Polyakov-Nambu-Jona-Lasinio model with a Vandermonde term

Sanjay K. Ghosh[*] and Tamal K. Mukherjee[+]
Department of Physics, Bose Institute, 93/1, A.P.C. Road, Kolkata 700 009, India

Munshi G. Mustafa[‡] and Rajarshi Ray[§]
Theory Division, Saha Institute of Nuclear Physics, 1/AF, Bidhannagar, Kolkata 700 064, India.
(Received 1 November 2007; published 29 May 2008)

We extend the Polyakov-Nambu-Jona-Lasinio model for two degenerate flavors by including the effect of the SU(3) measure with a Vandermonde term. This ensures that the Polyakov loop always remains in the domain [0, 1]. The pressure, energy density, specific heat, speed of sound, and conformal measure show small or negligible effects from this term. However various quark number and isospin susceptibilities are all found to approach their respective ideal gas limits around $2T_c$. We compare our methods with other similar approaches in Polyakov-Nambu-Jona-Lasinio model and also present a quantitative comparison with lattice QCD data.

DOI: 10.1103/PhysRevD.77.094024 PACS numbers: 12.38.Aw, 12.38.Mh, 12.39.−x

I. INTRODUCTION

Recently, there is a lot of interest in the studies of thermodynamics of strongly interacting matter using the Polyakov loop enhanced Nambu-Jona-Lasinio (PNJL) model [1–3]. This model couples the chiral and deconfinement order parameters through a simple-minded coupling of the NJL model [4] with the Polyakov loop model [5]. The two major thrusts in recent times have been to estimate various thermodynamic observables using this model (see e.g. [6–10]), and to make systematic improvements of the model [11–13]. Another set of important results has come from similar studies in chiral quark models that go beyond the mean-field treatment [14].

In this paper we deal with the improvement of the Polyakov loop model and describe some of its consequences, remaining within the domain of mean-field analysis. The Polyakov loop model used in much of the recent literature is the one given in Ref. [3]. The Polyakov loop Φ has been treated here as a Z(3) spin field [15]. Using this model we estimated [6] a very sensitive observable—the quark number susceptibility (QNS) and also the higher order coefficients in the Taylor expansion of pressure in quark number chemical potential μ_0. Comparison with the data from lattice QCD (LQCD) [16] showed that the QNS in the PNJL model and LQCD agree quite well both qualitatively and quantitatively. The fourth-order coefficient c_4 showed qualitative agreement but had a quantitative difference at high temperatures. Some of us further extended the PNJL model to include isospin chemical potential μ_I [8]. The isospin number susceptibility (INS) and its derivative with respect to μ_0 and μ_I were obtained. In this case the fourth-order derivative c_4^I was quite consistent with lattice data, but the INS was not. A possible reason for such departures is that the mean-field treatment of the PNJL model is insufficient. But then it should have affected the coefficients systematically *i.e.*, all the fourth-order coefficients should deviate further from LQCD data than the second-order coefficients.

There are, however, other *simpler* reasons that should be considered first. The PNJL model is only a model which can mimic some of the characteristics of a fundamental theory like QCD and its discretized version LQCD. Moreover, the parameters like the couplings and masses are quite different in the PNJL model and the LQCD simulations. Thus some quantitative difference is naturally expected. Apart from these we made an important observation in [8] that Φ has a big role to play in the behavior of these coefficients. We pointed out how the quantitative differences could be caused by the behavior of Φ as a function of temperature and chemical potentials. The most important *physical problem* in the simple-minded PNJL model is the following. Φ being the normalized trace of the Wilson line $\mathbf{L}$, which is an SU(3) matrix, should lie in the range $0 \leq \Phi \leq 1$. But it was found to be greater than 1 at temperatures above $2T_c$ (see Fig. 2 in Ref. [8]). The natural way to cure this problem is to consider a proper Jacobian of transformation from the matrix valued field L to the complex valued field Φ which will then constrain the value of Φ to $\Phi < 1$. This is quite a well-known construction in SU(N) matrix model (see e.g. [17–19]), in certain variations of Polyakov loop model ([11,20]), as well as in QCD motivated phenomenological models (see [21] and references therein). Also this is ubiquitous in various strong coupling effective theories of lattice QCD (see e.g. [22]).

Here we introduce the Vandermonde term in the Polyakov loop model in a conceptually different way than that in the earlier models. In the next section we

[*]sanjay@bosemain.boseinst.ac.in
[+]tamal@bosemain.boseinst.ac.in
[‡]munshigolam.mustafa@saha.ac.in
[§]rajarshi.ray@saha.ac.in

discuss our approach. In Sec. III we show the changes in measurements of the susceptibilities and various other quantities due to the Vandermonde (VdM) term. The final section contains our conclusions.

II. FORMALISM

At a temperature T, the SU(3) Wilson line is given by $\mathbf{L}(\mathbf{x}) = \mathcal{P}\exp(ig\int_0^{1/T} A_0^a(\mathbf{x})\lambda_a d\tau)$, where g is the gauge coupling, A_0^a ($a = 1, 2, \ldots 8$) are the timelike components of the gluon field, λ_a are the Gell-Mann matrices, and τ the imaginary time in the Euclidian field theory. The Polyakov loop is defined as $\Phi = \mathrm{tr}\mathbf{L}/3$ and its conjugate is $\bar{\Phi} = \mathrm{tr}\mathbf{L}^\dagger/3$. Since $\mathbf{L}$ is itself a SU(3) matrix so $\Phi, \bar{\Phi} \leq 1$. The gluon thermodynamics can be described as an effective theory of the Polyakov loops [5]. On the other hand quark thermodynamics can be effectively described in terms of NJL model [4], and the two are coupled to obtain the PNJL model (e.g., [3]). The thermodynamic potential in this model can be obtained in terms of the sigma and pion condensates and the thermal average of the Polyakov loop.

However the version of the PNJL model [3] leads to $\Phi > 1$ for $T > 2T_c$. To rectify this anomaly, the authors of Ref. [3] have recently proposed a complete modification of the Polyakov loop model [11], motivated from the strong coupling results used by Fukushima [2] Our aim in this work is also similar, but the approach is somewhat different. We retain the Polyakov loop potential of [3,6,8] but treat it as a matrix model. Also the way we define pressure is quite different as discussed below.

We first outline our scheme using an arbitrary matrix model for the Wilson line $\mathbf{L}$, which for simplicity is assumed to be a potential $\mathcal{V}[\mathbf{L}] \equiv \mathcal{V}[\Phi, \bar{\Phi}]$. In the following equation, we express the partition function for this theory first as a path integral over $\mathbf{L}$ and then over the fields Φ and $\bar{\Phi}$.

$$Z = \int \mathcal{D}\mathbf{L} e^{-(1/T)\mathcal{V}[\Phi,\bar{\Phi}]}$$

$$= \int \prod_{\mathbf{x}} d\mathbf{L}(\mathbf{x}) e^{-(1/T)\mathcal{V}[\Phi,\bar{\Phi}]} \quad (1a)$$

$$= \int \prod_{\mathbf{x}} J[\Phi(\mathbf{x}), \bar{\Phi}(\mathbf{x})] d\Phi(\mathbf{x}) d\bar{\Phi}(\mathbf{x}) e^{-(1/T)\mathcal{V}[\Phi,\bar{\Phi}]}, \quad (1b)$$

where $\mathcal{D}\mathbf{L}$ is the SU(3) Haar measure, $J[\Phi, \bar{\Phi}]$ is the Jacobian of transformation (also called Vandermonde determinant; see e.g. Ref. [23]) from $\mathbf{L}$ to $(\Phi, \bar{\Phi})$, and is given as $J[\Phi, \bar{\Phi}] \equiv (27/24\pi^2)(1 - 6\bar{\Phi}\Phi + 4(\bar{\Phi}^3 + \Phi^3) - 3(\bar{\Phi}\Phi)^2)$. Our interest then would be to obtain the pressure which is given by

$$P = T\frac{\partial \ln Z}{\partial v} = -\left\langle \frac{\partial \mathcal{V}}{\partial v} \right\rangle \simeq -\frac{1}{v}\langle \mathcal{V} \rangle, \quad (2)$$

where v denotes the physical volume of the system and $\langle\rangle$ denotes thermal averaging. The last approximation holds in the infinite volume limit.

The role of the Jacobian is to be understood as follows. First, it is a factor reweighting the field configurations and hence significantly affects all thermal averages. However the Jacobian is not explicitly space-time dependent; there is no extra term to be averaged in Eq. (2) as one might expect when redefining the path integration from $\mathbf{L}$ to Φ [Eqs. (1a) and (1b)]. A typical example of such a dependence would be if we were considering say a Fourier transform of the fields. In case of a free field this kind of dependence of the Jacobian on the volume and temperature is very important in obtaining the correct partition function.

Thus, in our mean-field treatment we have to carefully incorporate the effect of the Jacobian and this is the main aim of this paper. The effect of the Jacobian is reflected in the mean fields $\langle\Phi\rangle$ and $\langle\bar{\Phi}\rangle$, and we express the pressure as

$$P = -\frac{1}{v}\mathcal{V}(\langle\Phi\rangle, \langle\bar{\Phi}\rangle). \quad (3)$$

To relate to pure glue theory, we now replace the potential density $\mathcal{V}/v$ by a Landau-Ginzburg type functional $\mathcal{U}$, given by [3],

$$\frac{\mathcal{U}(\Phi, \bar{\Phi}, T)}{T^4} = -\frac{b_2(T)}{2}\bar{\Phi}\Phi - \frac{b_3}{6}(\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4}(\bar{\Phi}\Phi)^2, \quad (4)$$

with

$$b_2(T) = a_0 + a_1\left(\frac{T_0}{T}\right) + a_2\left(\frac{T_0}{T}\right)^2 + a_3\left(\frac{T_0}{T}\right)^3. \quad (5)$$

To make a saddle point approximation to the mean fields, the potential density $\mathcal{U}$ was minimized with respect to (w.r.t.) Φ and $\bar{\Phi}$ in Ref. [3]. These were then used to obtain pressure $P = -\mathcal{U}$. The coefficients a_i ($i = 0, 1, 2, 3$) and b_j ($j = 2, 3, 4$) were fitted from lattice data of pressure in pure gauge theory, and T_0 is precisely the transition temperature $T_c = 270$ MeV [24–26]. As $T \to \infty$, $P/T^4 \to 16\pi^2/90$. However, to take care of the effect of the Jacobian as discussed above, we now propose to minimize the following modified potential,

$$\frac{\mathcal{U}'(\Phi, \bar{\Phi})}{T^4} = \frac{\mathcal{U}(\Phi, \bar{\Phi})}{T^4} - \kappa \ln[J(\Phi, \bar{\Phi})], \quad (6)$$

where κ is a dimensionless parameter to be determined phenomenologically. The mean-field value of pressure is still obtained from the relation $P = -\mathcal{U}$. A very simple example of this approach is demonstrated in the Appendix. Note that the Jacobian term is considered as an extra effective term in the modified potential density implying a sort of normalized volume factor. This is quite natural as the form of Eq. (1b) implies that there is a Jacobian sitting at each and every space-time coordinate, depending on the value of Φ and $\bar{\Phi}$.

With the new minimization condition all the coefficients should be estimated afresh. Instead, we retain the values of a_i and b_j obtained in [3] and tune only the values of T_0 and

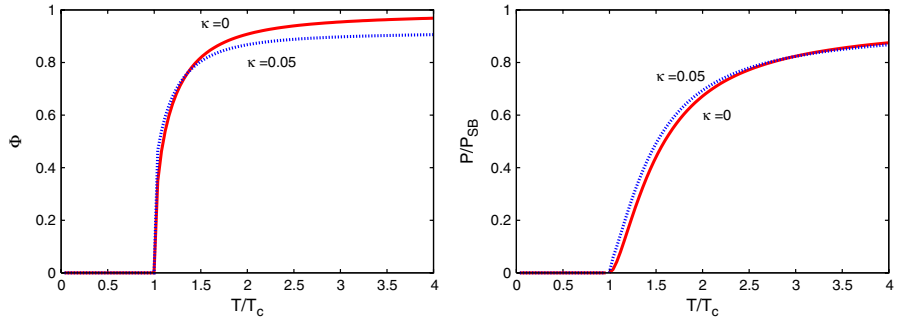

FIG. 1 (color online). Φ and P/P_{SB} for $\kappa = 0$ ($T_0 = 0.27$ GeV) and $\kappa = 0.5$ ($T_0 = 0.2555$ GeV). The value of T_c is 0.270 GeV.

κ. This is equivalent to a correlated modification of the a_i and b_j keeping T_0 fixed at 270 MeV.

We show the variation of the Polyakov loop and the pressure P normalized to Stefan-Boltzmann (SB) pressure P_{SB} for pure gauge theory, as a function of temperature. In Fig. 1 we have used a small nonzero constant value of $\kappa = 0.05$. In Fig. 2 we find similar behavior for a temperature dependent $\kappa = 0.22T_0^3/T^3$. In both the figures the $\kappa = 0$ curves are for the Polyakov loop model without the VdM term. Thus the parameter space of κ is quite open at this stage.

Within the range of temperatures ($T < 3T_c$) where the Polyakov loop model is supposed to be a good description of the system, our approach and that of Ref. [11] give similar results. The reason behind this is that one can suitably adjust the parameters in both approaches. However, our method for introducing the VdM potential as discussed above, is very much different from that of Ref. [11]. The main difference is that the pressure computed in Ref. [11] includes the VdM term. Thus the coefficient of the VdM term requires an inverse temperature dependence, so that on a naive extrapolation to high temperatures, the pressure does not blow up with the logarithm of the Jacobian. In that case another problem crops up with the remaining part of the thermodynamic potential, which at high temperatures has no bound, contrary to the claim that $\Phi \to 1$ as $T \to \infty$. Precisely because Φ should go to 1 as $T \to \infty$ we believe that the VdM term should be very important at high temperatures to constrain the maximum value of Φ to 1.

The exercise for introducing a VdM term for the Polyakov loop model itself has nothing new to offer. Even without it the potential $\mathcal{U}$ was able to describe the pure glue theory quite well. However its importance becomes evident in the PNJL model. The Polyakov loop has a coupling to the fermionic part as will be seen in the corresponding thermodynamic potential below, which forces the Φ to be greater than 1, and more so as the chemical potential is increased. The VdM term can inhibit such a behavior.

The thermodynamic potential of the PNJL model [3,6,8] is given as

$$\Omega = \mathcal{U}(\Phi, \bar{\Phi}, T) + 2G_1(\sigma_u^2 + \sigma_d^2) + 4G_2\sigma_u\sigma_d - \sum_{f=u,d} 2T \int \frac{\mathrm{d}^3p}{(2\pi)^3}$$
$$\times \{\ln[1 + 3(\Phi + \bar{\Phi}\mathrm{e}^{-(E_f-\mu_f)/T})\mathrm{e}^{-(E_f-\mu_f)/T} + \mathrm{e}^{-3(E_f-\mu_f)/T}]$$
$$+ \ln[1 + 3(\bar{\Phi} + \Phi\mathrm{e}^{-(E_f+\mu_f)/T})\mathrm{e}^{-(E_f+\mu_f)/T} + \mathrm{e}^{-3(E_f+\mu_f)/T}]\} - \sum_{f=u,d} 6 \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_f \theta(\Lambda^2 - \vec{p}^2). \quad (7)$$

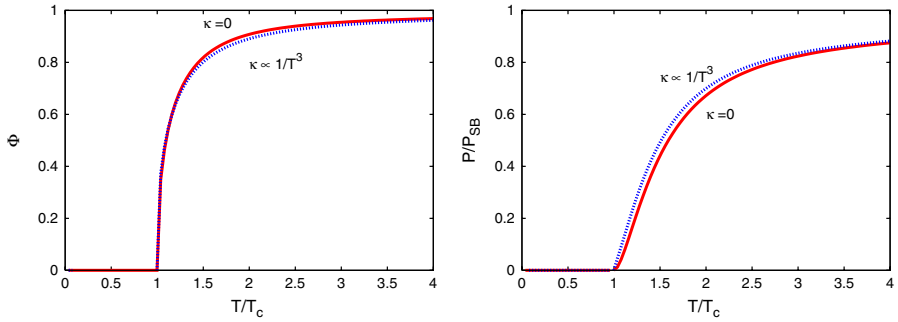

FIG. 2 (color online). Φ and P/P_{SB} for $\kappa = 0$ ($T_0 = 0.27$ GeV) and $\kappa = (0.22T_0^3/T^3)$ ($T_0 = 0.2555$ GeV). Here $T_c = 0.270$ GeV.

Here quark condensates for the two light flavors u and d are given by $\sigma_u = \langle \bar{u}u \rangle$ and $\sigma_d = \langle \bar{d}d \rangle$, respectively, and the respective chemical potentials are μ_u and μ_d. Note that $\mu_0 = (\mu_u + \mu_d)/2$ and $\mu_I = (\mu_u - \mu_d)/2$. The quasiparticle energies are $E_{u,d} = \sqrt{\vec{p}^2 + m_{u,d}^2}$, where $m_{u,d} = m_0 - 4G_1\sigma_{u,d} - 4G_2\sigma_{d,u}$ are the constituent quark masses and m_0 is the current quark mass (we assume flavor degeneracy). G_1 and G_2 are the effective coupling strengths of a local, chiral symmetric four-point interaction. We take $G_1 = G_2 = G/4$, where G is the coupling used in Ref. [3]. Λ is the 3-momentum cutoff in the NJL model. $\mathcal{U}(\Phi, \bar{\Phi}, T)$ is the effective potential for Φ and $\bar{\Phi}$ as given in Eq. (4). We locate the transition temperature in this model from the peaks in the temperature variation of $d\Phi/dT$ and $d\sigma_{u,d}/dT$.

Similar to the case of the Polyakov loop model we would now obtain the mean fields by minimizing

$$\frac{\Omega'}{T^4} = \frac{\Omega}{T^4} - \kappa \ln[J(\Phi, \bar{\Phi})]. \tag{8}$$

The coefficient κ in the VdM term can in general have some temperature and/or chemical potential dependence. Here we take a constant value $\kappa = 0.2$, which suffices for the purpose of the present work. To set this value we looked at the two important quantities affected by the VdM term. The first one is Φ, which decreases with the increase of κ and hence decreases the pressure. The second one is the transition temperature, which increases with κ. Thus we try to optimize κ to get both the pressure and the transition temperature as close as possible to the LQCD results for two quark flavors.

On a naive extrapolation of this model to large chemical potentials, the Φ and $\bar{\Phi}$ should grow towards 1 (deconfinement at large chemical potential) even at very low temperatures. Thus again the logarithmic term blows up. So if pressure is computed including the VdM term as is done in Ref. [11], an anomalous logarithmic divergence would come up. There may be some new physics that can obscure such terms by making $\kappa \to 0$ as $\mu \to \infty$. But that would again run into a problem in restricting Φ in the domain $0 \leq \Phi \leq 1$.

Apart from the difference in the treatment of the VdM term we would now remove the condition $\Phi = \bar{\Phi}$ used in [11], since it has important implications for susceptibilities.

Before going over to our results let us take a digression to the lattice computation of Φ. On the lattice Φ is computed from the relation [15]

$$\Phi(T) = \exp(-\triangle F_{q\bar{q}}(\infty, T)/2T), \tag{9}$$

where $\triangle F_{q\bar{q}}(\infty, T) = F_{q\bar{q}}(\infty, T) - F_{00}(T)$, and $F_{q\bar{q}}(r, T)$ is the free energy of a pair of heavy quark and antiquark at a separation r at a temperature T. This has been used to define a renormalized Polyakov loop in lattice simulations of both pure gluon [27,28] and full QCD [29]. In fact the data of [27] was used to obtain the different parameters of the Polyakov loop model in [3], and is being used by us here, and in that sense Φ is the renormalized Polyakov loop. But even in this exercise the Φ in the Polyakov loop model of [3] goes to 1 at large T and is thus different from lattice results for $T > T_c$. On the lattice the value of Φ goes above 1 for $T > T_c$. It has been argued that since the Φ measured in lattice simulations is a renormalized quantity, it is no longer a character of the group SU(3) and is thus not limited to values below 1. From Eq. (9), it is evident that $\Phi > 1$ only when $\triangle F_{q\bar{q}}(\infty, T) < 0$, and this can be very easily seen to be true in the lattice simulations and happens for $T > T_c$. Now, the free energy $F_{q\bar{q}}(r, T)$ can be considered to be composed of three components, namely, a confining potential, a screening potential, and an entropy part. For low temperatures the confining part is dominant and $\triangle F_{q\bar{q}}(\infty, T) > 0$. In the deconfined phase for large distances, the screening potential drops out so the entropy part is dominant, which could lead to $\triangle F_{q\bar{q}}(\infty, T) \simeq -T \triangle S_{q\bar{q}}(T) < 0$, where $\triangle S_{q\bar{q}}(T) = S_{q\bar{q}}(T) - S_{00}$, and $S_{q\bar{q}}(T)$ denotes the entropy of the system with a pair of quark and antiquark. However the heavy quarks as such are not expected to contribute significantly to the entropy and it seems natural to have $\triangle S_{q\bar{q}}(T) = 0$, and thus $\triangle F_{q\bar{q}}(\infty, T) = 0$ for $T > T_c$. Instead the value is negative on the lattice and $\triangle F_{q\bar{q}}(\infty, T) \to -\infty$ as $T \to \infty$, leading to $\Phi \to \infty$. One has to then worry about what can bend it down towards 1 at asymptotic temperatures as was observed by Gava and Jengo in perturbative evaluation of Φ [30]. However this perturbative calculation also points to the fact that as the temperature is lowered from asymptotic values the Φ is greater than 1. Also recent continuum estimates in chiral quark models [31] using dimensional reduction find close agreement with both lattice and perturbative calculations.

On the other hand another lattice computation of the Polyakov loop in pure glue theory uses a renormalization dependent on temperature instead on the lattice spacing and finds the values to remain below 1 at least up to $T \sim 3.5T_c$ [32]. We thus admit that the state of affairs with the lattice computation of Φ is not very clear to us at this stage. There is a missing link from quantum computations to our matrix model mean-field computations.

III. RESULTS AND DISCUSSIONS

A. PNJL Model: Pressure, specific heat, and speed of sound

Now we discuss the results for the PNJL model with the VdM term. Here the Ω' as given in Eq. (8) is minimized with respect to the fields and all the thermodynamic quantities are obtained using these values. The peaks of the $d\Phi/dT$ and $d\sigma_{u,d}/dT$ curves, as shown in Fig. 3(a), differ by 5 MeV. Their average position, which is at 230 MeV, is taken as the transition (or crossover) temperature T_c. In

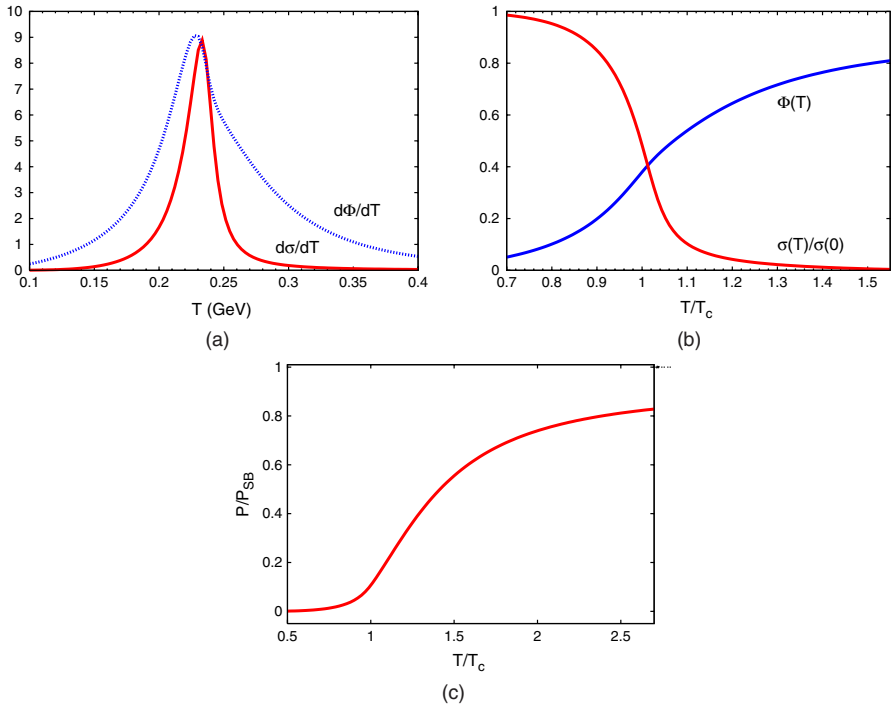

FIG. 3 (color online). (a) Peaks in $d\Phi/dT$ and $d\sigma/dT$ set the T_c at around 230 MeV. (b) Φ and σ as functions of T/T_c. *Note*: In this figure $\sigma = G(\sigma_u + \sigma_d)$.

spite of the significant difference of T_c in the PNJL model with the corresponding LQCD value of 192(7)(4) MeV [33], the thermodynamic quantities when plotted against the scaled temperature T/T_c show similar behavior. We shall henceforth show the temperature dependences in terms of T/T_c.

As mentioned earlier we are using an optimized value of $\kappa = 0.2$. The temperature dependence of the fields are shown in Fig. 3(b). It agrees reasonably with that of the LQCD results as shown in Fig. 1 of Ref. [34]. The scaled pressure P/P_{SB} is plotted in Fig. 3(c). It slightly overestimates the LQCD pressure [35]. However it agrees well with the recent LQCD results for $2 + 1$ flavors with almost physical quark masses [36].

Now, the energy density ϵ is obtained from the relation

$$\epsilon = -T^2 \frac{\partial}{\partial T}\left(\frac{\Omega}{T}\right)\bigg|_V = -T\frac{\partial \Omega}{\partial T}\bigg|_V + \Omega. \tag{10}$$

The rate of change of energy density ϵ with temperature at constant volume is the specific heat C_V which is given as

$$C_V = \frac{\partial \epsilon}{\partial T}\bigg|_V = -T\frac{\partial^2 \Omega}{\partial T^2}\bigg|_V. \tag{11}$$

The square of velocity of sound at constant entropy S is given by

$$v_s^2 = \frac{\partial P}{\partial \epsilon}\bigg|_S = \frac{\partial P}{\partial T}\bigg|_V \Big/ \frac{\partial \epsilon}{\partial T}\bigg|_V = \frac{\partial \Omega}{\partial T}\bigg|_V \Big/ T\frac{\partial^2 \Omega}{\partial T^2}\bigg|_V. \tag{12}$$

The conformal measure is given by

$$\mathcal{C} = \Delta/\epsilon; \qquad \Delta = \epsilon - 3P. \tag{13}$$

These quantities are plotted in Fig. 4. At higher temperatures the C_V is slightly lower than the values obtained in [6]. However, the velocity of sound and the conformal measure remain unaltered in the whole range of temperatures. Thus the VdM term affects C_V but not quantities involving ratios of pressure and energy density e.g. v_s^2 and $\mathcal{C}$. It is interesting to note that our earlier [6] as well as the present work, have been able to predict the value of v_s^2 quite well when compared to the recent LQCD results [36]. We hope similar encouraging results would be obtained on the lattice for the specific heat.

B. Taylor expansion of pressure

The Taylor expansion coefficients of pressure with respect to chemical potentials have been the focus of comparison of PNJL and LQCD results [6,8,11,37]. Here we have expanded the scaled pressure (P/T^4) in a Taylor series for the quark number and isospin number chemical potentials, μ_0 and μ_I, respectively,

$$\frac{P(T,\mu_0,\mu_I)}{T^4} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}c_n^{jk}(T)\left(\frac{\mu_0}{T}\right)^j\left(\frac{\mu_I}{T}\right)^k; \qquad k = n - j, \tag{14}$$

where

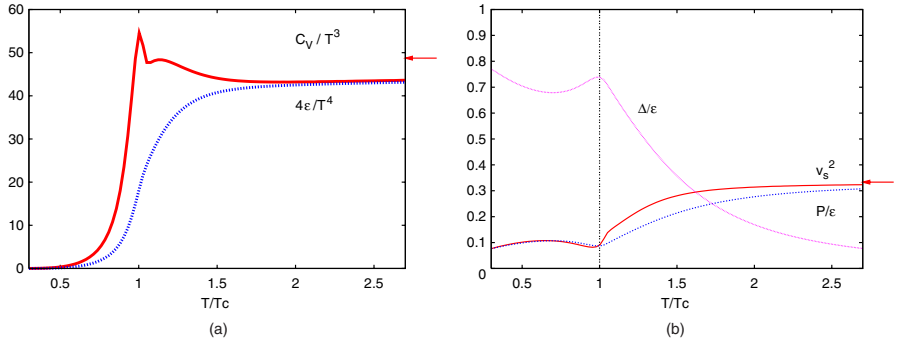

FIG. 4 (color online). (a) Temperature dependence of energy density ϵ and specific heat C_V. (b) Temperature dependence of squared speed of sound v_s^2 and conformal measure Δ/ϵ. The arrows on the right show the corresponding SB limit.

$$c_n^{jk}(T) = \frac{1}{n!}\frac{\partial^n(P(T,\mu_0,\mu_I)/T^4)}{\partial(\frac{\mu_0}{T})^j\partial(\frac{\mu_I}{T})^k}\bigg|_{\mu_0=0,\mu_I=0}. \quad (15)$$

The $n =$ odd terms vanish due to CP symmetry. Even for the $n =$ even terms, due to flavor degeneracy all the coefficients c_n^{jk} with j and k both odd vanish identically. We evaluate all the 10 nonzero coefficients (including the pressure at $\mu_0 = \mu_I = 0$) up to order $n = 6$ and compare them to LQCD data. These coefficients were evaluated in [6,8] and certain differences were found w.r.t. LQCD data. We shall now discuss the effects of the VdM term on these coefficients.

The coefficients we deal with are given by

$$c_n(T) = \frac{1}{n!}\frac{\partial^n(P(T,\mu_0)/T^4)}{\partial(\frac{\mu_0}{T})^n}\bigg|_{\mu_0=0} = c_n^{n0}, \quad (16)$$

$$c_n^I(T) = \frac{1}{n!}\frac{\partial^n(P(T,\mu_0,\mu_I)/T^4)}{\partial(\frac{\mu_0}{T})^{n-2}\partial(\frac{\mu_I}{T})^2}\bigg|_{\mu_0=0,\mu_I=0} = c_n^{(n-2)2}; \quad n > 1. \quad (17)$$

We present the QNS, INS, and their higher order derivatives with respect to μ_0 in Fig. 5. We have plotted the LQCD data from Ref. [16] for quantitative comparison. At the second order [Fig. 5(a)] we find that the QNS c_2 compares well with the LQCD data up to about $1.2T_c$. Thereafter the PNJL values rise up towards the SB limit, while the LQCD values saturate at about 80% of this limit. The INS c_2^I also shows similar behavior, but at lower temperatures it goes slightly above the corresponding LQCD values. There is no significant difference of c_2^I with and without the VdM term. However c_2 was close to the LQCD result without the VdM term [8], but now at high temperatures it goes above the LQCD values and approaches c_2^I. Thus at high temperatures these coefficients overestimate the LQCD results but both are almost equal to each other, similar to that observed on the lattice. This was not so without the VdM term [8].

Now we discuss the fourth-order coefficients [Fig. 5(b)]. The values of c_4 in the PNJL model with the VdM term matches closely with those of LQCD data for the full range of temperatures. This is in contrast to that found without the VdM term [6] where they were close only up to $T \sim 1.1T_c$. The VdM term does not affect the coefficient c_4^I which agrees well with LQCD data for the full range of T. Also both these coefficients approach each other as well as the corresponding SB limit. At the sixth-order [Fig. 5(c)] the coefficients do not seem to be affected by the VdM term.

Thus we write down the salient features regarding the Taylor coefficients in this modified PNJL model:

(i) All the coefficients start approaching their respective SB limit around $2T_c$.
(ii) Both the QNS and INS approach each other at $2T_c$. This is also true for their corresponding responses to quark chemical potential given by the fourth- and sixth-order coefficients.
(iii) At high temperatures, except c_2 and c_2^I, all the coefficients compare well quantitatively with the LQCD data.
(iv) The main effect of the VdM term is to move c_2 and c_4 close to their respective SB limits.

We have emphasized the role of the Polyakov loop in obtaining the values of the Taylor coefficients in our earlier works [6,8]. In those works we found firstly that the Polyakov loop goes above 1 at high temperatures and also has a significant dependence on μ_0 but not on μ_I. Here as shown in Fig. 6, the VdM term restricts the value of Φ within 1, and also the μ_0 dependence at higher temperatures is almost negligible. Thus even the splitting between Φ and $\bar{\Phi}$ has almost disappeared. We note here that though we let Φ and $\bar{\Phi}$ to be different, they come out to be almost equal at high temperatures. This is in contrast to imposing $\Phi = \bar{\Phi}$ for the full range of temperatures as done in Ref. [11]. The difference between Φ and $\bar{\Phi}$ is responsible for the difference of c_2 and c_2^I in the intermediate temperatures.

To complete the comparison with the LQCD data we have looked at the flavor diagonal (c_n^{uu}) and flavor off-diagonal (c_n^{ud}) susceptibilities defined as

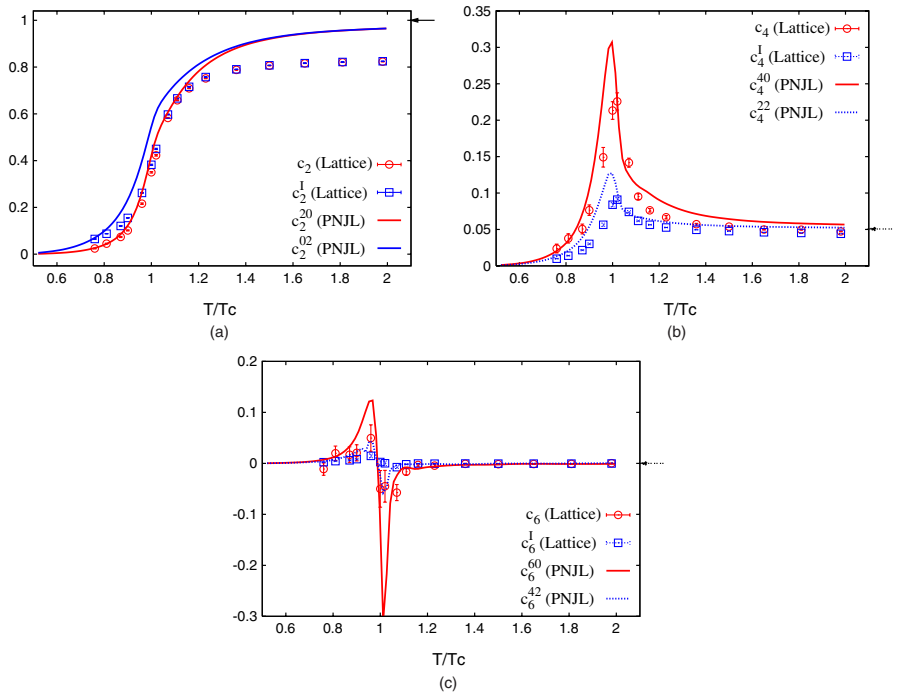

FIG. 5 (color online). The Taylor expansion coefficients of pressure in quark number and isospin chemical potentials as functions of T/T_c. Symbols are LQCD data [16]. Arrows on the right indicate the corresponding ideal gas values.

$$c_n^{uu} = \frac{c_n^{n0} + c_n^{(n-2)2}}{4}, \quad \text{and} \quad c_n^{ud} = \frac{c_n^{n0} - c_n^{(n-2)2}}{4}. \quad (18)$$

The second-order flavor diagonal and off-diagonal susceptibilities are given by

$$\frac{\chi_{uu}(T, \mu_u = 0, \mu_d = 0)}{T^2} = \frac{\partial^2 P(T, \mu_u, \mu_d)}{\partial \mu_u^2}\bigg|_{\mu_u=\mu_d=0} = 2c_2^{uu},$$

and

$$\frac{\chi_{ud}(T, \mu_u = 0, \mu_d = 0)}{T^2} = \frac{\partial^2 P(T, \mu_u, \mu_d)}{\partial \mu_u \partial \mu_d}\bigg|_{\mu_u=\mu_d=0} = 2c_2^{ud}.$$

These are shown in Fig. 7. Except c_2^{uu}, all the other LQCD diagonal and off-diagonal coefficients are close to their respective ideal gas values from $1.2T_c$ onwards. The most striking discrepancy without the VdM term w.r.t the LQCD data was (see [8]) in the second order flavour off-diagonal susceptibility c_2^{ud}. c_2^{ud} signifies the mixing of u and d quarks through the contribution of the two disconnected u and d quark loops. While the LQCD data shows that this kind of correlation between the $u - d$ flavors are almost zero just away form T_c, the PNJL model results remained nonzero even up to $2T_c$. Adding the VdM term this part of the PNJL physics is now consistent with LQCD results. Below $1.2T_c$ there is still a large quantitative difference between the PNJL and LQCD results for c_2^{ud}. Obviously the VdM term is not expected to affect the

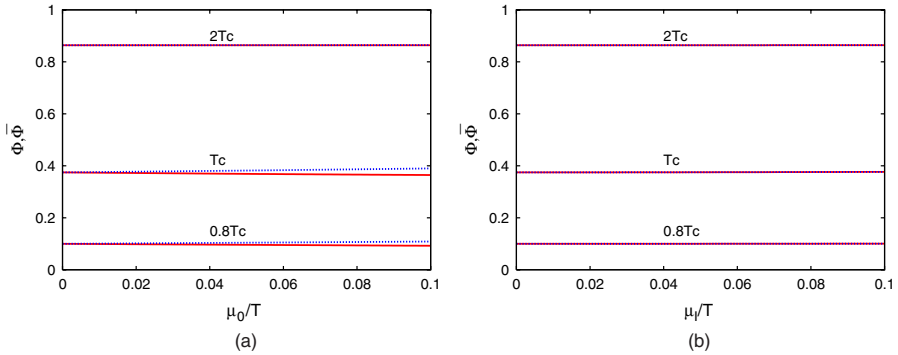

FIG. 6 (color online). (a) Φ (solid lines) decreases and $\bar{\Phi}$ (dotted lines) increases as a function of μ_0/T ($\mu_I = 0$) at low temperatures and almost equal and constant at high temperatures. (b) Φ (solid lines) and $\bar{\Phi}$ (dotted lines) are equal and almost constant as a function of μ_I/T ($\mu_0 = 0$).

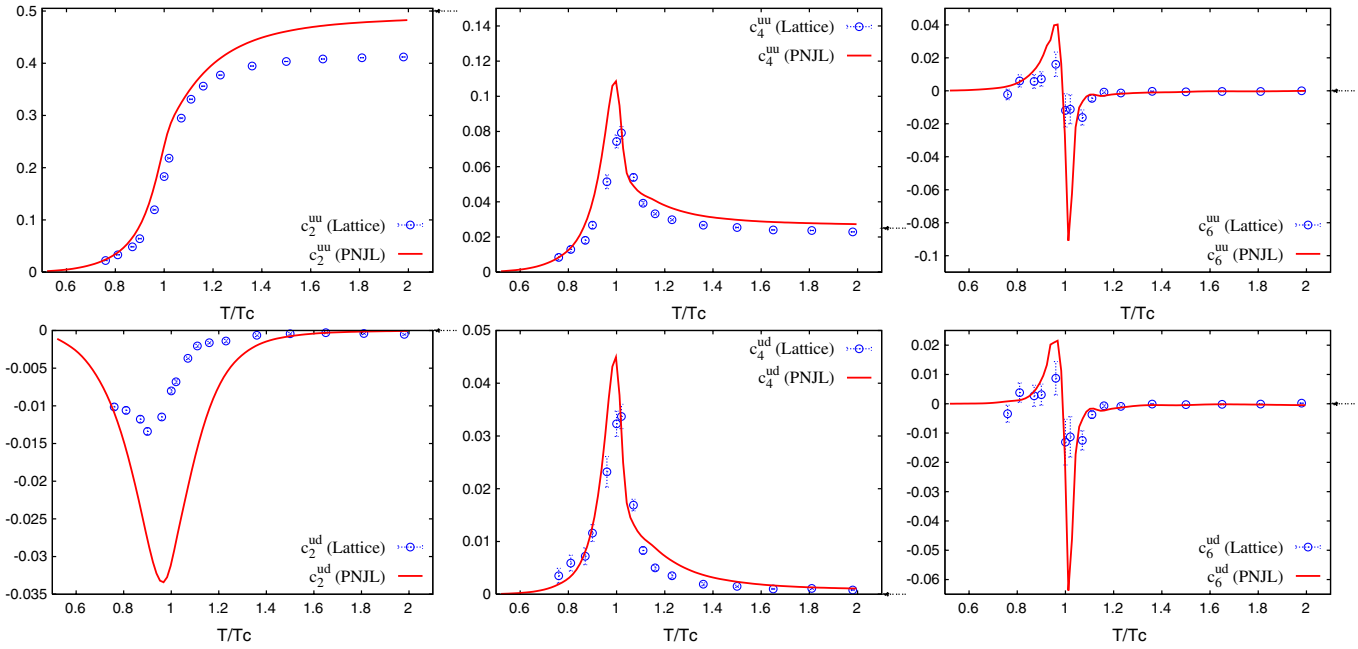

FIG. 7 (color online). The flavor diagonal (*upper row*) and flavor off-diagonal (*lower row*) susceptibilities for $n = 2$, 4, and 6 as functions of T/T_c. Symbols are LQCD data [16]. The arrows on the right indicate the respective ideal gas values.

results at low temperatures significantly. At the moment it is not clear what physics lie behind the difference between PNJL and LQCD results for c_2^{uu} at high temperatures and c_2^{ud} at low temperatures. Perhaps the quark masses may hold an answer.

IV. SUMMARY

In this work the PNJL model of Refs. [3,6,8] has been extended by introducing a VdM term. The important change it brings about is to set the upper limit of the Polyakov loop to 1. With this model we have studied some thermodynamic properties of strongly interacting matter with the light flavors u and d within a certain range of temperature T, and small values of chemical potentials μ_0 and μ_I. In principle the VdM term affects all thermodynamic quantities. We adjusted the parameters in the model so that the pressure and energy density is close to that computed in LQCD. We have then made estimates of the specific heat, the speed of sound, and conformal measure.

Further, we have extracted the Taylor expansion coefficients of pressure in the two chemical potentials up to sixth order. All the coefficients approach their respective SB limit above $2T_c$. A quantitative comparison with the LQCD results shows reasonable agreement, though the QNS c_2 and the INS c_2^I on the lattice are smaller by about 20%. In contrast our earlier estimates [6,8] of these coefficients without the VdM term showed that c_4 and c_I^2 differ from the LQCD results. Thus the main effect of the VdM term is to impose physical constraints on Φ and $\bar{\Phi}$ such that at large temperatures the coefficients of the same order approach each other. This is clearly visible from the flavor off-diagonal coefficients shown in Fig. 7. The remaining difference of the values of the QNS and INS in the model and lattice still needs to be addressed. Possible future steps to bring in better agreement could be to include beyond mean-field effects and/or to include some sort of temperature dependence to the coefficient of the VdM term. However the lattice quark masses may be important in bridging the gap. We already found that such data for pressure with almost physical quark masses [36] show an increase at any given temperature when compared to data with larger quark masses [35]. This would encourage us to believe that extraction of the susceptibilities with similar quark masses on the lattice may have a better agreement with our results. Another way to compare results would be to reestimate the parameters of the NJL model directly from the pion mass and decay constants from the lattice. We hope to undertake such studies in future.

In an alternative formulation of the PNJL model including the effect of the VdM term, the coefficients c_2, c_4, c_6, and c_8 have been calculated [11]. Surprisingly, we more or less agree with those results quantitatively. Apart from the fact that this may be possible due to various adjustable parameters in both the models, the main reason seems to be the small dependence of Φ and $\bar{\Phi}$ on the chemical potentials. The basic difference between the two approaches is in the use of the VdM potential. The VdM term is required to obtain the mean-field solution of Φ and $\bar{\Phi}$. But as we have explained in the formalism that it should not be included in the expression for pressure. On the other hand in Ref. [11]

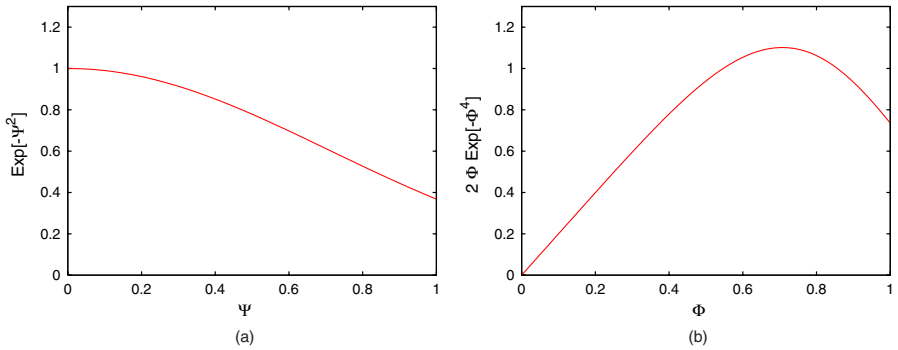

FIG. 8 (color online). (a) Φ and (b) pressure for $\kappa = 0$ ($T_0 = 0.27$ GeV) and $\kappa = 0.5$ ($T_0 = 0.2555$ GeV). The value of T_c is 0.270 GeV.

apart from obtaining the mean fields the VdM term is included while calculating the value of pressure. The difference in the mean-field treatment coupled by almost same final results provide hints to the fact that mean-field treatment has certain shortcomings and is unable to settle issues at hand. It would thus be worthwhile to look beyond.

ACKNOWLEDGMENTS

We would like to thank E. R. Arriola, A. Bhattacharya, S. Datta, S. Digal, E. M. Fernandez, S. Gupta, S. Mukherjee, P. B. Pal, and R. Pisarski for many useful discussions and comments. We are thankful to A. Dumitru and O. Kaczmarek for useful discussions on the lattice computation of the Polyakov loop.

APPENDIX A

We demonstrate the methodology of extracting the average value of a quantity from mean-field approximation. Suppose we have a variable Ψ with a probability distribution given by $\exp(-\Psi^2)$ and we have to obtain the average of the function in the exponential Ψ^2 [just like we have to obtain the average of $\mathcal{V}$ in Eq. (2)]. We define the distribution in the domain $0 \leq \Psi \leq 1$. The distribution is shown in Fig. 8(a). The average is obtained as

$$\langle \Psi^2 \rangle = \frac{\int_0^1 d\Psi \Psi^2 e^{-\Psi^2}}{\int_0^1 d\Psi e^{-\Psi^2}} = 0.2537, \tag{19}$$

where $Z = \int_0^1 d\Psi e^{-\Psi^2} = 0.747$ is like a partition function. In the given domain the distribution has no maximum and thus a mean-field solution cannot be obtained. Let us now make a change of variable from Ψ to Φ where $\Psi = \Phi^2$. The distribution becomes $2\Phi \exp(-\Phi^4)$ as shown in Fig. 8(b). Here, 2Φ is like the Jacobian in the main text. One can now easily check that corresponding to Eq. (19), we need to find the expectation value of Φ^4 given by

$$\langle \Phi^4 \rangle = \frac{\int_0^1 2\Phi d\Phi \Phi^4 e^{-\Phi^4}}{\int_0^1 2\Phi d\Phi e^{-\Phi^4}} = 0.2537. \tag{20}$$

In this case the distribution has a maximum and we can do a saddle point approximation. We thus minimize $\Phi^4 - \ln[2\Phi]$, which gives the mean-field value $\langle \Phi \rangle = 1/\sqrt{2}$. Using this value we find $\langle \Phi \rangle^4 = 0.25 \simeq \langle \Phi^4 \rangle$. So $\langle \Phi \rangle^4$ gives a good approximation to $\langle \Phi^4 \rangle$. On the other hand if we include the logarithm term we have $\langle \Phi \rangle^4 - \ln[2\langle \Phi \rangle] = -0.0966$, which is widely different.

[1] P. N. Meisinger and M. C. Ogilvie, Phys. Lett. B **379**, 163 (1996); Nucl. Phys. B, Proc. Suppl. **47**, 519 (1996).
[2] K. Fukushima, Phys. Lett. B **591**, 277 (2004).
[3] C. Ratti, M. A. Thaler, and W. Weise, Phys. Rev. D **73**, 014019 (2006).
[4] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).
[5] R. D. Pisarski, Phys. Rev. D **62**, 111501(R) (2000); R. D. Pisarski, in *Proceedings on Strong and Electroweak matter, Marseille, 2000*, edited by C. P. Korthals Altes (World Scientific, Singapore, 2001), pp. 107–117.
[6] S. K. Ghosh, T. K. Mukherjee, M. G. Mustafa, and R. Ray, Phys. Rev. D **73**, 114007 (2006).
[7] H. Hansen *et al.*, Phys. Rev. D **75**, 065004 (2007).
[8] S. Mukherjee, M. G. Mustafa, and R. Ray, Phys. Rev. D **75**, 094015 (2007).
[9] Z. Zhang and Yu-X. Liu, Phys. Rev. C **75**, 064910 (2007).
[10] C. Sasaki, B. Friman, and K. Redlich, Phys. Rev. D **75**, 074013 (2007).
[11] S. Roszner, C. Ratti, and W. Weise, Phys. Rev. D **75**, 034007 (2007).

[12] B-J. Schaefer, J. M. Pawlowski, and J. Wambach, Phys. Rev. D **76**, 074023 (2007).
[13] D. Blaschke, M. Buballa, A. E. Radzhabov, and M. K. Volkov, arXiv:hep-ph/0705.0384.
[14] E. Megias, E. R. Arriola, L. L. Salcedo, Phys. Rev. D **74**, 065005 (2006); **74**, 114014 (2006).
[15] L. D. McLerran and B. Svetitsky, Phys. Rev. D **24**, 450 (1981); B. Svetitsky and L. G. Yaffe, Nucl. Phys. **B210**, 423 (1982); B. Svetitsky, Phys. Rep. **132**, 1 (1986).
[16] C. R. Allton *et al.*, Phys. Rev. D **71**, 054508 (2005).
[17] A. Dumitru, R. D. Pisarski, and D. Zschiesche, Phys. Rev. D **72**, 065008 (2005).
[18] H. Steinacker, Nucl. Phys. **B679**, 66 (2004).
[19] G. Akemann, Int. J. Mod. Phys. A **22**, 1077 (2007).
[20] P. N. Meisinger, T. R. Miller, and M. C. Ogilvie, Phys. Rev. D **65**, 034009 (2002).
[21] M. G. Mustafa, Phys. Lett. B **318**, 517 (1993).
[22] E-M. Ilgenfritz and J. Kripfganz, Z. Phys. C **29**, 79 (1985).
[23] O. L. Trinhammer and G. Olafsson, arXiv:math-ph/9901005.
[24] G. Boyd *et al.*, Nucl. Phys. **B469**, 419 (1996); Y. Iwasaki, K. Kanaya, T. Kaneko, and T. Yoshie, Phys. Rev. D **56**, 151 (1997).
[25] B. Beinlich *et al.*, Eur. Phys. J. C **6**, 133 (1999); M. Okamoto *et al.*, Phys. Rev. D **60**, 094510 (1999).
[26] P. de Forcrand *et al.*, Nucl. Phys. **B577**, 263 (2000); Y. Namekawa *et al.*, Phys. Rev. D **64**, 074507 (2001).
[27] O. Kaczmarek, F. Karsch, P. Petreczky, and F. Zantow, Phys. Lett. B **543**, 41 (2002).
[28] O. Kaczmarek, S. Gupta, and K. Hubner, Proc. Sci., LATTICE2007 (2007) 195 [arXiv:0710.2277].
[29] O. Kaczmarek, Proc. Sci., CPOD07 (2007) 043 [arXiv:0710.0498].
[30] E. Gava and R. Jengo, Phys. Lett. B **105**, 285 (1981).
[31] E. Megias, E. R. Arriola, and L. L. Salcedo, J. High Energy Phys. 01 (2006) 073; Phys. Rev. D **75**, 105019 (2007).
[32] A. Dumitru *et al.*, Phys. Rev. D **70**, 034511 (2004).
[33] M. Cheng *et al.*, Phys. Rev. D **74**, 054507 (2006).
[34] P. Petreczky, arxiv:0705.2175.
[35] F. Karsch, E. Laermann, and A. Peikert, Phys. Lett. B **478**, 447 (2000).
[36] M. Cheng *et al.*, arxiv:0710.0354.
[37] C. Ratti, S. Roessner and W. Weise, Phys. Lett. B **649**, 57 (2007).