

Quasinormal modes and thermodynamic phase transitions

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It has recently been suggested that scalar, Dirac, and Rarita-Schwinger perturbations are related to thermodynamic phase transitions of charged (Reissner-Nordström) black holes. In this Brief Report we show that this result is probably a numerical coincidence, and that the conjectured correspondence does not straightforwardly generalize to other metrics, such as Kerr or Schwarzschild (anti-)de Sitter. Our calculations do not rule out a relation between dynamical and thermodynamical properties of black holes, but they suggest that such a relation is nontrivial.

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I. INTRODUCTION

Kerr-Newman black holes (BHs) in (anti-)de Sitter space are characterized by their mass M , dimensionless rotation parameter a/M , dimensionless charge Q/M , and possibly a nonzero cosmological constant Λ (here and in the following we adopt geometrical units, $G = c = 1$, and we write Einstein's equations as $G_{\mu\nu} = 3\Lambda g_{\mu\nu}$.)

Davies [1,2] pointed out that BHs can undergo a second-order phase transition in which their specific heat changes sign as a/M and/or Q/M are increased. For our considerations below, we only need to recall that the phase transition occurs when

$$\left(\frac{Q}{M}\right)^2 = \left(\frac{9}{4}\right)^2 \Lambda M^2 + \frac{3}{4} \quad (1)$$

for Reissner-Nordström (anti-)de Sitter BHs ($a = 0$) and

$$\left(\frac{a}{M}\right)^2 = \frac{\sqrt{5} - 1}{2} \quad (2)$$

for Kerr BHs ($\Lambda = Q = 0$).

BH perturbations due to external fields of different spin have been studied for decades [3]. After a transient phase, the decay of these perturbations can be described as a superposition of damped exponentials of the form $\exp(i\omega t)$. For a given spacetime (i.e. for a given cosmological constant Λ) the quasinormal mode (QNM) frequencies $\omega = \omega_R + i\omega_I$ depend only on the BH parameters M , a , and Q . They are complex numbers because oscillations are damped away by the emission of gravitational radiation, and they are usually labeled by three integers: $\omega = \omega_{lmn}$ (see [4] for a review, and [5] for a summary of the properties of the QNM spectrum.) Two integers (l, m)

correspond to the “angular quantum numbers” of the spin-weighted spheroidal harmonics used to separate the angular dependence of the perturbations. For spherically symmetric backgrounds, such as the Reissner-Nordström metric ($a = \Lambda = 0$), perturbations are degenerate with respect to the azimuthal index m , but this degeneracy is broken when $a \neq 0$. A third integer $n (= 0, 1, \dots)$, called the “overtone number,” sorts the frequencies by the magnitude of their imaginary parts.

For Reissner-Nordström BHs and $n < n_c$ (n_c being some critical value that depends on l and on the spin s of the perturbing field), ω_R and ω_I are usually monotonic functions of the charge Q/M . For $n = n_c$ the real part of the QNM frequency has an extremum as a function of the charge. When $n > n_c$, both ω_R and ω_I usually become oscillatory functions of Q/M [6–9]. A similar behavior has been observed for $m = 0$ QNM frequencies of Kerr BHs as functions of a/M [7,10,11] (see also [12] for a generalization to Kerr-Newman BHs.)

Jing and Pan [13] recently computed QNM frequencies for scalar ($s = 0$), Dirac ($s = 1/2$), and Rarita-Schwinger ($s = 3/2$) perturbations of Reissner-Nordström BHs. They noticed that the first maximum of $M\omega_R(Q/M)$, as computed numerically, matches within $\sim 2.5\%$ the value of $Q/M = \sqrt{3/4} \simeq 0.866$ predicted by Eq. (1) for Davies' second-order phase transition. They conjectured that this agreement implies a connection between dynamical and thermodynamic properties of Reissner-Nordström BHs. In this Brief Report we show that the observed agreement is probably a numerical coincidence, and that such a connection (if it exists) should be nontrivial.

Our results can be summarized as follows:

- (1) The approximate agreement found in Ref. [13] is usually observed for $n = n_c + 1$, not for $n = n_c$. We see no compelling reason why this mode should be singled out from the QNM spectrum as especially relevant.

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- (2) For $n = n_c + 1$, the numerical agreement with Davies' thermodynamic phase transition point is significantly worse for “electromagnetic-type” or “gravitational-type” perturbations of Reissner-Nordström BHs, which were not considered in Ref. [13].
- (3) Finally, and perhaps more convincingly, we show that the conjecture does not hold for (i) integer-spin, $m = 0$ perturbations of Kerr BHs, and (ii) integer-spin perturbations of Schwarzschild anti-de Sitter (SAdS) BHs.

In Sec. II we present details of our calculations. In Sec. III we argue that, if a relation between classical BH oscillations and their thermodynamic properties exists, it is not as simple as proposed in Ref. [13].

II. RESULTS

A. Reissner-Nordström black holes

Reference [13] considered scalar ($s = 0$), Dirac ($s = 1/2$), and Rarita-Schwinger ($s = 3/2$) perturbations of a Reissner-Nordström BH. The results presented in their Fig. 1 for $s = 0$ agree with those previously published in [12]. From the calculations shown in Fig. 1 of [13] (or Fig. 1 of [12], where modes are counted starting from $n = 1$ rather than from $n = 0$) one deduces that, for scalar perturbations with $l = 0$, $M\omega_R$ has the first extremum when $n = n_c = 0$ and $Q/M = 0.962$. Similarly, for scalar perturbations with $l = 1$, the first extremum occurs when $n = n_c = 1$ and $Q/M = 0.960$. Both values are quite far from Davies' phase transition point $Q/M \approx 0.866$.

For $s = 0$, the maximum in $M\omega_R(Q/M)$ gets in better agreement with the phase transition point if we consider the next overtone $n = n_c + 1$, as we show by an explicit calculation in Table I. For the $s = l = 0$, $n = 1$ mode, and accounting for their different choice of units, Ref. [13] determines the “critical” charge to be $Q/M = 0.876$. This is in good (if not perfect) agreement with our value

TABLE I. Critical charge Q/M corresponding to the first extremum of $M\omega_R(Q/M)$ for Reissner-Nordström BHs. We consider only the dominant multipolar component of the radiation, i.e. $l = |s|$. For each spin s , in the first row we list the critical charge for which an extremum in $M\omega_R$ first appears and the corresponding overtone number $n = n_c$ (in parentheses). In the second row, we show the charge corresponding to the first extremum in $M\omega_R$ for $n = n_c + 1$.

| Q/M | | | |
|----------------------|-----------|-----------|-----------|
| Perturbation | $l = 0$ | $l = 1$ | $l = 2$ |
| Scalar | 0.962 (0) | ... | ... |
| | 0.879 (1) | ... | ... |
| Electromagnetic-type | | 0.962 (2) | ... |
| | | 0.940 (3) | ... |
| Gravitational-type | | | 0.958 (1) |
| | | | 0.910 (2) |

$Q/M = 0.879$, which has a 1.5% disagreement with the phase transition point predicted by Davies.

Reference [13] did not consider gravitational ($s = 2$) and electromagnetic ($s = 1$) perturbations of Reissner-Nordström BHs. These perturbations are coupled to each other by the BH charge, but the scattering problem can still be reduced to a pair of wave equations (see discussions in [3,12,14].) The two equations are said to describe electromagnetic-type and gravitational-type perturbations, depending on whether they reduce to pure electromagnetic or pure gravitational perturbations of a Schwarzschild BH in the limit $Q/M \rightarrow 0$. In Table I we list the critical charge for the dominant multipoles of electromagnetic-type and gravitational-type perturbations. From the tabulated values it is quite clear that the agreement with Davies' phase transition point does not get better even if, following Ref. [13], we consider the QNM with $n = n_c + 1$ rather than that with $n = n_c$.

In our opinion, this is evidence that the conjectured correspondence is only a numerical coincidence. The value of the charge for which $M\omega_R$ first has an extremum is around $Q/M \approx 0.96$ for all integer spins $s = 0, 1, 2$, and this is quite far from the thermodynamic phase transition point $Q/M = 0.866$. One could still argue that the dynamical and thermodynamic properties of BHs are not simplest when one considers coupled electromagnetic and gravitational perturbations: see e.g. [14] for a discussion. To show that this is not the only reason for the discrepancy with Davies' critical point, in the remainder of this paper we show that the conjectured correspondence does *not* hold for other metrics as well. In particular, below we consider Kerr and SAdS BHs.

B. Kerr black holes

For Kerr BHs, according to Eq. (2) the thermodynamic phase transition corresponds to a rotation parameter $a/M \approx 0.786$. For moderate values of n , only Kerr perturbations with $m = 0$ are oscillatory functions of a/M [5,11]. For this reason, in our discussion we only consider QNM frequencies with $m = 0$.

The first occurrence of local extrema in $M\omega_R(a/M)$ for $m = 0$ and different values of l and s is listed in Table II. For $n = n_c$ there is no agreement with the thermodynamic

TABLE II. Dimensionless angular momentum parameter a/M corresponding to the first extremum of $M\omega_R(a/M)$ for Kerr perturbations with $m = 0$. We only consider the two lowest allowed values of l : $l = |s|$, $|s| + 1$. In parentheses we list n_c , the overtone number for which an extremum first appears.

| a/M | | | | |
|-----------------|-----------|-----------|-----------|-----------|
| Perturbation | $l = 0$ | $l = 1$ | $l = 2$ | $l = 3$ |
| Scalar | 0.819 (0) | 0.898 (1) | ... | ... |
| Electromagnetic | | 0.922 (1) | 0.925 (2) | ... |
| Gravitational | | | 0.947 (2) | 0.937 (2) |

TABLE III. BH horizon radius corresponding to the first extremum of $\omega_R(r_+)/\sqrt{|\Lambda|}$ for SAdS BHs. The number in parentheses is n_c .

| $\sqrt{ \Lambda }r_+$ Perturbation | $l = 0$ | $l = 1$ | $l = 2$ | $l = 3$ |
|---------------------------------------|----------|---------|----------|----------|
| Scalar | 0.39 (0) | 0.61(0) | 0.66 (0) | 0.78 (0) |
| Electromagnetic | | | 0.92 (0) | 0.92 (0) |
| Odd gravitational | | | 0.75 (1) | 1.26 (1) |
| Even gravitational | | | 0.73 (0) | 1.04 (0) |

phase transition point predicted by Davies. As in the Reissner-Nordström case, we verified that the agreement would not improve if we considered the next overtone $n = n_c + 1$. For example, for $s = l = m = 0$ and $n = 1$ the first extremum in $M\omega_R(a/M)$ occurs when $a/M = 0.503$.

C. Schwarzschild (anti-)de Sitter black holes

Setting $Q = 0$ in Eq. (1) we get $0 = (9/4)^2 \Lambda M^2 + 3/4$, $r_+ = 3M/2$. A phase transition occurs when $\sqrt{|\Lambda|M} = -2/(3\sqrt{3})$ and $\sqrt{|\Lambda|r_+} = 1/\sqrt{3} \approx 0.577$ for negative cosmological constant (anti-de Sitter space.) For positive Λ (de Sitter space), there is no phase transition.

The perturbation equations for SAdS spacetimes have been worked out in the literature, and QNM frequencies have been studied in many papers (see e.g. [15–18].) We used numerical codes we developed in the past to look for extrema in the QNM frequencies. We have two choices to make the frequency dimensionless:

- (i) We can rescale by the BH mass and compute $M\omega_R(r_+)$. This function has no extrema for SAdS BHs.
- (ii) We can rescale frequencies by the cosmological constant Λ . Then $\omega_R(r_+)/\sqrt{|\Lambda|}$ does have an extre-

um for the values or r_+ listed in Table III: compare with Fig. 1 of Ref. [17]. It is clear from the table that these extrema are strongly dependent on l and s , and that in general they do not agree with Davies' phase transition point. The fact that QNM frequencies *do* have an extremum close to the phase transition point is intriguing, but the physical reason for this extremum is unclear.

III. CONCLUSIONS

In the last few years, remarkable relations between classical and thermodynamic properties of black objects have been uncovered. For instance, a correspondence between classical and thermodynamic instabilities of a large number of black branes conjectured by Gubser and Mitra [19,20] was proved by Reall [21] (see [22] for a review.) Manifestations of this duality are expected to appear in the QNM spectra. Indeed, some indications that phase transitions do show up in the QNM spectrum were provided in specific cases by various authors [23–27]. The main purpose of this Brief Report is to show that the correspondence is not as simple as proposed in [13].

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