor t_{ab} written only in terms of the local metric and such that it satisfies $\nabla^a t_{ab} = 0$ without either affecting the equations of motion for the matter or violating the first four axioms introduced and discussed in Wald's paper. The conservation equation for t_{ab} is not the unique constraint we may wish to impose on such a tensor, and, in particular, a further natural requirement would be that t_{ab} behaves as T_{ab} under scale transformations. In other words, this implies that t_{ab} arises out of the following variation:

$$t_{ab} = \frac{\delta}{\delta g^{ab}} \int A\sqrt{g}R^2 + B\sqrt{g}R_{ab}R^{ab},$$

A and B being just arbitrary real numbers. Leaving the details of the above construction and analysis to [9–11,17], we shall stress only that the trace of t_{ab} turns out to be proportional to $\Box R$ independently from the choice of A and B. This is an unavoidable arbitrariness in the employed scheme, and, as a by-product, it leads us to think of A and B as renormalization constants on their own. We are now able to compute the trace of the whole quantum modified stress-energy tensor:

$$\langle T\rangle_\omega := \left(-3\left(\frac{1}{6} - \xi\right)\Box - m^2\right)\frac{\langle\phi^2\rangle_\omega}{8\pi^2} + \frac{2[v_1]}{8\pi^2} + c\Box R,$$

where c is a linear combination of A and B and it represents the freedom in the renormalization procedure that we exploited. In order to carry on our analysis, we shall now require Wald's fifth axiom to hold true. We stress to a potential reader that, while performing such a step, we are referring, in particular, to the analysis in [9], where it was shown that such an axiom can be implemented only in a sense weaker than first envisaged in [8]. To wit, there must be no derivative of the metric of degree higher than 2 in the expectation value of the trace of the stress-energy tensor. On a practical ground, such a concept can be implemented by choosing in the preceding formula the constant c in such a way to exactly cancel all of the terms proportional to $\Box R$ arising in $\langle T\rangle_\omega$. It is always possible to perform such a choice, and, particularly, when $\zeta = \frac{1}{6}$, i.e., the scalar field is conformally coupled to scalar curvature, then $c = -\frac{1}{2880\pi^2}$. Such a procedure fixes only $A + B/3$, whereas, in our specific model, the remaining freedom for A and B could eventually be fixed by requiring the validity of the constraint $G_{00} = 8\pi\langle T_{00}\rangle_\omega$ at the initial time t_0. Further renormalization ambiguities are encoded in the expectation value of the field $\langle\phi^2\rangle_\omega$; we shall come back later to this point by fixing the ambiguity by physical motivation.

We stress that a similar observation brought interest in the so-called modified theory of gravity also known as $f(R)$ gravity. Nonetheless, the view we wish to push home is the following: Adding t_{ab} does not come from a modified gravitational action, but it originates only from the employed renormalization scheme; i.e., it must be an effect coming from quantum matter. Naturally, this does not exclude that such a perspective cannot provide hints on how a candidate theory of quantum gravity interacts with quantum matter. As a final comment, we stress that the above is the subtlest point in the whole construction. We used an expression for the stress-energy tensor which is suitable in order to deal with the semiclassical Einstein equation. Nonetheless, such a modification is not artificial, corresponding as a matter of fact just to a specific choice of the renormalization constants arising out of the employed scheme.

III. EVOLUTION EQUATION OF THE MODEL

In the case of conformal coupling $\xi = 1/6$, Eq. (2), written in terms of $H = \dot{a}/a$, becomes

$$-6(\dot{H} + 2H^2) = -8\pi G m^2\langle\phi^2\rangle_\omega + \frac{G}{\pi}\left(-\frac{1}{30}(\dot{H}H^2 + H^4) + \frac{m^4}{4}\right). \quad (6)$$

The aim of this section is to analyze in detail the possible solutions of (6) under some specific hypotheses on the expectation value for $\langle\phi^2\rangle_\omega$. Particularly, we shall show that a de Sitter space with a specific curvature will appear as a stable solution.

A. Conformal invariant case: Stability of de Sitter phase

As a starting point, we shall discuss the $m = 0$ scenario, already considered in Starobinsky's paper [4] (see also [5]). We also stress to a potential reader that most of the results of Sec. III A have been already derived in an earlier paper of Wald [18]. As remarked above, this case is rather special, since there is no need to select a specific state and an ordinary differential equation rules the evolution of H. Hence, by setting $m = 0$ in (6), we end up with

$$\dot{H}(H^2 - H_0^2) = -H^4 + 2H_0^2H^2. \quad (7)$$

Here $H_0^2 = \frac{180\pi}{G}$ depends on the Newton constant, and it has an order of magnitude of 24 times the inverse Planck time. Let us notice that, out of the right-hand side of (7), we can extract two critical points; therefore, (7) admits two constant solutions, namely, $H(t) = 0$ and $H(t) = H_+ = \sqrt{2}H_0$, corresponding, respectively, to a Minkowski spacetime and to a de Sitter one. Suppose now that we assign an initial condition at a fixed time t_0 such that $H(t_0) \neq 0$ and $H(t_0) \neq H_+$; we are interested to realize if the solution interpolating such an initial condition flows at large times either to 0 or to H_+, i.e., in order words, whether these two critical points are stable or not. To bring such a task to a good end, we simply need to notice that (7) is integrable as

$$Ke^{4t} = e^{2/H}\left|\frac{H + H_+}{H - H_+}\right|^{1/H_+}, \quad (8)$$

where K stands for the integration constant to be fixed out of the initial condition H_0. Depending on such a last value, all of the solutions $H(t)$ flow either to 0 or to H_+. Hence, both critical points turn out to be stable. This result is different from the classical outcome of the analysis due to Starobinsky [4] (see also Vilenkin and Ford [5,6]). The price to pay, in order to achieve such a result, is a choice by hand of a renormalization constant. It turns out to be an addition of a tensor written only in terms of the metric, and such an operation introduces in the theory a scale length, as already discussed by Wald in [9]. We have to stress that, on the dark side, the above de Sitter solution cannot describe the present-day form of the Universe being $H_+ \simeq 6.4 \times 10^{44}\ \mathrm{s}^{-1}$, i.e., many orders of magnitude bigger then the present measured Hubble constant $(2.6 \pm 0.2) \times 10^{-18}\ \mathrm{s}^{-1}$. On the bright side, instead, we have shown that, by encompassing the full quantum effects, we are led to find a stable de Sitter solution even if no cosmological constant is present in the equations.

B. Massive case with $\xi = 1/6$: Stability of the de Sitter phase, effective cosmological constant

In this section we switch from the massless to the massive case. The most important difference is the following: The right-hand side of (6) depends explicitly upon the state via the expectation value of ϕ^2. The expectation value of $\langle\phi^2\rangle_\omega$ on a general Hadamard state ω is $\frac{[w]}{8\pi^2} + \alpha m^2 + \beta R$, where α and β are renormalization constants encoding the ambiguities still present in the procedure. We assume for the moment the existence of a set of Hadamard states $\tilde{\omega}$, one for each spacetime whose metric is of the form (1) being $H = \dot{a}/a$ and $\langle\phi^2\rangle_{\tilde{\omega}} = \alpha m^2 + \beta R$. We shall see later that this assumption turns out to be an approximation of the expectation values of the fields computed on the adiabatic states of FRW in the limit where $m^2 \gg R$ and $m \gg H$. Moreover, by the principle of general local covariance [10–12], we are entitled to fix the renormalization constants once and in the same way for every spacetime that we are considering. Then the expectation value of $\langle\phi^2\rangle_{\tilde{\omega}}$ on the states we are considering takes the following values:

$$\langle\phi^2\rangle_{\tilde{\omega}} = \alpha m^2 + \beta R \tag{9}$$

on all of the considered FRW spacetimes. Therefore, by taking into account these remarks, (6) takes the following form:

$$\dot{H}(H^2 - H_0^2) = -H^4 + 2H_0^2 H^2 + M, \tag{10}$$

where H_0 and M are the following two constants with the following values:

$$H_0^2 = \frac{180\pi}{G} - 8\pi^2 180 m^2 \beta,$$
$$M = \frac{15}{2} m^4 - 240\pi^2 m^4 \alpha.$$

As in the previous section, the right-hand side of (10) displays at most two critical points amounting to

$$H_\pm^2 = H_0^2 \pm \sqrt{H_0^4 + M}, \tag{11}$$

both corresponding either to a de Sitter phase or to a Minkowski phase, under the assumption that α and β have been chosen in such a way that both H_-^2 and H_+^2 are greater than or equal than 0.

A straightforward analysis shows that both $H(t) = H_\pm$ appear to be stable since all of the solutions flow to either one of the two fixed points. It is remarkable that the existence and the stability behavior of the latter are left unchanged whether the right-hand side of (9) is modified by adding a term such as $Aa^{-\lambda}(t)$, $\lambda \in \mathbb{R}$ and A a constant of suitable dimension. It is also interesting to notice that a formula similar to (11) already appeared in [19], although, in the cited paper, a classical cosmological constant has been introduced from the beginning. At this stage, our simple model depends on three parameters α, β, and m. A minimal and, to a certain extent, compulsory choice is to require Minkowski as a solution of our system. This amounts to fixing $\alpha = (32\pi^2)^{-1}$, which, on the other hand, entails $M = 0$. The form of the solution is then equal to that of the massless case (8), where one of the fixed points corresponds to a Minkowski space—$H(t) = 0$—while the other fixed point $H(t) = H_+$ corresponds to de Sitter. With respect to the massless conformal factor, here we can fine-tune the parameters β and m in such a way for H_+ to be small enough in order to account for the present measured value of the Hubble constant. Hence, heuristically speaking, our system behaves as if an effective cosmological constant enters the fray without even being present at the beginning, and this is a strict consequence of encompassing the full quantum properties of the field. As a further remark, we notice that (8) displays, for a large class of initial conditions, an early time phase of rapid expansion which is a prerequisite feature of modern models for studying the early stages of evolution of the Universe. This is in sharp contrast with the canonical paradigm according to which quantum effects should account only for small fluctuations with respect to the classical behavior. On the opposite, even in the most simple example of a massive scalar field and with the most simple assumptions, our system displays a behavior which drastically differs from the one we could *a priori* expect only from a classical analysis. Hence this suggests that, when dealing with scalar fields on a FRW background, one should always perform a full-fledged analysis of the semiclassical behavior of the system since the quantum contributions appear to be hardly negligible as one can also infer from Fig. 1.

As a final comment, we stress that, in a neighborhood of $H = H_+$, the found solution (8) looks rather similar to the one of a classical flat universe with the cosmological constant filled with radiation. As a matter of fact, in that

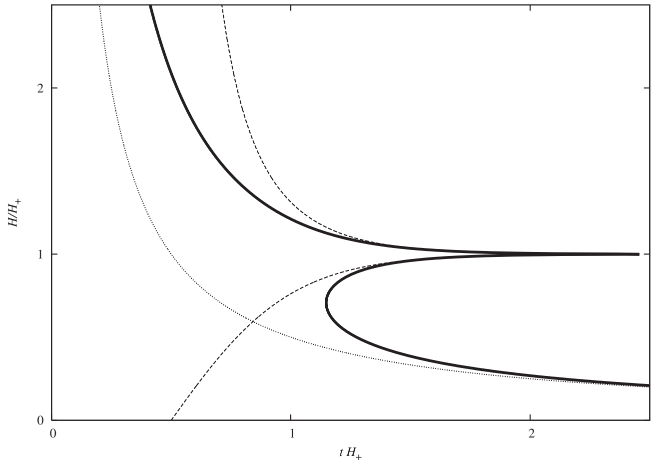

FIG. 1. Here the dashed line corresponds to the behavior of $\frac{H}{H_+}$ as a function of time t (normalized with respect to $1/H_+$) in a FRW universe with a nonvanishing cosmological constant and filled with radiation, while the dotted line stands for the lone classical contribution coming from radiation. Conversely, the continuous line depicts the outcome of our model where quantum effects are also taken into account.

case $H_{\mathrm{rad},\Lambda}(t) = A\tanh(2(t-t_0)A)$, where A is a constant related to the cosmological constant, and it can be inverted as

$$Ke^{4t} = \left|\frac{H+A}{H-A}\right|^{1/A},$$

which looks very similar to (8) when $H \sim A$ and $H_+ = A$; this corresponds to the dashed line in Fig. 1. The quantum effects are not important only around $H = 0$ where (8) looks like $H_{\mathrm{rad}}(t)$ in a flat universe filled only with radiation, namely, the dotted line in Fig. 1. Eventually, we would like to stress that, considering the upper branch of the solution, in the past, it displays the behavior of a classical flat universe with a kind of matter such that $\rho = Aa(t)^{-2}$. Even in this regime, quantum effects are not negligible. As a further remarkable consequence of the analytic form of $H(t)$, it turns out that the singularity at $t = t_0$ coincides with null past infinity in the flat spacetime conformally related to (1); hence, it descends that the particle horizon is not present. Therefore, any pair of points in the underlying background was casually related in the past, and, thus, as a by-product, such a property of our model could provide a solution to the problem of homogeneity.

IV. EXPECTATION VALUE OF ϕ^2 ON THE ADIABATIC VACUUM

In the preceding section, we have seen that, by assuming a suitable form of $\langle\phi^2\rangle$, two stable de Sitter phases can arise as solutions of the semiclassical Einstein equation. We would like to give a justification for our assumption, namely, we shall show that there is a regime in which it is valid. Here we restrict our attention to the case of a massive scalar field with a conformal coupling to the metric. The first observation is that, if we select the Bunch-Davies state ω_B [20] on a de Sitter spacetime and if we compute the renormalized version of the expectation value of ϕ^2, we obtain a constant that depends only on the mass m and on H. With this observation, we can immediately conclude that the two fixed points $H(t) = H_+$ and $H(t) = H_-$ discussed above are really exact solutions of the semiclassical Einstein equation. In the next we shall select a class of states that, in the limit of a large mass, shows an expectation value for $\langle\phi^2\rangle_\omega$ that is of the type $\alpha m^2 + \beta R$.

Adiabatic states and large mass expansion

We would like to select here the class of adiabatic states, i.e., those introduced by Parker [21] to minimize particle creation (see also [22] for a derivation of the expectation values of the stress tensor). Much work has been done also recently in order to make the definition of these states precise [23–25]. In order to write the two-point function of these states, we follow the construction as in Parker [21]. In the case of conformal coupling, it is convenient to use the conformal time τ defined as $\tau - \tau_0 = \int_{t_0}^t \frac{dt'}{a(t)}$. Therefore, the two-point function of such a kind of states is

$$\omega(x_1, x_2) = \frac{1}{8\pi^3}\frac{1}{a(\tau_1)a(\tau_2)} \times \int d^3\mathbf{k}\,\overline{\Psi_k(\tau_1)}\Psi_k(\tau_2)e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)};$$

x_i k_i are four vectors and $\mathbf{x}_i$ are three vectors, whereas k stands for the length of the spatial vector $\mathbf{k}$. The functions $\Psi_k(\tau)$ are solutions of a differential equation with a suitable normalization condition:

$$\left(\frac{d^2}{d\tau^2} + k^2 + m^2a(\tau)^2\right)\Psi_k(\tau) = 0,$$
$$\overline{\Psi_k(\tau)}\frac{d}{d\tau}\Psi_k(\tau) - \Psi_k(\tau)\frac{d}{d\tau}\overline{\Psi_k(\tau)} = i.$$

Each $\Psi_k(\tau)$ can alternatively be written in the following way:

$$\Psi_k(\tau) = \frac{1}{\sqrt{2\Omega_k(\tau)}}e^{i\int_{\tau_0}^{\tau}\Omega_k(\tau)}.$$

In the adiabatic approximation $\Omega_k(\tau)$ is a function constructed recursively in the following way:

$$\Omega_k^{(0)^2}(\tau) = k^2 + m^2a(\tau)^2,$$

and

$$\Omega_k^{(n+1)^2}(\tau) = k^2 + m^2 a(\tau)^2 + \frac{3}{4}\left(\frac{\Omega_k^{(n)'}(\tau)}{\Omega_k^{(n)}(\tau)}\right)^2 - \frac{1}{2}\frac{\Omega_k^{(n)''}(\tau)}{\Omega_k^{(n)}(\tau)}, \tag{12}$$

where the prime stands for the derivation with respect to τ. The nth order approximation consists then in the substitution of Ω_k with $\Omega_k^{(n)}$ in $\Psi_k(\tau)$, and we shall indicate with $\omega_2^{(n)}$ the counterpart for the two-point function of the state. Nonetheless, one should bear in mind that this recursive procedure does not have nice convergence properties, though, thanks to the work of Junker and Schrohe [24], we know that the state constructed in this way is an adiabatic state in the sense that $\omega_2^{(n)}$ have a certain Sobolev wave-front set. Hence, if n is large enough, we can use the approximated state in order to build the stress-energy tensor or the expectation value of ϕ^2. In particular, we can compute the approximated expectation value $\langle\phi^2\rangle_{(n)} = \lim_{x\to y}(\omega_2^{(n)}(x,y) - H(x,y))$, which, more explicitly, becomes

$$\langle\phi^2\rangle_{(n)} = \frac{1}{4\pi^2 a(\tau)^2}\int_0^\infty dkk^2\left(\frac{1}{\Omega_k^{(n)}(\tau)} - \frac{1}{\Omega_k^{(0)}(\tau)}\right) + \alpha' R + \beta' m^2.$$

Above, α' and β' need to be interpreted as renormalization constants. An exact computation of this integral can be very difficult to perform; hence, we will show only how to compute an expectation value in the limit of a large mass, namely, by assuming that $H(t)$ is a smooth function and $m^2 \gg R$. In this case, if furthermore $n \geq 2$, it is possible to expand the integral in powers of $1/m^2$ as

$$\langle\phi^2\rangle_{(n)} = \alpha m^2 + \beta R + O\left(\frac{1}{m^2}\right),$$

where α and β are slightly different from the ones written before. In the large mass limit we shall simply consider $\langle\phi^2\rangle_{(n)} = \alpha m^2 + \beta R$. The result should be read as a confirmation for the approximation that we have done in the preceding section.

V. INTERPRETATION OF THE RESULTS AND FINAL COMMENTS

In the present paper, we have shown that, when dealing with cosmological models, quantum effects are not negligible even when we consider basic models. As a matter of fact, our analysis displays that, from a careful analysis of the expectation values of the renormalized stress-energy tensor, there arises an effective cosmological constant which can be interpreted as dark energy.

Such a feature is manifest if we take into account a massive scalar field propagating in a curved background, although we envisage that similar effects would be present if we consider other kinds of fields. Furthermore, we have seen that a de Sitter solution appears as a stable fixed point of the semiclassical Einstein equation, and, to a certain extent, also a phase of rapid expansion can be foreseen in the model. We also believe that, since the found results, and particularly the stability of the de Sitter solution, are based upon a modification of the point-splitting procedure by a pure gravitational term, this could be read as a hint for future study of quantum gravitational models interacting with matter. To this avail, it also seems interesting to pinpoint that, even considering the one-loop corrections to the action of an $f(R)$ theory, one is led to a stable de Sitter solution [26,27]. Furthermore, also in this last case, stability is a joint effect of quantum theory and classical gravity, and this is a behavior which a lone $f(R) = R^2$ term does not display.

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APPENDIX A: POINT-SPLITTING REGULARIZATION OF THE STRESS-ENERGY TENSOR

Let ω_2 be the two-point function of a quasifree Hadamard state. The expectation value of the stress-energy tensor regularized by means of the point-splitting procedure is

$$\begin{aligned}\langle T_{ab}\rangle_\omega(z) := \lim_{(y,x)\to(z,z)}\Big[&\partial_a\partial'_b - \frac{1}{6}g_{ab}(g^{cd}\partial_d\partial'_c + m^2) \\ &- 2\xi(\nabla_a\partial_b + \partial_a\partial'_b) + \xi\left(R_{ab}(z) - \frac{R(z)}{6}g_{ab}\right) \\ &+ \left(\xi - \frac{1}{6}\right)g_{ab}(2\nabla^c\partial_c + 2g^{dc}(z)\partial_d\partial'_c)\Big]\frac{1}{2} \\ &\times(\omega_2(y,x) - H(y,x) + \omega_2(x,y) - H(x,y)),\end{aligned}$$

where the prime stands for a derivative in y whereas the one without a prime is a derivative with respect to x. A reader should notice that, in the last part of the equation, there is a symmetrization done at the level of the two-point function and that $H(x,y)$ is the singular part of the Hadamard series.

$[v_1]$ coefficient in the cosmological case

Since it is a relevant datum in our procedure, we provide the explicit expression for $2[v_1] = [a_2]/2$, a_2 being the

Schwinger-de Witt coefficient as derived at page 194 in [16] with the choice of $V = \xi R + m^2$ (see also [28])

$$2[v_1] = \frac{1}{360}\left(C_{ijkl}C^{ijkl} + R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{1}{4}\left(\frac{1}{6} - \xi\right)^2 R^2 + \frac{m^4}{4} - \frac{1}{2}\left(\frac{1}{6} - \xi\right)m^2 R + \frac{1}{12}\left(\frac{1}{6} - \xi\right)\Box R. \quad \text{(A1)}$$

Furthermore, by assuming that the metric has the form of a flat FRW universe (1) and writing $H = \dot{a}/a$, $[v_1]$ takes the following form:

$$2[v_1] = -\frac{1}{30}(\dot{H}H^2 + H^4) + \frac{1}{12}\left(\frac{1}{5} - \xi\right)\Box R + 9\left(\frac{1}{6} - \xi\right)^2(\dot{H}^2 + 4H^2\dot{H} + 4H^4) + \frac{m^4}{4} - 3\left(\frac{1}{6} - \xi\right)m^2(\dot{H} + 2H^2).$$

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