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Role of the σ resonance in determining the convergence of chiral perturbation theory

D. J. Cecile and Shailesh Chandrasekharan

Department of Physics, Box 90305, Duke University, Durham, North Carolina 27708

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The dimensionless parameter $\xi = M_\pi^2/(16\pi^2 F_\pi^2)$, where F_π is the pion decay constant and M_π is the pion mass, is expected to control the convergence of chiral perturbation theory applicable to QCD. Here we demonstrate that a strongly coupled lattice gauge theory model with the same symmetries as two-flavor QCD but with a much lighter σ-resonance is different. We first confirm that the leading low-energy constants appearing in the chiral Lagrangian are the same when calculated from the p-regime and the ϵ-regime as expected. However, $\xi \lesssim 0.002$ is necessary before 1-loop chiral perturbation theory predicts the data within 1%. For $\xi > 0.0035$ the data begin to deviate dramatically from 1-loop chiral perturbation theory predictions. We argue that this qualitative change is due to the presence of a light σ-resonance in our model. Our findings may be useful for lattice QCD studies.

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I. INTRODUCTION

Chiral perturbation theory captures the chiral symmetry properties of QCD, while its dynamical properties are encoded through a series of low-energy constants in the chiral Lagrangian [1]. At the leading order there are two low-energy constants: F the pion decay constant and Σ the chiral condensate, both evaluated in the chiral limit. One of the important topics of research today is to compute these and other higher order low-energy constants from first principles using lattice QCD [2–5]. Interestingly, the effects of a small quark mass m can also be taken into account and physical quantities can be expressed as a power series in a dimensionless parameter $\xi = M_\pi^2/(16\pi^2 F_\pi^2)$ where M_π is the physical pion mass and F_π is the physical pion decay constant. This series, which we refer to as the "chiral expansion," not only contains powers of ξ but also powers of $\xi \log\xi$, $\xi^2 \log\xi$, and so on. In QCD we can estimate $\xi \sim 0.015$ assuming $M_\pi \sim 140$ MeV and $F_\pi \sim 90$ MeV. For later convenience we also define $\xi' = M^2/(16\pi^2 F^2)$ where $M^2 = m\Sigma/F^2$ is the square of the pion mass to the leading order in the quark mass. It is easily verified that $\xi \approx \xi' + \mathcal{O}(\xi'^2)$, which means that to the first order in ξ we can ignore the difference between ξ and ξ'.

An important question in the field is to find the range in ξ where 1-loop perturbation theory will be valid up to a given error say 1% (or 5%)[6–8]. This will help lattice QCD calculations to extract reliably the low-energy constants. Current lattice calculations use 2-loop chiral perturbation theory in the region $0.02 < \xi < 0.1$ to fit the data in order to extract the low-energy constants of QCD [9–12]. Given that one usually has few data points with large errors at small quark masses and a large number of low-energy constants in the fits, it is undesirable to use 2-loop chiral perturbation theory for chiral extrapolations. The presence of many fitting parameters decreases their reliability. Fitting 2-loop results to lattice QCD data is still important for determining the higher order low-energy constants which can greatly increase the predictive power of chiral perturbation theory. However, to reliably extract these constants, it is important to identify the region where 1-loop results are applicable. If we understand the physics that controls the convergence properties of the chiral expansion, we can determine more confidently the region where 1-loop chiral perturbation theory will be applicable. Typically one believes that it is the ρ-meson resonance that puts the limit on pion masses where chiral perturbation theory will be valid. In order to avoid physically important singularities in $\pi - \pi$ scattering, it is reasonable to expect $M_\pi < M_\rho/2$ is necessary for chiral perturbation theory to be reliable. Experts believe that perhaps one needs at least $M_\pi < M_\rho/3$ [7].

In principle, there is another resonance that can limit the convergence of the chiral expansion. This is the so-called σ-resonance and arises in $\pi - \pi$ scattering in a channel with vacuum quantum numbers. Recently, the properties of this resonance in the physical world were estimated using experimental input, dispersion theory and chiral perturbation theory. It was estimated that $M_\sigma \simeq 440$ MeV and $\Gamma_\sigma \simeq 544$ MeV [13]. See [14–17] for a discussion of errors in the analysis. Thus the sigma is a broad and perhaps not so interesting resonance in the context of the convergence of chiral perturbation theory. On the other hand, in lattice QCD, as the pion masses increase, this resonance could become sharper and σ could become a stable physical particle, a bound state of two pions. Recent studies find that the properties of the σ-resonance do depend strongly on the quark mass [18,19]. It is interesting to ask if this dependence can affect the chiral expansion. Although this is a difficult question to answer in QCD, it may be possible to explore it with simpler models. Here we show that the σ-resonance can in principle affect the chiral expansion.

It is easy to argue that a light σ-resonance can indeed trigger the breakdown of chiral perturbation theory.

Consider a nonlinear sigma model which contains a coupling T that can be tuned such that for $T < T_c$ it is in a phase where the global symmetry is spontaneously broken and for $T > T_c$ it is in a symmetric phase. Chiral perturbation theory must be useful in describing the low-energy properties of the theory in the broken phase, but not in the symmetric phase. This means, as T is tuned towards T_c in the broken phase, chiral perturbation theory must become poorly convergent. Close to T_c, if the phase transition is second order, the linear sigma model becomes a good description of the physics, and in that model, as we will see later, the breakdown of chiral perturbation theory can be traced to the fact that M_σ/F_π becomes small. Note that at the critical point the sigma and the pions become degenerate and chiral symmetry is completely restored. The motivation of our current work is to show explicitly how the above scenario plays out in a specific model. Here we study a QCD-like lattice field theory model which has the same symmetries as two-flavor QCD. Hence $SU(2) \times SU(2)$ chiral perturbation theory is applicable. Our model also contains a parameter equivalent to the coupling T of the nonlinear sigma model discussed above. We tune this coupling to be close to the critical point and hence know that our model contains a light sigma resonance although we do not know its exact properties *a priori*. We then find evidence that indeed chiral perturbation theory breaks down when $M_\pi > M_\sigma/3$ is roughly satisfied.

II. MODEL AND PREDICTIONS

Our model involves two flavors of staggered fermions interacting strongly with abelian gauge fields. We recently developed an efficient cluster algorithm for this model and studied it in the ϵ-regime [20]. Here we will focus on the p-regime. The action of the model is given by

$$S = -\sum_x \sum_{\mu=1}^{5} \eta_{\mu,x}[\mathrm{e}^{i\phi_{\mu,x}}\bar{\psi}_x\psi_{x+\hat{\mu}} - \mathrm{e}^{-i\phi_{\mu,x}}\bar{\psi}_{x+\hat{\mu}}\psi_x] - \sum_x\Big[m\bar{\psi}_x\psi_x + \frac{\tilde{c}}{2}(\bar{\psi}_x\psi_x)^2\Big], \quad (1)$$

where x denotes a lattice site on a $4+1$-dimensional hypercubic lattice $L_t \times L^4$. Here L^4 is the usual Euclidean space-time box while L_t represents a fictitious temperature direction whose role will be discussed below. The two component Grassmann fields, $\bar{\psi}_x$ and ψ_x, represent the two quark (u, d) flavors of mass m, and $\phi_{\mu,x}$ is the compact $U(1)$ gauge field through which the quarks interact. Here $\mu = 1, 2, .., 5$ runs over the $4+1$ directions. The $\mu = 1$ direction will denote the fictitious temperature direction, while the remaining directions represent Euclidean space-time. The usual staggered fermion phase factors $\eta_{\mu,x}$ obey the relations: $\eta_{1,x}^2 = T$ and $\eta_{i,x}^2 = 1$ for $i = 2$, 3, 4, 5. The parameter T controls the fictitious temperature. The four fermion coupling $\tilde{c}$ sets the strength of the anomaly. As explained in [20], the above model has the same symmetries as $N_f = 2$ QCD. In this work we fix $L_t = 2$ and $\tilde{c} = 0.3$. For these parameters the temperature T can be tuned so that the model is in a spontaneously broken phase for $T < T_c$ or in the symmetric phase for $T > T_c$, where $T_c = 1.737\,79(4)$ was determined in the earlier work [20]. Since the phase transition is second order, close to T_c the pion decay constant in the chiral limit F is small in lattice units. Further, tuning T close to T_c also makes the σ-resonance light as discussed above. For these reasons, we chose to fix $T = 1.7$ in this work.

We focus on three observables: The vector current susceptibility Y_v, the chiral current susceptibility Y_c, and the chiral condensate susceptibility χ_σ. The current susceptibilities are defined as

$$Y_{v,c} = \frac{1}{dL^d}\Big\langle \sum_{\mu=1}^{d}\Big(\sum_x J_\mu^{v,c}(x)\Big)^2\Big\rangle, \quad (2)$$

where $J_\mu^v(x)$ and $J_\mu^c(x)$ denote one of the components of the vector and the chiral current, respectively. The condensate susceptibility is defined as

$$\chi_\sigma = \frac{1}{L^d}\sum_{x,y}\langle\bar{\psi}_x\psi_x\bar{\psi}_y\psi_y\rangle. \quad (3)$$

For a detailed discussion of our algorithm and observables, we refer the reader to [20].

The behavior of these observables for large L and small m is governed by chiral perturbation theory, which is described by the Euclidean chiral Lagrangian density

$$\mathcal{L} = \frac{F^2}{4}\mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U) - \frac{m\Sigma}{4}\mathrm{Tr}(U + U^\dagger), \quad (4)$$

where F is the chiral pion decay constant, Σ is the chiral condensate, and $U \in SU(2)$ is the pion field. Using this Lagrangian, the finite-size scaling formulas for many quantities have been found in the literature [21–25]. The predictions for Y_c, Y_v, and χ_σ in the p-regime can be found in [22,25]

$$Y_c = (F_\pi)^2[1 - 2\tilde{g}_1(LM_\pi)\xi + \mathcal{O}(\xi^2)], \quad (5a)$$

$$Y_v = (F_\pi)^2\Big[-2L\frac{\partial\tilde{g}_1(LM_\pi)}{\partial L}\xi + \mathcal{O}(\xi^2)\Big], \quad (5b)$$

$$\chi_\sigma = (\langle\bar{q}q\rangle)^2L^4[1 - 3\tilde{g}_1(LM_\pi)\xi + \mathcal{O}(\xi^2)], \quad (5c)$$

where M_π is the pion mass, F_π is the pion decay constant, and $\langle\bar{q}q\rangle$ is the chiral condensate at a given quark mass m. The function $\tilde{g}_1$ arises due to pions constrained to be inside a periodic box and is given by

$$\tilde{g}_1(\lambda) = \sum_{n_1,n_2,n_3,n_4\neq 0}^{\infty}\frac{4}{\lambda\sqrt{n}}K_1(\lambda\sqrt{n}), \quad (6)$$

where K_1 is a Bessel function of the second kind and $n = n_1^2 + n_2^2 + n_3^2 + n_4^2$.

III. RESULTS

We have varied the quark mass in the interval $0.0002 \leq m \leq 0.01$ for lattices in the range $12 \leq L \leq 32$. As an illustration, we show the data at $m = 0.000\,65$, $m = 0.0035$, $m = 0.002$, and $m = 0.001$ in Fig. 1. Our data fit well to the above predictions of chiral perturbation theory for $0.0002 < m \leq 0.0035$. The detailed results are summarized in Table I.

Thus, we are able to extract F_π, M_π, and $\langle \bar{q}q \rangle$ as functions of the quark mass. Note that the fit is not as reliable at the lowest mass ($m = 0.0002$) as compared to higher masses. It is possible that our lattices are not sufficiently large at this tiny quark mass to allow us to fit to 1-loop results.

At $m \geq 0.002$ the fits converge only if we exclude almost all the curvature in Y_c and χ_σ. In particular, we are not sensitive to the $\tilde{g}_1(\lambda)$ function for these two observables and the data fit well even to a constant as shown in Table II.

This issue can be seen in Fig. 1 at $m = 0.0035$ and 0.0065. Comparing the results from the two different fits we see that the error bars for $\langle \bar{q}q \rangle$ are underestimated by a factor of 2 or 3 at the higher masses. We find that M_π can be calculated very accurately by a one-parameter fit of Y_v which may be a useful observation for lattice QCD calculations. Interestingly, Y_v continues to fit well to the 1-loop formula even at higher masses, but n [in Eq. (6)] could be restricted to small values (typically less than 3).

The quark mass dependence of F_π, $\langle \bar{q}q \rangle$, and M_π have been computed up to 1-loop in [21,22]

$$F_\pi = F[1 - \xi' \log\xi' + 2\xi' c_F], \quad (7a)$$

$$\langle \bar{q}q \rangle = \Sigma[1 - \tfrac{3}{2}\xi' \log\xi' + 3\xi' c_\Sigma], \quad (7b)$$

$$M_\pi^2 = M^2[1 + \tfrac{1}{2}\xi' \log\xi' - \xi' c_M], \quad (7c)$$

where c_F, c_Σ, and c_M are higher order low-energy constants and are usually defined in the literature as $c_i = \log(\Lambda_i/4\pi F)$. We have performed a combined fit of all the values of F_π, $\langle \bar{q}q \rangle$, and M_π quoted in Table I in the region $0.0002 \leq m \leq 0.001$ to the above three relations. The result is tabulated in the first row of the Table III. We note that the values of F and Σ agree nicely with $F = 0.2327(1)$ and $\Sigma = 0.4346(2)$ computed earlier at $m = 0$ [20]. Further, in the ϵ-regime we find $c_M + 4c_\Sigma = 80(6)$, while in the p-regime (from Table III) we see that this number is 87(4). Thus, we confirm that the p-regime and the ϵ-regime are described by the same low-energy constants as expected.

In order to isolate the region where 1-loop corrections are a good description of the data we define the following rescaled and subtracted quantities:

$$R_F \equiv F_\pi/F - 1 + \xi' \log\xi', \quad (8a)$$

$$R_\Sigma \equiv \langle \bar{q}q \rangle/\Sigma - 1 + 3\xi' \log\xi'/2, \quad (8b)$$

$$R_M \equiv M_\pi^2/M^2 - 1 - \xi' \log(\xi')/2. \quad (8c)$$

We use chiral values $F = 0.2329$ and $\Sigma = 0.4354$ obtained from our fits (see Table III first row) to compute the R's and ξ'. By definition, the R's must be linear in ξ' in the region where 1-loop results are valid. In Fig. 2 we plot the R's as a function of ξ'. Assuming errors of 1%(5%) or

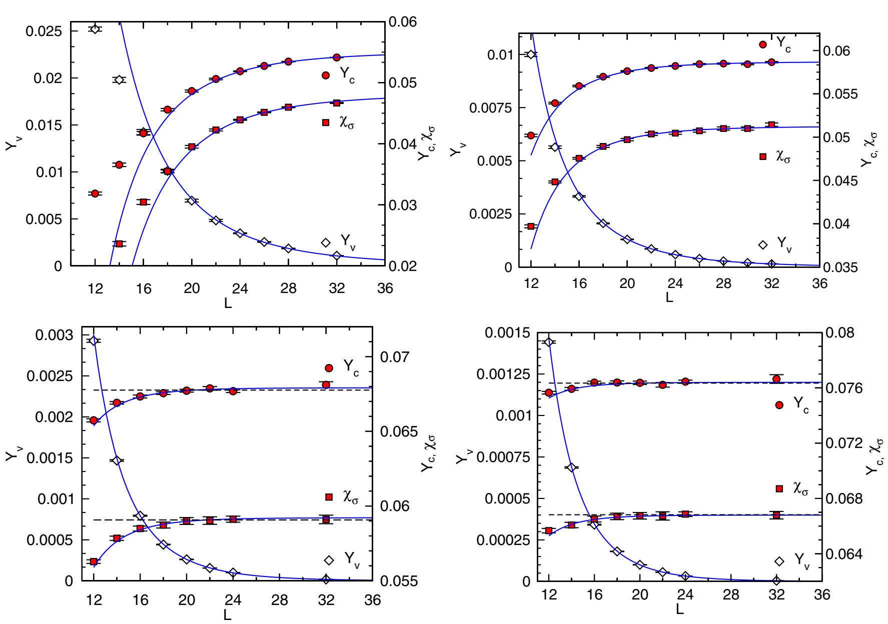

FIG. 1 (color online). Finite-size scaling of Y_v, Y_c, and χ_σ at $m = 0.0002$ (top left), $m = 0.001$ (top right), $m = 0.0035$ (bottom left), and $m = 0.0065$ (bottom right). The solid lines are fits of the data to the expected finite-size scaling form from chiral perturbation theory while dashed lines are fits to a constant.

TABLE I. Results from fitting Y_v, Y_c, and χ_σ as a function of L to the finite-size 1-loop chiral perturbation theory. The χ^2 quoted is per degree of freedom.

m	$\langle\bar{q}q\rangle$	F_π	M_π	χ^2	Fit range
0.0002	0.4392(2)	0.2348(1)	0.0400(2)	2.5	$24 \leq L \leq 32$
0.0005	0.4441(2)	0.2377(1)	0.0627(2)	1.1	$24 \leq L \leq 32$
0.0008	0.4499(2)	0.2406(1)	0.0789(1)	0.9	$22 \leq L \leq 32$
0.0010	0.4528(2)	0.2423(1)	0.0878(1)	0.8	$18 \leq L \leq 32$
0.0015	0.4606(2)	0.2467(1)	0.1070(2)	1.3	$18 \leq L \leq 32$
0.0020	0.4678(2)	0.2501(1)	0.1220(2)	1.8	$20 \leq L \leq 32$
0.0025	0.4740(2)	0.2538(1)	0.1356(2)	1.6	$16 \leq L \leq 32$
0.0035	0.4867(2)	0.2606(1)	0.1584(2)	0.9	$16 \leq L \leq 32$

TABLE II. Results from fitting Y_c and χ_σ to a constant while Y_v is fit to 1-loop chiral perturbation theory.

m	$\langle\bar{q}q\rangle$	χ^2	F_π	χ^2	M_π	χ^2
0.0020	0.4668(3)	1.2	0.2498(1)	0.1	0.1226(2)	0.6
0.0025	0.4728(3)	0.7	0.2536(2)	0.9	0.1356(2)	1.6
0.0035	0.4861(3)	0.1	0.2603(1)	1.5	0.1584(2)	1.7
0.0050	0.5024(3)	0.2	0.2690(2)	1.1	0.1860(3)	0.7
0.0065	0.5170(3)	0.1	0.2764(2)	0.7	0.2083(4)	0.5
0.0075	0.5247(3)	0.2	0.2807(2)	1.6	0.2219(4)	0.9
0.0100	0.5433(2)	0.7	0.2912(2)	0.1	0.2521(5)	1.8

less can be tolerated, Fig. 2 shows that 1-loop chiral perturbation theory describes the data for $\xi' \lesssim 0.002(0.006)$. Interestingly, there is also an approximately linear region for $\xi' \gtrsim 0.006$ but with a completely different slope. This is shown as the dashed line in Fig. 2. This behavior suggests that chiral perturbation theory begins to break down. We will argue below that the σ-resonance is responsible for this break down. Note that $\xi' \approx 0.0035$ is the rough location of the "knee" that separates the low ξ' and high ξ' regions.

IV. DISCUSSION AND CONCLUSIONS

The unnaturally large values of c_F, c_Σ, and c_M are clearly responsible for the break down of the chiral expansion at very small values of ξ'. What is the physics behind these large values? It has been argued in the context of the $O(4)$ linear sigma model, that the physics in the sigma channel is directly related to these terms. In particular, perturbative calculations show that [26–28]

$$c_\Sigma = \log(M_R/4\pi F) - \frac{7}{6} + \frac{8\pi^2}{3g_R}, \qquad (9a)$$

$$c_M = \log(M_R/4\pi F) - \frac{7}{3} + \frac{8\pi^2}{g_R}, \qquad (9b)$$

where

$$M_\sigma^2 = M_R^2\left[1 + \frac{g_R}{16\pi^2}(3\pi\sqrt{3} - 13)\right]. \qquad (10)$$

Here M_σ is that mass of the σ particle and g_R is the corresponding renormalized coupling, $g_R = M_R^2/2F^2$. We believe that in our model the above relations must be valid at least as a good approximation because we are close to the critical point where g_R is expected to be small and the perturbative $O(4)$ linear sigma model is a good description of the low-energy physics. Indeed, using $c_\Sigma = 12$ we find that $M_\sigma/F \sim 2$ while using $c_M = 39$ we again find that $M_\sigma/F \sim 2$. The fact that these two agree with each other is a clear confirmation of our belief. Assuming $M_\sigma/F \sim 2$ and setting the scale of our lattice with $F = 90$ MeV we estimate $M_\sigma \sim 180$ MeV in our model. At $\xi' \sim 0.0035$ we find that $M_\pi \sim 60$ MeV. Hence, we conclude that when $M_\pi > M_\sigma/3$ chiral perturbation theory begins to break down and the physics is better described by the linear sigma model.

Among the many differences between our model and QCD, the most significant is the presence of a light and narrow σ-resonance. This difference was responsible for the large low energy constants and poor convergence of the chiral expansion in our model. Despite the differences, it is

TABLE III. Results from a combined fit of the data in Table I to Eqs. (7). The first row uses data in the range $0.0002 \leq m \leq 0.001$ while the second row excludes $m = 0.0002$ from the fit.

Σ	F	c_Σ	c_F	c_M	χ^2
0.4354(3)	0.2329(2)	11.9(3)	19.3(5)	39(3)	1.1
0.4351(5)	0.2331(4)	12.3(5)	18.9(9)	37(3)	1.6

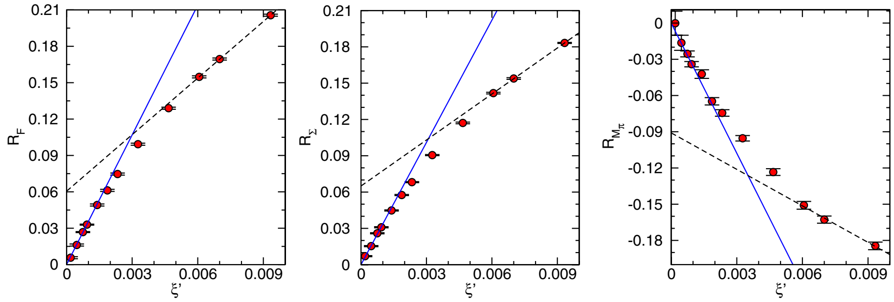

FIG. 2 (color online). Rescaled and subtracted quantities R_{F_π}, $R_{\langle\bar{q}q\rangle}$, and $R_{M_\pi^2}$ defined in Eqs. (8). The solid lines are plots of the fits discussed in the text. The dashed lines show the linear region for larger values of ξ'. The knee is estimated roughly as the point where the two lines cross.

indeed encouraging that our model supports the most important expectations of chiral perturbation theory, namely, chiral perturbation theory is applicable in a region of small quark masses and that the properties of resonances play an important role in determining this region. In particular, we have found evidence that the properties of σ-resonance are encoded in the low-energy constants that control the chiral logarithms and hence can play an important role in determining the region where 1-loop chiral perturbation theory is valid. The resonance properties of course change with the quark masses. Thus, it is safe to assume that chiral perturbation theory can only become reliable in the region of the quark mass where the properties of the σ-resonance (and other resonances) vary little. A rough estimate based on [19] suggests that $M_\pi \lesssim 250$ MeV may be necessary. Thus, it should not be very surprising that 1-loop chiral perturbation theory may be applicable at a few percent accuracy only at realistic pion masses.

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