# Stability of an isotropic cosmological singularity in higher-order gravity

Jonathan Middleton and John D. Barrow
*DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
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We study the stability of the isotropic vacuum Friedmann universe in gravity theories with higher-order curvature terms of the form $(R_{ab}R^{ab})^n$ added to the Einstein-Hilbert Lagrangian of general relativity on approach to an initial cosmological singularity. Earlier, we had shown that, when $n = 1$, a special isotropic vacuum solution exists which behaves like the radiation-dominated Friedmann universe and is stable to anisotropic and small inhomogeneous perturbations of scalar, vector, and tensor type. This is completely different to the situation that holds in general relativity, where an isotropic initial cosmological singularity is unstable in vacuum and under a wide range of nonvacuum conditions. We show that when $n \neq 1$, although a special isotropic vacuum solution found by Clifton and Barrow always exists, it is no longer stable when the initial singularity is approached. We find the particular stability conditions under the influence of tensor, vector, and scalar perturbations for general $n$ for both solution branches. On approach to the initial singularity, the isotropic vacuum solution with scale factor $a(t) = t^{P_-/3}$ is found to be stable to tensor perturbations for $0.5 < n < 1.1309$ and stable to vector perturbations for $0.861\,425 < n \leq 1$, but is unstable as $t \to 0$ otherwise. The solution with scale factor $a(t) = t^{P_+/3}$ is not relevant to the case of an initial singularity for $n > 1$ and is unstable as $t \to 0$ for all $n$ for each type of perturbation.

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## I. INTRODUCTION

The study of the very early universe leads us to investigate what happens to our assumptions about the truth of Einstein's general theory of relativity when the curvature of space and the density of matter and radiation approach the fundamental Planck values defined by the constants of nature, $G$, $c$, and $h$. The most natural extensions to explore as generalizations of general relativity are the higher-order theories of gravity that arise when the Einstein-Hilbert Lagrangian is extended by adding powers of the scalar curvature or the square of the Ricci tensor. As the Planck epoch is reached, or passed, on approach to a cosmological singularity, these higher-order terms are expected to dominate the behavior of simple cosmological models. Any evaluation of what are likely initial conditions during the preinflationary era of a cosmological model should therefore be based on a full understanding of the general behavior of cosmological models in the presence of higher-order gravity terms.

Contributions to the Lagrangian from powers of the scalar curvature, $R^n$, are conformally equivalent to the presence of a self-interacting scalar field and are understood [1]. In an earlier paper [2], we considered the effect on cosmological singularities of adding the quadratic Ricci invariant $R_{ab}R^{ab}$ to the Einstein-Hilbert action of general relativity. The purely quadratic Lagrangian gravity theories that contain this invariant, but not the Einstein-Hilbert ($R$) term, possess an isotropic vacuum cosmological solution, in which the expansion scale factor, $a(t)$, behaves as in the flat Friedmann-Robertson-Walker (FRW) radiation-dominated universe of general relativity, with $a(t) = t^{1/2}$ [3,4]. In the case of zero spatial curvature,[1] this vacuum solution of the pure $R_{ab}R^{ab}$ theory therefore has the exact metric:

$$ds^2 = -dt^2 + t(dx^2 + dy^2 + dz^2). \tag{1}$$

Thus, we see that the higher-order Ricci stresses induce a behavior that mimics the effect of an isotropic blackbody radiation stress, even though no physical stress of this sort is present. Earlier studies of anisotropic, spatially homogeneous universes of Bianchi types I, II [4], and IX [5] showed that this special isotropic solution is stable against homogeneous anisotropic distortions as $t \to 0$. This surprising situation is completely different to that encountered in general relativity (GR), when the $R_{ab}R^{ab}$ term is absent from the action. In GR, the expansion and 3-curvature anisotropies dominate the vacuum dynamics as $t \to 0$ so as to produce anisotropic [6,7], and even chaotic [8], dynamics. For all perfect fluids with pressure, $p$, and density, $\rho$, satisfying $-\rho/3 < p < \rho$, the isotropic solution is unstable as $t \to 0$ and hence such isotropic solutions are special in GR [9]. This instability does not occur when the $R_{ab}R^{ab}$ term is present. On approach to the cosmological singularity, the higher-order curvature terms render the isotropic solution stable. This has all sorts of consequences for physical cosmology. For example, it ensures that a preinflationary state will likely be isotropic and it removes the need for the introduction of an extra physical principle,

[1] The Friedmann radiation solutions are also exact solutions of the pure $R_{ab}R^{ab}$ theory in the cases of nonzero spatial curvature [4].

 

like the minimization of a "gravitational entropy" [10,11], in order to enforce a special isotropic initial state. However, it does suggest that a stable state of isotropic contraction will be produced on approach to any future singularity in a closed universe and that may be an awkward conclusion for any theory of a gravitational entropy governed by its own gravitational "Second Law".

The addition of quadratic Ricci terms can also create unusual evolutionary behavior, not seen in general relativity. Barrow and Hervik found exact solutions which display anisotropic inflation [4,11]. These solutions do not have a general-relativistic limit and are intrinsically nonlinear with respect to the space-time curvature.

In our first paper [2], we extended the study of the effects of an $R_{ab}R^{ab}$ addition to the Einstein-Hilbert action to the situation of anisotropic and inhomogeneous cosmologies. Specifically, we investigated the behavior of small scalar, vector, and tensor perturbations to the metric (1) as $t \to 0$. We found that there were *no* growing metric perturbation modes of scalar, vector, or tensor sorts as $t \to 0$. Thus, a small perturbation of the isotropic cosmological solution forms part of the general solution of the gravitational field equations when the $R_{ab}R^{ab}$ term is present: it is an open property of the initial data space of the quadratic theory.

These results immediately suggest that we should investigate whether or not the stability of isotropic singularities is maintained to higher order when we introduce additions to the Einstein-Hilbert action of the form $(R_{ab}R^{ab})^n$. We expect the situation for $n \neq 1$ to be more complicated because there will no longer be a simple Gauss-Bonnet invariant underlying the field equations. This question of the stability of the $n \neq 1$ theories is the subject of this paper. In the absence of the Einstein-Hilbert term, there is a counterpart to the simple isotropic vacuum solution of equation (1) in the case of general $n$, which was found by Clifton and Barrow [12]. This reduces to the solution (1) as $n \to 1$. It is the stability of this isotropic power-law solution for general $n$ that we shall investigate.

In Sec. II we give the field equations for the gravity theory with an $R + A(R_{ab}R^{ab})^n$ Lagrangian and give the exact isotropic vacuum solutions. These solutions have two branches. We identify the physically interesting one that describes an expanding universe and show that as $n \to \infty$ the exact vacuum solution approaches that of a dust-filled general relativity solution with $a(t) = t^{2/3}$.

In Sec. III, we present the formalism for studying small tensor, vector, and scalar perturbations of this special vacuum solution in order to determine the conditions on $n$ for which it is stable as $t \to 0$ and the initial singularity is approached. In Secs. IV, V, and VI these stability analyses are carried out for tensor, vector, and scalar perturbation modes, respectively. The results are summarized and discussed in Sec. VII. A collection of useful quantities is derived in the Appendices.

## II. FIELD EQUATIONS

Consider a higher-order gravity theory with action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{\chi}(R + A(R_{ab}R^{ab})^n) + L_m\right],$$

where $\chi$, $A$, and $n$ are constants. The field equations are obtained using the general formula from Clifton and Barrow [12] which expresses the higher-order contributions as an additional effective stress tensor:

$$G^a_b + AP^a_b = \frac{\chi}{2}T^a_b, \tag{2}$$

where

$$\begin{aligned} P^a_b \equiv {} & -\tfrac{1}{2}Y^n g^a_b + nR^a_b\Box(Y^{n-1}) + nY^{n-1}\Box R^a_b + 2ng^{cd}(Y^{n-1})_{,c}R^a_{b;d} + ng^a_b(Y^{n-1}{}_{;cd}R^{cd} + 2(Y^{n-1})_{,c}R^{cd}{}_{;d} + \tfrac{1}{2}Y^{n-1}\Box R) \\ & - n((Y^{n-1})_{;b}{}^{c}R^a_c + (Y^{n-1})_{;c}{}^{a}R^c_b + (Y^{n-1})_{;b}R^{ca}{}_{;c} + g^{ad}(Y^{n-1})_{;d}R^c_{b;c} + (Y^{n-1})_{,c}R^{ca}{}_{;b} + (Y^{n-1})_{,c}g^{ad}R^c_{b;d} \\ & + Y^{n-1}(g^{ad}R_{;db} + 2R^a{}_{cdb}R^{cd})), \end{aligned}$$

with $Y = R^{ab}R_{ab}$, and $G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab}$ is the usual Einstein tensor.

We consider perturbations about a spatially flat, homogeneous and isotropic FRW spacetime with metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{3}$$

with aforementioned scale factor $a(t)$ and associated Hubble expansion rate $H \equiv \frac{\dot{a}}{a}$.

In the limit where the Ricci term dominates, $A \to \infty$, which we expect to be appropriate in the neighborhood of the cosmological singularity where $a \to 0$, provided $n > \frac{1}{2}$, the vacuum field equations reduce to $P^a_b = 0$. To background order, we have

$$\begin{aligned} P^0_0 = {} & -Y^{n-2}\{\tfrac{1}{2}Y^2 + 6nY(2H\ddot{H} - 2\dot{H}^2 + 3H^2\dot{H} - 3H^4) \\ & + 6n(n-1)\dot{Y}(2H\dot{H} + 3H^3)\}, \end{aligned} \tag{4}$$

$$P^0_\alpha = 0 = P^\alpha_0, \tag{5}$$

$$\begin{aligned} P^\alpha_\beta = {} & -Y^{n-3}\delta^\alpha_\beta\{\tfrac{1}{2}Y^3 + nY^2(4\dddot{H} + 24H\ddot{H} + 12\dot{H}^2 \\ & + 18H^2\dot{H} - 18H^4) + n(n-1)Y\dot{Y}(8\ddot{H} + 36H\dot{H} \\ & + 12H^3) + 2n(n-1)(2\dot{H} + 3H^2)((n-2)\dot{Y}^2 + Y\ddot{Y})\}. \end{aligned} \tag{6}$$

Substituting for $Y$ and $\dot{Y}$ in terms of $H, \dot{H}, \ldots$ gives

$$P_0^0 = -72(12\dot{H}^2 + 36H^2\dot{H} + 36H^4)^{n-2}\{\dot{H}^4 + 6H^2\dot{H}^3 + 15H^4\dot{H}^2 + 18H^6\dot{H} + 9H^8 + n(-2H\dot{H}^2\ddot{H} - 6H^3\dot{H}\ddot{H} - 3H^5\ddot{H} - 2\dot{H}^4 - 15H^2\dot{H}^3 - 42H^4\dot{H}^2 - 36H^6\dot{H} - 9H^8) + n^2(4H\dot{H}^2\ddot{H} + 12H^3\dot{H}\ddot{H} + 9H^5\ddot{H} + 12H^2\dot{H}^3 + 42H^4\dot{H}^2 + 36H^6\dot{H})\}.$$

Hence, the Friedmann-like equation for this theory in vacuum is

$$0 = H^5\ddot{H}(9n^2 - 3n) + H^3\dot{H}\ddot{H}(12n^2 - 6n) + H\dot{H}^2\ddot{H}(4n^2 - 2n) + \dot{H}^4(1 - 2n) + H^2\dot{H}^3(6 - 15n + 12n^2) + H^4\dot{H}^2(15 - 42n + 42n^2) + H^6\dot{H}(18 - 36n + 36n^2) + H^8(9 - 9n). \quad (7)$$

For power-law scale factors, $a = t^k$, and general values of $n \neq 1$, this implies

$$k = 0, \qquad k = \frac{1}{2} \pm \frac{i}{6}\sqrt{3}, \qquad \text{or } k = \frac{P}{3},$$

where the possible values of $P$ are given by the two roots of a quadratic:

$$P = P_\pm \equiv \frac{3(1 - 3n + 4n^2) \pm \sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)}. \quad (8)$$

The variation with $n$ is displayed in Figs. 1 and 2. In the limit $n \to 1$, $P_- \to \frac{3}{2}$, and we obtain the special $a = t^{1/2}$ vacuum solution of the quadratic ($n = 1$) case studied in Ref. [2]. Note also that $P_-$ rapidly asymptotes towards 2 as $n \to \infty$,

$$P_- \to 2 - \frac{1}{3n} - \frac{1}{18n^2} - \frac{13}{216n^3} + O(n^{-4}), \quad (9)$$

and the vacuum solution rapidly approaches the behavior of the GR dust solution with $a = t^{2/3}$, see Eq. (9). $P$ (or its real part) is greater than 3 only for the range $-\frac{1}{2} < n < -0.390\,388$. For the choices $k = 0$, $k = \frac{1}{2} \pm \frac{i}{6}\sqrt{3}$, we must have $n > 0$, since $12\dot{H}^2 + 36H^2\dot{H} + 36H^4$ also vanishes. The physically interesting cases relevant to an initial singularity are those with $k > 0$, i.e. solutions which are expanding to the future. Finally, we note that an exponential scale factor with $H =$ constant is possible iff $n = 1$.

For comparison, in a perfect fluid-filled universe with equation of state $p = w\mu$, there is a flat FRW exact solution of the $(R_{ab}R^{ab})^n$ theory where the scale factor is given by

$$a(t) = t^{4n/[3(w+1)]}. \quad (10)$$

## III. INHOMOGENEOUS PERTURBATIONS

We will now develop the formalism for studying small perturbations of the spatially flat isotropic FRW solutions of the $(R_{ab}R^{ab})^n$ theory, which generalizes the formalism developed by Noh and Hwang for the quadratic ($n = 1$) theory [13–15]. We are interested in the stability of the spatially flat isotropic background FRW solution

$$a(t) = t^{P/3}, \quad (11)$$

where

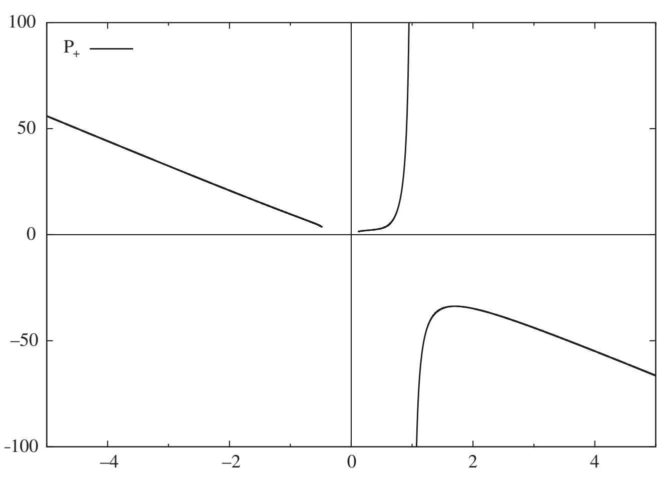


FIG. 1. The variation of $P_+$ with $n$.

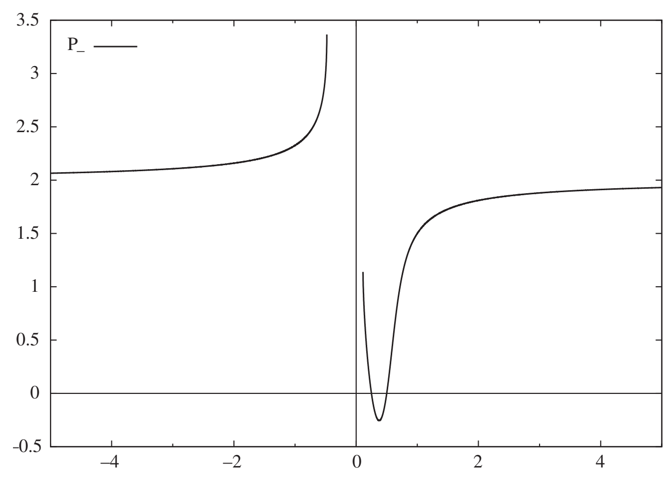


FIG. 2. The variation of $P_-$ with $n$.

$$P = P_{\pm} = \frac{3(1 - 3n + 4n^2) \pm \sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)}. \tag{12}$$

The general perturbed metric may be written as

$$ds^2 = -a^2(1 + 2\alpha)d\eta^2 - a^2\tilde{B}_\alpha d\eta dx^\alpha + a^2(\delta_{\alpha\beta} + \tilde{C}_{\alpha\beta})dx^\alpha dx^\beta, \tag{13}$$

where $\eta$ is a conformal time coordinate that is related to the comoving proper time, $t$, by $dt = ad\eta$. We can decompose the perturbation variables into their scalar, vector, and tensor parts in the standard way, as in [2], by writing

$$\tilde{B}_\alpha = 2\beta_{,\alpha} + 2B_\alpha,$$
$$\tilde{C}_{\alpha\beta} = 2\phi\delta_{\alpha\beta} + 2\gamma_{,\alpha\beta} + 2C_{(\alpha,\beta)} + 2C_{\alpha\beta}.$$

There are four scalar perturbation variables, $\alpha$, $\beta$, $\phi$, and $\gamma$, two vector variables, $B_\alpha$ and $C_\alpha$, and one tensor, $C_{\alpha\beta}$. The quantities $B_\alpha$ and $C_\alpha$ are divergence-free, i.e. $B^\alpha{}_{,\alpha} \equiv 0 \equiv C^\alpha{}_{,\alpha}$, and $C_{\alpha\beta}$ is transverse and trace-free. These three types of perturbation evolve independently of each other at linear order. We will determine the equations which describe their time evolution and then solve each of them to determine whether the metric perturbations to the special solution are stable as $t \to 0$. In the $n = 1$ case the problem, the equations, and their solutions will reduce to those of [2]. In this way we establish the ranges of $n$ values for which the special isotropic vacuum solution is a stable initial condition for the higher-order theory.

## IV. TENSOR (GRAVITATIONAL-WAVE) PERTURBATIONS

The expansion of the metric around the spatially flat Friedmann solution now takes the form

$$ds^2 = -dt^2 + a^2(\delta_{\alpha\beta} + 2C_{\alpha\beta})dx^\alpha dx^\beta.$$

The tensor $C_{\alpha\beta}$ is trace-free and transverse, i.e.

$$C^\alpha_\alpha = 0 = C^\alpha_{\beta,\alpha} \tag{14}$$

and $C = C(\mathbf{x}, t)$.

The $n = 1$ case was solved exactly in [2] for perturbations about $a(t) = t^{1/2}$. Here, we want to perturb an isotropic background solution which has $a(t) = t^{P/3}$ with

$$P = P_{\pm} = \frac{3(1 - 3n + 4n^2) \pm \sqrt{3(48n^4 - 40n^3 - 5n^2 + 10n - 1)}}{2(1 - n)}. \tag{15}$$

The important quantities to linear order in the perturbation are given in Appendix A. In the limit where the higher-order terms dominate, the perturbed field equation is

$$\delta P^\alpha_\beta = -nY^{n-1}\Big(\dddot{C}^\alpha_\beta + 6H\ddot{C}^\alpha_\beta + 3H^2\ddot{C}^\alpha_\beta - (3\ddot{H} + 21H\dot{H} + 18H^3)\dot{C}^\alpha_\beta - 2\frac{\Delta}{a^2}\ddot{C}^\alpha_\beta - 2H\frac{\Delta}{a^2}\dot{C}^\alpha_\beta + \Big(4\dot{H} + 8H^2 + \frac{\Delta}{a^2}\Big)\frac{\Delta}{a^2}C^\alpha_\beta\Big) - n(Y^{n-1})^{\cdot}\Big(2\ddot{C}^\alpha_\beta + 9H\ddot{C}^\alpha_\beta + 3H^2\dot{C}^\alpha_\beta - 2\frac{\Delta}{a^2}\dot{C}^\alpha_\beta + H\frac{\Delta}{a^2}C^\alpha_\beta\Big) - n(Y^{n-1})^{\cdot\cdot}\Big(\ddot{C}^\alpha_\beta + 3H\dot{C}^\alpha_\beta - \frac{\Delta}{a^2}C^\alpha_\beta\Big). \tag{16}$$

### A. Large scales

In the long-wavelength limit, on superhorizon scales, we can neglect terms involving $\Delta C$, $\Delta^2 C$, and $\Delta\dot{C}$. For $a(t) = t^{P/3}$, we have $H = \frac{P}{3t}$ and $Y(t) \propto t^{-4}$, so the equation for the perturbations becomes

$$0 = \delta P^\alpha_\beta = -nY^{n-1}\Big\{\ddddot{C}^\alpha_\beta + (-8n + 8 + 2P)\frac{\dddot{C}^\alpha_\beta}{t} + \Big(16n^2 - 28n + 12 + 12P - 12Pn + \frac{P^2}{3}\Big)\frac{\ddot{C}^\alpha_\beta}{t^2} + \Big(16Pn^2 - 28Pn + 10P + \frac{11P^2}{3} - \frac{4P^2n}{3} - \frac{2P^3}{3}\Big)\frac{\dot{C}^\alpha_\beta}{t^3}\Big\} \tag{17}$$

and so

$C \propto t^\lambda$

$$0 = \lambda\left(\lambda^3 + (-8n + 2 + 2P)\lambda^2 + \left(-4n - 1 + 16n^2 + 6P(1 - 2n) + \frac{P^2}{3}\right)\lambda - 2 + 12n - 16n^2 + P(2 - 16n + 16n^2) + P^2\left(\frac{10}{3} - \frac{4n}{3}\right) - \frac{2}{3}P^3\right). \tag{18}$$

The four roots of this are

$$\lambda = 0, \lambda_1, \lambda_\pm, \tag{19}$$

where

$$\lambda_\pm \equiv \tfrac{1}{2}(\lambda_1 \pm \sqrt{\lambda_2}), \tag{20}$$

$$\lambda_1 \equiv -1 - P + 4n \tag{21}$$

$$\begin{aligned}\lambda_2 &\equiv \frac{11}{3}P^2 - 14P + 8Pn + 9 - 24n + 16n^2 \\ &= \frac{464n^4 - 392n^3 + 9n^2 + 64n - 13 \pm (36n^2 - 11n - 3)\sqrt{3(48n^4 - 40n^3 - 5n^2 + 10n - 1)}}{2(1 - n)^2}.\end{aligned} \tag{22}$$

$\lambda_1$ and $\lambda_2$ are real whenever $P$ is real, i.e. $n \notin (-0.479\,42, 0.110\,873)$.[2] For $\lambda_\pm$ to be real, we need $\lambda_2 \geq 0$.

We are interested in the signs of the possible values of $\lambda$ in order to determine the behavior of gravitational-wave perturbations of the isotropic solution as $t \rightarrow 0$. If any $\Re(\lambda_i) < 0$, the solution is unstable as $t \rightarrow 0$. Otherwise, we need to look at the stability problem to second order, due to the presence of the zero eigenvalue. For reference, we recall that for the $n = 1$ theory, studied earlier [2], we had

$$C(\mathbf{x}, t) \propto \alpha + \beta t^{1/2} + \gamma t + \delta t^{3/2}$$

and there were no diverging metric perturbation modes as $t \rightarrow 0$. Let us now analyze the situation in the more complicated $n \neq 1$ case.

### B. Solutions with $P = P_+$

First consider the case $P = P_+$, so that

$$P = \frac{3(1 - 3n + 4n^2) + \sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)},$$

for which the stability is decided by the quantities

$$\lambda_1 = \frac{-20n^2 + 19n - 5 - \sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)},$$

$$\lambda_2 = \frac{464n^4 - 392n^3 + 9n^2 + 64n - 13 + (36n^2 - 11n - 3)\sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)^2}.$$

[2]The values here and in the tables that follow are approximate numerical roots of the appropriate polynomials.

Values of $\lambda_i$ for different values of $n \neq 1$, taking $P = P_+$

| | $P_+$ | $\lambda_1$ | $\lambda_2$ | $\lambda_+$ | $\lambda_-$ | Remarks |
|---|---|---|---|---|---|---|
| $n < -1.300\,84$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $-1.300\,84 < n < -0.479\,42$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $-0.479\,42 < n < 0.110\,873$ | $\mathbb{C}$, $\mathcal{R}(P_+) > 0$ | $\mathbb{C}$, $\mathcal{R}(\lambda_1) < 0$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | Unstable as $t \to 0$. |
| $0.110\,873 < n < 0.452\,692$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | $\mathbb{C}$, $\mathcal{R}(\lambda_+) < 0$ | $\mathbb{C}$, $\mathcal{R}(\lambda_-) < 0$ | Unstable as $t \to 0$. |
| $0.452\,692 < n < 0.5$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $0.5 < n < 1$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $n > 1$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | Universe contracts. $P_+ < 0$ |

From the above table and Fig. 3 we note that, for the solutions with $a(t) = t^{P_+/3}$, there is always a negative eigenvalue, so are unstable for any $n$ as $t \to 0$. For $n > 1$, $P_+ < 0$, so this corresponds to a contracting universe, in which we are not interested here. However, we have $\Re(P_+) > 3$ for $1 > n > \frac{1}{2}$ and $n < \frac{1-\sqrt{17}}{8}$, so we need to be careful that the instability for these $n$ is not arising from the negative curvature contribution characteristic of the Milne universe. We expect the overall assumption that the higher-order Ricci terms dominate the GR terms in the neighborhood of the initial cosmological singularity to hold so long as $n > 1/2$.

### C. Solutions with $P = P_-$

Now consider the second case, with $P = P_-$, which turns out to be the most physically relevant for consideration of the effects of higher-order ($n > 1$) corrections. We have

$$P = \frac{3(1 - 3n + 4n^2) - \sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)},$$

with the stability decided by

$$\lambda_1 = \frac{-20n^2 + 19n - 5 + \sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)},$$

$$\lambda_2 = \frac{464n^4 - 392n^3 + 9n^2 + 64n - 13 - (36n^2 - 11n - 3)\sqrt{3(-1 + 10n - 5n^2 - 40n^3 + 48n^4)}}{2(1 - n)^2}.$$

Values of $\lambda_i$ for different values of $n \neq 1$, taking $P = P_-$

| | $P_-$ | $\lambda_1$ | $\lambda_2$ | $\lambda_+$ | $\lambda_-$ | Remarks |
|---|---|---|---|---|---|---|
| $n < -0.479\,42$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $-0.479\,42 < n < 0.110\,873$ | $\mathbb{C}$, $\mathcal{R}(P_-) > 0$ | $\mathbb{C}$, $\mathcal{R}(\lambda_1) < 0$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | Unstable as $t \to 0$. |
| $0.110\,873 < n < 0.159\,452$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | $\mathbb{C}$, $\mathcal{R}(\lambda_+) < 0$ | $\mathbb{C}$, $\mathcal{R}(\lambda_-) < 0$ | Unstable as $t \to 0$. |
| $0.159\,452 < n < 0.169\,938$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $0.169\,938 < n < 0.25$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $0.25 < n < 0.5$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | Universe contracts. $P_- < 0$ |
| $0.5 < n < 0.520\,752$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | Stable as $t \to 0$. |
| $0.520\,752 < n < 0.989\,666$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{C}$, $\mathcal{R}(\lambda_+) > 0$ | $\mathbb{C}$, $\mathcal{R}(\lambda_-) > 0$ | Stable as $t \to 0$. |
| $0.989\,666 < n < 1.1309$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | Stable as $t \to 0$. |
| $n > 1.1309$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |

We saw that for $n > 1$, $P_+ < 0$, so for an expanding universe, the only relevant value for $P$ with $n > 1$ is $P = P_-$. As shown in Fig. 4 and the above table, this solution can only be stable as $t \to 0$ if

$$\frac{1}{2} < n < \sqrt{2}\cos\left[\frac{1}{3}\arccos\left(\frac{-\sqrt{2}}{4}\right)\right] \approx 1.1309. \quad (23)$$

In particular, for all integers $n > 1$, the exact isotropic solution with $a(t) = t^{P_-/3}$ is *not* a past attractor as $t \to 0$. Thus, it appears that the quadratic ($n = 1$) case studied earlier was exceptional and the stability of the isotropic singularity found for that case does not extend to higher-order corrections to general relativity with $n > 1$.

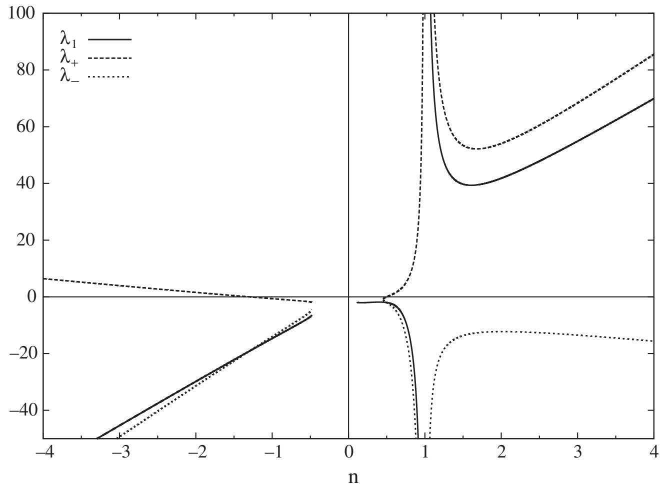


FIG. 3. Power-law exponents, $\lambda_1$, $\lambda_\pm$, versus $n$ for tensor perturbations with $P = P_+$.

## V. VECTOR (VORTICAL) PERTURBATIONS

We have shown that for gravitational-wave perturbations there is a very small range of values of $n$, given in Eq. (23), for which the perturbations are stable as $t \to 0$. We now consider the vortical perturbations, which are of vector-type. The metric is

$$ds^2 = -dt^2 - 2aB_\alpha dt dx^\alpha + a^2(\delta_{\alpha\beta} + 2C_{(\alpha,\beta)})dx^\alpha dx^\beta,$$

where $B^\alpha{}_{,\alpha} \equiv 0 \equiv C^\alpha{}_{,\alpha}$.

The energy-momentum tensor is decomposed as usual [16],

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{24}$$

where $h_{ab} \equiv g_{ab} + u_a u_b$, $q_a u^a \equiv 0 \equiv \pi_{ab} u^b$, and $\pi^a_a \equiv 0$. The fluid four-velocity $u_a$ and the energy flux $q_a$ are decomposed as (using $t$ as the time variable, index "0" denotes $t$)

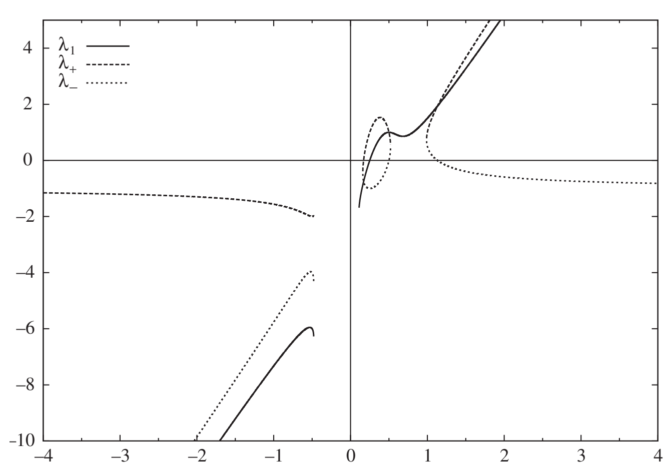


FIG. 4. Power-law exponents, $\lambda_1$, $\lambda_\pm$, versus $n$ for tensor perturbations with $P = P_-$.

$$u^0 \equiv 1, \qquad u_0 = -1, \qquad u^\alpha \equiv a^{-1}V^\alpha,$$
$$u_\alpha = a(V_\alpha - B_\alpha), \qquad q_0 = 0, \qquad q_\alpha \equiv aQ_\alpha. \tag{25}$$

The energy-momentum tensor is then

$$T^0_\alpha = (\mu + p)\left(V_\alpha + \frac{Q_\alpha}{\mu + p} - B_\alpha\right), \qquad \delta T^\alpha_\beta = \Pi^\alpha_\beta. \tag{26}$$

We can also decompose the perturbation variables as

$$B_\alpha(\mathbf{x}, t) \equiv b(t)Y_\alpha(\mathbf{x}), \qquad C_\alpha \equiv cY_\alpha,$$
$$\Delta Y_\alpha \equiv -k^2 Y_\alpha, \qquad V_\alpha \equiv vY_\alpha,$$
$$Q_\alpha \equiv qY_\alpha, \qquad \Pi^\alpha_\beta \equiv p\pi_T Y^\alpha_\beta \tag{27}$$

and introduce the gauge-invariant variables [14]:

$$V_\alpha + \frac{Q_\alpha}{\mu + p} - B_\alpha = \left(v + \frac{q}{\mu + p} - b\right)Y_\alpha \equiv v_\omega Y_\alpha, \tag{28}$$

$$V_\alpha + \frac{Q_\alpha}{\mu + p} + C'_\alpha = \left(v + \frac{q}{\mu + p} + c'\right)Y_\alpha \equiv v_\sigma Y_\alpha, \tag{29}$$

$$B_\alpha + C'_\alpha = (v_\sigma - v_\omega)Y_\alpha \equiv \Psi Y_\alpha, \tag{30}$$

where a prime denotes a derivative with respect to the conformal time variable $\eta$; $v_\omega$ and $v_\sigma$ may be interpreted as the velocity variables related to the vorticity and the shear, respectively.

We will work in the " $C$-gauge"' i.e. we set $C_\alpha = 0$, which completely fixes the gauge condition. Then, using the quantities presented in Appendix B, we find that the perturbed parts of the tensor $P^a_b$ are

$$\delta P^0_0 = 0, \tag{31}$$

$$P^0_\alpha = nY^{n-1}\left\{\frac{\Delta}{2a}\ddot{B}_\alpha + \frac{H}{2}\frac{\Delta}{a}\dot{B}_\alpha - (2\dot{H} + 4H^2)\frac{\Delta}{a}B_\alpha - \frac{\Delta^2}{2a^3}B_\alpha\right\} + n(Y^{n-1})^{\cdot}\left\{\frac{\Delta}{2a}\dot{B}_\alpha + H\frac{\Delta}{a}B_\alpha\right\}, \tag{32}$$

$$\delta P^\alpha_\beta = -nY^{n-1}\frac{1}{a}\left\{\dddot{B}^{(\alpha}{}_{,\beta)} + 3H\ddot{B}^{(\alpha}{}_{,\beta)} - \left(3\dot{H} + 6H^2 + \frac{\Delta}{a^2}\right)\dot{B}^{(\alpha}{}_{,\beta)} - 4(\ddot{H} + 6H\dot{H} + 4H^3)B^{(\alpha}{}_{,\beta)}\right\} - n(Y^{n-1})^{\cdot}\frac{1}{a}\left\{2\ddot{B}^{(\alpha}{}_{,\beta)} + 5H\dot{B}^{(\alpha}{}_{,\beta)} - \left(2\dot{H} + 4H^2 + \frac{\Delta}{a^2}\right)B^{(\alpha}{}_{,\beta)}\right\} - n(Y^{n-1})^{\cdot\cdot}\frac{1}{a}\{\dot{B}^{(\alpha}{}_{,\beta)} + 2HB^{(\alpha}{}_{,\beta)}\}. \tag{33}$$

Combining the equations (26)–(33), we have

$$T^0_\alpha = -\frac{\Delta B_\alpha}{2a} + A\bigg[nY^{n-1}\bigg\{\frac{\Delta}{2a}\ddot{B}_\alpha + \frac{1}{2}\bigg((n-1)\frac{\dot{Y}}{Y} + H\bigg) \times\frac{\Delta}{a}\dot{B}_\alpha + \bigg((n-1)H\frac{\dot{Y}}{Y} - (2\dot{H} + 4H^2)\bigg)\frac{\Delta}{a}B_\alpha - \frac{\Delta^2}{2a^3}B_\alpha\bigg\}\bigg] = a(\mu + p)v_\omega Y_\alpha, \quad (34)$$

$$-\frac{1}{k}p\pi_T = \frac{2}{k^2a^3}[a^4(\mu + p)v_\omega]^{\cdot}, \quad (35)$$

and so for vanishing anisotropic pressure of the matter part, $p\pi_T = 0$, angular momentum is conserved exactly as in the quadratic case and $a^3T^0_\alpha \equiv \Omega Y_\alpha(\mathbf{x})$ is a constant in time.

For $a = t^{P/3}$, $H = \frac{P}{3t}$, and $Y = \frac{4P^2}{9t^4}(3 - 3P + P^2) \propto t^{-4}$, so we have therefore

$$P^0_\alpha = nY^{n-1}\frac{\Delta}{2t^{P/3}}\bigg\{\ddot{B}_\alpha + \frac{1}{t}\bigg(4 + \frac{P}{3} - 4n\bigg)\dot{B}_\alpha + \frac{1}{t^2}\bigg(4P - \frac{8P^2}{9} - \frac{8Pn}{3}\bigg)B_\alpha - \frac{\Delta}{t^{2P/3}}B_\alpha\bigg\}. \quad (36)$$

## A. The $A \to \infty$ limit

In the limit $A \to \infty$, where the GR term can be neglected and the higher-order Ricci terms dominate, we have

$$\tilde{\Omega}t^{4n-4-(2P/3)} = \ddot{\Psi} + \frac{1}{t}\bigg(4 - 4n + \frac{P}{3}\bigg)\dot{\Psi} + \frac{4P}{9t^2}(9 - 6n - 2P)\Psi + \frac{k^2}{t^{2P/3}}\Psi, \quad (37)$$

where we have defined the constant

$$\tilde{\Omega} \equiv -\frac{2\Omega}{nk^2}\bigg(\frac{9}{4P^2(3 - 3P + P^2)}\bigg)^{n-1}.$$

If we take the long-wavelength limit, i.e. we drop the last term on the right-hand side, then we have to solve

$$\tilde{\Omega}t^{4n-2-(2P/3)} = t^2\ddot{\Psi} + t\bigg(-4(n-1) + \frac{P}{3}\bigg)\dot{\Psi} + \frac{2}{3}\bigg(-4Pn + 6P - \frac{4P^2}{3}\bigg)\Psi.$$

The complementary function (left-hand side = 0) is solved by $\Psi = t^\xi$, where

$$0 = \xi^2 + \bigg(3 - 4n + \frac{P}{3}\bigg)\xi + \frac{2}{3}\bigg(-4Pn + 6P - \frac{4P^2}{3}\bigg)$$

$$\Rightarrow \xi = \xi_\pm \equiv \frac{1}{6}(-9 - P + 12n \pm \sqrt{3(27 - 72n + 48n^2 - 42P + 24nP + 11P^2)}) = \lambda_\pm + \frac{P}{3} - 1,$$

where the $\lambda_i$ were defined in (21) and (22), and for stability as $t \to 0$, we need $\Re(\xi_i) \geq 0$. The additional mode from the particular solution has $\Psi \sim t^{\xi_1}$, where $\xi_1 \equiv 4n - 2 - \frac{2P}{3} = \lambda_1 + \frac{P}{3} - 1$. The signs of these exponents for different values of $n$ are summarized in the tables which follow and their values are plotted in Figs. 5 and 6 for $P_+$ and $P_-$ respectively.

### 1. *Solutions with $P = P_+$*

For $P = P_+$, we have $\Re(\xi_+) \geq 0$ for $n \geq \frac{1}{2}$ and $n \leq -\frac{13}{18}$, while $\Re(\xi_-) < 0$ for all $n$. Finally, for the particular solution, the exponent $\xi_1$ is positive for $n > 1$ and negative (and hence unstable as $t \to 0$) for $n < 1$. Thus, for the solution branch $P = P_+$, the vector modes are unstable as $t \to 0$ for all values of $n$.

Values of $\xi_i$ for different values of $n \neq 1$, taking $P = P_+$

| | $P_+$ | $\xi_+$ | $\xi_-$ | $\xi_1$ | Remarks |
|---|---|---|---|---|---|
| $n < -\frac{13}{18}$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $-\frac{13}{18} < n < -0.479\,42$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $-0.479\,42 < n < 0.110\,873$ | $\mathbb{C}$, $\mathcal{R}(P_+) > 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_+) < 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_-) < 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_1) < 0$ | Unstable as $t \to 0$. |
| $0.110\,873 < n < 0.452\,692$ | $\mathbb{R} > 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_+) < 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_-) < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $0.452\,692 < n < 0.5$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $0.5 < n < 1$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $1 < n$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | Universe contracts. $P_+ < 0$ |

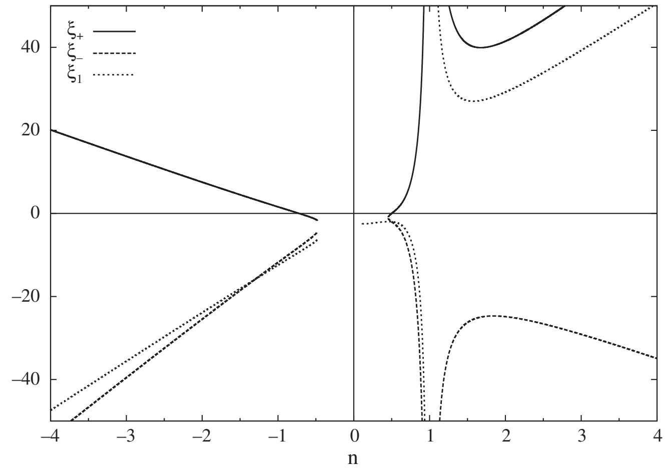

FIG. 5. Power-law exponents, $\xi_1$, $\xi_\pm$, versus $n$ for vector perturbations with $P = P_+$.

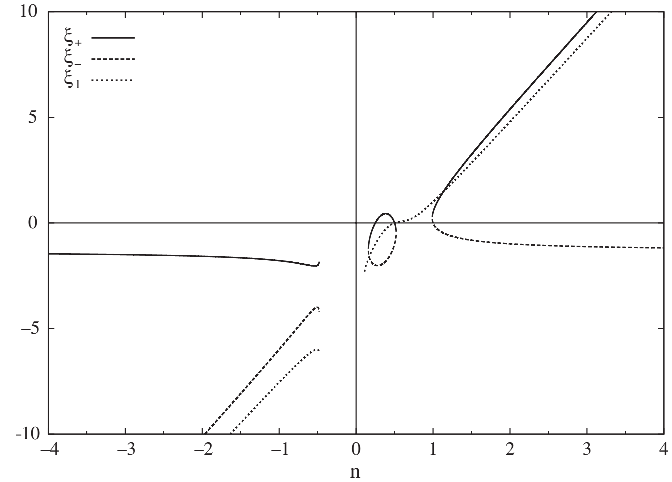

FIG. 6. Power-law exponents, $\xi_1$, $\xi_\pm$, versus $n$ for vector perturbations with $P = P_-$.

### *2. Solutions with $P = P_-$*

For $P = P_-$, $\Re(\xi_+) \geq 0$ for $\frac{1}{4} \leq n \leq \frac{1}{2}$ and $n \geq 0.861\,425$, $\Re(\xi_-) \geq 0$ for $0.861\,425 \leq n \leq 1$, while $\xi_1$ is positive for $n > 0.5$ and negative for $n < 0.5$. For this branch, the vector perturbations are stable to linear order as $t \to 0$ for

$$1 \geq n \geq \frac{1}{36}\left(25 + 2\sqrt{23}\sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{316}{23\sqrt{23}}\right)\right]\right) \approx 0.861\,425 \tag{38}$$

and are unstable for all other $n$.

| Values of $\xi_i$ for different values of $n \neq 1$, taking $P = P_-$ | | | | | |
|---|---|---|---|---|---|
| | $P_-$ | $\xi_+$ | $\xi_-$ | $\xi_1$ | Remarks |
| $n < -0.479\,42$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $-0.479\,42 < n < 0.110\,873$ | $\mathbb{C}$, $\mathcal{R}(P_-) > 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_+) < 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_-) < 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_-) < 0$ | Unstable as $t \to 0$. |
| $0.110\,873 < n < 0.159\,452$ | $\mathbb{R} > 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_+) < 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_-) < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $0.159\,452 < n < 0.25$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Unstable as $t \to 0$. |
| $0.25 < n < 0.5$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | Universe contracts. $P_- < 0$ |
| $0.5 < n < 0.520\,752$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | Unstable as $t \to 0$. |
| $0.520\,752 < n < 0.861\,425$ | $\mathbb{R} > 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_+) < 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_-) < 0$ | $\mathbb{R} > 0$ | Unstable as $t \to 0$. |
| $0.861\,425 < n < 0.989\,666$ | $\mathbb{R} > 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_+) > 0$ | $\mathbb{C}$, $\mathcal{R}(\xi_-) > 0$ | $\mathbb{R} > 0$ | Stable as $t \to 0$. |
| $0.989\,666 < n < 1$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | Stable as $t \to 0$. |
| $1 < n$ | $\mathbb{R} > 0$ | $\mathbb{R} > 0$ | $\mathbb{R} < 0$ | $\mathbb{R} > 0$ | Unstable as $t \to 0$. |

## VI. SCALAR PERTURBATIONS

We will now consider scalar perturbations. The metric for the general scalar-type perturbation takes the form

$$ds^2 = -(1 + 2\alpha)dt^2 - 2a\beta_{,\alpha}dtdx^\alpha + a^2(\delta_{\alpha\beta}(1 + 2\phi) + 2\gamma_{,\alpha\beta})dx^\alpha dx^\beta. \tag{39}$$

We use the proper time, $t$, as the time variable and also define the quantities, $\chi \equiv a(\beta + a\dot{\gamma})$, $f \equiv Y^{n-1} \equiv (R_{ab}R^{ab})^{n-1}$. We use overbars and deltas to denote background and perturbed quantities, so that in general, $A = \bar{A} + \delta A$, and in particular $\bar{f} = \bar{Y}^{n-1}$, $\delta f = (n-1)\bar{Y}^{n-2}\delta Y$. The important quantities to linear order in the perturbation are given in Appendix C.

In the unperturbed background metric, we use a synchronous time coordinate, i.e. we relate time to the worldlines of fictitious freely falling dust particles, and so we are, in effect, referring our results for the perturbations to them. Under gauge transformations, the perturbed order variables $\alpha(\mathbf{x}, t)$, $\phi(\mathbf{x}, t)$, and $\chi(\mathbf{x}, t)$ depend only on the temporal gauge transformation; they are spatially gauge invariant [15]. Using the gravitational field equations, we obtain the complete set of equations for the perturbed variables which are presented here *without* imposing the temporal gauge condition, i.e. the equations are presented in a gauge-ready form. In the discussion that follows, we will take advantage of the right to impose the temporal gauge condition to make a useful simplification, in so doing completely fixing the temporal gauge degree of freedom.

Energy:

$$\delta P^0_0 = n\overline{f}\Big[-\frac{1}{2}\delta Y + 12H^2\ddot{\alpha} + 6(2H\dot{H} + 3H^3)\dot{\alpha} + 24(2H\ddot{H} - 2\dot{H}^2 + 3H^2\dot{H} - 3H^4)\alpha - 2(8\dot{H} + 17H^2)\frac{\Delta}{a^2}\alpha - 2\frac{\Delta^2}{a^4}\alpha - 12H\dddot{\phi} + 6(4\dot{H} - 3H^2)\ddot{\phi} - 12(\ddot{H} + 3H\dot{H} - 6H^3)\dot{\phi} + 4\frac{\Delta}{a^2}\ddot{\phi} + 24H\frac{\Delta}{a^2}\dot{\phi} - 4(\dot{H} + 5H^2)\frac{\Delta}{a^2}\phi - 2\frac{\Delta^2}{a^4}\phi - 4H\frac{\Delta}{a^2}\ddot{\chi} + 2(4\dot{H} + 5H^2)\frac{\Delta}{a^2}\dot{\chi} - 4(\ddot{H} + 5H\dot{H} - 5H^3)\frac{\Delta}{a^2}\chi + 2\frac{\Delta^2}{a^4}\dot{\chi} + 2H\frac{\Delta^2}{a^4}\chi\Big] + n\dot{\overline{f}}\Big[12H^2\dot{\alpha} + 24H(2\dot{H} + 3H^2)\alpha + 4H\frac{\Delta}{a^2}\alpha - 12H\ddot{\phi} - 6(2\dot{H} + 9H^2)\dot{\phi} + 4H\frac{\Delta}{a^2}\phi - 4H\frac{\Delta}{a^2}\dot{\chi} - 2(2\dot{H} + 5H^2)\frac{\Delta}{a^2}\chi\Big] + n\Big[6(-2H\ddot{H} + 2\dot{H}^2 - 3H^2\dot{H} + 3H^4)\delta f - 6H(2\dot{H} + 3H^2)\dot{\delta f} + 2(2\dot{H} + 3H^2)\frac{\Delta}{a^2}\delta f\Big]$$

Momentum:

$$P^0_\alpha = n\overline{f}\Big[-4H\ddot{\alpha} - 12(\dot{H} + H^2)\dot{\alpha} - 12(\ddot{H} + 4H\dot{H})\alpha - 2\frac{\Delta}{a^2}\dot{\alpha} + 4H\frac{\Delta}{a^2}\alpha + 4\dddot{\phi} + 12H\ddot{\phi} + 24\dot{H}\dot{\phi} - 2\frac{\Delta}{a^2}\dot{\phi} + 8H\frac{\Delta}{a^2}\phi + 2\frac{\Delta}{a^2}\ddot{\chi} - 2H\frac{\Delta}{a^2}\dot{\chi} + 2(\dot{H} - 4H^2)\frac{\Delta}{a^2}\chi\Big]_{,\alpha} + n\dot{\overline{f}}\Big[-4H\dot{\alpha} - 6(2\dot{H} + 3H^2)\alpha - 2\frac{\Delta}{a^2}\alpha + 4\ddot{\phi} + 12H\dot{\phi} - 2\frac{\Delta}{a^2}\phi + 2\frac{\Delta}{a^2}\dot{\chi} + 2H\frac{\Delta}{a^2}\chi\Big]_{,\alpha} + 2n(2\dot{H} + 3H^2)\dot{\delta f}_{,\alpha} + 2n(2\ddot{H} + 6H\dot{H} - 3H^3)\delta f_{,\alpha}$$

Trace:

$$\delta P^\alpha_\alpha = n\ddot{\overline{f}}\Big[12H\dot{\alpha} + 24(2\dot{H} + 3H^2)\alpha + 4\frac{\Delta}{a^2}\alpha - 12\ddot{\phi} - 36H\dot{\phi} + 4\frac{\Delta}{a^2}\phi - 4\frac{\Delta}{a^2}\dot{\chi} - 4H\frac{\Delta}{a^2}\chi\Big] + n\dot{\overline{f}}\Big[24H\ddot{\alpha} + 42(2\dot{H} + 3H^2)\dot{\alpha} + 48(2\ddot{H} + 9H\dot{H} + 3H^3)\alpha + 8\frac{\Delta}{a^2}\dot{\alpha} - 12H\frac{\Delta}{a^2}\alpha - 24\dddot{\phi} - 108H\ddot{\phi} - 108(\dot{H} + H^2)\dot{\phi} + 16\frac{\Delta}{a^2}\dot{\phi} - 16H\frac{\Delta}{a^2}\phi - 8\frac{\Delta}{a^2}\ddot{\chi} - 4H\frac{\Delta}{a^2}\dot{\chi} - 4(5\dot{H} - H^2)\frac{\Delta}{a^2}\chi\Big] + n\overline{f}\Big[12H\dddot{\alpha} + 24(2\dot{H} + 3H^2)\ddot{\alpha} + 36(2\ddot{H} + 9H\dot{H} + 3H^3)\dot{\alpha} + 24(2\dddot{H} + 12H\ddot{H} + 9\dot{H}^2 + 18H^2\dot{H})\alpha + 4\frac{\Delta}{a^2}\ddot{\alpha} - 16H\frac{\Delta}{a^2}\dot{\alpha} - 4(5\dot{H} + 7H^2)\frac{\Delta}{a^2}\alpha - 2\frac{\Delta^2}{a^4}\alpha - 12\ddddot{\phi} - 72H\dddot{\phi} - 108(\dot{H} + H^2)\ddot{\phi} - 72(\ddot{H} + 3H\dot{H})\dot{\phi} + 16\frac{\Delta}{a^2}\dddot{\phi} + 16H\frac{\Delta}{a^2}\dot{\phi} - 8(2\dot{H} + H^2)\frac{\Delta}{a^2}\phi - 6\frac{\Delta^2}{a^4}\phi - 4\frac{\Delta}{a^2}\dddot{\chi} - 12(\dot{H} - H^2)\frac{\Delta}{a^2}\dot{\chi} - 8(2\ddot{H} - H^3)\frac{\Delta}{a^2}\chi + 2\frac{\Delta^2}{a^4}\dot{\chi} + 6H\frac{\Delta^2}{a^4}\chi\Big] - 6n[\ddot{\delta f}(2\dot{H} + 3H^2) + 2\dot{\delta f}(2\ddot{H} + 9H\dot{H} + 3H^3) + \delta f(2\dddot{H} + 12H\ddot{H} + 6\dot{H}^2 + 9H^2\dot{H} - 9H^4)] + 4n(\dot{H} + 3H^2)\frac{\Delta}{a^2}\delta f$$

Trace-free:

$$P^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta P^\alpha_\alpha = \frac{n}{a^2}\Big(\nabla^\alpha\nabla_\beta - \frac{1}{3}\delta^\alpha_\beta\Delta\Big)\Big[\ddot{\overline{f}}[\alpha + \phi - \dot{\chi} - H\chi] + \dot{\overline{f}}[2\dot{\alpha} + 3H\alpha - 2\dot{\phi} - H\phi - 2\ddot{\chi} - H\dot{\chi} + (4\dot{H} + 7H^2)\chi] + \overline{f}\Big[\ddot{\alpha} + 5H\dot{\alpha} + (4\dot{H} + 8H^2)\alpha + \frac{\Delta}{a^2}\alpha - 5\ddot{\phi} - 17H\dot{\phi} - (4\dot{H} + 8H^2)\phi + 3\frac{\Delta}{a^2}\phi - \dddot{\chi} + (6\dot{H} + 9H^2)\dot{\chi} + (5\ddot{H} + 21H\dot{H} + 8H^3)\chi - \frac{\Delta}{a^2}\dot{\chi} - 3H\frac{\Delta}{a^2}\chi\Big] - 2(\dot{H} + 3H^2)\delta f\Big],$$

with

$$\delta Y = -12H(2\dot{H} + 3H^2)\dot{\alpha} - 48(\dot{H}^2 + 3H^2\dot{H} + 3H^4)\alpha - 4(2\dot{H} + 3H^2)\frac{\Delta}{a^2}\alpha + 12(2\dot{H} + 3H^2)\ddot{\phi} + 72H(\dot{H} + 2H^2)\dot{\phi} - 8(\dot{H} + 3H^2)\frac{\Delta}{a^2}\phi + 4(2\dot{H} + 3H^2)\frac{\Delta}{a^2}\dot{\chi} + 8H(\dot{H} + 3H^2)\frac{\Delta}{a^2}\chi. \tag{40}$$

The special unperturbed solution has $a = t^{P/3}$, $H = \frac{P}{3t}$, and $\bar{Y} = \frac{4P^2}{9t^4}(3 - 3P + P^2) \propto t^{-4}$.

### Linearizing about the special solution in the zero-shear gauge

We now take the large-scale limit and choose the zero-shear gauge ($\chi \equiv 0$) and linearize about the special flat FRW solution with $a = t^{P/3}$. For $P = P_\pm$, the equations simplify to

Energy:

$$\delta P^0_0 = n\left(\frac{4P^2}{9t^4}(3 - 3P + P^2)\right)^{n-2}\left\{\frac{8P^4}{27t^7}(-6 + 6P - P^2 + n(12 - 12P + 3P^2))\right\}\left\{t\ddot{\alpha} + (2 - 4n + P)\dot{\alpha} - \frac{3}{P}(t^2\dddot{\phi} + (5 - 4n + P)t\ddot{\phi} + (4 - 8n + 2P)\dot{\phi})\right\}$$

Momentum:

$$\delta P^0_\alpha = n\left(\frac{4P^2}{9t^4}(3 - 3P + P^2)\right)^{n-2}\left\{\frac{8P^3}{27t^6}(6 - 6P + P^2 - n(12 - 12P + 3P^2))\right\}\nabla_\alpha\left\{t\ddot{\alpha} + (1 - 4n + P)\dot{\alpha} - \frac{3}{P}(t^2\dddot{\phi} + (4 - 4n + P)t\ddot{\phi} + (2 - 8n + 2P)\dot{\phi})\right\}$$

Trace:

$$\delta T^\alpha_\alpha = n\left(\frac{4P^2}{9t^4}(3 - 3P + P^2)\right)^{n-2}\left[\frac{8P^3}{9t^7}(-6 + 6P - P^2 + 3n(P - 2)^2)\right]\left[t^2\dddot{\alpha} + (4 - 8n + 2P)t\ddot{\alpha} + (2 - 4n + P)(1 - 4n + P)\dot{\alpha} - \frac{3}{P}(t^3\ddddot{\phi} + (8 - 8n + 2P)t^2\dddot{\phi} + (7 - 4n + P)(2 - 4n + P)t\ddot{\phi} + 2(2 - 4n + P)(1 - 4n + P)\dot{\phi})\right]$$

Trace-free propagation:

$$\delta T^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta\delta T^\alpha_\alpha = n\frac{1}{a^2}\left(\frac{4P^2}{9t^4}(3 - 3P + P^2)\right)^{n-2}\left(\nabla^\alpha\nabla_\beta - \frac{1}{3}\delta^\alpha_\beta\Delta\right)\left[\frac{4P^2}{9t^4}(3 - 3P + P^2)\ddot{\alpha} + \frac{4P^2}{27t^5}(72 - 69P + 27P^2 - P^3 + 6n(-12 + 14P - 7P^2 + P^3))\dot{\alpha} + \frac{16P^2}{81t^6}(3 - 3P + P^2)(27 + 12P - 4P^2 + 3n(-21 - 5P + 2P^2) + 36n^2)\alpha + \frac{4P^2}{9t^4}(-3 - 3P + P^2 + 6n(-2 + 3P - P^2))\ddot{\phi} + \frac{4P^2}{27t^5}(-72 + 57P - 33P^2 + 7P^3 + 12n(2 - P)(3 - 3P + 2P^2))\dot{\phi} + \frac{16P^2}{81t^6}(3 - 3P + P^2)(27 - 2P^2 + n(-63 + 3P) + 36n^2)\phi\right].$$

The energy and momentum equations together imply that

$$\alpha = \frac{3}{P}(t\dot{\phi} + \phi + \alpha_0), \tag{41}$$

where $\alpha_0$ is a free constant. In addition, this also satisfies the trace equation.

Finally, we use the trace-free equation to calculate that

$$\phi = \phi_0 + \phi_1 t^{\rho_1} + \phi_+ t^{\rho_+} + \phi_- t^{\rho_-}, \tag{42}$$

$$\alpha = \frac{3}{P}[(1 + \rho_1)\phi_1 t^{\rho_1} + (1 + \rho_+)\phi_+ t^{\rho_+} + (1 + \rho_-)\phi_- t^{\rho_-}] - \frac{27 - 2P^2 + n(-63 + 3P) + 36n^2}{27 + 12P - 4P^2 + 3n(-21 - 5P + 2P^2) + 36n^2}\phi_0, \tag{43}$$

$$\rho_1 \equiv -3 + 4n - \frac{P}{3} = \xi_1 + \frac{P}{3} - 1 = \lambda_1 + 2\left(\frac{P}{3} - 1\right) \tag{44}$$

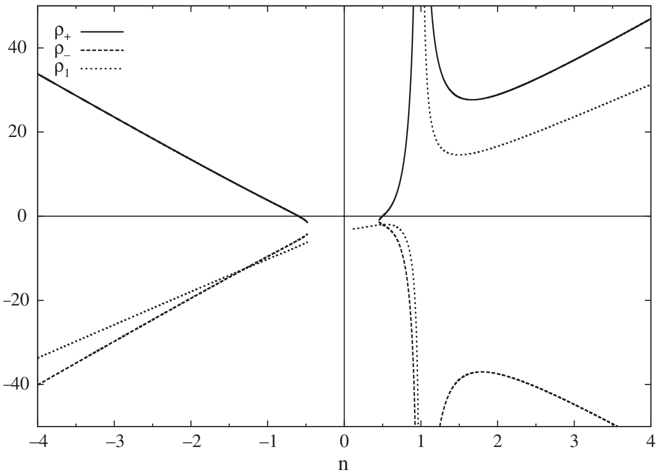


FIG. 7. Power-law exponents, $\rho_1$, $\rho_\pm$, versus $n$ for scalar perturbations with $P = P_+$.

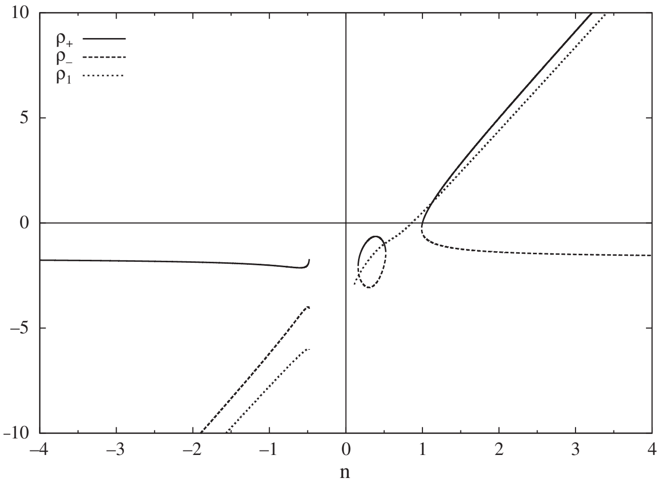


FIG. 8. Power-law exponents, $\rho_1$, $\rho_\pm$, versus $n$ for scalar perturbations with $P = P_-$.

$$\rho_\pm \equiv \frac{1}{6}(-15 + 12n + P \pm \sqrt{3(27 - 42P + 11P^2 - 72n + 24nP + 48n^2)}) = \xi_\pm + \frac{P}{3} - 1 = \lambda_\pm + 2\left(\frac{P}{3} - 1\right), \quad (45)$$

where $\phi_0$, $\phi_1$, and $\phi_\pm$ are free constants, and $\xi_\pm$ were defined in (38). We note that these are the same power-law exponents as for the vorticity perturbations plus $\frac{P}{3} - 1$. These exponents are shown in Figs. 7 and 8. For all $n$, and either choice of $P = P_\pm$, at least one of these exponents has negative real part, and hence the isotropic vacuum solution is unstable as $t \to 0$.

## VII. SUMMARY

We can now summarize the results for the linearized tensor, vector, and scalar perturbations about the spatially flat vacuum FRW solution (11) of the theory with Lagrangian $(R_{ab}R^{ab})^n$. The general perturbed metric in the neighborhood of the isotropic vacuum solution $a = t^{P/3}$, given by Eq. (11), has the form

$$ds^2 = -(1 + 2\alpha)dt^2 - a\tilde{B}_\alpha dt dx^\alpha + a^2(\delta_{\alpha\beta} + \tilde{C}_{\alpha\beta})dx^\alpha dx^\beta, \quad (46)$$

and the perturbation variables may be decomposed into their scalar, vector, and tensor parts by writing

$$\tilde{B}_\alpha = 2\beta_{,\alpha} + 2B_\alpha,$$
$$\tilde{C}_{\alpha\beta} = 2\phi\delta_{\alpha\beta} + 2\gamma_{,\alpha\beta} + 2C_{(\alpha,\beta)} + 2C_{\alpha\beta}.$$

In the gauge defined by $\beta \equiv 0 \equiv \gamma$ and $C_\alpha \equiv 0$, the general solution of the linearized equations is given by

$$C^\alpha_\beta = a^\alpha_\beta(\mathbf{x}) + t^{\lambda_1} b^\alpha_\beta(\mathbf{x}) + t^{\lambda_+} c^\alpha_\beta(\mathbf{x}) + t^{\lambda_-} d^\alpha_\beta(\mathbf{x}) \quad (47)$$

$$B_\alpha = t^{\xi_+} Y^{(1)}_\alpha(\mathbf{x}) + t^{\xi_-} Y^{(2)}_\alpha(\mathbf{x}) - \frac{3\tilde{\Omega}}{2(3 - 6P + P^2 - 6n + 6nP)} t^{\xi_1} Y^{(3)}_\alpha(\mathbf{x}) \quad (48)$$

$$\phi = \phi_0(\mathbf{x}) + \phi_1(\mathbf{x})t^{\rho_1} + \phi_+(\mathbf{x})t^{\rho_+} + \phi_-(\mathbf{x})t^{\rho_-} \quad (49)$$

$$\alpha = \alpha_0\phi_0 + \frac{3}{P}\{(\rho_1 + 1)\phi_1 t^{\rho_1} + (\rho_+ + 1)\phi_+ t^{\rho_+} + (\rho_- + 1)\phi_- t^{\rho_-}\}, \quad (50)$$

where

$$\lambda_1 = -1 - P + 4n$$
$$\lambda_2 = 9 - 14P + \frac{11}{3}P^2 - 24n + 8Pn + 16n^2$$
$$\lambda_\pm = \frac{1}{2}(\lambda_1 \pm \sqrt{\lambda_2})$$
$$\xi_i = \lambda_i + \frac{P}{3} - 1$$
$$\rho_i = \lambda_i + 2\left(\frac{P}{3} - 1\right)$$
$$\alpha_0 = -\frac{27 - 2P^2 + n(-63 + 3P) + 36n^2}{27 + 12P - 4P^2 + 3n(-21 - 5P + 2P^2) + 36n^2}.$$

For the solution branch defined by $P = P_+$, each type of perturbation (tensor, vector, and scalar) is unstable as $t \to 0$ for all values of $n$. For $P = P_-$, which is the only physically relevant value of $P$ for $n > 1$, the tensor perturbations are stable to linear order as $t \to 0$ for

$$\frac{1}{2} < n < \sqrt{2}\cos\left[\frac{1}{3}\arccos\left(\frac{-\sqrt{2}}{4}\right)\right] \approx 1.1309$$

and the vector perturbations are stable to linear order as $t \to 0$ for

$$1 \geq n \geq \frac{1}{36}\left(25 + 2\sqrt{23}\sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{316}{23\sqrt{23}}\right)\right]\right) \approx 0.861\,425.$$

For all other $n$, these perturbations are unstable as $t \to 0$. The scalar perturbations are unstable as $t \to 0$ for all $n$.

In conclusion, in our earlier work [2], we discovered that isotropic cosmological models in theories of gravity formed with a quadratic Ricci term added to the Einstein-Hilbert action are stable on approach to an initial "big bang" singularity. This is quite different to the behavior of general-relativistic cosmological models, where isotropy is strongly unstable in this limit in vacuum [6,7]. In this paper, we have analyzed the more complicated problem of cosmological evolution in the presence of arbitrary powers of the Ricci term in the Lagrangian. We have found that the behavior displayed in the quadratic case was special. Isotropic power-law solutions of the sort found by Barrow and Clifton [12,17] still exist in vacuum and with a perfect fluid when a term proportional to $(R_{ab}R^{ab})^n$ is added to the Einstein-Hilbert action for general $n \neq 1$. However, both solution branches of these special isotropic solutions are unstable to the growth of small metric perturbations as $t \to 0$, and so the quadratic case with $n = 1$, in which these perturbations are bounded in this limit, is special.

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## APPENDIX A: RELEVANT TENSOR QUANTITIES USING $t$ AS THE TIME VARIABLE

Using $t$ as the time variable and $H = \frac{\dot{a}}{a}$, the important quantities to linear order in the gravitational-wave-type perturbation are

$$\Gamma^0{}_{\alpha\beta} = a^2[H(\delta_{\alpha\beta} + 2C_{\alpha\beta}) + \dot{C}_{\alpha\beta}], \qquad \Gamma^\alpha{}_{0\beta} = H\delta^\alpha_\beta + \dot{C}^\alpha_\beta, \qquad \Gamma^\alpha{}_{\beta\gamma} = C^\alpha{}_{\beta,\gamma} + C^\alpha{}_{\gamma,\beta} - C_{\gamma\beta,}{}^\alpha, \qquad \Gamma = 0 \text{ otherwise},$$

$$g^{cd}\Gamma^0{}_{cd} = 3H, \qquad R^0_0 = 3(\dot{H} + H^2), \qquad R^\alpha_0 = R^0_\alpha = 0, \qquad R^\alpha_\beta = (\dot{H} + 3H^2)\delta^\alpha_\beta + \ddot{C}^\alpha_\beta + 3H\dot{C}^\alpha_\beta - \frac{\Delta}{a^2}C^\alpha_\beta,$$

$$R = 6(\dot{H} + 2H^2), \qquad Y \equiv R^a_b R^b_a = 12(\dot{H}^2 + 3\dot{H}H^2 + 3H^4), \qquad R^{cd}R^0{}_{cd0} = -3(\dot{H} + H^2)(\dot{H} + 3H^2),$$

$$R^{cd}R^0{}_{cd\alpha} = 0 = R^{cd}R^\alpha{}_{cd0}, R^{cd}R^\alpha{}_{cd\beta} = -(3\dot{H}^2 + 8\dot{H}H^2 + 9H^4)\delta^\alpha_\beta - (3\dot{H} + 2H^2)\ddot{C}^\alpha_\beta - (7\dot{H} + 6H^2)H\dot{C}^\alpha_\beta$$
$$+ (\dot{H} + 2H^2)\frac{\Delta}{a^2}C^\alpha_\beta, \qquad \nabla_0 R^0_0 = 3(\ddot{H} + 2H\dot{H}), \qquad \nabla_0 R^\alpha_0 = 0 = \nabla_0 R^0_\alpha = \nabla_\alpha R^0_0,$$

$$\nabla_0 R^\alpha_\beta = (\ddot{H} + 6H\dot{H})\delta^\alpha_\beta + \dddot{C}^\alpha_\beta + 3H\ddot{C}^\alpha_\beta + 3\dot{H}\dot{C}^\alpha_\beta - \frac{\Delta}{a^2}\dot{C}^\alpha_\beta + 2H\frac{\Delta}{a^2}C^\alpha_\beta,$$

$$\nabla_\beta R^0_\alpha = a^2\left[-2H\dot{H}\delta_{\alpha\beta} + H\ddot{C}_{\alpha\beta} + (3H^2 - 2\dot{H})\dot{C}_{\alpha\beta} - H\left(4\dot{H} + \frac{\Delta}{a^2}\right)C_{\alpha\beta}\right], \qquad \nabla_\beta R^\beta_0 = 6H\dot{H},$$

$$\nabla_c R^c_0 = 3(\ddot{H} + 4H\dot{H}), \qquad \nabla_\beta R^\beta_\alpha = 0, \qquad \Box R^0_0 = -3(\dddot{H} + 5H\ddot{H} + 2\dot{H}^2 + 2H^2\dot{H}), \qquad \Box R^0_\alpha = 0 = \Box R^\alpha_0,$$

$$\Box R^\alpha_\beta = -(\dddot{H} + 9H\ddot{H} + 6\dot{H}^2 + 22H^2\dot{H})\delta^\alpha_\beta - \left(\ddot{D}^\alpha_\beta + 3H\dot{D}^\alpha_\beta - \left(2H^2 + \frac{\Delta}{a^2}\right)D^\alpha_\beta\right) - 8H\dot{H}\dot{C}^\alpha_\beta,$$

$$D^\alpha_\beta \equiv \ddot{C}^\alpha_\beta + 3H\dot{C}^\alpha_\beta - \frac{\Delta}{a^2}C^\alpha_\beta, \qquad \Box R = -6(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H}),$$

and for any scalar function $f(t)$ of time only, it holds that

$$f(t)_{;}{}^\alpha{}_\beta = -(H\delta^\alpha{}_\beta + \dot{C}^\alpha{}_\beta)\dot{f}, \qquad f(t)_{;}{}^0{}_0 = -\ddot{f}, \qquad f(t)_{;}{}^\alpha{}_0 = 0 = f(t)_{;}{}^0{}_\alpha, \qquad \Box f(t) = -\ddot{f} - 3H\dot{f}.$$

Thus, since $Y = Y(t)$,

$$\Box(Y^{n-1}) = (1 - n)Y^{n-3}((n - 2)\dot{Y}^2 + Y\ddot{Y} + 3HY\dot{Y}).$$

## APPENDIX B: RELEVANT VECTOR QUANTITIES USING $t$ AS THE TIME VARIABLE

Using $t$ as the time variable, $H = \frac{\dot{a}}{a}$ and $B^{(\alpha}{}_{,\beta)} \equiv \frac{1}{2}(B^{\alpha}{}_{,\beta} + B_{\beta}{}^{|\alpha})$, the important vector quantities to linear order in the perturbation are

$$\Gamma^0_{00} = 0, \qquad \Gamma^0_{0\alpha} = -aHB_\alpha, \qquad \Gamma^0_{\alpha\beta} = a^2 H\delta_{\alpha\beta} + aB_{(\alpha,\beta)}, \qquad \Gamma^\alpha_{00} = -\frac{1}{a}(HB^\alpha + \dot{B}^\alpha),$$

$$\Gamma^\alpha_{0\beta} = H\delta^\alpha_\beta - \frac{1}{2a}(B^\alpha{}_{,\beta} - B_\beta{}^{|\alpha}), \qquad \Gamma^\alpha_{\beta\gamma} = aHB^\alpha\delta_{\beta\gamma}, \qquad \Gamma^c_{\alpha c} = 0 \qquad R^0_0 = 3(\dot{H} + H^2), \qquad R^0_\alpha = -\frac{\Delta B_\alpha}{2a},$$

$$R^\alpha_0 = \left(2\dot{H} + \frac{\Delta}{2a^2}\right)\frac{B^\alpha}{a}, \qquad R^\alpha_\beta = (\dot{H} + 3H^2)\delta^\alpha_\beta + \frac{1}{a^3}(a^2 B^{(\alpha}{}_{,\beta)})^{\cdot}, \qquad R = 6(\dot{H} + 2H^2),$$

$$Y \equiv R^a_b R^b_a = 12(\dot{H}^2 + 3\dot{H}H^2 + 3H^4), \qquad R^{cd}R^0{}_{cd0} = -3(\dot{H} + H^2)(\dot{H} + 3H^2), \qquad R^{cd}R^0{}_{cd\alpha} = H^2\frac{\Delta}{a}B_\alpha,$$

$$R^{cd}R^\alpha{}_{cd\beta} = -(3\dot{H}^2 + 8\dot{H}H^2 + 9H^4)\delta^\alpha_\beta - \frac{1}{a}(3\dot{H} + 2H^2)\dot{B}^{(\alpha}{}_{,\beta)} - \frac{4}{a}(H\dot{H} + H^3)B^{(\alpha}{}_{,\beta)},$$

$$\nabla_0 R^0_0 = 3(\ddot{H} + 2H\dot{H}), \qquad \nabla_0 R^0_\alpha = 2aH\dot{H}B_\alpha - \frac{\Delta}{2a}\dot{B}_\alpha + H\frac{\Delta}{a}B_\alpha, \qquad \nabla_0 R^\alpha_0 = O(B),$$

$$\nabla_0 R^\alpha_\beta = (\ddot{H} + 6H\dot{H})\delta^\alpha_\beta + \frac{1}{a^3}(a^2 B^{(\alpha}{}_{,\beta)})^{\cdot\cdot} - \frac{3H}{a^3}(a^2 B^{(\alpha}{}_{,\beta)})^{\cdot}, \qquad \nabla_\beta R^0_0 = aH\left(2\dot{H} + \frac{\Delta}{a^2}\right)B_\beta,$$

$$\nabla_\beta R^0_\alpha = -2a^2 H\dot{H}\delta_{\alpha\beta} + a\left[H\dot{B}_{(\alpha,\beta)} - 2(\dot{H} - H^2)B_{(\alpha,\beta)} - \frac{\Delta}{2a^2}B_{\alpha,\beta}\right], \qquad \nabla_\beta R^\beta_0 = 6H\dot{H}, \qquad \nabla_c R^c_0 = 3(\ddot{H} + 4H\dot{H}),$$

$$\nabla_\alpha R^\beta_\gamma = \frac{1}{a^3}(a^2 B^{(\beta}{}_{,\gamma),\alpha})^{\cdot} - aH\left(2\dot{H}B^\beta\delta_{\alpha\gamma} + \frac{\Delta}{2a^2}(B^\beta\delta_{\alpha\gamma} + B_\gamma\delta^\beta_\alpha)\right), \qquad \nabla_\beta R^\beta_\alpha = \frac{\Delta}{2a}\dot{B}_\alpha - aH\left(2\dot{H} + \frac{\Delta}{a^2}\right)B_\alpha,$$

$$\nabla_c R^c_\alpha = 0, \qquad \Box R^0_0 = -3(\dddot{H} + 5H\ddot{H} + 2\dot{H}^2 + 2H^2\dot{H}), \qquad \Box R^0_\alpha = \frac{\Delta}{2a}\ddot{B}_\alpha + H\frac{\Delta}{2a}\dot{B}_\alpha - 2(\dot{H} + H^2)\frac{\Delta}{a}B_\alpha - \frac{\Delta^2}{2a^3}B_\alpha,$$

$$\Box R^\alpha_\beta = -(\dddot{H} + 9H\ddot{H} + 6\dot{H}^2 + 22H^2\dot{H})\delta^\alpha_\beta - \frac{1}{a^3}(a^2 B^{(\alpha}{}_{,\beta)})^{\cdot\cdot\cdot} + \frac{3H}{a^3}(B^{(\alpha}{}_{,\beta)})^{\cdot\cdot} + \frac{1}{a^3}\left(3\dot{H} + 2H^2 + \frac{\Delta}{a^2}\right)(a^2 B^{(\alpha}{}_{,\beta)})^{\cdot} - \frac{1}{a}$$
$$\times\left(8H\dot{H} + 2H\frac{\Delta}{a^2}\right)B^{(\alpha}{}_{,\beta)}$$

and for any scalar function $f(t)$ of time only,

$$f(t)_{;}{}^0{}_0 = -\ddot{f}, \quad f(t)_{;}{}^0{}_\alpha = 0, \quad f(t)_{;}{}^\alpha{}_0 = -\frac{1}{a}B^\alpha(\ddot{f} - H\dot{f}), \quad f(t)_{;}{}^\alpha{}_\beta = -\left(H\delta^\alpha{}_\beta + \frac{1}{a}B^{(\alpha}{}_{,\beta)}\right)\dot{f}, \quad \Box f(t) = -\ddot{f} - 3H\dot{f}.$$

Note that $R = R(t)$ and $Y = Y(t)$ are the same as in the gravitational-wave case.

## APPENDIX C: RELEVANT SCALAR PERTURBATION QUANTITIES

Taking $t$ as the time variable, the metric takes the form

$$ds^2 = -(1 + 2\alpha)dt^2 - 2a\beta_{,\alpha}dtdx^\alpha + a^2(t)(\delta_{\alpha\beta}(1 + 2\phi) + 2\gamma_{,\alpha\beta})dx^\alpha dx^\beta.$$

The important scalar quantities to linear order in the perturbation (using $H = \frac{\dot{a}}{a}$, $\chi \equiv a(\beta + a\dot{\gamma})$) are

$$\Gamma^0_{00} = \dot{\alpha},$$
$$\Gamma^0_{0\alpha} = (\alpha - aH\beta)_{,\alpha},$$
$$\Gamma^0_{\alpha\beta} = a^2\Big[\delta_{\alpha\beta}(H - 2H\alpha + 2H\phi + \dot{\phi}) + \Big(2H\gamma + \frac{\chi}{a^2}\Big)_{,\alpha\beta}\Big],$$
$$\Gamma^\alpha_{00} = \frac{1}{a^2}(\alpha - aH\beta - a\dot{\beta})^{|\alpha},$$
$$\Gamma^\alpha_{0\beta} = H\delta^\alpha_\beta + \dot{\phi}\delta^\alpha_\beta + \dot{\gamma}^{|\alpha}{}_\beta,$$
$$\Gamma^\alpha_{\beta\gamma} = aH\beta^{|\alpha}\delta_{\beta\gamma} + \phi_{,\gamma}\delta^\alpha_\beta + \phi_{,\beta}\delta^\alpha_\gamma - \phi^{|\alpha}\delta_{\beta\gamma} + \gamma^\alpha_{,\beta\gamma},$$
$$\Gamma^c_{0c} = 3H + \dot{\alpha} + 3\dot{\phi} + \Delta\dot{\gamma},$$
$$\Gamma^c_{\alpha c} = (\alpha + 3\phi + \Delta\gamma)_{,\alpha},$$
$$g^{cd}\Gamma^0_{cd} = 3H - \dot{\alpha} - 6H\alpha + 3\dot{\phi} + \frac{\Delta}{a^2}\chi,$$
$$g^{cd}\Gamma^\alpha_{cd} = \frac{1}{a^2}(-\alpha + a\dot{\beta} + 2aH\beta - \phi + \Delta\gamma)^{|\alpha},$$
$$R^0_0 = 3(\dot{H} + H^2) - 3H\dot{\alpha} - 6(\dot{H} + H^2)\alpha - \frac{\Delta}{a^2}\alpha + 3\ddot{\phi} + 6H\dot{\phi} + \frac{\Delta}{a^2}\dot{\chi},$$
$$R^0_\alpha = -2(H\alpha - \dot{\phi})_{,\alpha},$$
$$R^\alpha_0 = \frac{2}{a^2}(H\alpha - \dot{\phi} + a\dot{H}\beta)^{|\alpha},$$
$$R^\alpha_\beta = \Big(\dot{H} + 3H^2 - H\dot{\alpha} - 2(\dot{H} + 3H^2)\alpha + \ddot{\phi} + 6H\dot{\phi} - \frac{\Delta}{a^2}\phi + H\frac{\Delta}{a^2}\chi\Big)\delta^\alpha_\beta + \frac{1}{a^2}(-\alpha - \phi + \dot{\chi} + H\chi)^{|\alpha}{}_\beta,$$
$$R = 6(\dot{H} + 2H^2) - 6H\dot{\alpha} - 12(\dot{H} + 2H^2)\alpha - 2\frac{\Delta}{a^2}\alpha + 6\ddot{\phi} + 24H\dot{\phi} - 4\frac{\Delta}{a^2}\phi + 2\frac{\Delta}{a^2}\dot{\chi} + 4H\frac{\Delta}{a^2}\chi,$$

$$\begin{aligned} Y &\equiv R^a_b R^b_a \\ &= 12(\dot{H}^2 + 3H^2\dot{H} + 3H^4) - 12H(2\dot{H} + 3H^2)\dot{\alpha} - 48(\dot{H}^2 + 3H^2\dot{H} + 3H^4)\alpha - 4(2\dot{H} + 3H^2)\frac{\Delta}{a^2}\alpha \\ &\quad + 12(2\dot{H} + 3H^2)\ddot{\phi} + 72H(\dot{H} + 2H^2)\dot{\phi} - 8(\dot{H} + 3H^2)\frac{\Delta}{a^2}\phi + 4(2\dot{H} + 3H^2)\frac{\Delta}{a^2}\dot{\chi} + 8H(\dot{H} + 3H^2)\frac{\Delta}{a^2}\chi \\ &\equiv \bar{Y} + \delta Y, \end{aligned}$$
with $\bar{Y} \equiv 12(\dot{H}^2 + 3H^2\dot{H} + 3H^4)$,

$$\begin{aligned} R^{cd}R^0{}_{cd0} &= -3(\dot{H}^2 + 4H^2\dot{H} + 3H^4) + 6H(\dot{H} + 2H^2)\dot{\alpha} + 12(\dot{H}^2 + 4H^2\dot{H} + 3H^4)\alpha + 2(\dot{H} + 2H^2)\frac{\Delta}{a^2}\alpha \\ &\quad - 6(\dot{H} + 2H^2)\ddot{\phi} - 12H(2\dot{H} + 3H^2)\dot{\phi} + 4(\dot{H} + H^2)\frac{\Delta}{a^2}\phi - 2(\dot{H} + 2H^2)\frac{\Delta}{a^2}\dot{\chi} - 4H(\dot{H} + H^2)\frac{\Delta}{a^2}\chi, \end{aligned}$$
$$R^{cd}R^0{}_{cd\alpha} = 4H^2(H\alpha - \dot{\phi})_{,\alpha},$$
$$\begin{aligned} R^{cd}R^\alpha{}_{cd\beta} &= -(3\dot{H}^2 + 8\dot{H}H^2 + 9H^4)\delta^\alpha_\beta + \delta^\alpha_\beta\Big[(6H\dot{H} + 8H^3)\dot{\alpha} + (12\dot{H}^2 + 32H^2\dot{H} + 36H^4)\alpha + (\dot{H} + 2H^2)\frac{\Delta}{a^2}\alpha \\ &\quad - (6\dot{H} + 8H^2)\ddot{\phi} - (16H\dot{H} + 36H^3)\dot{\phi} + (\dot{H} + 6H^2)\frac{\Delta}{a^2}\phi - (\dot{H} + 2H^2)\frac{\Delta}{a^2}\dot{\chi} - (H\dot{H} + 6H^3)\frac{\Delta}{a^2}\chi\Big] \\ &\quad + \frac{1}{a^2}[(3\dot{H} + 2H^2)\alpha + (\dot{H} + 2H^2)\phi - (3\dot{H} + 2H^2)\dot{\chi} - H(\dot{H} + 2H^2)\chi]^{|\alpha}{}_\beta, \end{aligned}$$

$$\nabla_0 R^0_0 = 3(\ddot{H} + 2H\dot{H}) - 3H\ddot{\alpha} - (9\dot{H} + 6H^2)\dot{\alpha} - 6(\ddot{H} + 2H\dot{H})\alpha - \frac{\Delta}{a^2}\dot{\alpha} + 2H\frac{\Delta}{a^2}\alpha + 3\dddot{\phi} + 6H\ddot{\phi} + 6\dot{H}\dot{\phi} + \frac{\Delta}{a^2}\ddot{\chi} - 2H\frac{\Delta}{a^2}\dot{\chi},$$

$$\nabla_0 R^0_\alpha = [-2H\dot{\alpha} + (2H^2 - 4\dot{H})\alpha + 2\ddot{\phi} - 2H\dot{\phi} + 2aH\dot{H}\beta]_{,\alpha},$$

$$\nabla_0 R^\alpha_0 = \frac{1}{a^2}[2H\dot{\alpha} + (4\dot{H} - 2H^2)\alpha - 2\ddot{\phi} + 2H\dot{\phi} + 2a(\ddot{H} - H\dot{H})\beta]^{|\alpha},$$

$$\nabla_0 R^\alpha_\beta = (\ddot{H} + 6H\dot{H})\delta^\alpha_\beta + \delta^\alpha_\beta\left[-H\ddot{\alpha} - 3(\dot{H} + 2H^2)\dot{\alpha} - 2(\ddot{H} + 6H\dot{H})\alpha + \dddot{\phi} + 6H\ddot{\phi} + 6\dot{H}\dot{\phi} - \frac{\Delta}{a^2}\dot{\phi} + 2H\frac{\Delta}{a^2}\phi + H\frac{\Delta}{a^2}\dot{\chi} + (\dot{H} - 2H^2)\frac{\Delta}{a^2}\chi\right] + \frac{1}{a^2}[-\dot{\alpha} + 2H\alpha - \dot{\phi} + 2H\phi + \ddot{\chi} - H\dot{\chi} + (\dot{H} - 2H^2)\chi]^{|\alpha}{}_\beta,$$

$$\nabla_\alpha R^0_0 = \left[-3H\dot{\alpha} - (6\dot{H} + 2H^2)\alpha - \frac{\Delta}{a^2}\alpha + 3\ddot{\phi} + 2H\dot{\phi} + \frac{\Delta}{a^2}\dot{\chi} + 2aH\dot{H}\beta\right]_{,\alpha},$$

$$\nabla_\alpha R^0_\beta = a^2\delta_{\alpha\beta}\left[-2H\dot{H} + 2H^2\dot{\alpha} + 8H\dot{H}\alpha + H\frac{\Delta}{a^2}\alpha - 2H\ddot{\phi} - 2\dot{H}\dot{\phi} - 4H\dot{H}\phi - H\frac{\Delta}{a^2}\phi - H\frac{\Delta}{a^2}\dot{\chi} + H^2\frac{\Delta}{a^2}\chi\right] + [-3H\alpha + 2\dot{\phi} - H\phi + H\dot{\chi} + (H^2 - 2\dot{H})\chi - 4a^2H\dot{H}\gamma]_{,\alpha\beta},$$

$$\nabla_\alpha R^\beta_0 = \delta^\beta_\alpha\left[2H\dot{H} - 2H^2\dot{\alpha} - 4H\dot{H}\alpha - H\frac{\Delta}{a^2}\alpha + 2H\ddot{\phi} + 2\dot{H}\dot{\phi} + H\frac{\Delta}{a^2}\phi + H\frac{\Delta}{a^2}\dot{\chi} - H^2\frac{\Delta}{a^2}\chi\right] + \frac{1}{a^2}[3H\alpha - 2\dot{\phi} + H\phi - H\dot{\chi} + (2\dot{H} - H^2)\chi]^{|\beta}{}_\alpha,$$

$$\nabla_c R^c_0 = 3(\ddot{H} + 4H\dot{H}) - 3H\ddot{\alpha} - (9\dot{H} + 12H^2)\dot{\alpha} - 6(\ddot{H} + 4H\dot{H})\alpha - \frac{\Delta}{a^2}\dot{\alpha} + 2H\frac{\Delta}{a^2}\alpha + 3\dddot{\phi} + 12H\ddot{\phi} + 12\dot{H}\dot{\phi} - 2\frac{\Delta}{a^2}\dot{\phi} + 4H\frac{\Delta}{a^2}\phi + \frac{\Delta}{a^2}\ddot{\chi} + (2\dot{H} - 4H^2)\frac{\Delta}{a^2}\chi,$$

$$\nabla_\gamma R^\alpha_\beta = \delta^\alpha_\beta\left[-H\dot{\alpha} - 2(\dot{H} + 3H^2)\alpha + \ddot{\phi} + 6H\dot{\phi} - \frac{\Delta}{a^2}\phi + H\frac{\Delta}{a^2}\chi\right]_{,\gamma} - 2H\delta_{\beta\gamma}(H\alpha - \dot{\phi} + a\dot{H}\beta)^{|\alpha} - 2H\delta^\alpha_\gamma(H\alpha - \dot{\phi})_{,\beta} + \frac{1}{a^2}(-\alpha - \phi + \dot{\chi} + H\chi)^{|\alpha}{}_{\beta\gamma},$$

$$\nabla_c R^c_\alpha = \left[-3H\dot{\alpha} - 6(\dot{H} + 2H^2)\alpha - \frac{\Delta}{a^2}\alpha + 3\ddot{\phi} + 12H\dot{\phi} - 2\frac{\Delta}{a^2}\phi + \frac{\Delta}{a^2}\dot{\chi} + 2H\frac{\Delta}{a^2}\chi\right]_{,\alpha},$$

$$\Box R^0_0 = -3(\dddot{H} + 5H\ddot{H} + 2\dot{H}^2 + 2H^2\dot{H}) + 3H\dddot{\alpha} + (12\dot{H} + 15H^2)\ddot{\alpha} + (18\ddot{H} + 57H\dot{H} + 6H^3)\dot{\alpha} + 12(\dddot{H} + 5H\ddot{H} + 2\dot{H}^2 + 2H^2\dot{H})\alpha + \frac{\Delta}{a^2}\ddot{\alpha} - 4H\frac{\Delta}{a^2}\dot{\alpha} - (8\dot{H} + 4H^2)\frac{\Delta}{a^2}\alpha - \frac{\Delta^2}{a^4}\alpha - 3\ddddot{\phi} - 15H\dddot{\phi} - 6(2\dot{H} + H^2)\ddot{\phi} - 3(5\ddot{H} + 4H\dot{H})\dot{\phi} + 3\frac{\Delta}{a^2}\ddot{\phi} - 2H\frac{\Delta}{a^2}\dot{\phi} + 8H^2\frac{\Delta}{a^2}\phi - \frac{\Delta}{a^2}\dddot{\chi} + H\frac{\Delta}{a^2}\ddot{\chi} + 2(\dot{H} + 3H^2)\frac{\Delta}{a^2}\dot{\chi} - (3\ddot{H} - 2H\dot{H} + 8H^3)\frac{\Delta}{a^2}\chi + \frac{\Delta^2}{a^4}\dot{\chi},$$

$$\Box R^0_\alpha = \bigg[2H\ddot{\alpha} + 6(\dot{H} + H^2)\dot{\alpha} + (6\ddot{H} + 12H\dot{H} - 16H^3)\alpha - 2H\frac{\Delta}{a^2}\alpha - 2\dddot{\phi} - 6H\ddot{\phi} + 16H^2\dot{\phi} + 2\frac{\Delta}{a^2}\dot{\phi} - 4H\frac{\Delta}{a^2}\phi + (4H^2 - 2\dot{H})\frac{\Delta}{a^2}\chi\bigg]_{,\alpha},$$

$$\Box R^\alpha_\beta = -(\dddot{H} + 9H\ddot{H} + 6\dot{H}^2 + 22H^2\dot{H})\delta^\alpha_\beta + \delta^\alpha_\beta\bigg[H\dddot{\alpha} + (4\dot{H} + 9H^2)\ddot{\alpha} + (6\ddot{H} + 39H\dot{H} + 22H^3)\dot{\alpha} + (4\dddot{H} + 36H\ddot{H} + 24\dot{H}^2 + 88H^2\dot{H})\alpha - H\frac{\Delta}{a^2}\dot{\alpha} - (2\dot{H} + 4H^2)\frac{\Delta}{a^2}\alpha - \ddddot{\phi} - 9H\dddot{\phi} - (12\dot{H} + 22H^2)\ddot{\phi} - (9\ddot{H} + 44H\dot{H})\dot{\phi} + 2\frac{\Delta}{a^2}\ddot{\phi} + 5H\frac{\Delta}{a^2}\dot{\phi} - (2\dot{H} + 4H^2)\frac{\Delta}{a^2}\phi - \frac{\Delta^2}{a^4}\phi - H\frac{\Delta}{a^2}\ddot{\chi} - (2\dot{H} + H^2)\frac{\Delta}{a^2}\dot{\chi} - (2\ddot{H} + 3H\dot{H} - 4H^3)\frac{\Delta}{a^2}\chi + H\frac{\Delta^2}{a^4}\chi\bigg] + \frac{1}{a^2}\bigg[\ddot{\alpha} - H\dot{\alpha} - (2\dot{H} + 12H^2)\alpha - \frac{\Delta}{a^2}\alpha + \ddot{\phi} + 7H\dot{\phi} - (2\dot{H} + 4H^2)\phi - \frac{\Delta}{a^2}\phi - \ddot{\chi} + 5H^2\dot{\chi} - (\ddot{H} + 5H\dot{H} - 4H^3)\chi + \frac{\Delta}{a^2}\dot{\chi} + H\frac{\Delta}{a^2}\chi\bigg]^{|\alpha}{}_{\beta},$$

$$\Box R = -6\dddot{H} - 42H\ddot{H} - 24\dot{H}^2 - 72H^2\dot{H} + 6H\dddot{\alpha} + (24\dot{H} + 42H^2)\ddot{\alpha} + (36\ddot{H} + 174H\dot{H} + 72H^3)\dot{\alpha} + 24(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H})\alpha + 2\frac{\Delta}{a^2}\ddot{\alpha} - 8H\frac{\Delta}{a^2}\dot{\alpha} - (16\dot{H} + 28H^2)\frac{\Delta}{a^2}\alpha - 2\frac{\Delta^2}{a^4}\alpha - 6\ddddot{\phi} - 42H\dddot{\phi} - 24(2\dot{H} + 3H^2)\ddot{\phi} - (42\ddot{H} + 144H\dot{H})\dot{\phi} + 10\frac{\Delta}{a^2}\ddot{\phi} + 20H\frac{\Delta}{a^2}\dot{\phi} - 8(\dot{H} + H^2)\frac{\Delta}{a^2}\phi - 4\frac{\Delta^2}{a^4}\phi - 2\frac{\Delta}{a^2}\dddot{\chi} - 2H\frac{\Delta}{a^2}\ddot{\chi} - (4\dot{H} - 8H^2)\frac{\Delta}{a^2}\dot{\chi} - (10\ddot{H} + 12H\dot{H} - 8H^3)\frac{\Delta}{a^2}\chi + 2\frac{\Delta^2}{a^4}\dot{\chi} + 4H\frac{\Delta^2}{a^4}\chi,$$

and, for a general scalar function $f(\mathbf{x}, t)$,

$$f_{;00} = \ddot{f} - \dot{\alpha}\dot{f},$$
$$f_{;0\alpha} = \dot{f}_{,\alpha} - (\alpha - aH\beta)_{,\alpha}\dot{f} - Hf_{,\alpha},$$
$$f_{;\alpha\beta} = f_{,\alpha\beta} - a^2\bigg[\delta_{\alpha\beta}(H - 2H\alpha + 2H\phi + \dot{\phi}) + \Big(2H\gamma + \frac{\chi}{a^2}\Big)_{,\alpha\beta}\bigg]\dot{f},$$
$$f^{;0}{}_0 = -\ddot{f} + 2\alpha\ddot{f} + \dot{\alpha}\dot{f},$$
$$f^{;0}{}_\alpha = -\dot{f}_{,\alpha} + \alpha_{,\alpha}\dot{f} + Hf_{,\alpha},$$
$$f^{;\alpha}{}_0 = \frac{1}{a^2}[\dot{f}^{|\alpha} - \alpha^{|\alpha}\dot{f} + aH\beta^{|\alpha}\dot{f} - a\beta^{|\alpha}\ddot{f} - Hf^{|\alpha}],$$
$$f^{;\alpha}{}_\beta = \frac{1}{a^2}f^{|\alpha}{}_\beta - \bigg[\delta^\alpha_\beta(H - 2H\alpha + \dot{\phi}) + \frac{1}{a^2}\chi^{|\alpha}{}_\beta\bigg]\dot{f},$$
$$\Box f = -\ddot{f} + 2\alpha\ddot{f} + \frac{\Delta}{a^2}f - \Big(3H - \dot{\alpha} - 6H\alpha + 3\dot{\phi} + \frac{\Delta}{a^2}\chi\Big)\dot{f}.$$


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