Improved determination of color-singlet nonrelativistic QCD matrix elements for *S*-wave charmonium

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We present a new computation of *S*-wave color-singlet nonrelativistic QCD matrix elements for the J/ψ and the η_c. We compute the matrix elements of leading order in the heavy-quark velocity v and the matrix elements of relative order v^2. Our computation is based on the electromagnetic decay rates of the J/ψ and the η_c and on a potential model that employs the Cornell potential. We include relativistic corrections to the electromagnetic decay rates, resumming a class of corrections to all orders in v, and find that they significantly increase the values of the matrix elements of leading order in v. This increase could have important implications for theoretical predictions for a number of quarkonium decay and production processes. The values that we find for the matrix elements of relative order v^2 are somewhat smaller than the values that one obtains from estimates that are based on the velocity-scaling rules of nonrelativistic QCD.

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I. INTRODUCTION

In the nonrelativistic quantum chromodynamics (NRQCD) factorization formalism [1], heavy-quarkonium decay and production rates are expressed as sums of short-distance coefficients times NRQCD operator matrix elements. The matrix elements in these sums scale as powers of v, the typical heavy-quark (or antiquark) velocity in the quarkonium rest frame. Hence, the sum in the NRQCD factorization expression can be thought of as an expansion in powers of v. The term that is proportional to the matrix element of leading order in v often gives the dominant contribution in decay and production processes. The leading-order matrix element involves the production or annihilation of a heavy quark-antiquark ($Q\bar{Q}$) pair in a color-singlet state. The term that is proportional to the matrix element of relative order v^2 gives the first relativistic correction. This order-v^2 matrix element also involves the production or annihilation of a heavy $Q\bar{Q}$ pair in a color-singlet state.

In the vacuum-saturation approximation [1] for decay matrix elements, one keeps only the vacuum intermediate state, while, in the vacuum-saturation approximation for production matrix elements, one keeps only the heavy $Q\bar{Q}$ intermediate state. The vacuum-saturation approximation is valid up to corrections of relative order v^4 [1]. In this approximation, the color-singlet decay matrix elements are equal to color-singlet production matrix elements. These vacuum-saturation matrix elements are also the relevant ones for purely electromagnetic decay and production processes and for exclusive decay and production processes. The vacuum-saturation matrix element at leading order in v is proportional to the square of the quarkonium wave function at the origin. In this paper, we compute the vacuum-saturation matrix elements of leading order in v and of relative order v^2 for the J/ψ and η_c states.

The analysis of these matrix elements for the J/ψ state differs in several respects from a previous one involving some of the authors [2]. In that analysis, the matrix element at leading order in v was obtained by comparing the theoretical expression for the decay width $\Gamma[J/\psi \to e^+e^-]$ with the experimental measurement. The theoretical expression that was used in that analysis included the order-α_s correction, but not the relativistic corrections. In the present paper, we include those relativistic corrections. The matrix element of relative order v^2 is determined from a potential-model calculation [2] that uses the leading-order matrix element as an input. The relativistic corrections to $\Gamma[J/\psi \to e^+e^-]$ in turn depend upon that order-v^2 matrix element. Hence, the leading-order and order-v^2 matrix elements are related through a coupled pair of nonlinear equations, which we solve numerically.

We obtain values for the η_c matrix elements in two different ways and average the results. First, we obtain a set of values by making use of the comparison between theory and experiment for the width $\Gamma[\eta_c \to \gamma\gamma]$. This comparison gives one nonlinear equation for the matrix elements. As in the J/ψ case, we make use of a potential-model calculation of the order-v^2 matrix element to obtain a second nonlinear equation, and we solve the coupled nonlinear equations numerically to obtain a set of values for the η_c matrix elements. We obtain a second set of values by making use of the fact that, because of the approximate heavy-quark spin symmetry of NRQCD [1], the η_c and J/ψ matrix elements are equal, up to corrections of relative order v^2. We define this second set of values for the η_c matrix elements simply by taking the values that we obtain for the J/ψ matrix elements and

appending additional error bars that take into account the order-v^2 corrections to the heavy-quark-spin-symmetry relation.

Because the two sets of values for the η_c matrix elements that we obtain in this way have input parameters (such as the heavy-quark mass and the string tension) in common, the uncertainties in these matrix elements are highly correlated between sets and between matrix elements within a set. Therefore, we carry out a covariance-matrix analysis to compute the average. The J/ψ and η_c matrix elements are also highly correlated. Such correlations could be important in applications of our results to calculations involving both the J/ψ and the η_c and/or order-v^2 corrections. Therefore, we present tables showing the variations of each of the matrix elements with respect to the various sources of uncertainty and also give the covariance matrix that corresponds to these variations.

A further refinement that we include in this work is to resum a class of relativistic corrections [2] to $\Gamma[J/\psi \to e^+e^-]$ and $\Gamma[\eta_c \to \gamma\gamma]$. First, we consider all corrections that arise from matrix elements involving only color-singlet $Q\bar{Q}$ Fock states. By making use of a generalization of the Gremm-Kapustin relation [2,3], we can determine all of these matrix elements, up to corrections of relative order v^2, from the leading-order and order-v^2 matrix elements. The simple expressions that result can easily be summed to all orders in v. This resummation is equivalent to retaining all of the relativistic corrections that are contained in a potential-model $Q\bar{Q}$ wave function, up to the ultraviolet cutoff of the NRQCD matrix elements.

Because the expressions for the matrix elements of order v^2 and higher are accurate only up to corrections of relative order v^2, the uncertainty in the resummed expression is of order v^4 relative to the leading-order expression. That is, the nominal accuracy in v is no higher than that of a fixed-order calculation through relative order v^2. However, if the relativistic corrections to a given process that arise from the $Q\bar{Q}$ Fock-state wave function have particularly large coefficients in the v expansion, then the use of the resummed expression may improve the numerical accuracy. Furthermore, the resummation may give an indication of the rate of convergence of the v expansion. In any case, it is generally desirable to include in a calculation a well-defined, if incomplete, set of contributions whenever possible.

The remainder of this paper is organized as follows. In Sec. II, we review the definitions of the S-wave NRQCD matrix elements at the leading and higher orders in v, and we give the relations of these matrix elements to the quarkonium wave functions. We also introduce the generalized Gremm-Kapustin relation for an S-wave quarkonium state, which expresses matrix elements of higher order in v in terms of the matrix element of leading order in v and the binding energy. The generalized Gremm-Kapustin relation allows us to resum a class of relativistic corrections to quarkonium decay to all orders in v. In Sec. III, we present the resummed formulas for the electromagnetic decay widths of the J/ψ and the η_c. Section IV contains a description of the potential-model method that we use to compute the binding energy of the S-wave states and, through the generalized Gremm-Kapustin relation, the NRQCD matrix elements of higher order in v. In Sec. V, we compute the numerical values of the NRQCD matrix elements for the J/ψ and the η_c. We compare our results for the matrix elements with those from previous determinations in Sec. VI. Finally, we summarize our results in Sec. VII.

II. NRQCD MATRIX ELEMENTS

A. Decay and production matrix elements

In the cases of the inclusive decays of spin-singlet and spin-triplet S-wave quarkonium states, such as the η_c and the J/ψ, the matrix elements at the leading power in v are

$$\langle \mathcal{O}_1(^1S_0)\rangle_H = \langle H(^1S_0)|\psi^\dagger\chi\chi^\dagger\psi|H(^1S_0)\rangle, \tag{1a}$$

$$\langle \mathcal{O}_1(^3S_1)\rangle_H = \langle H(^3S_1)|\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi|H(^3S_1)\rangle, \tag{1b}$$

where H is a quarkonium state, $^{2s+1}S_J$ is the standard spectroscopic notation for a state with spin angular momentum s, orbital angular momentum zero, and total angular momentum J, ψ is a two-component Pauli spinor that annihilates a heavy quark, χ is a two-component Pauli spinor that creates a heavy antiquark, and σ^i is a Pauli matrix. The subscript 1 on a NRQCD operator $\mathcal{O}$ indicates that it is a color-singlet operator.

Similarly, in the case of the inclusive production of spin-singlet and spin-triplet S-wave quarkonium states, the matrix elements at the leading power in v are

$$\langle \mathcal{O}_1^H(^1S_0)\rangle = \langle 0|\chi^\dagger\psi\Big(\sum_{X,\text{pol.}}|H(^1S_0)+X\rangle\langle H(^1S_0)+X|\Big)\times\psi^\dagger\chi|0\rangle, \tag{2a}$$

$$\langle \mathcal{O}_1^H(^3S_1)\rangle = \langle 0|\chi^\dagger\sigma^i\psi\Big(\sum_{X,\text{pol.}}|H(^3S_1)+X\rangle\langle H(^3S_1)+X|\Big)\times\psi^\dagger\sigma^i\chi|0\rangle, \tag{2b}$$

where the sum is over the light degrees of freedom X and the $2J+1$ quarkonium polarizations.

In the vacuum-saturation approximation [1], which is valid up to corrections of relative order v^4, the decay matrix elements in Eq. (1) and $1/(2J+1)$ times the production matrix elements in Eq. (2) both reduce to

$$\langle \mathcal{O}_1(^1S_0)\rangle_H^{\text{VS}} = |\langle 0|\chi^\dagger\psi|H(^1S_0)\rangle|^2, \tag{3a}$$

$$\langle \mathcal{O}_1(^3S_1)\rangle_H^{\text{VS}} = |\langle 0|\chi^\dagger\boldsymbol{\sigma}\psi|H(^3S_1)\rangle|^2 = |\langle 0|\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*\psi|H(^3S_1)\rangle|^2, \tag{3b}$$

in the spin-singlet and spin-triplet cases, respectively. In Eq. (3b), the quarkonium polarization vector is denoted by

$\boldsymbol{\epsilon}$, and there is no sum over the polarization states of the quarkonium. In the cases of purely electromagnetic decay or production or exclusive decay or production, the matrix elements in Eq. (3) are the relevant ones at leading order in v.

The first relativistic corrections to inclusive S-wave decay and production involve operators that are analogous to those in Eqs. (1) and (2), but that contain a factor of $(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}})^2$ between either $\psi^\dagger$ and χ or $\chi^\dagger$ and ψ. Here, $\overleftrightarrow{\boldsymbol{D}}$ is the spatial part of the covariant derivative acting to the left and right antisymmetrically: $\chi^\dagger \overleftrightarrow{\boldsymbol{D}} \psi \equiv \chi^\dagger (\boldsymbol{D}\psi) - (\boldsymbol{D}\chi)^\dagger \psi$. These operators are of order v^2 relative to those in Eqs. (1) and (2). The corresponding matrix elements reduce in the vacuum-saturation approximation to

$$\langle \mathcal{P}_1(^1S_0)\rangle_H^{\rm VS} = {\rm Re}\Big[\langle H(^1S_0)|\psi^\dagger\chi|0\rangle \times \Big\langle 0\Big|\,\chi^\dagger\Big(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\Big)^2\psi\,\Big|H(^1S_0)\Big\rangle\Big], \tag{4a}$$

$$\langle \mathcal{P}_1(^3S_1)\rangle_H^{\rm VS} = {\rm Re}\Big[\langle H(^3S_1)|\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi|0\rangle \times \Big\langle 0\Big|\,\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*\Big(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\Big)^2\psi\,\Big|H(^3S_1)\Big\rangle\Big]. \tag{4b}$$

In the cases of purely electromagnetic production or decay or exclusive production or decay, the matrix elements in Eq. (4) are the relevant ones.

Corrections of still higher orders in v^2 involve, among other matrix elements, those in which higher powers of $(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}})^2$ appear. It is convenient to define ratios of these matrix elements to the matrix elements of lowest order in v:

$$\langle \boldsymbol{q}^{2r}\rangle_{H(^1S_0)} = \frac{\langle 0|\chi^\dagger(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}})^{2r}\psi|H(^1S_0)\rangle}{\langle 0|\chi^\dagger\psi|H(^1S_0)\rangle}, \tag{5a}$$

$$\langle \boldsymbol{q}^{2r}\rangle_{H(^3S_1)} = \frac{\langle 0|\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}})^{2r}\psi|H(^3S_1)\rangle}{\langle 0|\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*\psi|H(^3S_1)\rangle}, \tag{5b}$$

where $\boldsymbol{q}$ is half the relative three-momentum of the Q and $\bar{Q}$ in the quarkonium rest frame.

In this paper, we compute the quantities $\langle \mathcal{O}_1(^1S_0)\rangle_{\eta_c}^{\rm VS}$ and $\langle \mathcal{O}_1(^3S_1)\rangle_{J/\psi}^{\rm VS}$, which are given by Eq. (3), and the quantities $\langle \boldsymbol{q}^{2r}\rangle_{\eta_c}$ and $\langle \boldsymbol{q}^{2r}\rangle_{J/\psi}$, which are given by Eq. (5). As we shall see, the higher-order ratios in Eq. (5) can be related to the lowest-order ones by making use of a generalization of the Gremm-Kapustin relation [2,3].

As is discussed in Ref. [2], the higher-order matrix elements in Eq. (5) contain power ultraviolet divergences and require regularization. In this paper, we regulate these power divergences dimensionally at the one-loop level. One-loop dimensional regularization of the matrix elements is appropriate for use in conjunction with one-loop calculations of the short-distance coefficients.

B. Relations of NRQCD matrix elements to quarkonium wave functions

In the rest frame of an S-wave heavy quarkonium H in a spin-singlet ($^1S_0$) or spin-triplet ($^3S_1$) state, one can express the wave function at the origin of the leading $Q\bar{Q}$ Fock state in terms of the color-singlet NRQCD matrix elements [1]:

$$\psi_{H(^1S_0)}(0) = \int \frac{d^3q}{(2\pi)^3}\tilde{\psi}_{H(^1S_0)}(\boldsymbol{q}) = \frac{1}{\sqrt{2N_c}}\langle 0|\chi^\dagger\psi|H(^1S_0)\rangle, \tag{6a}$$

$$\boldsymbol{\epsilon}\psi_{H(^3S_1)}(0) = \boldsymbol{\epsilon}\int \frac{d^3q}{(2\pi)^3}\tilde{\psi}_{H(^3S_1)}(\boldsymbol{q}) = \frac{1}{\sqrt{2N_c}}\langle 0|\chi^\dagger\boldsymbol{\sigma}\psi|H(^3S_1)\rangle. \tag{6b}$$

$\tilde{\psi}_H(\boldsymbol{q})$ is the momentum-space wave function for the leading $Q(\boldsymbol{q})\bar{Q}(-\boldsymbol{q})$ Fock state of the quarkonium. The wave function is, of course, gauge dependent. Throughout this paper, we work in the Coulomb gauge. The normalization factor $1/\sqrt{2N_c}$ accounts for the traces in the SU(2)-spin and SU(3)-color spaces.

Relativistic corrections to the production and decay rates for a heavy quarkonium involve matrix elements that are related to derivatives of the wave function at the origin:

$$\psi^{(2r)}_{H(^1S_0)}(0) \equiv \int \frac{d^3q}{(2\pi)^3}\boldsymbol{q}^{2r}\tilde{\psi}_{H(^1S_0)}(\boldsymbol{q}) = \frac{1}{\sqrt{2N_c}}\langle 0|\chi^\dagger\Big(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\Big)^{2r}\psi|H(^1S_0)\rangle, \tag{7a}$$

$$\boldsymbol{\epsilon}\psi^{(2r)}_{H(^3S_1)}(0) \equiv \boldsymbol{\epsilon}\int \frac{d^3q}{(2\pi)^3}\boldsymbol{q}^{2r}\tilde{\psi}_{H(^3S_1)}(\boldsymbol{q}) = \frac{1}{\sqrt{2N_c}}\langle 0|\chi^\dagger\boldsymbol{\sigma}\Big(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\Big)^{2r}\psi|H(^3S_1)\rangle. \tag{7b}$$

Usually, these operator matrix elements are written in terms of the covariant derivative $\overleftrightarrow{\boldsymbol{D}}$ (Ref. [1]), as in Eqs. (4) and (5), rather than $\overleftrightarrow{\boldsymbol{\nabla}}$. However, in the Coulomb gauge, the difference between the $\overleftrightarrow{\boldsymbol{D}}$ and $\overleftrightarrow{\boldsymbol{\nabla}}$ is suppressed as v^2 (Ref. [1]). We emphasize again that the derivatives of the wave function at the origin, as defined in Eq. (7), are ultraviolet-divergent quantities, which must be regulated. We also note that $\psi_H^{(2r)}(0)$ is different from the expectation value of $\boldsymbol{q}^{2r}$:

$$\psi_H^{(2r)}(0) \neq \int \frac{d^3q}{(2\pi)^3}\boldsymbol{q}^{2r}\tilde{\psi}_H^*(\boldsymbol{q})\tilde{\psi}_H(\boldsymbol{q}). \tag{8}$$

Comparing Eq. (7) with Eq. (5), we see that

$$\langle \boldsymbol{q}^{2r}\rangle_H = \frac{\psi_H^{(2r)}(0)}{\psi_H(0)}[1 + \mathcal{O}(v^2)]. \tag{9}$$

One can also define matrix elements of powers of the heavy-quark velocity in terms of matrix elements of

powers of the heavy-quark momentum:

$$\langle v^{2r}\rangle_H = \langle \boldsymbol{q}^{2r}\rangle_H / m_Q^{2r}, \tag{10}$$

where m_Q is the heavy-quark mass.

C. The generalized Gremm-Kapustin relation

From the effective field theory known as potential NRQCD (pNRQCD) [4], it follows that one can compute the wave functions at the origin and derivatives of wave functions at the origin in Eq. (9), up to errors of relative order v^2, from the Schrödinger wave function for a heavy $Q\bar{Q}$ pair interacting through the leading (static) spin-independent $Q\bar{Q}$ potential. It was shown in Ref. [2] that, in the case of a spin-independent potential and for dimensionally regulated matrix elements, the ratios in Eq. (9) are related through the generalized Gremm-Kapustin relation:

$$\langle \boldsymbol{q}^{2r}\rangle_H = (m\epsilon_{nS})^r[1 + \mathcal{O}(v^2)], \tag{11}$$

where ϵ_{nS} is the binding energy of the $Q\bar{Q}$ pair in the quarkonium state H with principal quantum number n and orbital angular momentum S, and m is the heavy-quark mass in the effective theory pNRQCD. The relation (11) follows from the equations of motion of the $Q\bar{Q}$ pair and from dimensional regularization of the matrix elements at the one-loop level, provided that the potential is parametrized as a sum of constants times powers of the $Q\bar{Q}$ separation. The Cornell potential [5], which we will employ later in this paper, is parametrized in this way. We note that Eq. (11) implies that

$$\langle \boldsymbol{q}^{2r}\rangle_H = \langle \boldsymbol{q}^{2}\rangle_H^r, \tag{12}$$

up to corrections of relative order v^2.

We will use Eq. (11) to determine the quantities $\langle \boldsymbol{q}^{2r}\rangle_{\eta_c}$ and $\langle \boldsymbol{q}^{2r}\rangle_{J/\psi}$. In order to evaluate the ground-state binding energy ϵ_{1S}, we will make use of a potential model that is based on the Cornell potential.

III. FORMULAS FOR ELECTROMAGNETIC DECAYS OF S-WAVE HEAVY QUARKONIA

In this section, we present the NRQCD factorization expressions for the electromagnetic decay widths $\Gamma[H(^3S_1) \to e^+e^-]$ and $\Gamma[H(^1S_0) \to \gamma\gamma]$. In subsequent parts of this paper, we will apply these formulas to the decays $J/\psi \to e^+e^-$ and $\eta_c \to \gamma\gamma$.

A. $\Gamma[H(^3S_1) \to e^+e^-]$

The NRQCD factorization formula for the amplitude for the decay $H(^3S_1) \to e^+e^-$ is

$$\mathcal{A}\,[H(^3S_1) \to e^+e^-] = \sqrt{2m_H}\sum_n d_n(^3S_1)\langle 0|\mathcal{O}_n|H(^3S_1)\rangle, \tag{13}$$

where m_H is the quarkonium mass, the $d_n(^3S_1)$ are short-distance coefficients, and the $\mathcal{O}_n$ are NRQCD operators. The prefactor $\sqrt{2m_H}$ compensates for the fact that the hadronic NRQCD operator matrix elements conventionally have nonrelativistic normalization, while we choose the amplitude on the left side of Eq. (13) to have relativistic normalization.

Now we approximate the formula (13) by retaining only those operator matrix elements that connect the vacuum to the color-singlet, $Q\bar{Q}$ Fock state of the quarkonium H. Then, we have

$$\mathcal{A}[H(^3S_1) \to e^+e^-] = \sqrt{2m_H}\sum_n c_n^i(^3S_1) \times \left\langle 0 \left| \chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)^{2n}\sigma^i\psi \right| H(^3S_1)\right\rangle, \tag{14}$$

where the short-distance coefficients $c_n^i(^3S_1)$ are a subset of the short-distance coefficients d_n in Eq. (13). We will clarify the meaning of the approximation that we have taken to arrive at Eq. (14) below.

Because the $c_n^i(^3S_1)$ are insensitive to the long-distance nature of the hadronic state, we can calculate them by replacing the initial hadronic state $\sqrt{2m_H}|H(^3S_1)\rangle$ in Eq. (14) with a perturbative spin-triplet S-wave $Q\bar{Q}$ state:

$$\mathcal{A}[Q\bar{Q}_1(^3S_1) \to e^+e^-] = \sum_n c_n^i(^3S_1)\left\langle 0 \left| \chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)^{2n}\sigma^i\psi \right| Q\bar{Q}_1(^3S_1)\right\rangle. \tag{15}$$

The factor $\sqrt{2m_H}$ is absent in Eq. (15) because we use the same (relativistic) normalization for the $Q\bar{Q}$ state on both sides of Eq. (15).

In the rest frame of the quarkonium, the perturbative amplitude on the left side of Eq. (15) at order α_s^0 is [6]

$$\mathcal{A}[Q\bar{Q}_1(^3S_1) \to e^+e^-] = \sqrt{2N_c}2E(q)\frac{e^2e_Q}{m_H^2}\boldsymbol{L}\cdot\boldsymbol{\epsilon} \times\left(1 - \frac{\boldsymbol{q}^2}{3E(q)[E(q) + m_Q]}\right), \tag{16}$$

where e is the electromagnetic coupling constant, e_Q is the electric charge of the heavy quark, $\boldsymbol{\epsilon}$ is the polarization vector for the spin-triplet state, L is the leptonic current, and $E(q) = \sqrt{m_Q^2 + \boldsymbol{q}^2}$. In the expression (16), we have neglected the electron mass in comparison with the quarkonium mass. The factor $1/m_H^2$ arises from the photon propagator. The perturbative matrix elements on the right side of Eq. (15) are given by

$$\left\langle 0 \left| \chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)^{2n}\sigma^i\psi \right| Q\bar{Q}_1(^3S_1)\right\rangle = \sqrt{2N_c}2E(q)\boldsymbol{q}^{2n}\boldsymbol{\epsilon}^i. \tag{17}$$

The factor $2E(q)$ arises from the relativistic normalization of the $Q\bar{Q}$ state. By comparing Eqs. (15) and (16), one can

read off the short-distance coefficients $c_n^i({}^3S_1)$:

$$c_n^i({}^3S_1) = \frac{e^2 e_Q}{m_H^2} L^i \left[\frac{1}{n!} \left(\frac{\partial}{\partial \boldsymbol{q}^2} \right)^n \left(1 - \frac{\boldsymbol{q}^2}{3E(q)[E(q)+m_Q]} \right) \right]_{\boldsymbol{q}^2=0}. \tag{18}$$

Substituting the $c_n^i({}^3S_1)$ in Eq. (18) into Eq. (14), using Eq. (12), and including the order-α_s correction to the amplitude [7,8], we obtain

$$\begin{aligned} &\mathcal{A}[H({}^3S_1) \to e^+ e^-] \\ &= \sqrt{2m_H} \frac{e^2 e_Q}{m_H^2} L^i \left[1 - f(\langle \boldsymbol{q}^2 \rangle_H / m_Q^2) - 2C_F \frac{\alpha_s}{\pi} \right] \\ &\quad \times \langle 0 | \chi^\dagger \sigma^i \psi | H({}^3S_1) \rangle, \end{aligned} \tag{19}$$

where $f(x)$ is defined by

$$f(x) = \frac{x}{3(1 + x + \sqrt{1+x})}. \tag{20}$$

Now we can clarify the meaning of the approximation that was taken to arrive at Eq. (14). Suppose that we specialize to the Coulomb gauge. Then, we can drop the gauge fields in covariant derivatives in the matrix elements in Eq. (14), making errors of relative order v^2. The matrix elements are then proportional to derivatives of the Coulomb-gauge color-singlet $Q\bar{Q}$ quarkonium wave function at the origin [1]. (See Sec. II B.) That is, they are proportional to the moments of the momentum-space wave function with respect to the wave-function momentum (the relative momentum of the Q and $\bar{Q}$). From Eq. (18), we see that the short-distance coefficients $c_n^i({}^3S_1)$, when contracted into ϵ^i, are the coefficients of the Taylor expansion of $\mathcal{A}[Q\bar{Q}_1({}^3S_1) \to e^+ e^-]/[\sqrt{2N_c} 2E(q)]$ with respect to the wave-function momentum. Hence, Eq. (14) has the interpretation of the convolution of the short-distance amplitude with the momentum-space quarkonium wave function, where the short-distance coefficients have been Taylor expanded with respect to the wave-function momenta. Therefore, we see that the approximate NRQCD expansion in Eq. (14) includes all of the relativistic corrections that are contained in the color-singlet $Q\bar{Q}$ quarkonium wave function, up to the ultraviolet cutoff of the NRQCD matrix elements.[1]

We note that, in the quarkonium rest frame, the square of the spatial part of the leptonic factor L, summed over lepton spins, is given by

$$\sum_{\text{spins}} L^i L^{*j} = 2m_H^2 (\delta^{ij} - \hat{\boldsymbol{k}}^i \hat{\boldsymbol{k}}^j), \tag{21}$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ and $\boldsymbol{k}$ is the three-momentum of the e^- in the quarkonium rest frame. The temporal parts of $\sum_{\text{spins}} L^\mu L^{*\nu}$ vanish in the quarkonium rest frame.

We obtain the leptonic decay width of the spin-triplet S-wave heavy quarkonium by taking the square of the amplitude (19), summing over lepton spins using Eq. (21), averaging over the $H({}^3S_1)$ polarization states, and multiplying by the two-body phase space and the normalization $(2m_H)^{-1}$. The result is [6,9]

$$\begin{aligned} &\Gamma[H({}^3S_1) \to e^+ e^-] \\ &= \frac{8\pi e_Q^2 \alpha^2}{3m_H^2} \left[1 - f(\langle \boldsymbol{q}^2 \rangle_H / m_Q^2) - 2C_F \frac{\alpha_s}{\pi} \right]^2 \langle \mathcal{O}_1 \rangle_H, \end{aligned} \tag{22}$$

where $\alpha = e^2/(4\pi)$. In Eq. (22), the explicit relativistic corrections are contained in the term $-f(\langle \boldsymbol{q}^2/m_Q^2 \rangle_H)$. In addition, there are implicit relativistic corrections that are contained in the factors m_H. Strictly speaking, if one were to compute the decay amplitude completely with the framework of NRQCD, then m_H would be written as $2E(q)$ and expanded in powers of $|\boldsymbol{q}|/m_Q$ to obtain additional relativistic corrections. (See, for example, Refs. [6,10].) However, we note that the factor $1/m_H^2$ in Eq. (22) is clearly identifiable as arising from the photon propagator and the leptonic current, and, so, it is not necessary to treat that factor within the framework of NRQCD. We choose not to apply the nonrelativistic expansion of NRQCD to the factor $1/m_H^2$. That is, we apply NRQCD only to the heavy-quark factor in the amplitude. This choice reduces the theoretical uncertainties by making use of the fact that the quarkonium masses are known very precisely.

The order-α_s^2 corrections to $\Gamma[H({}^3S_1) \to e^+ e^-]$ (Refs. [11,12]) contain a strong dependence on the NRQCD factorization scale. If one were to include those corrections in the expression (22) and use it to determine $\langle \mathcal{O}_1 \rangle_H$, then $\langle \mathcal{O}_1 \rangle_H$ would also contain a strong dependence on the NRQCD factorization scale. If one were to make use of $\langle \mathcal{O}_1 \rangle_H$ in calculating other quarkonium decay and production processes, then the factorization-scale dependence would cancel only if the short-distance coefficients for those processes were calculated through relative order α_s^2. Generally, short-distance coefficients for quarkonium processes have not been calculated beyond relative order α_s. For this reason, we have chosen to omit the order-α_s^2 corrections to the leptonic width in Eq. (22).

B. $\Gamma[H({}^1S_0) \to \gamma\gamma]$

Employing a method analogous to that which is given in Sec. III A, one can obtain the NRQCD factorization formula for the relativistic corrections to the two-photon decay of a spin-singlet S-wave quarkonium state $H({}^1S_0)$.

The NRQCD factorization formula for the amplitude for the decay $H({}^1S_0) \to \gamma\gamma$ is

[1] We note that, in the case of dimensionally regulated NRQCD matrix elements, pure power ultraviolet divergences in the matrix elements are set to zero. Hence, the effects of purely power-divergent contributions are absent in the resummation.

$$\mathcal{A}\,[H(^1S_0) \to \gamma\gamma] = \sqrt{2m_H}\sum_n d_n(^1S_0)\langle 0|\mathcal{O}_n|H(^1S_0)\rangle, \tag{23}$$

where the $d_n(^1S_0)$ are short-distance coefficients. We approximate this expression by keeping only those matrix elements that connect the vacuum to the color-singlet $Q\bar{Q}$ Fock state in the quarkonium. Then, we have

$$\mathcal{A}[H(^1S_0) \to \gamma\gamma] = \sqrt{2m_H}\sum_n c_n(^1S_0) \times \left\langle 0 \left| \chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)^{2n}\psi \right| H(^1S_0)\right\rangle. \tag{24}$$

As in the spin-triplet case, this modified NRQCD factorization formula retains all of the relativistic corrections that are contained in a potential-model $Q\bar{Q}$ wave function, up to the ultraviolet cutoff of the NRQCD matrix elements.

We can calculate the short-distance coefficients $c_n(^1S_0)$ by replacing the initial hadronic state $\sqrt{2m_H}|H(^1S_0)\rangle$ in Eq. (24) with a perturbative spin-singlet S-wave $Q\bar{Q}$ state:

$$\mathcal{A}[Q\bar{Q}_1(^1S_0) \to \gamma\gamma] = \sum_n c_n(^1S_0) \times \left\langle 0 \left| \chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)^{2n}\psi \right| Q\bar{Q}_1(^1S_0)\right\rangle. \tag{25}$$

In the rest frame of the quarkonium, the perturbative amplitude on the left side of Eq. (25) at order α_s^0 is given by [6]

$$\mathcal{A}[Q\bar{Q}_1(^1S_0) \to \gamma\gamma] = \sqrt{2N_c}e^2e_Q^2\frac{\boldsymbol{k}_1\cdot\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*}{|\boldsymbol{k}_1|}\frac{m_Q}{|\boldsymbol{q}|} \times \log\frac{E(q)+|\boldsymbol{q}|}{E(q)-|\boldsymbol{q}|}, \tag{26}$$

where k_i and $\boldsymbol{\epsilon}_i$ are the momentum and the polarization of the ith photon. The perturbative NRQCD matrix elements on the right side of Eq. (25) are given by

$$\left\langle 0 \left| \chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)^{2n}\psi \right| Q\bar{Q}_1(^1S_0)\right\rangle = \sqrt{2N_c}2E(q)\boldsymbol{q}^{2n}. \tag{27}$$

By comparing Eqs. (25) and (26), one can read off the short-distance coefficients $c_n(^1S_0)$:

$$c_n(^1S_0) = e^2e_Q^2\frac{\boldsymbol{k}_1\cdot\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*}{|\boldsymbol{k}_1|}\left[\frac{1}{n!}\left(\frac{\partial}{\partial\boldsymbol{q}^2}\right)^n\frac{m_Q}{2E(q)|\boldsymbol{q}|} \times \log\frac{E(q)+|\boldsymbol{q}|}{E(q)-|\boldsymbol{q}|}\right]_{\boldsymbol{q}^2=0}. \tag{28}$$

Substituting the $c_n(^1S_0)$ in Eq. (28) into Eq. (24), using Eq. (12), and including the order-α_s correction to the amplitude [13–15], we obtain

$$\mathcal{A}[H(^1S_0) \to \gamma\gamma] = \frac{\sqrt{2m_H}}{m_Q}e^2e_Q^2\frac{\boldsymbol{k}_1\cdot\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*}{|\boldsymbol{k}_1|} \times\left[1 - g(\langle\boldsymbol{q}^2\rangle_H/m_Q^2) - \frac{20-\pi^2}{8}C_F\frac{\alpha_s}{\pi}\right]\langle 0|\chi^\dagger\psi|H(^1S_0)\rangle, \tag{29}$$

where $g(x)$ is defined by

$$g(x) = 1 - \frac{1}{2\sqrt{x(1+x)}}\log\left[\frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}-\sqrt{x}}\right] = 1 - \frac{1}{2\sqrt{x(1+x)}}\log[1+2\sqrt{x(1+x)}+2x]. \tag{30}$$

Squaring the amplitude (29), summing over the photon polarizations, multiplying by the phase space and $1/2!$ for the two identical particles in the final state, and dividing by the normalization $2m_H$, we obtain the two-photon decay width of a spin-singlet S-wave heavy quarkonium:

$$\Gamma[H^{(^1S_0)} \to \gamma\gamma] = \frac{2\pi\alpha^2e_Q^4}{m_Q^2}\left[1 - g(\langle\boldsymbol{q}^2\rangle_H/m_Q^2) - \frac{20-\pi^2}{8}C_F\frac{\alpha_s}{\pi}\right]^2\langle\mathcal{O}_1\rangle_H. \tag{31}$$

In the formula (31), we have omitted the order-α_s^2 corrections to the decay amplitude [16]. As we discussed in the case of the leptonic width of a spin-triplet S-wave quarkonium, the order-α_s^2 corrections contain a dependence on the NRQCD factorization scale and can only be used consistently in conjunction with calculations of other quarkonium processes through order α_s^2.

IV. POTENTIAL MODEL

As we have explained earlier, in order to compute the higher-order matrix elements that appear in Eq. (5), we need to compute the ground-state binding energy ϵ_{1S} that appears in the generalized Gremm-Kapustin relation (11). In this section, we describe briefly the potential model that we use to compute ϵ_{1S}. For details of the model, we refer the reader to Refs. [2,5].

The model makes use of the Cornell potential [5], which parametrizes the $Q\bar{Q}$ potential as a linear combination of the Coulomb and linear potentials:

$$V(r) = -\frac{\kappa}{r} + \sigma r, \tag{32}$$

where κ is a dimensionless model parameter for the Coulomb strength and σ is the string tension, which is of mass dimension two. In the original formulation of the Cornell potential model [5], the strength of the linear potential was given in terms of a parameter a, where

$$a = 1/\sqrt{\sigma}. \tag{33}$$

By varying the parameters in the Cornell potential, one can obtain good fits to lattice measurements of the $Q\bar{Q}$ static potential [17]. Therefore, we assume that the use of the Cornell parametrization of the $Q\bar{Q}$ potential results in errors that are much less than the order-v^2 errors (about 30%) that are inherent in the leading-potential approximation to NRQCD.

The Schrödinger equation for the radial wave function $R_{n\ell}(r)$ with the radial and orbital angular-momentum quantum numbers n and ℓ is

$$\left[-\frac{1}{mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+\frac{\ell(\ell+1)}{mr^2}+V(r)\right]R_{n\ell}(r) = \epsilon_{n\ell}R_{n\ell}(r), \tag{34}$$

where m is the quark mass and $\epsilon_{n\ell}$ is the binding energy of the $n\ell$ state. We treat m as a parameter of the potential model and note that it is, in general, different from the heavy-quark mass m_Q, which appears in the short-distance coefficients of NRQCD factorization formulas. As usual, for an S-wave state, the wave function is $\psi_{nS}(r) = R_{nS}(r)/\sqrt{4\pi}$.

Introducing the scaled radius ρ and scaled coupling λ [5],

$$\rho = (\sigma m)^{1/3} r, \tag{35a}$$

$$\lambda = \frac{\kappa}{(\sigma/m^2)^{1/3}}, \tag{35b}$$

which are dimensionless, one can rewrite the radial equation (34) as [5]

$$\left[\frac{d^2}{d\rho^2}-\frac{\ell(\ell+1)}{\rho^2}+\frac{\lambda}{\rho}-\rho+\zeta_{n\ell}\right]u_{n\ell}(\rho)=0, \tag{36}$$

where $u_{n\ell}(\rho)$ and $\zeta_{n\ell}$ are the dimensionless radial wave function and the dimensionless energy eigenvalue of the $n\ell$ state. The relation between $R_{n\ell}(r)$ and $u_{n\ell}(\rho)$ is

$$R_{n\ell}(r) = \sqrt{\sigma m}\frac{u_{n\ell}(\rho)}{\rho}, \tag{37}$$

where the wave functions are normalized according to

$$\int_0^\infty |u_{n\ell}(\rho)|^2 d\rho = \int_0^\infty |R_{n\ell}(r)|^2 r^2 dr = 1. \tag{38}$$

The binding energy is related to the dimensionless eigenvalue $\zeta_{n\ell}$ as

$$\epsilon_{n\ell} = [\sigma^2/m]^{1/3}\zeta_{n\ell}(\lambda). \tag{39}$$

Now let us specialize to the S-wave case. In order to compute ϵ_{nS} from Eq. (39), we must fix the model parameters σ, m, and λ and solve Eq. (36), with $\ell = 0$, for $\zeta_{nS}(\lambda)$. Our strategy is to fix σ from lattice measurements and to use the measured $1S$-$2S$ mass splitting and $|\psi_{nS}(0)|^2$, as determined from the electromagnetic decay widths, to solve for m and λ. Using Eq. (39), we can express m in terms of the $1S$-$2S$ mass splitting:

$$m(\lambda) = \sigma^2\left[\frac{\zeta_{2S}(\lambda)-\zeta_{1S}(\lambda)}{m_{2S}-m_{1S}}\right]^3. \tag{40}$$

For S-wave states, the wave function at the origin $\psi_{nS}(0) = R_{nS}(0)/\sqrt{4\pi}$ can be expressed as [5]

$$|\psi_{nS}(0)|^2 = \frac{m}{4\pi}\int d^3r|\psi_{nS}(r)|^2\frac{\partial V(r)}{\partial r} = \frac{\sigma m(\lambda)}{4\pi}[1+\lambda F_{nS}(\lambda)], \tag{41}$$

where $F_{nS}(\lambda)$ is the expectation value of $1/\rho^2$ for the nS state:

$$F_{nS}(\lambda) = \int_0^\infty \frac{d\rho}{\rho^2}|u_{nS}(\rho)|^2. \tag{42}$$

The first equality in Eq. (41) can be obtained by multiplying the radial Schrödinger equation (34) on the left by $R^*_{nS}(r)$ and integrating by parts.

For purposes of computation of the NRQCD matrix elements, it is convenient to express those matrix elements in terms of the potential-model parameters. From Eqs. (39) and (41) and the generalized Gremm-Kapustin relation (11), we find that

$$\langle\mathcal{O}_1\rangle_H = 2N_c|\psi(0)|^2 = \frac{\sigma N_c m(\lambda)}{2\pi}[1+\lambda F_{1S}(\lambda)], \tag{43a}$$

$$\langle\boldsymbol{q}^2\rangle_H = m(\lambda)\epsilon_{1S}(\lambda) = [\sigma m(\lambda)]^{2/3}\zeta_{1S}(\lambda), \tag{43b}$$

where $m(\lambda)$ is given in Eq. (40) and $F_{1S}(\lambda)$ is given in Eq. (42).

V. COMPUTATION OF THE NRQCD MATRIX ELEMENTS

In this section, we determine the numerical values of the NRQCD matrix elements for the J/ψ and the η_c. In this and subsequent discussions, we drop the superscript VS on $\langle\mathcal{O}_1\rangle_H^{\rm VS}$ because other sources of uncertainty, which we will describe, are much larger than the error in the vacuum-saturation approximation.

A. Method of computation

Were it not for the relativistic corrections in the decay widths in Eqs. (22) and (31), we could simply solve those equations for $\langle\mathcal{O}_1\rangle_H$. Then we could use the value for $\langle\mathcal{O}_1\rangle_H$ that we would obtain to solve Eq. (43a) for λ and use that value of λ to solve Eq. (43b) for $\langle\boldsymbol{q}^2\rangle_H$. Because the relativistic corrections in Eqs. (22) and (31) couple those equations weakly to Eq. (43b), we must carry out the more difficult task of solving Eq. (22) or Eq. (31) simultaneously with Eqs. (43a) and (43b).

First, we express $\Gamma[J/\psi \to e^+e^-]$ and $\Gamma[\eta_c \to \gamma\gamma]$ in terms of the potential-model parameters by substituting Eq. (43) into Eqs. (22) and (31), respectively. We equate those expressions to the experimental values of the electromagnetic widths [18]:

$$\Gamma[J/\psi \to e^+e^-] = 5.55 \pm 0.14 \pm 0.02 \text{ keV}, \quad (44a)$$

$$\Gamma[\eta_c \to \gamma\gamma] = 7.2 \pm 0.7 \pm 2.0 \text{ keV}. \quad (44b)$$

Then, we solve the resulting equations numerically for the model parameter λ. In computing the solution, we express the eigenvalues $\zeta_{1S}(\lambda)$ and $\zeta_{2S}(\lambda)$ and the expectation value $F_{1S}(\lambda)$ [Eq. (42)] as functions of λ by fitting interpolating polynomials to computations of the eigenvalues and expectation value at fixed values of λ. Once we have obtained a value for λ, we substitute it into Eq. (43) to obtain values for the NRQCD matrix elements.

In carrying out the numerical computation, we need values for the charm-quark mass m_c, the string tension σ, and the $1S$-$2S$ mass splitting. In order to maintain consistency with the calculations of the electromagnetic decay widths of the J/ψ and the η_c at NLO in α_s, we take m_c to be the pole mass. The specific numerical value that we use is[2]

$$m_c = 1.4 \pm 0.2 \text{ GeV}. \quad (45)$$

We fix the string tension σ by making use of lattice measurements. From Ref. [19], we find that $\sigma a_{\rm L}^2 = 0.0114(2)$ at a lattice coupling $\beta = 6.5$, where $a_{\rm L}$ is the lattice spacing. Lattice calculations of the hadron spectrum at $\beta = 6.5$ yield values for $1/a_{\rm L}$ of 3.962(127) GeV (Refs. [20,21]) and 3.811(59) GeV (Refs. [20,22]). These result in values for the string tension of $\sigma = 0.1790 \pm 0.0119$ GeV^2 and $\sigma = 0.1656 \pm 0.0059$ GeV^2, respectively. Combining these two values, we obtain

$$\sigma = 0.1682 \pm 0.0053 \text{ GeV}^2. \quad (46)$$

For the $1S$-$2S$ mass splitting, we take the mass difference between the J/ψ and $\psi(2S)$ [18]:

$$m_{2S} - m_{1S} = 589.177 \pm 0.036 \text{ MeV}. \quad (47)$$

We use $m_{J/\psi} = 3.096\,916$ GeV and $m_{\eta_c} = 2.9798$ GeV [18]. We also need values for α_s. In the case of $J/\psi \to e^+e^-$, we choose the scale of α_s to be that of the momentum transfer at the virtual-photon-charm-quark vertex, namely, $m_{J/\psi}$. In the case of $\eta_c \to \gamma\gamma$, we choose the scale of α_s to be that of the momentum transfer at either of the photon-charm-quark vertices, namely, $m_{\eta_c}/2$.[3] In order to take into account uncertainties in the scale and omitted corrections to the decay rates of next-to-next-to-leading order (NNLO) in α_s, we attach an uncertainty to α_s whose relative size is α_s. Then, we have

$$\alpha_s(m_{J/\psi}) = 0.25 \pm 0.06, \quad (48a)$$

$$\alpha_s(m_{\eta_c}/2) = 0.35 \pm 0.12. \quad (48b)$$

We choose the scales for the running QED coupling α to be the same as those for α_s:

$$\alpha(m_{J/\psi}) = \frac{1}{132.6}, \quad (49a)$$

$$\alpha(m_{\eta_c}/2) = \frac{1}{133.6}, \quad (49b)$$

where we ignore the uncertainties in α.

In the case of the η_c matrix elements, we actually make use of two methods of computation. One method is to compute the η_c matrix elements from $\Gamma[\eta_c \to \gamma\gamma]$, as we have outlined above. A second method is to equate the η_c matrix elements to the J/ψ matrix elements that we determine from $\Gamma[J/\psi \to e^+e^-]$. Owing to the approximate heavy-quark spin symmetry of NRQCD [1], this equality is valid up to corrections of relative order v^2. By combining these two methods of determining the η_c matrix elements, we can reduce the uncertainties. This approach is useful because the experimental result for $\Gamma[\eta_c \to \gamma\gamma]$ [Eq. (44b)] has a relative uncertainty that is comparable to the corrections to the spin-symmetry relation, which are of order $v^2 \approx 30\%$. In principle, we could apply a similar approach to the J/ψ matrix elements, but we would not gain a significant reduction in the uncertainties because the relative uncertainty in the experimental result for $\Gamma[J/\psi \to e^+e^-]$ [Eq. (44a)] is small compared to the corrections to the spin-symmetry relation. In averaging the two sets of η_c matrix elements, we must take into account the fact that many of the uncertainties are correlated between the two sets. We describe the procedure that we use for doing this in detail in the next section.

B. Sources of uncertainties

Let us now list the various uncertainties that enter into the calculations of the matrix elements. There is a theoretical uncertainty in the value of $\langle \boldsymbol{q}^2 \rangle_H$ that arises from the fact that the leading-potential approximation is accurate only up to corrections of relative order v^2. For the computation that is based on $\Gamma[J/\psi \to e^+e^-]$, we denote this uncertainty by $\Delta\langle \boldsymbol{q}^2 \rangle_{J/\psi}$, and for the computation that is based on $\Gamma[\eta_c \to \gamma\gamma]$, we denote this uncertainty by $\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$. We take these uncertainties to be $v^2 \approx 30\%$ times the central values. The uncertainties that arise from the scale uncertainties in α_s and from neglecting NNLO corrections to the J/ψ and η_c electromagnetic widths are denoted by $\Delta\text{NNLO}_{J/\psi}$, $\Delta\text{NNLO}_{\eta_c}$, respectively. As we have explained above, we parametrize these uncertainties as uncertainties in α_s [Eq. (48)]. However, we take $\Delta\text{NNLO}_{J/\psi}$ and $\Delta\text{NNLO}_{\eta_c}$, to be uncorrelated. There are also uncertainties that are associated with the charm-quark mass m_c [Eq. (45)], the string tension σ [Eq. (46)], and the

[2]The most recent compilation of the Particle Data Group [18] suggests that the actual uncertainty in m_c may be a factor of 2 smaller than the uncertainty that we use here. However, since it is not clear that the systematic errors are well understood in the various determinations that enter into that compilation, we make a conservative choice of error bars.

[3]We compute α_s and α at each scale by making use of the code GLOBAL ANALYSIS OF PARTICLE PROPERTIES (GAPP) [23].

uncertainties in the experimental measurements of $\Gamma[J/\psi \to e^+e^-]$ and $\Gamma[\eta_c \to \gamma\gamma]$ [Eqs. (44a) and (44b)]. We denote these uncertainties by Δm_c, $\Delta\sigma$, $\Delta\Gamma_{J/\psi}$, and $\Delta\Gamma_{\eta_c}$, respectively. When we combine the values of the η_c matrix elements that we obtain from $\Gamma[\eta_c \to \gamma\gamma]$ with those that we obtain from $\Gamma[J/\psi \to e^+e^-]$ by invoking the heavy-quark spin symmetry, there is an uncertainty from corrections to the spin symmetry, which applies to the latter set of matrix elements. We take it to be $v^2 \approx 30\%$ times the values of that set of matrix elements. Since $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ already has an uncertainty $\Delta\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ of order v^2, we apply this additional order-v^2 uncertainty only to $\langle \mathcal{O}_1 \rangle_{J/\psi}$. We denote it by Δv^2.

In making these uncertainty estimates, we have assumed that the standard NRQCD power-counting (velocity-scaling) rules [1] hold. Various alternatives to the NRQCD power-counting rules have been suggested [4,24,25]. Application of these alternative rules would affect our estimate of the correction to the heavy-quark spin symmetry, Δv^2, and our estimates of the corrections to the static potential, $\Delta\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ and $\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$. In the standard NRQCD power-counting rules, Δv^2 is of relative order v^2. In the strong-coupling regime of Refs. [4,24] and in the power-counting rules of Ref. [25], Δv^2 is of relative order $\Lambda_{\rm QCD}/m_c$, which is actually smaller numerically than v^2. The leading correction to the static potential is denoted by $V^{(1)}/m$ (Ref. [4]). In the standard NRQCD power-counting rules, $V^{(1)}/m$ is suppressed as v^2 relative to the static potential. In the strong-coupling regime of Refs. [4,24] and in the power-counting rules of Ref. [25], $V^{(1)}/m$ is of the same order as the static potential [24,26]. In the lattice calculation of Ref. [27], $V^{(1)}/m$ corrects the string tension by about 17%, which is numerically smaller than v^2. Other lattice calculations [28–30] also suggest that terms of higher order in the standard NRQCD power counting are suppressed at least as much as would be expected from the standard power counting. Therefore, we believe that the standard NRQCD power-counting rules give an upper bound on the uncertainties, and we use them for our uncertainty estimates. One could implement the alternative power-counting rules by equating Δv^2 to $\Lambda_{\rm QCD}/m_c$ times the central value and by equating $\Delta\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ and $\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ to 100% of the central value.

C. Numerical results

1. Computations using $\Gamma[J/\psi \to e^+e^-]$ and $\Gamma[\eta_c \to \gamma\gamma]$

The results of our computations of matrix elements from $\Gamma[J/\psi \to e^+e^-]$ and $\Gamma[\eta_c \to \gamma\gamma]$ are shown in Tables I and II, respectively. In each table, in the first row below the headings, we give the central values for the potential-model parameter λ, the matrix element $\langle \mathcal{O}_1 \rangle_H$, and the ratio $\langle \boldsymbol{q}^2 \rangle_H$. Subsequent rows contain the values for λ, the matrix element, and the ratio that result from shifting each uncertain quantity in the calculation by plus or minus its uncertainty. We put a superscript $\gamma\gamma$ on the matrix element and the ratio for the η_c that are shown in Table II, in order to specify that these numbers are the result of a fit to $\Gamma[\eta_c \to \gamma\gamma]$.

TABLE I. The potential-model parameter λ, the NRQCD matrix element $\langle \mathcal{O}_1 \rangle_{J/\psi}$, and the ratio $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$, as obtained from $\Gamma[J/\psi \to e^+e^-]$. The first row below the headings contains the central values for λ, the matrix element, and the ratio. Subsequent rows contain the maximum and minimum values for these quantities that are obtained by varying them with respect to each uncertainty.

Case	λ	$\langle \mathcal{O}_1 \rangle_{J/\psi}$ (GeV^3)	$\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ (GeV^2)
central	1.243	0.440	0.441
$+\Delta\langle \boldsymbol{q}^2 \rangle_{J/\psi}$	1.256	0.450	0.573
$-\Delta\langle \boldsymbol{q}^2 \rangle_{J/\psi}$	1.230	0.430	0.308
$+\Delta m_c$	1.233	0.433	0.443
$-\Delta m_c$	1.258	0.451	0.437
$+\Delta\sigma$	1.191	0.443	0.482
$-\Delta\sigma$	1.297	0.437	0.400
$+\Delta\mathrm{NNLO}_{J/\psi}$	1.325	0.504	0.419
$-\Delta\mathrm{NNLO}_{J/\psi}$	1.166	0.387	0.459
$+\Delta\Gamma_{J/\psi}$	1.258	0.451	0.437
$-\Delta\Gamma_{J/\psi}$	1.228	0.429	0.444

The matrix elements and ratios of matrix elements, along with their uncertainties, are as follows:

$$\langle \mathcal{O}_1 \rangle_{J/\psi} = 0.440^{+0.009+0.011+0.003+0.064+0.011}_{-0.010-0.008-0.003-0.053-0.011}\ \mathrm{GeV}^3 = 0.440^{+0.067}_{-0.055}\ \mathrm{GeV}^3, \quad (50a)$$

$$\langle \boldsymbol{q}^2 \rangle_{J/\psi} = 0.441^{+0.132+0.003+0.041+0.018+0.004}_{-0.132-0.004-0.040-0.022-0.004}\ \mathrm{GeV}^2 = 0.441^{+0.140}_{-0.140}\ \mathrm{GeV}^2, \quad (50b)$$

TABLE II. The potential-model parameter λ, the NRQCD matrix element $\langle \mathcal{O}_1 \rangle_{\eta_c}$, and the ratio $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$, as obtained from $\Gamma[\eta_c \to \gamma\gamma]$. The first row below the headings contains the central values for λ, the matrix element, and the ratio. Subsequent rows contain the maximum and minimum values for these quantities that are obtained by varying them with respect to each uncertainty.

Case	λ	$\langle \mathcal{O}_1 \rangle^{\gamma\gamma}_{\eta_c}$ (GeV^3)	$\langle \boldsymbol{q}^2 \rangle^{\gamma\gamma}_{\eta_c}$ (GeV^2)
central	1.234	0.434	0.443
$+\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$	1.291	0.476	0.576
$-\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$	1.175	0.393	0.310
$+\Delta m_c$	1.340	0.517	0.415
$-\Delta m_c$	1.129	0.364	0.467
$+\Delta\sigma$	1.195	0.446	0.481
$-\Delta\sigma$	1.276	0.422	0.406
$+\Delta\mathrm{NNLO}_{\eta_c}$	1.340	0.517	0.415
$-\Delta\mathrm{NNLO}_{\eta_c}$	1.134	0.368	0.466
$+\Delta\Gamma_{\eta_c}$	1.374	0.546	0.405
$-\Delta\Gamma_{\eta_c}$	1.041	0.315	0.484

$$\langle \mathcal{O}_1 \rangle_{\eta_c}^{\gamma\gamma} = 0.434^{+0.042+0.083+0.012+0.083+0.112}_{-0.040-0.069-0.012-0.066-0.118}\ \mathrm{GeV}^3 = 0.434^{+0.169}_{-0.158}\ \mathrm{GeV}^3, \quad (51a)$$

$$\langle \boldsymbol{q}^2 \rangle_{\eta_c}^{\gamma\gamma} = 0.443^{+0.133+0.024+0.038+0.023+0.041}_{-0.133-0.028-0.037-0.028-0.038}\ \mathrm{GeV}^2 = 0.443^{+0.148}_{-0.149}\ \mathrm{GeV}^2. \quad (51b)$$

In the first equalities in Eqs. (50) and (51), the uncertainties are presented in the same order as in Tables I and II. In the last equalities in each of these equations, we have added the uncertainties in quadrature. However it must be kept in mind for many applications that the individual uncertainties are correlated between the matrix elements. The correlations can be determined from the tabulations in Tables I and II.

From Eqs. (10), (50), and (51) and the uncertainty in m_c in Eq. (45), it can be deduced that

$$\langle v^2 \rangle_{J/\psi} = 0.225^{+0.106}_{-0.088}, \quad (52a)$$

$$\langle v^2 \rangle_{\eta_c}^{\gamma\gamma} = 0.226^{+0.123}_{-0.098}. \quad (52b)$$

The central values of these results are somewhat smaller than an estimate, based on the NRQCD velocity-scaling rules [1], that $\langle v^2 \rangle$ should be equal approximately to $v^2 \approx 0.3$. However, they are consistent with being of order v^2.

We can see the effect of resummation by repeating our analysis, but keeping only the order-v^2 corrections in the formulas for the decay rates in Eqs. (22) and (31). The results are that the central values are shifted to $\langle \mathcal{O}_1 \rangle_{J/\psi} = 0.446\,269\ \mathrm{GeV}^3$, $\langle \boldsymbol{q}^2 \rangle_{J/\psi} = 0.438\,520\ \mathrm{GeV}^2$, $\langle \mathcal{O}_1 \rangle_{\eta_c}^{\gamma\gamma} = 0.459\,867\ \mathrm{GeV}^3$, and $\langle \boldsymbol{q}^2 \rangle_{\eta_c}^{\gamma\gamma} = 0.433\,879\ \mathrm{GeV}^2$. Hence, the effects of the resummation on these quantities are -1.4%, $+0.5\%$, -5.7%, and $+2.1\%$, respectively. The small effects from resummation suggest that the v expansion of NRQCD converges well for the widths $\Gamma[J/\psi \to e^+e^-]$ and $\Gamma[\eta_c \to \gamma\gamma]$.

2. Average values of η_c matrix elements

Because some of the uncertainties in Tables I and II are correlated, we must take care in combining the results in these tables to obtain average values for the η_c matrix element and the η_c ratio of matrix elements. First, we construct a two-by-two covariance matrix for the quantities $\langle \mathcal{O}_1 \rangle_{J/\psi}$ and $\langle \mathcal{O}_1 \rangle_{\eta_c}^{\gamma\gamma}$ from the deviations from the central values that correspond to the uncertainties listed in Tables I and II. Then, we use the inverse of the covariance matrix to construct χ^2 for the deviation of the average value of $\langle \mathcal{O}_1 \rangle_{\eta_c}$ from the two input values. We fix the average value of $\langle \mathcal{O}_1 \rangle_{\eta_c}$ by minimizing this χ^2 with respect to it. The minimum value of χ^2 is 8.9×10^{-4}. This small value of χ^2 reflects the fact that $\langle \mathcal{O}_1 \rangle_{J/\psi}$ and $\langle \mathcal{O}_1 \rangle_{\eta_c}^{\gamma\gamma}$ are much closer in value than one would expect from the velocity-scaling rules of NRQCD. Once we have obtained the average value of $\langle \mathcal{O}_1 \rangle_{\eta_c}$, we use it as an input to the potential model to compute the average value of $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$. We carry out this computation of the average values of $\langle \mathcal{O}_1 \rangle_{\eta_c}$ and $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ for values of the input parameters that correspond to each of the uncertainties that we have described. (The effect of the uncertainty $\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ on the average value of $\langle \mathcal{O}_1 \rangle_{\eta_c}$ has already been taken into account through the inputs to that average. We obtain the effect of $\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ on the average value of $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ by varying the central value of the average value of $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ by $v^2 \approx 30\%$.) The average values of $\langle \mathcal{O}_1 \rangle_{\eta_c}$ and $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ that result from these computations are shown in Table III. The first row after the headings in Table III gives the central values of the averages of the η_c matrix element $\langle \mathcal{O}_1 \rangle_{\eta_c}$ and ratio of η_c matrix elements $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$. Subsequent rows show the effects of the various uncertainties on the average values. The central values and uncertainties in Table III can be summarized as follows:

TABLE III. Average values of the NRQCD matrix element $\langle \mathcal{O}_1 \rangle_{\eta_c}$ and the ratio $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$. The method of averaging is described in the text. The first row below the headings contains the central values for the matrix element and the ratio. Subsequent rows contain the maximum and minimum values for these quantities that are obtained by varying them with respect to each uncertainty.

Case	$\langle \mathcal{O}_1 \rangle_{\eta_c}$ (GeV^3)	$\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ (GeV^2)
central	0.437	0.442
$+\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$	0.461	0.574
$-\Delta\langle \boldsymbol{q}^2 \rangle_{\eta_c}$	0.414	0.309
$+\Delta m_c$	0.470	0.430
$-\Delta m_c$	0.413	0.450
$+\Delta\sigma$	0.444	0.482
$-\Delta\sigma$	0.431	0.403
$+\Delta\mathrm{NNLO}_{J/\psi}$	0.473	0.429
$-\Delta\mathrm{NNLO}_{J/\psi}$	0.408	0.452
$+\Delta\Gamma_{J/\psi}$	0.443	0.440
$-\Delta\Gamma_{J/\psi}$	0.431	0.444
$+\Delta v^2$	0.511	0.417
$-\Delta v^2$	0.364	0.467
$+\Delta\mathrm{NNLO}_{\eta_c}$	0.474	0.429
$-\Delta\mathrm{NNLO}_{\eta_c}$	0.408	0.452
$+\Delta\Gamma_{\eta_c}$	0.487	0.425
$-\Delta\Gamma_{\eta_c}$	0.385	0.460

$$\langle \mathcal{O}_1 \rangle_{\eta_c} = 0.437^{+0.024+0.033+0.007+0.036+0.006+0.073+0.037+0.050}_{-0.023-0.025-0.007-0.029-0.006-0.073-0.029-0.053}\ \mathrm{GeV}^3 = 0.437^{+0.111}_{-0.105}\ \mathrm{GeV}^3, \quad (53a)$$

$$\langle \boldsymbol{q}^2 \rangle_{\eta_c} = 0.442^{+0.132+0.009+0.040+0.010+0.002+0.026+0.010+0.018}_{-0.132-0.011-0.039-0.012-0.002-0.025-0.013-0.017}\ \mathrm{GeV}^2 = 0.442^{+0.143}_{-0.143}\ \mathrm{GeV}^2. \quad (53b)$$

In the first equalities in Eq. (53), the uncertainties are presented in the same order as in Table III. In the last equalities, we have added the uncertainties in quadrature. As we have mentioned, it must be kept in mind for many applications that the

individual uncertainties are correlated between the matrix elements. The correlations can be determined from the tabulations in Table III.

The correlated errors can also be expressed conveniently in terms of a correlation matrix. We construct a (symmetric) correlation matrix whose rows and columns correspond to $\langle \mathcal{O}_1 \rangle_{J/\psi}$, $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$, $\langle \mathcal{O}_1 \rangle_{\eta_c}$, and $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$, respectively, taking the deviations from the central values from Tables I and III. The result is

$$C_1 = \begin{pmatrix} 3.71 \times 10^{-3} & 1.64 \times 10^{-4} & 1.94 \times 10^{-3} & 8.18 \times 10^{-4} \\ 1.64 \times 10^{-4} & 1.96 \times 10^{-2} & 2.84 \times 10^{-3} & 1.93 \times 10^{-2} \\ 1.94 \times 10^{-3} & 2.84 \times 10^{-3} & 1.16 \times 10^{-2} & -3.71 \times 10^{-4} \\ 8.18 \times 10^{-4} & 1.93 \times 10^{-2} & -3.71 \times 10^{-4} & 2.04 \times 10^{-2} \end{pmatrix}, \tag{54}$$

where the quantity in ith row and jth column is expressed in units of $\mathrm{GeV}^{n_i+n_j}$, with $n_1 = n_3 = 3$ and $n_2 = n_4 = 2$. In charmonium decay and production processes, the NRQCD short-distance coefficients typically depend on m_c. Hence, there may be correlations between the matrix elements and short-distance coefficients with respect to the uncertainty in m_c. Therefore, we also give the correlation matrix for $\langle \mathcal{O}_1 \rangle_{J/\psi}$, $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$, $\langle \mathcal{O}_1 \rangle_{\eta_c}$, and $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$, respectively, in which we omit the uncertainties that arise from m_c:

$$C_2 = \begin{pmatrix} 3.63 \times 10^{-3} & 1.93 \times 10^{-4} & 2.21 \times 10^{-3} & 7.27 \times 10^{-4} \\ 1.93 \times 10^{-4} & 1.96 \times 10^{-2} & 2.75 \times 10^{-3} & 1.94 \times 10^{-2} \\ 2.21 \times 10^{-3} & 2.75 \times 10^{-3} & 1.08 \times 10^{-2} & -8.86 \times 10^{-5} \\ 7.27 \times 10^{-4} & 1.94 \times 10^{-2} & -8.86 \times 10^{-5} & 2.03 \times 10^{-2} \end{pmatrix}, \tag{55}$$

where the dimensions of the elements of C_2 are the same as those of the corresponding elements of C_1 in Eq. (54). We note that both correlation matrices C_1 and C_2 contain large off-diagonal elements that correspond to a correlation between the uncertainty in $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ and the uncertainty in $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$. Most of this correlation arises from the uncertainty in the string tension σ.

VI. COMPARISONS WITH PREVIOUS CALCULATIONS

Our results for the matrix elements can be compared with those in Ref. [10]. In that paper, the values $\langle \mathcal{O}_1 \rangle^{\mathrm{BL}}_{J/\psi} = 0.335 \pm 0.024\ \mathrm{GeV}^3$ and $\langle \mathcal{O}_1 \rangle^{\mathrm{BL}}_{\eta_c} = 0.297 \pm 0.032\ \mathrm{GeV}^3$ are given. In the case of $\langle \mathcal{O}_1 \rangle_{J/\psi}$, our result is 31% larger than that in Ref. [10]. Approximately 6% of that change is the result of the change in the experimental value of $\Gamma[J/\psi \to e^+e^-]$ from 5.26 ± 0.37 keV [31] to $5.55 \pm 0.14 \pm 0.02$ keV [18]. An implicit relativistic correction of about 22% arises from the use of $m_{J/\psi}$ in Eq. (22), rather than $2m_c$. The use of $\alpha(m_{J/\psi}) = 1/132.6$ [Eq. (49a)], rather than $\alpha = 1/137$, decreases $\langle \mathcal{O}_1 \rangle_{J/\psi}$ by approximately 6%. The remaining change of about 9% is the result of including the explicit relativistic corrections in Eq. (22). In the case of $\langle \mathcal{O}_1 \rangle^{\gamma\gamma}_{\eta_c}$, our result in Eq. (51) is 46% larger than the value $\langle \mathcal{O}_1 \rangle^{\mathrm{BL}}_{\eta_c} = 0.297 \pm 0.032\ \mathrm{GeV}^3$ that is given in Ref. [10]. In this case, there is a decrease in the value of $\langle \mathcal{O}_1 \rangle^{\gamma\gamma}_{\eta_c}$ of 4%, owing to the change in the experimental value of $\Gamma[\eta_c \to \gamma\gamma]$ from 7.5 ± 0.8 keV [31] to $7.2 \pm 0.7 \pm 2.0$ keV [18]. The use of $\alpha(m_{\eta_c}/2) = 1/133.6$ [Eq. (49b)], rather than $\alpha = 1/137$, decreases $\langle \mathcal{O}_1 \rangle^{\gamma\gamma}_{\eta_c}$ by approximately 5%. The use of $\alpha_s(m_{\eta_c}/2) = 0.35$ [Eq. (48b)], rather than $\alpha_s(m_{J/\psi}) = 0.25$ [Eq. (48a)], which is used for the process $\eta_c \to \gamma\gamma$ in Ref. [10], enhances the matrix element by approximately 14%. The remaining change of about 41% arises from the relativistic corrections in Eq. (31).

In Ref. [10], the values $\langle \boldsymbol{q}^2 \rangle_{J/\psi} = 0.43\ \mathrm{GeV}^2$ and $\langle \boldsymbol{q}^2 \rangle^{\gamma\gamma}_{\eta_c} = 0.25\ \mathrm{GeV}^2$ were obtained by making use of the Gremm-Kapustin [3] relation for the physical quarkonium mass and m_c. While these results are not far from those in Eqs. (50) and (51), the uncertainties given in Ref. [10] are on the order 100%, owing to the uncertainty in m_c. In our calculation, we have been able to reduce the uncertainties significantly by making use of the Gremm-Kapustin relation (11) for the binding energy in the potential model. This leads to much smaller uncertainties than the use of the Gremm-Kapustin relation for the physical quarkonium mass and m_c because we compute the binding energy directly in the potential model, instead of expressing it as a difference between m_H and $2m_c$.

In Ref. [2], the result $\langle \boldsymbol{q}^2 \rangle_{J/\psi} = 0.50 \pm 0.09 \pm 0.15\ \mathrm{GeV}^2$ was obtained from a potential-model calculation, which also made use of the Cornell potential. That result agrees, within errors, with the result in Eq. (50). In Ref. [2], the value of the matrix element $\langle \mathcal{O}_1 \rangle_{J/\psi}$ was taken from Ref. [10], in which the relativistic correction to $\Gamma[J/\psi \to e^+e^-]$ was not taken into account. The inclusion of that correction in the present work, along with a more precise determination of the potential-model parameter λ, accounts for the difference in the value of $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ between Ref. [2] and Eq. (50).

We can also compare our results with those in Ref. [32]. In that work, the following values are reported:

$\langle \mathcal{O}_1 \rangle^{\rm HFC}_{J/\psi} = 0.573\ {\rm GeV}^3$, $\langle \mathcal{O}_1 \rangle^{\rm HFC}_{\eta_c} = 0.432\ {\rm GeV}^3$, and $\langle \mathcal{P}_1 \rangle^{\rm HFC}_{J/\psi}/m_c^2 = \langle \mathcal{P}_1 \rangle^{\rm HFC}_{\eta_c}/m_c^2 = 0.0514\ {\rm GeV}^3$. These values were obtained by comparing the theoretical formulas for $\Gamma[J/\psi \to e^+e^-]$, $\Gamma[\eta_c \to \gamma\gamma]$, and $\Gamma[J/\psi \to$ light hadrons] with the experimental results and by assuming that $\langle \mathcal{P}_1 \rangle^{\rm HFC}_{\eta_c} = \langle \mathcal{P}_1 \rangle^{\rm HFC}_{J/\psi}$. In the case of $\Gamma[J/\psi \to$ light hadrons], processes involving an intermediate virtual photon were excluded in both the theoretical formula and the experimental rate. The theoretical expressions that were used in Ref. [32] to obtain these results contain the QCD corrections of relative order α_s and the relativistic corrections of order v^2. Taking $m_c = 1.5$ GeV, which is the value that is used in Ref. [32], we find that the results given in Ref. [32] yield $\langle \boldsymbol{q}^2 \rangle^{\rm HFC}_{J/\psi} = 0.202\ {\rm GeV}^2$ and $\langle \boldsymbol{q}^2 \rangle^{\rm HFC}_{\eta_c} = 0.268\ {\rm GeV}^2$. These values are considerably below the values in Eqs. (50) and (51) and considerably below the expectations from the velocity-scaling rules of NRQCD. The small values of $\langle \boldsymbol{q}^2 \rangle^{\rm HFC}_{J/\psi}$ and $\langle \boldsymbol{q}^2 \rangle^{\rm HFC}_{\eta_c}$ are traceable to the use of the theoretical expression for $\Gamma[J/\psi \to$ light hadrons]. In that expression, the coefficient of the contribution that is proportional to $\langle \boldsymbol{q}^2 \rangle_{J/\psi}/m_c^2$ is about -5.32 relative to the leading contribution. Because of this large negative coefficient, the quantity $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ must be much less than the values that we obtain in order for the decay width to be positive. We regard this as an indication that the v expansion is not reliable for the rate $\Gamma[J/\psi \to$ light hadrons]. It is possible that the resummation methods that we have used in the present work could be used to tame the v-expansion for $\Gamma[J/\psi \to$ light hadrons]. The value of $\langle \mathcal{O}_1 \rangle^{\rm HFC}_{J/\psi}$ is about 30% larger than the value in Eq. (50) while the value of $\langle \mathcal{O}_1 \rangle^{\rm HFC}_{\eta_c}$ is about 1% smaller than the value in Eq. (51). Some of this difference is accounted for by the smaller values of $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ and $\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ in Ref. [32]. Reference [32] also makes use of slightly different values of m_c (1.5 GeV) and α_s (0.26) than those employed in the present work. A further difference is that the expressions for $\Gamma[J/\psi \to e^+e^-]$ and $\Gamma[\eta_c \to \gamma\gamma]$ in Ref. [32] are expanded to first order in α_s and v^2, rather than expressed as exact squares of amplitudes, as in Eqs. (22) and (31).

Finally, there are quenched lattice computations [28] of the ground-state S-wave charmonium matrix elements that yield $\langle \mathcal{O}_1 \rangle_{J/\psi\text{-}\eta_c} = 0.3312 \pm 0.0006 \pm 0.0030^{+0.0681}_{-0.0483}\ {\rm GeV}^3$ and $\langle \boldsymbol{q}^2 \rangle_{J/\psi\text{-}\eta_c} = 0.07$–$0.82\ {\rm GeV}^2$. In $\langle \mathcal{O}_1 \rangle_{J/\psi\text{-}\eta_c}$, the first error bar is from lattice statistics, the second error bar is from lattice systematics, and the third error bar is from the uncertainty in the one-loop perturbative computation that relates the lattice-regulated matrix elements to the continuum $\overline{\rm MS}$ matrix elements. The lattice computations do not distinguish between the J/ψ state and the η_c state. The lattice results are in agreement with our results, within uncertainties, but the lattice uncertainties are much larger than ours. These large uncertainties arise from the uncertainty in the perturbative conversion from lattice to continuum $\overline{\rm MS}$ matrix elements.

VII. SUMMARY

For many S-wave heavy-quarkonium decay and production processes, the color-singlet S-wave NRQCD matrix elements of leading order in v enter into the dominant theoretical contribution. The first relativistic corrections to these processes involve the matrix elements of relative order v^2.

We have computed the color-singlet S-wave NRQCD matrix elements of leading order and next-to-leading order in v^2 for the J/ψ and the η_c. For each of these quarkonium states, we have determined the values of these matrix elements by comparing the theoretical expressions for the electromagnetic decay rates ($\Gamma[J/\psi \to e^+e^-]$ or $\Gamma[\eta_c \to \gamma\gamma]$) with the experimental measurements and by using a potential model to compute the matrix elements of relative order v^2. If the static, spin-independent $Q\bar{Q}$ potential were known exactly, then the potential-model calculation would be accurate up to corrections of relative order v^2. We made use of the Cornell potential and fixed its parameters by using as inputs the lattice measurements of the string tension, the J/ψ-$\psi(2S)$ mass splitting, and the quarkonium wave function at the origin, which corresponds to the NRQCD matrix element of leading order in v. Because the potential-model calculation of the order-v^2 matrix element depends on the leading-order NRQCD matrix element and the decay widths depend on both of these matrix elements, we obtained the matrix elements for the J/ψ and the η_c by solving, in each case, two coupled nonlinear equations.

In the theoretical expressions for the electromagnetic decay widths, we made use of the generalized Gremm-Kapustin relation (11) (Ref. [2]) to resum a class of relativistic corrections. This resummation includes all of the relativistic corrections that are contained in the leading-potential approximation to the quarkonium $Q\bar{Q}$ color-singlet wave function, up to the ultraviolet cutoff of the NRQCD matrix elements.

There are many sources of uncertainties in our calculation. Some of these are correlated among the matrix elements. Therefore, we reported the variations of the matrix elements with respect to each source of uncertainty.

The experimental measurement of the width $\Gamma[\eta_c \to \gamma\gamma]$ has relatively large uncertainties, which translate into large uncertainties in the η_c matrix elements. Owing to the heavy-quark spin symmetry [1], the J/ψ and η_c matrix elements are equal, up to corrections of relative order v^2. Therefore, we were able to reduce the uncertainties in the η_c matrix elements by averaging the values that we obtained from $\Gamma[\eta_c \to \gamma\gamma]$ plus the potential model with the values that we obtained from $\Gamma[J/\psi \to e^+e^-]$ plus the potential model. In performing this average, we took into account the additional uncertainty of relative

order v^2 that arises from equating η_c matrix elements to J/ψ matrix elements.

Our principal results are given in Tables I, II, and III and are summarized in Eqs. (50), (51), and (53) for the matrix elements that were determined from $\Gamma[J/\psi \to e^+e^-]$, the matrix elements that were determined from $\Gamma[\eta_c \to \gamma\gamma]$, and the average of the two, respectively. We consider the results in Table III and Eq. (53) to be our best values for the η_c matrix elements. In applying these results to calculations of quarkonium decay and production rates, it should be kept in mind that the uncertainties are highly correlated between matrix elements and that there are correlations between matrix elements and short-distance coefficients with respect to the uncertainties in m_c. Therefore, it may be necessary to use all of the information that is contained in Tables I, II, and III, rather than to rely on the summaries in Eqs. (50), (51), and (53).

Our results in Tables I and II and Eqs. (50) and (51) conform to the expectation, from the heavy-quark spin symmetry, that the J/ψ and η_c matrix elements are equal, up to corrections of relative order $v^2 \approx 30\%$. In fact, the leading-order matrix elements differ by about 1.5%, while $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ and $\langle \boldsymbol{q}^2 \rangle_{\eta_c}^{\gamma\gamma}$ differ by only about 0.5%. The velocity-scaling rules of NRQCD [1] state that the quantities $\langle v^2 \rangle_H = \langle \boldsymbol{q}^2 \rangle_H / m_c^2$ should be of order $v^2 \approx 0.3$. From Eq. (52), it can be seen that our results satisfy this expectation, although they are somewhat smaller than the nominal value of v^2.

As we have discussed in Sec. V C 1, the effects from resummation on our results are small, ranging from -5.7% for $\langle \mathcal{O}_1 \rangle_{\eta_c}^{\gamma\gamma}$ to 2.1% for $\langle \boldsymbol{q}^2 \rangle_{\eta_c}^{\gamma\gamma}$. The small effects from resummation suggest that the v expansion of NRQCD converges well for the widths $\Gamma[J/\psi \to e^+e^-]$ and $\Gamma[\eta_c \to \gamma\gamma]$.

Our results for $\langle \mathcal{O}_1 \rangle_{J/\psi}$ and $\langle \mathcal{O}_1 \rangle_{\eta_c}^{\gamma\gamma}$ are considerably larger than those in Ref. [10], primarily because we have included relativistic corrections to the electromagnetic decay rates in the present work. The changes in the values of these matrix elements would significantly increase the rate for the process $e^+e^- \to J/\psi + \eta_c$ that is calculated in Ref. [10].

Our result for $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ agrees, within uncertainties, with that in Ref. [2], but is slightly smaller. Most of this difference arises from the fact that, in Ref. [2], the value of $\langle \mathcal{O}_1 \rangle_{J/\psi}$ was taken from Ref. [10].

In Ref. [32], a much smaller value for $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ was reported. ($\langle \boldsymbol{q}^2 \rangle_{\eta_c}$ was assumed to be equal to $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ in this work.) The smallness of $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ in Ref. [32] can be traced to the use of the width $\Gamma[J/\psi \to \text{light hadrons}]$ to constrain the matrix elements. The theoretical expression for that width contains large order-v^2 corrections that, in our opinion, make the reliability of the expression suspect. It is possible that the resummation technique that we have employed in this paper could be used to bring the v expansion for $\Gamma[J/\psi \to \text{light hadrons}]$ under control.

Our results for $\langle \mathcal{O}_1 \rangle_{J/\psi}$ and $\langle \boldsymbol{q}^2 \rangle_{J/\psi}$ are in agreement with those from lattice calculations [28], although the lattice uncertainties are much larger than ours.

We believe that the values that we have obtained for the J/ψ and η_c color-singlet NRQCD matrix elements are the most precise ones that are available to date. The new values for the matrix elements of leading order in v should have a significant impact on the calculations of a number of charmonium decay and production processes [33]. For quite a few charmonium processes, it is clear that relativistic corrections are important. Within the framework of NRQCD, the matrix elements of order v^2 are essential ingredients in calculating those corrections. We have also attempted to quantify all of the significant theoretical uncertainties in our determination of the J/ψ and η_c color-singlet matrix elements. Our treatment of uncertainties could provide the basis for more reliable estimates of theoretical uncertainties in future calculations of charmonium decay and production rates.

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