

## Note on a sigma model connection with instanton dynamics

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It is well known that the instanton approach to QCD generates an effective term which looks like a three-flavor determinant of quark bilinears. This has the right behavior to explain the unusual mass and mixing of the  $\eta(958)$  meson, as is often simply illustrated with the aid of a linear SU(3) sigma model. It is less well known that the instanton analysis generates another term which has the same transformation property but does not have a simple interpretation in terms of this usual linear sigma model. Here we point out that this term has an interpretation in a generalized linear sigma model containing two chiral nonets. The second chiral nonet is taken to correspond to mesons having two quarks and two antiquarks in their makeup. The generalized model seems to be useful for learning about the spectrum of low-lying scalar mesons which have been emerging in the last few years. The physics of the new term is shown to be related to the properties of an excited  $\eta'$  state present in the generalized model and for which there are some experimental candidates.

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### I. INTRODUCTION

The instanton approach to QCD (see Refs. [1–9] for some of the many interesting references) has played an important role in understanding the origin of the  $U(1)_A$  violation in that theory. Specifically, 't Hooft showed [2] that a new quark term arises which conserves the  $SU(N_f)_L \times SU(N_f)_R$  symmetry, where  $N_f$  denotes the number of relevant low-energy quark flavors, but violates  $U(1)_A$ . In the case  $N_f = 2$ , the  $U(1)_A$  violating term is a  $2 \times 2$  determinant of quark bilinears. If this is generalized to  $N_f = 3$ , the resulting  $3 \times 3$  determinant has the right transformation properties to explain the unusually high mass as well as the mixing pattern of the puzzling pseudoscalar meson,  $\eta'(958)$ . The relevant calculation is often performed using an “effective low energy” linear SU(3) sigma model containing both a pseudoscalar nonet as well as an additional scalar nonet. Such a model gives the usual “precision” current algebra results for the pion (and to some extent the kaon) interactions and an acceptable description of the  $\eta'(958)$ . It also contains information about the scalars although they are often “integrated out.” That procedure converts the model to a nonlinear sigma model.

It is amusing to note [4] that in the  $N_f = 3$  case, the instanton calculation gives not only the determinant type  $U(1)_A$  violation term but also another  $U(1)_A$  violation term of nondeterminant type. That term will be of interest in the present paper. We have been studying a generalized linear sigma model [10–15] containing two chiral nonets which are allowed to mix with each other. Related models for

thermodynamic properties of QCD are discussed in Refs. [16]. The underlying motivation arises from the increasing likelihood [17] of the existence of light scalar mesons which show up, for instance, in the analysis of pion-pion scattering data. Note that, at present, the scalars below 1 GeV appear to fit into a nonet as

$$\begin{aligned}
 I = 0: m[f_0(600)] &\approx 500 \text{ MeV} \\
 I = 1/2: m[\kappa] &\approx 800 \text{ MeV} \\
 I = 0: m[f_0(980)] &\approx 980 \text{ MeV} \\
 I = 1: m[a_0(980)] &\approx 980 \text{ MeV.}
 \end{aligned} \tag{1}$$

This level ordering is seen to be flipped compared to that of the standard vector meson nonet. It was pointed out a long time ago in Ref. [18], that the level order is automatically flipped when mesons are made of two quarks and two antiquarks instead of a single quark and antiquark. That argument was given for a diquark-antidiquark structure but is easily seen to also hold for a meson-meson, “molecule”-type structure which was advocated, at least for a partial nonet, in Ref. [19]. Thus, on empirical grounds a four-quark structure for the light scalars seems plausible. Of course, one expects higher mass scalars related to p-wave quark-antiquark composites to also exist. It is natural to expect mixing between states with the same quantum numbers, and there is some phenomenological evidence for this as noted in Refs [20,21]. Thus, it seems reasonable to construct a generalized linear sigma model containing a chiral “four quark” nonet as well as the usual chiral “two quark” nonet. The study of such a model in fact yields a plausible explanation of the main experimental facts. Of relevance to the instanton physics is that the two nonets are

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distinguished from each other by having different  $U(1)_A$  transformation properties. Furthermore, the treatment of the model in [13–15] brings in an additional  $U(1)_A$  violation term which seems to have the same structure as the additional term arising from the instanton analysis [4].

In Sec. II, we repeat, for the reader’s convenience, the notation [10] being used for schematic quark field combinations transforming like chiral nonets with quark, anti-quark and various two-quark, two-antiquark structures.

In Sec. III, we demonstrate that the schematic molecule-type chiral nonet can be written as a linear combination of two diquark, antidiquark type nonets (which have different  $SU(3)$  color representations for the diquarks). All of these “four quark” configurations have the same  $U(1)_A$  transformation property. Here it will be sufficient to assume that an unspecified “four quark” configuration is bound.

In Sec. IV, we give a brief outline of the linear sigma model containing both a “two quark” chiral nonet and a “four quark” chiral nonet. It is convenient to introduce the  $U(1)_A$  violation in such a way that the classical Lagrangian mocks up the anomaly exactly. This leads to  $\ln$ ’s of the violation operators. It has the advantage that the  $\eta$ ’s essentially decouple from the rest of the particles. In order to compare with the instanton analysis, we thus calculate the leading terms which are linear in the violation operators.

In Sec. V, we quote the known three-flavor effective quark Lagrangian arising from the instanton analysis. We rewrite it using Fierz transformations so that the desired “four quark” fields become manifest. They are presented as a linear combination of a molecule-type field and a field made from a color  $\bar{3}$  diquark combined with its corresponding antiquark.

In Sec. VI, we compare the relative strengths of the two  $U(1)_A$  violation terms as obtained from the instanton analysis to the ones obtained from the generalized linear sigma model. The linear sigma model relative term strengths are obtained from comparing the properties of the  $\eta(958)$  with those of the apparently best candidate to be its partner, the  $\eta(1475)$ . To convert the quark instanton Lagrangian to one involving only mesons we need a way to characterize our ignorance of the quark wave functions inside the meson states. This is done via a parameter denoted  $\omega$ , which is estimated.

Section VII contains some concluding remarks.

## II. NOTATION

Even though one cannot write down the exact QCD wave functions of the low-lying mesons it is easy to write down schematic descriptions of how quark fields may combine to give particles with specified transformation properties. For spinor notations we employ the Pauli conventions. We work in a representation where the  $\gamma$  matrices and the charge conjugation matrix have the form:

$$\begin{aligned} \gamma_i &= \begin{bmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{bmatrix}, & \gamma_4 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \gamma_5 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, & C &= \begin{bmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}. \end{aligned} \quad (2)$$

A nonet  $M(x)$  realizing the  $q\bar{q}$  structure can be written as

$$M_a^b = (q_{bA})^\dagger \gamma_4 \frac{1 + \gamma_5}{2} q_{aA}, \quad (3)$$

where  $a$  and  $A$  are, respectively, flavor and color indices. Our convention for matrix notation is  $M_a^b \rightarrow M_{ab}$ . Then  $M$  transforms under chiral  $SU(3)_L \times SU(3)_R$ , charge conjugation  $C$  and parity  $P$  as

$$\begin{aligned} M &\rightarrow U_L M U_R^\dagger & C: M &\rightarrow M^T, \\ P: M(\mathbf{x}) &\rightarrow M^\dagger(-\mathbf{x}). \end{aligned} \quad (4)$$

Here  $U_L$  and  $U_R$  are unitary, unimodular matrices associated with the transformations on the left-handed ( $q_L = \frac{1}{2}(1 + \gamma_5)q$ ) and right-handed ( $q_R = \frac{1}{2}(1 - \gamma_5)q$ ) quark projections. For the  $U(1)_A$  transformation one has

$$M \rightarrow e^{2i\nu} M. \quad (5)$$

Next consider nonets with “four quark,”  $qq\bar{q}\bar{q}$  structures. One possibility is that the four-quark states are “molecules” made out of two quark-antiquark states. This leads to the following schematic form:

$$M_a^{(2)b} = \epsilon_{acd} \epsilon^{bef} (M^\dagger)_e^c (M^\dagger)_f^d. \quad (6)$$

Another possibility is that the four-quark states may be bound states of a diquark and an antiquark. There are two choices if the diquark is required to belong to a  $\bar{3}$  representation of flavor  $SU(3)$ . In the first case it belongs to a  $\bar{3}$  of color and is a spin singlet with the structure

$$\begin{aligned} L^{gE} &= \epsilon^{gab} \epsilon^{EAB} q_{aA}^T C^{-1} \frac{1 + \gamma_5}{2} q_{bB}, \\ R^{gE} &= \epsilon^{gab} \epsilon^{EAB} q_{aA}^T C^{-1} \frac{1 - \gamma_5}{2} q_{bB}. \end{aligned} \quad (7)$$

Then the matrix  $M$  has the form

$$M_g^{(3)f} = (L^{gA})^\dagger R^{fA}. \quad (8)$$

In the second case the diquark belongs to a 6 representation of color and has spin 1. It has the schematic chiral realization

$$L_{\mu\nu,AB}^g = L_{\mu\nu,BA}^g = \epsilon^{gab} q_{aA}^T C^{-1} \sigma_{\mu\nu} \frac{1 + \gamma_5}{2} q_{bB}, \quad (9)$$

$$R_{\mu\nu,AB}^g = R_{\mu\nu,BA}^g = \epsilon^{gab} q_{aA}^T C^{-1} \sigma_{\mu\nu} \frac{1 - \gamma_5}{2} q_{bB},$$

where  $\sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu]$ . The corresponding  $M$  matrix has the form

$$M_g^{(4)f} = (L_{\mu\nu,AB}^g)^\dagger R_{\mu\nu,AB}^f, \quad (10)$$

where the dagger operation includes a factor  $(-1)^{\delta_{\mu 4} + \delta_{\nu 4}}$ . The nonets  $M^{(2)}$ ,  $M^{(3)}$  and  $M^{(4)}$  transform like  $M$  under all of  $SU(3)_L \times SU(3)_R$ ,  $C$ ,  $P$ . Under  $U(1)_A$  they transform as

$$M^{(2)} \rightarrow e^{-4i\nu} M^{(2)}. \quad (11)$$

It is seen that the  $U(1)_A$  transformation distinguishes the ‘‘four quark’’ from the ‘‘two quark’’ states.

### III. DIFFERENT FOUR-QUARK STRUCTURES

Now we will show that  $M^{(2)}$ ,  $M^{(3)}$  and  $M^{(4)}$  are related by a Fierz transformation; thus only two of them are linearly independent. For this purpose it is convenient to express the four component spinors in terms of the two component chiral projections in the basis given above:

$$q_{aA} = \begin{bmatrix} q_{LaA} \\ q_{RaA} \end{bmatrix}. \quad (12)$$

The quark-antiquark field,  $M$  has the schematic structure  $M_a^b = q_{RbA}^\dagger q_{LaA}$  while  $(M^\dagger)_b^a = q_{LaA}^\dagger q_{RbA}$ . Similarly, the schematic molecule-type field  $M^{(2)}$  takes the form

$$M_g^{(2)f} = \epsilon_{gab} \epsilon^{fde} [q_{LaA}^\dagger q_{RdA}] [q_{LbB}^\dagger q_{ReB}]. \quad (13)$$

Using the definition  $(\sigma_2)_{\alpha\beta} = -i\epsilon_{\alpha\beta}$  and the anticommutativity of the Fermi fields we readily obtain the decomposition of  $M^{(3)}$  as

$$M_g^{(3)f} = 2\epsilon_{gab} \epsilon^{fde} ([q_{LaA}^\dagger q_{RdA}] [q_{LbB}^\dagger q_{ReB}] - [q_{LaA}^\dagger q_{RdB}] [q_{LbB}^\dagger q_{ReA}]). \quad (14)$$

To simplify  $M^{(4)}$  we make use of the well-known identity  $\sigma_2 \sigma_k^* \sigma_2 = -\sigma_k$  and also the Fierz-type relation,

$$(\sigma_k \sigma_2)_{\beta\alpha} (\sigma_2 \sigma_k)_{\eta\rho} = \delta_{\beta\rho} \delta_{\alpha\eta} + \delta_{\beta\eta} \delta_{\alpha\rho}. \quad (15)$$

Then we find,

$$M_g^{(4)f} = -4\epsilon_{gab} \epsilon^{fde} ([q_{LaA}^\dagger q_{RdA}] [q_{LbB}^\dagger q_{ReB}] + [q_{LaA}^\dagger q_{RdB}] [q_{LbB}^\dagger q_{ReA}]). \quad (16)$$

Now it is easy to see that the molecule-type field  $M^{(2)}$  may be expressed as a linear combination of  $M^{(3)}$  and  $M^{(4)}$ :

$$M_a^{(2)b} = \frac{2M_a^{(3)b} - M_a^{(4)b}}{8}. \quad (17)$$

Thus, at a naive quark model level, there is no absolute distinction between the molecule-type field and a linear combination of two different diquark-antidiquark configurations. It may be amusing to note that, in the MIT bag model approach [22] to four-quark scalars, the relevant eigenstates of the hyperfine splitting Hamiltonian also emerge as a linear combination of two diquark-antidiquark configurations. Of course there may be differences which would emerge if the full QCD dynamics could be solved. Some dynamical arguments are discussed in Ref. [23].

There are no external quantum numbers to differentiate  $M^{(2)}$ ,  $M^{(3)}$ , and  $M^{(4)}$  from each other. Thus we just assume that the dynamics selects a particular but unknown linear combination of (any two of) them to be a bound ‘‘four quark’’ field,  $M'$ . Note, however, that  $M$  and  $M'$  are distinguished from each other by their different  $U(1)_A$  transformation properties.

### IV. EFFECTIVE POTENTIAL

In our model Lagrangian we use scalar fields with the transformation properties of the schematic fields  $M$  and  $M'$  just discussed. These fields may be decomposed into Hermitian scalar ( $S$ ) and pseudoscalar ( $\phi$ ) nonets as

$$M = S + i\phi, \quad M' = S' + i\phi'. \quad (18)$$

The Lagrangian density for our model is taken to have the simple form

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(\partial_\mu M \partial_\mu M^\dagger) - \frac{1}{2} \text{Tr}(\partial_\mu M' \partial_\mu M'^\dagger) - V_0(M, M') - V_{\text{SB}}, \quad (19)$$

with nonderivative interaction terms. Here  $V_0(M, M')$  stands for a general function made from  $SU(3)_L \times SU(3)_R$  but not necessarily  $U(1)_A$  invariants formed out of  $M$  and  $M'$ . Furthermore  $V_{\text{SB}}$  is a flavor symmetry breaking term designed to model the quark mass terms in QCD.

Generally one has the situation where nonzero vacuum values of the diagonal components of  $S$  and  $S'$  may exist. These will be denoted by

$$\langle S_a^b \rangle = \alpha_a \delta_a^b, \quad \langle S'^b_a \rangle = \beta_a \delta_a^b. \quad (20)$$

In the isospin invariant limit,  $\alpha_1 = \alpha_2$  and  $\beta_1 = \beta_2$ , while in the  $SU(3)$  invariant limit,  $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$  and  $\beta_1 = \beta_2 = \beta_3 \equiv \beta$ .

The model is an upgrading of the single-M  $SU(3)$  linear sigma model to one containing two chiral nonets. However, it is much more complicated. For example, the renormalizable version of the present model has (see Appendix A of [12] and Appendix A of [14]) twenty-one invariant terms in  $V_0$  while the renormalizable version of the single-M model has only four terms. To make progress we suggested first

including only those terms with no more than a total of eight (quark plus antiquark) lines in the underlying schematic interaction. This led to the predictions ([14,15]):i) a very light singlet scalar which might be identified with the  $f_0(600)$ , ii) large four quark content of the lighter scalars, iii) improved s-wave pion-pion isosinglet scattering length. The model included two  $U(1)_A$  violating, but chiral  $SU(3)$  conserving, terms. These were chosen to mock up the  $U(1)_A$  anomaly of QCD. That is a reasonable requirement in the present context since the  $U(1)_A$  symmetry distinguishes the two-quark from the four-quark mesons.

In the single-M model, it was noted (see Appendix of [24]) even before QCD, that a determinant type  $U(1)_A$  violating piece was needed to explain the  $\eta$  mesons. After QCD, 't Hooft [2] showed that a quark level term of the required sort would arise from instanton contributions. Actually, he did not completely present the relevant three-flavor version of his model. Other authors [4] later gave this result, and one can see that there is an additional  $U(1)_A$  violation term present. Here we will show that the additional term has the same structure as the one we added on the basis of the quark counting just mentioned.

In the effective Lagrangian framework the axial anomaly was first “exactly” modeled [25] by including a term proportional to  $G(\ln \det M - \ln \det M^\dagger)$ , where  $G$  represents the pseudoscalar Yang Mills invariant  $\text{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$ , constructed from the field strength tensor. It is necessary to include a wrong sign mass term for  $G$  which is then integrated out. Then one obtains a form like

$$\mathcal{L}_\eta = -c_3 \left[ \ln \left( \frac{\det M}{\det M^\dagger} \right) \right]^2, \quad (21)$$

where  $c_3$  is a numerical parameter. In the present model with two chiral nonets this form is not unique and the most plausible modification [13] is to replace  $\ln \left( \frac{\det M}{\det M^\dagger} \right)$  by

$$\gamma_1 \left[ \ln \left( \frac{\det(M)}{\det(M^\dagger)} \right) \right] + (1 - \gamma_1) \left[ \ln \left( \frac{\text{Tr}(MM^\dagger)}{\text{Tr}(M'M^\dagger)} \right) \right], \quad (22)$$

where  $\gamma_1$  is a dimensionless parameter. A similar additional term was discussed in Refs. [26]. In that model, however, the additional term is somewhat different and involves a three-quark–three-antiquark state rather than the product of a two-quark–two-antiquark state with a conventional quark-antiquark state in the present model.

For the purpose of comparison with instanton results in the next section we will approximate this somewhat complicated form by its leading term. With the assumption that  $\frac{\det(M)}{\langle \det(M) \rangle} = 1 + \text{small}$ , we write:

$$\ln[\det(M)] \approx \langle \det(M) \rangle + \left[ \frac{\det(M)}{\langle \det(M) \rangle} - 1 \right]. \quad (23)$$

Then,

$$\begin{aligned} (\ln[\det(M)] - \ln[\det(M^\dagger)])^2 &\approx \left[ \frac{\det(M) - \det(M^\dagger)}{\langle \det(M) \rangle} \right]^2 \\ &= \frac{1}{\alpha^6} [(\det(M) + \det(M^\dagger))^2 \\ &\quad - 4 \det(MM^\dagger)] \\ &\approx \frac{4}{\alpha^3} (\det(M) + \det(M^\dagger)). \end{aligned} \quad (24)$$

In this procedure a purely numerical constant has been dropped and the  $U(1)_A$  invariant piece,  $\det(MM^\dagger)$ , was considered small compared to other  $U(1)_A$  invariant pieces. Similarly,

$$\left[ \ln \left( \frac{\text{Tr}(MM^\dagger)}{\text{Tr}(M'M^\dagger)} \right) \right]^2 \approx \frac{4}{3\alpha\beta} [\text{Tr}(MM^\dagger) + \text{Tr}(M'M^\dagger)]. \quad (25)$$

Cross terms from squaring Eq. (22) are neglected in the same approximation. Summarizing these steps we write:

$$\begin{aligned} \mathcal{L}_\eta &= -c_3 \left( \gamma_1 \ln \frac{\det(M)}{\det(M^\dagger)} + (1 - \gamma_1) \ln \frac{\text{Tr}(MM^\dagger)}{\text{Tr}(M'M^\dagger)} \right)^2 \\ &\approx -4c_3 \left( \frac{\gamma_1^2}{\alpha^3} [\det(M) + \det(M^\dagger)] + \frac{(1 - \gamma_1)^2}{3\alpha\beta} \right. \\ &\quad \left. \times [\text{Tr}(MM^\dagger) + \text{Tr}(M'M^\dagger)] \right). \end{aligned} \quad (26)$$

In contrast to these approximations, keeping the  $\ln$ 's in the calculations involving the mesonic Lagrangian leads to a desirable simplifying decoupling of the  $\eta'$  sector of the model from the parts which conserve  $U(1)_A$ . This was previously discussed in [13–15].

Another way to see that the linearized form in Eq. (26) is a reasonable approximation is to refer to some work on the single-M model, corresponding to Eq. (21). In Eq. (5) of the second paper in Ref. [25] it is shown that Eq. (21) above may be written as a function of  $J = \det M + \det M^\dagger$  as well as a  $U(1)_A$  invariant. Thus we may expand,

$$\mathcal{L}_\eta = \left\langle \frac{\partial \mathcal{L}_\eta}{\partial J} \right\rangle J + \dots, \quad (27)$$

which is the analog of Eq. (26). This captures the essential low-energy physics in the pseudoscalar isosinglet sector; for example, Eqs. (41) and (42) in Ref. [24] show that the isosinglet squared masses depend on the quantity  $\left\langle \frac{\partial \mathcal{L}_\eta}{\partial J} \right\rangle$  denoted as  $\langle V_4 \rangle$  there.

## V. $U(1)_A$ VIOLATION FROM INSTANTONS

The 't Hooft effective Lagrangian for the three-flavor case [4] can be presented [27] as

$$\begin{aligned} \mathcal{L}(q) = & \text{const} \frac{1}{6N_c(N_c^2 - 1)} \epsilon_{gab} \epsilon^{fde} \left( \frac{2N_c + 1}{2N_c + 4} \left[ \bar{q}_{gA} \frac{1 + \gamma_5}{2} q_{fA} \right] \left[ \bar{q}_{aB} \frac{1 + \gamma_5}{2} q_{dB} \right] \left[ \bar{q}_{bC} \frac{1 + \gamma_5}{2} q_{eC} \right] + \frac{3}{8(N_c + 2)} \right. \\ & \times \left. \left[ \bar{q}_{gA} \frac{1 + \gamma_5}{2} q_{fA} \right] \left[ \bar{q}_{aB} \frac{1 + \gamma_5}{2} \sigma_{\mu\nu} q_{dB} \right] \left[ \bar{q}_{bC} \frac{1 + \gamma_5}{2} \sigma_{\mu\nu} q_{eC} \right] + \left[ \frac{1 + \gamma_5}{2} \rightarrow \frac{1 - \gamma_5}{2} \right]. \end{aligned} \quad (28)$$

Here the kinematics were modified from Euclidean space, appropriate for the path integral derivation, to ordinary Minkowski space. The overall constant contains a function of the QCD running coupling constant which essentially cuts it off at higher energies.

The quantities like  $[\bar{q}_{gA} \frac{1 + \gamma_5}{2} q_{fA}]$  which appear in this equation clearly can be identified with the usual quark-antiquark meson field  $M_f^g$  defined in Eq. (3). The quantities involving  $\sigma_{\mu\nu}$  on the second line are less familiar. Using the identity

$$(\sigma_k)_{\beta\alpha} (\sigma_k)_{\eta\rho} = 2\delta_{\beta\rho} \delta_{\alpha\eta} - \delta_{\beta\alpha} \delta_{\eta\rho} \quad (29)$$

we find

$$\begin{aligned} \epsilon_{gab} \epsilon^{fde} \left[ \bar{q}_{aB} \frac{1 + \gamma_5}{2} \sigma_{\mu\nu} q_{dB} \right] \left[ \bar{q}_{bC} \frac{1 + \gamma_5}{2} \sigma_{\mu\nu} q_{eC} \right] \\ = 4\epsilon_{gab} \epsilon^{fde} (2[q_{RaB}^\dagger q_{LdC}][q_{RbC}^\dagger q_{LeB}] \\ - [q_{RaA}^\dagger q_{LdA}][q_{RbD}^\dagger q_{LeD}]) \\ = 4[(M^{(2)\dagger})_g^f - (M^{(3)\dagger})_g^f], \end{aligned} \quad (30)$$

where Eqs. (13), (14), and (17) were used in the last step. Putting these identifications back into Eq. (28) finally yields

$$\begin{aligned} \mathcal{L}(q) = & \frac{\text{const}}{2N_c(N_c^2 - 1)(N_c + 2)} \left[ (2N_c + 1) \det(M(q)) \right. \\ & \left. + \frac{1}{2} \text{Tr}[M(q)(M^{(2)\dagger}(q) - M^{(3)\dagger}(q))] \right] + \text{H.c.} \end{aligned} \quad (31)$$

Here the determinant and trace refer to the three-flavor space.

## VI. COMPARISON OF SIGMA MODEL AND INSTANTON APPROACHES

It is immediately clear that the  $U(1)_A$  violating instanton generated Lagrangian of Eq. (31) has the same structure as  $\mathcal{L}_\eta$ , the linearized  $U(1)_A$  violating Lagrangian in Eq. (26). Of course, the sigma model expression is constructed out of physical meson fields while the instanton expression is constructed out of schematic combinations of quark fields with the same transformation properties. Presumably the schematic quark combinations will be dominated by, or at least have substantial overlap with, the corresponding meson fields. This similarity seems to be the strongest point of our discussion. It is especially interesting to us in the context of building linear sigma models to learn about possible mixing of quark-antiquark and two-quark plus

two-antiquark mesons. As noted in Sec. IV, even the *renormalizable* linear sigma model potential would have too many terms for practical analysis. We therefore suggested a simplifying scheme in which terms with the smallest number of underlying (quark plus antiquark) fields be retained. On this basis, the two dominant  $U(1)_A$  violating terms are expected to be the  $\det(M)$  and  $\text{Tr}(MM^\dagger)$  ones, each representing six underlying fermions. This is apparently confirmed by the leading instanton calculation. The four-quark structure appearing in Eq. (31) is seen to contain  $M^{(2)} - M^{(3)}$ , a linear combination with equal strengths of molecule-type and diquark plus antidiquark components. This does not guarantee, naturally, that such a combination is the one which is dynamically bound.

To get a rough indication of what is happening, we introduce a prescription for obtaining the leading  $U(1)_A$  violating terms in the meson Lagrangian given in Eq. (26); we simply make the replacements,

$$M(q) \rightarrow -\Lambda^2 M, \quad M^{(2)}(q) - M^{(3)}(q) \rightarrow \pm \omega \Lambda^5 M', \quad (32)$$

in the instanton Lagrangian of Eq. (31). Here,  $M$  and  $M'$  are the meson fields while  $M(q)$  etc. represent corresponding schematic quark structures with the same chiral transformation properties. The positive quantity  $\Lambda$  is a QCD-type scale with dimension of mass. The dimensionless, positive quantity  $\omega$  is a phenomenological parameter introduced to account for our ignorance of which linear combination of the possible four-quark states is actually bound, the possibility of other hadronized field combinations appearing in Eq. (30) as well as other QCD effects. Finally the sign choice measures the sign of the vacuum value of the left-hand side operator.

We may estimate  $\Lambda$  above by taking the ground state expectation value of the (11) matrix element:

$$\langle M_{11} \rangle = \alpha = \frac{-1}{2\Lambda^2} \langle \bar{q}_{1A} q_{1A} \rangle. \quad (33)$$

Using  $\alpha = 0.0606$  GeV, as obtained in the  $SU(3)$  limit for either the massless or massive pion cases in [14] or [15], together with  $\langle \bar{q}_{1A} q_{1A} \rangle \approx -0.016$  GeV<sup>3</sup> yields for the scale factor,  $\Lambda \approx 0.36$  GeV. As a check on this procedure, if the ‘‘quark mass’’ factor,  $A$ , in the symmetry breaking term  $\mathcal{L}_{\text{SB}} = 2 \text{Tr}(AS)$  is also scaled as  $A = \Lambda^2 m_q$ , where  $m_q$  is the diagonal matrix of quark masses, then the current algebra mass formula for the pseudoscalars is converted to the corresponding linear sigma model mass formula.

Notice that the overall factor in Eq. (31) is heavily suppressed for large  $N_c$  as expected for instanton effects.

Thus we shall set  $N_c$  to be three, for our world. Then the substitutions of Eq. (32) result in a meson  $U(1)_A$  violating Lagrangian of the form

$$\mathcal{L}_\eta = \text{const}[-7 \det(M) \mp \frac{\omega \Lambda}{2} \text{Tr}(MM'^\dagger)] + \text{H.c.} \quad (34)$$

Comparing Eq. (34) with Eq. (26) gives a relation between  $\gamma_1$ , a measure of the relative strengths of the det and Tr terms which are being used to model the  $U(1)_A$  anomaly, and the scaling factor  $\omega$  introduced in Eq. (32):

$$\frac{\gamma_1^2}{(1 - \gamma_1)^2} = \frac{\pm 14\alpha^2}{3\beta\Lambda\omega}. \quad (35)$$

Since the left-hand side of this equation must be positive and  $\beta$  ( $\approx 0.0249$  GeV) is positive, consistency requires us to keep only the + sign in Eq. (32). This corresponds to a positive vacuum value for  $M^{(2)}(q) - M^{(3)}(q)$ , as opposed to the negative vacuum value for  $M(q)$  shown in Eq. (33). Defining  $Q(\omega) = 14\alpha^2/(3\beta\Lambda\omega)$ , this quadratic equation has the solutions

$$\gamma_1(\omega) = \frac{-Q \pm \sqrt{Q}}{1 - Q}. \quad (36)$$

Note that the  $\pm$  sign here is related to solving the quadratic equation rather than to the possible choice displayed in Eq. (32). Figure 1 shows that, for the + sign choice,  $\gamma_1(\omega)$  is positive and slowly decreasing as  $\omega$  increases. On the other hand for the — sign choice,  $\gamma_1(\omega)$  can be either positive or negative, as seen in Fig. 2. For either sign choice  $\gamma_1(\omega)$  goes to one as  $\omega$  goes to zero.

In our mesonic level sigma model the value of  $\gamma_1$  affects the masses and mixings of the four pseudoscalar isosinglets which appear. The lowest-lying ones are the  $\eta(547)$  and the  $\eta(958)$  while there are four experimental candidates (with masses in MeV at 1295, 1405, 1475 and 1760) for the two higher-lying states. This is, in general, a complicated mixing problem with some experimental ambiguities. In

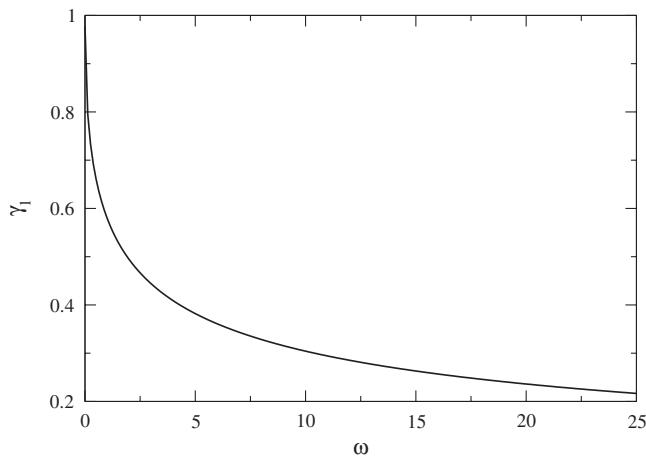


FIG. 1.  $\gamma_1$  vs.  $\omega$  for the positive sign choice in Eq. (36)

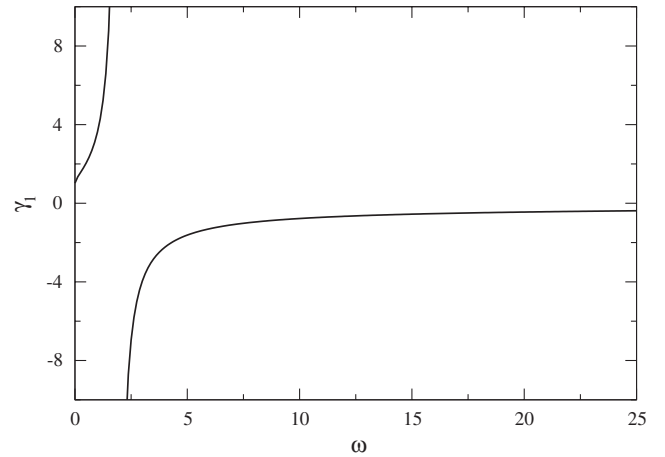


FIG. 2.  $\gamma_1$  vs.  $\omega$  for the negative sign choice in Eq. (36)

Refs [14] (see Sec. 4 and Appendix B) and [15] (see Appendix A) we examined flavor  $SU(3)$  symmetric situations and considered the favored scenario to be the one in which the  $\eta(958)$  mixed with the  $\eta(1475)$ . Furthermore there were two possible solutions with different “four-quark contents.” The preferred solution with mainly “two-quark content” for the  $\eta'$  (denoted I2 in Appendix B of [14]) gave  $\gamma_1 \approx 0.25$  while the somewhat less favored solution (labeled I1) with less “two-quark content” for the  $\eta'$  gave  $\gamma_1 \approx 0.54$ . These are seen to result, respectively, in values of about 18 and 1.3 for  $\omega$ . Clearly,  $\omega$  is sensitive to the value of  $\gamma_1$ , although it might be fairer to compare the values of  $\omega^{1/5}$  which differ less for the two alternatives.

Consider also the effect of choosing a different candidate for the second  $\eta'$ . As discussed in Refs. [14,15], the only other consistent choice of candidate, in our present  $SU(3)$  symmetric calculation, is the  $\eta(1760)$ . The favored solution, corresponding to  $\eta(958)$  being mainly of  $q\bar{q}$  type, yields  $\gamma_1 = -0.08$  compared to  $\gamma_1 = 0.25$  when the  $\eta(1475)$  is selected. Qualitatively,  $\gamma_1$  is small for both the favored cases.

It may be interesting to present these results in terms of quantities associated with the quark level instanton Lagrangian in Eq. (28). Using Eq. (32) and comparing with Eq. (26) gives for the overall constant:

$$\text{const} = \frac{960c_3\gamma_1^2}{7\alpha^3\Lambda^6}. \quad (37)$$

Similarly, the vacuum value of  $M^{(2)}(q) - M^{(3)}(q)$  can be estimated as

$$\langle M'(q) \rangle = \langle [M^{(2)}(q) - M^{(3)}(q)]_{11} \rangle = \omega\beta\Lambda^5. \quad (38)$$

The overall constant and the “four quark” vacuum value are listed in Table I for each of the two considered scenarios. For comparison, the square of the “two quark” vacuum value given in Eq. (33) is  $3.7 \times 10^{-3}$  GeV<sup>6</sup>, similar in order of magnitude to the four-quark ones.

TABLE I. Estimated values of the overall constant, const and the vacuum value,  $\langle M'(q) \rangle$  for the quark-level Lagrangian.

	I1	I2
$c_3(\text{GeV}^4)$	$-2.42 \times 10^{-4}$	$-2.42 \times 10^{-4}$
$\gamma_1$	$5.4 \times 10^{-1}$	$2.5 \times 10^{-1}$
$\text{const}(\text{GeV}^{-5})$	$-1.99 \times 10^4$	$-0.43 \times 10^4$
$\langle M'(q) \rangle(\text{GeV}^6)$	$1.96 \times 10^{-4}$	$2.7 \times 10^{-3}$

Of course, the results just discussed will be modified to some extent by the inclusion of SU(3) flavor symmetry breaking effects. This work, which is under study, involves at minimum the consideration of a  $4 \times 4$  mixing matrix for the isoscalar pseudoscalar mesons in the present framework.

## VII. SUMMARY AND DISCUSSION

We have shown that an extra term in the effective instanton generated Lagrangian has a natural interpretation as a mixing term between quark-antiquark spin zero mesons and spin zero mesons made from two quarks and two antiquarks (in some unspecified combination). Since the fields of the two kinds of mesons have different  $U(1)_A$  quantum numbers (before quark masses and spontaneous symmetry breaking are taken into account) this term also violates the  $U(1)_A$  symmetry.

On the question of what is the correct bound state of “four quark” mesons, we showed that at the zero quark mass kinematical level the molecule type could be rewritten as a linear combination of two different diquark-antidiquark types.

We worked at the level of a generalized linear SU(3) sigma model which contains two scalar nonets and two pseudoscalar nonets. The mixings between the two scalar nonets play an important role in explaining the properties which seem to be emerging from analysis of experimental data. The “extra” term of interest, on the other hand, primarily affects the mixing of the pseudoscalar SU(3) singlets. Indeed, we used a variation of the model in which the axial anomaly was exactly modeled, which has the effect of decoupling the pseudoscalar SU(3) singlets.

Using the masses of the  $\eta'(958)$  and the  $\eta(1475)$  in the sigma model we made numerical estimates of the overall constant for the instanton Lagrangian and the vacuum value of the “four quark” operator which appears in it.

An interesting treatment of the relation between instanton physics and the pattern of light scalar meson decay widths has been recently given in Ref. [28]. Some other discussions were given in Refs. [20,21]. Further discussion is planned in the present framework in the near future when our present work on the inclusion of flavor SU(3) breaking effects is completed.

The generalized linear sigma model in question is actually a very complicated one, describing many different particles and potentially having many different relevant terms. Thus while, due to chiral symmetry, it gives a good description of near threshold pion-pion scattering; for example, it is probably best regarded as a toy model for learning when it comes to describing a nonet’s worth of heavy pseudoscalars.

From the point of view of truncating the terms of this linear sigma model to a more manageable number we had made [13–15] a provisional ansatz that terms representing more than eight underlying fermion lines be discarded. This gave two  $U(1)_A$  violating terms and corresponded nicely to the instanton effective Lagrangian, which has two terms with six such lines. One might ask about going beyond this approximation for an effective model. If one allows 10 underlying fermion lines, the  $U(1)_A$  violating terms with coefficients  $e_3^b$ ,  $e_4^d$  and  $e_4^i$  in Eq. (A1) of [12] are possible. If one allows 12 underlying fermion lines, the terms with coefficients  $d_3$  and  $e_4^c$  kick in. Finally, if one allows 14 underlying fermion lines, the terms with coefficients  $e_4^e$  and  $e_4^f$  become possible. This could conceivably also be an interesting expansion in the instanton approach.

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