

# Yang-Mills vacuum in Coulomb gauge in $D = 2 + 1$ dimensions

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The variational approach to the Hamilton formulation of Yang-Mills theory in Coulomb gauge developed by the present authors previously is applied to Yang-Mills theory in  $2 + 1$  dimensions and is confronted with the existing lattice data. We show that the resulting Dyson-Schwinger equations (DSE) yield consistent solutions in  $2 + 1$  dimensions only for infrared divergent ghost form factor and gluon energy. The obtained numerical solutions of the DSE reproduce the analytic infrared results and are in satisfactory agreement with the existing lattice data in the whole momentum range.

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## I. INTRODUCTION

Recently, there has been renewed interest in the formulation of Yang-Mills theory in Coulomb gauge, both in the continuum theory [1–4] and on the lattice [5–9]. In the continuum formulation the Hamilton approach turns out to be very appealing, in particular, for the description of the confining properties of the theory [3]. The reason is that in Coulomb gauge Gauss' law can be explicitly resolved resulting in a static potential between color charges [10]. Recently, several papers have been devoted to a variational solution of the Yang-Mills Schrödinger equation in Coulomb gauge [2–4]. In particular, in Refs. [3,4] the present authors have developed a variational approach to Yang-Mills theory in Coulomb gauge which properly includes the curvature of the space of (transversal) gauge orbits. Using a Gaussian type of ansatz for the Yang-Mills vacuum wave functional minimization of the energy density results in a set of coupled Dyson-Schwinger equations which can be solved analytically in the infrared [1,11] and has been solved numerically in the full momentum range [3,12] in  $D = 3 + 1$  dimensions. An infrared divergent gluon energy and a linearly rising static quark potential was found [3,12], both signaling confinement. Furthermore, within this approach the spatial 't Hooft loop [13,14] was calculated in the vacuum state and a perimeter law was found [15], which is the behavior expected for this disorder parameter in a confinement phase. In the present paper we apply this approach to  $2 + 1$  dimensions and confront the results with the existing lattice data [9].

The  $2 + 1$  dimensional Yang-Mills theory is interesting in several aspects: When a Higgs field is included, it represents the high temperature limit of the  $3 + 1$  dimensional Yang-Mills theory, thereby the temporal component of the gauge field  $A_0$  becomes the Higgs field in the dimensional reduced theory. In many respects, the  $2 + 1$  dimensional theory is easier to treat than its  $3 + 1$  dimensional counterpart; in particular, much larger lattices can be afforded in  $2 + 1$  dimensions. This will be crucial for a comparison of the continuum results with the lattice data, since the lattices affordable in  $3 + 1$  dimensions are by far

too small to allow for a reliable extraction of the infrared properties of the Greens functions [16,17]. In addition,  $2 + 1$  dimensional Yang-Mills theory is super-renormalizable.

The balance of the paper is as follows: In Sec. II we briefly summarize the essential ingredients of the Hamilton approach to Yang-Mills theory in Coulomb gauge [10] and of the variational solution of the corresponding Yang-Mills Schrödinger equation [3]. In Sec. III the Dyson-Schwinger equations resulting from minimizing the energy density are studied in the ultraviolet and their renormalization is carried out. Some exact statements on their solutions are given in Sec. IV, where we also solve these equations analytically in the infrared. In Sec. V we present our numerical results and compare them with the existing lattice data. A short summary and some concluding remarks are given in Sec. VI.

## II. THE HAMILTON APPROACH TO YANG-MILLS THEORY IN COULOMB GAUGE

Canonical quantization of Yang-Mills theory is usually performed in Weyl gauge  $A_0 = 0$  to avoid the problem arising from the vanishing of the canonical momentum conjugate to  $A_0$ .<sup>1</sup> The prize one pays by choosing Weyl gauge is that one loses Gauss' law as equation of motion. To ensure gauge invariance, one has to impose Gauss' law as a constraint on the wave functional. Instead of using gauge invariant wave functionals [18–20], it is simpler to explicitly resolve Gauss' law by fixing the residual (time-independent) gauge invariance, and Coulomb gauge  $\partial_i A_i = 0$  is a particularly convenient gauge for this purpose. After resolving Gauss' law in Coulomb gauge, the Yang-Mills Hamiltonian reads [10]

$$\begin{aligned}
 H = & \frac{1}{2} \int d^2x [\mathcal{J}^{-1}[A^\perp] \Pi_i^{\perp a}(\mathbf{x}) \mathcal{J}[A^\perp] \Pi_i^{\perp a}(\mathbf{x}) + B_i^a(\mathbf{x})^2] \\
 & + \frac{g^2}{2} \int d^2x \int d^2x' \mathcal{J}^{-1}[A^\perp] \rho^a(\mathbf{x}) \\
 & \times F^{ab}(\mathbf{x}, \mathbf{x}') \mathcal{J}[A^\perp] \rho^b(\mathbf{x}'), \quad (1)
 \end{aligned}$$

<sup>1</sup>An alternative to the Weyl gauge is the light-cone gauge  $A_0 \pm A_1 = 0$ .

where

$$\Pi_i^{\perp a}(\mathbf{x}) = t_{ik}(\mathbf{x}) \frac{\delta}{i\delta A_k^a(\mathbf{x})} \equiv \frac{\delta}{i\delta A_i^{\perp a}(\mathbf{x})} \quad (2)$$

is the canonical momentum operator conjugate to the transverse gauge field  $A^\perp$ . Furthermore

$$\mathcal{J}[A^\perp] = \det(-\partial_i \hat{D}_i(A^\perp)) \quad (3)$$

is the Faddeev-Popov determinant with  $\hat{D}(A) = \partial + g\hat{A}$ ,  $\hat{A} = A^a \hat{T}_a$ ,  $(\hat{T}_a)^{cb} = f^{cab}$  being the covariant derivative in the adjoint representation. Furthermore

$$B = \frac{i}{2g} \epsilon^{ij} [D_i, D_j], \quad D_i = \partial_i + igA_i \quad (4)$$

is the magnetic field, which is a scalar in  $2 + 1$  dimensions ( $\epsilon_{ij} = -\epsilon_{ji}$ ,  $\epsilon_{12} = 1$ ). Finally,

$$F^{ab}(\mathbf{x}, \mathbf{x}') = \langle \mathbf{x}a | (-\hat{D}_i \partial_i)^{-1} (-\partial^2) (-\hat{D}_j \partial_j)^{-1} | \mathbf{x}'b \rangle \quad (5)$$

is the non-Abelian Coulomb propagator which mediates a static interaction between the color charge density of the gluons

$$\rho^a(\mathbf{x}) = -\hat{A}_i^{\perp ab}(\mathbf{x}) \Pi_i^{\perp b}(\mathbf{x}). \quad (6)$$

In the presence of external color charges, for example in the presence of quarks, their charge density has to be added to the gluon charge density. The kinetic term in the Hamilton has the form of a (variational extension of the) Laplace-Beltrami operator in curved space with the Faddeev-Popov determinant (3) corresponding to the determinant of the metric of the space of transversal gauge orbits. The Jacobian (3) also enters the scalar product in the Hilbert space of the Yang-Mills wave functionals in Coulomb gauge

$$\langle \Psi_1 | O | \Psi_2 \rangle = \int \mathcal{D}A^\perp \mathcal{J}[A^\perp] \Psi_1^*[A^\perp] O[A^\perp, \Pi^\perp] \Psi_2[A^\perp]. \quad (7)$$

We will solve the Yang-Mills Schrödinger equation by the variational principle

$$E = \langle \Psi | H | \Psi \rangle \rightarrow \min \quad (8)$$

using the following ansatz for the vacuum wave functional [3,4]:

$$\begin{aligned} \Psi[A^\perp] &= \langle \Psi | A^\perp \rangle \\ &= \mathcal{N} \mathcal{J}[A^\perp]^{-\alpha} \exp \left[ -\frac{1}{2} \right. \\ &\quad \left. \times \int d^2x \int d^2x' A_i^{\perp a}(\mathbf{x}) \omega(\mathbf{x}, \mathbf{x}') A_i^{\perp a}(\mathbf{x}') \right], \quad (9) \end{aligned}$$

where  $\omega(\mathbf{x}, \mathbf{x}')$  is the variational kernel, which by translational and rotational invariance of the vacuum depends only on  $|\mathbf{x} - \mathbf{x}'|$ , and, by the isotropy of color space, is independent of color. The ansatz (9) with  $\alpha = \frac{1}{2}$  is moti-

vated by the wave functional of a particle moving in an  $s$ -state in a spherically symmetric potential. In principle,  $\alpha$  could be used as a variational parameter to minimize the energy density. However, it turns out that, up to two loops in the energy, stationarity of the energy density with respect to  $\omega$ , i.e.  $\delta E / \delta \omega = 0$ , implies also stationarity with respect to  $\alpha$ , i.e.  $dE / d\alpha = 0$  [4]. Thus, we are free to choose  $\alpha$  for convenience, and as in Ref. [3] we will choose  $\alpha = \frac{1}{2}$ , which removes the Faddeev-Popov determinant from the integration measure in Eq. (7). Furthermore with the choice  $\alpha = \frac{1}{2}$  the gluon propagator is given by

$$\langle \Psi | A_i^\perp(x) A_j^\perp(y) | \Psi \rangle = \frac{1}{2} t_{ij}(x) \omega^{-1}(x, y), \quad (10)$$

so that the Fourier transform  $\omega(k)$  has the meaning of the gluon energy.

The calculation of the vacuum expectation value of the Coulomb Hamiltonian in the state (9) proceeds in the same way as in  $3 + 1$ -dimensions and we just quote the result. We find for the kinetic energy

$$E_k = \frac{N_C^2 - 1}{4} \delta^{(2)}(\mathbf{0}) \int d^2k \frac{[\omega(\mathbf{k}) - \chi(\mathbf{k})]^2}{\omega(\mathbf{k})}, \quad (11)$$

the potential energy

$$\begin{aligned} E_p &= \frac{N_C^2 - 1}{4} \delta^{(2)}(\mathbf{0}) \int d^2k \frac{\mathbf{k}^2}{\omega(\mathbf{k})} + \frac{N_C(N_C^2 - 1)}{16} g^2 \delta^{(2)}(\mathbf{0}) \\ &\quad \times \int \frac{d^2k d^2k'}{(2\pi)^2} \frac{1}{\omega(\mathbf{k}) \omega(\mathbf{k}')} \left( 1 - \frac{(\mathbf{k} \mathbf{k}')^2}{\mathbf{k}^2 \mathbf{k}'^2} \right) \quad (12) \end{aligned}$$

and for the Coulomb energy

$$\begin{aligned} E_c &= \frac{N_C(N_C^2 - 1)}{16} \delta^{(2)}(\mathbf{0}) \int \frac{d^2k d^2k'}{(2\pi)^2} \frac{(\mathbf{k} \mathbf{k}')^2}{\mathbf{k}^2 \mathbf{k}'^2} \\ &\quad \times \frac{d(\mathbf{k} - \mathbf{k}')^2 f(\mathbf{k} - \mathbf{k}')}{(\mathbf{k} - \mathbf{k}')^2} \\ &\quad \cdot \frac{([\omega(\mathbf{k}) - \chi(\mathbf{k})] - [\omega(\mathbf{k}') - \chi(\mathbf{k}')])^2}{\omega(\mathbf{k}) \omega(\mathbf{k}')}, \quad (13) \end{aligned}$$

where  $\delta^2(\mathbf{0}) = V / (2\pi)^2$  with  $V = \int d^2x$  being the 2-dimensional spatial volume. Furthermore the quantity

$$\chi(\mathbf{x}, \mathbf{x}') = -\frac{1}{N_C^2 - 1} \left\langle \frac{\delta^2 \ln \mathcal{J}}{\delta A_i^{\perp a}(\mathbf{x}) \delta A_i^{\perp a}(\mathbf{x}')} \right\rangle_\Psi \quad (14)$$

is referred to as ‘‘curvature’’ of the space of gauge orbits. Introducing the ghost propagator by

$$G = \langle \Psi | (-\partial_i \hat{D}_i)^{-1} | \Psi \rangle = (-\partial^2)^{-1} \frac{d(-\partial^2)}{g}, \quad (15)$$

with  $d(k)$  being the ghost form factor, the curvature (14) can be expressed as

$$\chi(\mathbf{k}) = \frac{N_C}{2} \int \frac{d^2q}{(2\pi)^2} \left( 1 - \frac{(\mathbf{k} \mathbf{q})^2}{\mathbf{k}^2 \mathbf{q}^2} \right) \frac{d(\mathbf{k} - \mathbf{q})d(\mathbf{q})}{(\mathbf{k} - \mathbf{q})^2}. \quad (16)$$

Finally,  $f(k)$  denotes the Coulomb form factor, which is

defined by

$$\langle \Psi | (-\partial_i \hat{D}_i)^{-1} (-\partial^2) (-\partial_i \hat{D}_i)^{-1} | \Psi \rangle = G(-\partial^2) f G. \quad (17)$$

A few comments are here in order: Within our variational ansatz (9) for the vacuum wave functional, the Faddeev-Popov determinant drops out from the integration measure in the matrix elements of operators not containing the momentum operator. This is an advantage of our ansatz. Furthermore for expectation values within the Gaussian wave functional Wick's theorem holds exactly. As a consequence expectation values of polynomials of the gauge field can be exactly evaluated by Wick's theorem while further approximations are necessary for the expectation values of nonpolynomial functions like the Faddeev-Popov determinant or its derivative occurring in the kinetic energy and in the Coulomb term. As a consequence, further approximations have to be introduced in the evaluation of the expectation values of these operators. In deriving the above expressions for the expectation value of the Yang-Mills Hamiltonian we have confined ourselves to two loop order. This implies that even within our ansatz for the vacuum wave functional Eqs. (11) and (13) are approximate while Eq. (12) is exact. Minimization of the energy density  $(E_k + E_p + E_c)/V$  results in the so-called gap equation, which can be cast into the form of a dispersion relation of a relativistic particle (gluon)

$$\omega(\mathbf{k})^2 = \mathbf{k}^2 + \chi(\mathbf{k})^2 + I_\omega(\mathbf{k}) + I_\omega^0. \quad (18)$$

Here  $I_\omega^0$  is an irrelevant uv-divergent constant which arises from the gluon tadpole. The latter vanishes in dimensional regularization. Furthermore, the quantity  $I_\omega(k)$ , which arises from the expectation value of the Coulomb term (13), can be expressed as

$$I_\omega(\mathbf{k}) = \frac{N_C}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{(\mathbf{k}\mathbf{q})^2}{\mathbf{k}^2 \mathbf{q}^2} \frac{d(\mathbf{k}-\mathbf{q})^2 f(\mathbf{k}-\mathbf{q})}{(\mathbf{k}-\mathbf{q})^2} \cdot \frac{[\omega(\mathbf{q}) - \chi(\mathbf{q}) + \chi(\mathbf{k})]^2 - \omega(\mathbf{k})^2}{\omega(\mathbf{q})}. \quad (19)$$

Both integrals  $I_\omega(k)$  and  $I_\omega^0$  are uv-divergent. By adding appropriate counterterms to the Hamiltonian, one can remove the divergent parts of these integrals [21]. Furthermore for dimensional reasons the linearly uv-diverging constant  $I_\omega^0 \sim \Lambda$  ( $\Lambda$ -cutoff) does not have a finite part and is thus completely removed after renormalization. In principle  $G$  or  $d$ ,  $\chi$ , and  $f$  are defined uniquely once the trial wave functional  $\Psi(A)$  is fixed. However, the exact evaluation of these expectation values (even with the above chosen trial wave function) is not feasible and one has to resort to further approximations. We shall adopt here the same approximation as in Ref. [3] used in  $3 + 1$  dimensions, approximating the full ghost-gluon vertex by its bare one. This approximation has been justified in Landau gauge [22] and, in fact, has received recently strong support by lattice calculations [23]. In this approximation the

ghost and Coulomb form factors satisfy the following Dyson-Schwinger equations:

$$\frac{1}{d(\mathbf{k})} = 1 - I_d(\mathbf{k}), \quad (20)$$

$$I_d(\mathbf{k}) = \frac{N_C}{2} \int \frac{d^2 q}{(2\pi)^2} \left( 1 - \frac{(\mathbf{k}\mathbf{q})^2}{\mathbf{k}^2 \mathbf{q}^2} \right) \frac{d(\mathbf{k}-\mathbf{q})}{(\mathbf{k}-\mathbf{q})^2 \omega(\mathbf{q})}, \quad (21)$$

$$f(\mathbf{k}) = 1 + I_f(\mathbf{k}), \quad (22)$$

$$I_f(\mathbf{k}) = \frac{N_C}{2} \int \frac{d^2 q}{(2\pi)^2} \left( 1 - \frac{(\mathbf{k}\mathbf{q})^2}{\mathbf{k}^2 \mathbf{q}^2} \right) \frac{d(\mathbf{k}-\mathbf{q})^2 f(\mathbf{k}-\mathbf{q})}{(\mathbf{k}-\mathbf{q})^2 \omega(\mathbf{q})}. \quad (23)$$

As described in Ref. [3] to the considered order (bare ghost-gluon vertex and 2-loop order in energy), the Coulomb form factor can be put  $f(k) = 1$ .

In  $D = 2 + 1$  the coupling constant  $g$  has the dimension of the square root the momentum and it is convenient to use the coupling constant to rescale all quantities by suitable powers of  $g$  to render them dimensionless. The coupling constant then disappears from the Dyson-Schwinger equations. Denoting the dimensionless quantities by a bar, we have

$$\bar{k} = \frac{k}{g^2}, \quad \bar{\omega}(\bar{k}) = \frac{\omega(g^2 \bar{k})}{g^2}, \quad \bar{\chi}(\bar{k}) = \frac{\chi(g^2 \bar{k})}{g^2} \quad (24)$$

$$\bar{d}(\bar{k}) = \frac{d(g^2 \bar{k})}{g}, \quad \bar{f}(\bar{k}) = f(g^2 \bar{k}).$$

In the following we will skip the bar and, unless stated otherwise, all quantities will be understood as the dimensionless ones.

In Sec. IVA we prove that for any solution of the coupled Dyson-Schwinger equations (18) and (20) in  $D = 2 + 1$  the ghost form factor and the gluon energy are infrared divergent

$$d^{-1}(k=0) = 0 \quad (25)$$

$$\omega^{-1}(k \rightarrow 0) = 0. \quad (26)$$

The first relation is the so-called horizon condition which had to be imposed *ad hoc* in  $D = 3 + 1$  but is a strict consequence of the Dyson-Schwinger equations in  $D = 2 + 1$ . The second condition (26) signals gluon confinement. This is again different from  $D = 3 + 1$  where the Dyson-Schwinger equations (DSE) allow for solutions with infrared finite  $\omega(k)$  when the curvature  $\chi(k)$  is ignored.

### III. ULTRAVIOLET BEHAVIOR AND RENORMALIZATION

In the following we will investigate the ultraviolet behavior of the solutions of the Dyson-Schwinger equations

and perform their renormalization, thereby following the procedure presented in Refs. [3,15] for the  $D = 3 + 1$ -dimensional case. Since most of the considerations will parallel the  $3 + 1$ -dimensional case, we will be very brief.

For the uv-analysis it is sufficient to use the angular approximation

$$h(|\mathbf{k} - \mathbf{q}|) = h(k)\Theta(k - q) + h(q)\Theta(q - k). \quad (27)$$

The angular integral in the ghost Dyson-Schwinger equation and also in the curvature then becomes trivial  $\int_0^{2\pi} d\varphi \sin^2 \varphi = \pi$  and we obtain for the corresponding momentum integrals:

$$I_d(k) = \frac{N_C}{8\pi} \left[ \frac{d(k)}{k^2} \int_0^k dq \frac{q}{\omega(q)} + \int_k^\infty dq \frac{d(q)}{q\omega(q)} \right] \quad (28)$$

$$\chi(k) = \frac{N_C}{8\pi} \left[ \frac{d(k)}{k^2} \int_0^k dq q d(q) + \int_k^\Lambda dq \frac{d(q)^2}{q} \right]. \quad (29)$$

The asymptotic analysis of the Dyson-Schwinger equations is simplified by taking the derivative with respect to the external momentum:

$$I'_d(k) = \frac{N_C}{8\pi} \frac{1}{k^2} \left[ d'(k) - 2 \frac{d(k)}{k} \right] \int_0^k dq \frac{q}{\omega(q)} \quad (30)$$

$$\chi'(k) = \frac{N_C}{8\pi} \frac{1}{k^2} \left[ d'(k) - 2 \frac{d(k)}{k} \right] \int_0^k dq q d(q). \quad (31)$$

As we will see below, the remaining momentum integrals are ultraviolet convergent.<sup>2</sup> The derivative of the ghost Dyson-Schwinger equation (20) yields

$$d'(k) \left[ \frac{1}{d(k)^2} - \frac{N_C}{8\pi} \frac{R(k)}{k^2} \right] = - \frac{N_C}{4\pi} \frac{d(k)}{k} \frac{R(k)}{k^2}, \quad (32)$$

where

$$R(k) = \int_0^k dq \frac{q}{\omega(q)} \quad (33)$$

and the derivative of the curvature (16) yields

$$\chi'(k) = \frac{N_C}{8\pi} \frac{1}{k^2} \left[ d'(k) - 2 \frac{d(k)}{k} \right] S(k), \quad (34)$$

where

$$S(k) = \int_0^k dq q d(q). \quad (35)$$

We now discuss the ultraviolet behavior of the relevant quantities.

### A. Ultraviolet behavior

For large momenta the  $k^2$  term on the right-hand side of the gap equation (18) dominates [3] and the gluon energy  $\omega(k)$  behaves like

$$\omega(k) \rightarrow \sqrt{k^2}, \quad k \rightarrow \infty \quad (36)$$

in accordance with asymptotic freedom. This behavior will in fact be confirmed by the numerical solutions presented in Sec. VI. Assuming (36), we will investigate the uv-behavior of the remaining quantities: the ghost form factor  $d(k)$ , the curvature  $\chi(k)$ , and the Coulomb form factor  $f(k)$ .

For large  $k \rightarrow \infty$  the  $k$ -dependence of the integral (33) is independent of the infrared behavior of  $\omega(k)$  and with Eq. (36) we find

$$R(k) = k, \quad k \rightarrow \infty \quad (37)$$

and the derivative of the ghost DSE (32) reduces to

$$\frac{d'(k)}{d(k)^2} \left[ 1 - \frac{N_C}{8\pi} \frac{d(k)^2}{k} \right] = \frac{N_C}{4\pi} \frac{d(k)}{k^2}. \quad (38)$$

To solve this equation, let us assume for the moment  $\frac{d(k)^2}{k} \ll 1$  for  $k \rightarrow \infty$ . Then the differential equation (38) reduces to

$$\frac{d'(k)}{d(k)^2} = - \frac{N_C}{4\pi} \frac{d(k)}{k^2}, \quad (39)$$

whose solution is given by

$$d(k) = \frac{1}{\sqrt{c^2 - \frac{N_C}{8\pi} \frac{1}{k}}}, \quad (40)$$

where  $c$  is an integration constant. Indeed, for large  $k \rightarrow \infty$  this solution satisfies  $\frac{d(k)^2}{k} \ll 1$ . To determine the integration constant  $c$ , we consider the asymptotic behavior of the integral  $I_d(k)$  (28) for  $k \rightarrow \infty$ , where  $\omega(k) \simeq k$  and  $d(k) \simeq c$ . This yields

$$I_d(k) \simeq \frac{N_C}{8\pi} \left[ \frac{c}{k^2} \int_0^k dq \frac{q}{\omega(q)} + c \int_k^\infty dq \frac{1}{q\omega(q)} \right] \simeq \frac{N_C}{4\pi} \frac{c}{k}. \quad (41)$$

Since  $I_d(k \rightarrow \infty) \rightarrow 0$ , we obtain from the DSE of the ghost form factor  $d(k \rightarrow \infty) \rightarrow 1$ , which fixes the integration constant in Eq. (40) to  $c = 1$ , so that the asymptotic form of the ghost form factor becomes

$$d(k) = \frac{1}{\sqrt{1 - \frac{N_C}{8\pi} \frac{1}{k}}}, \quad k \rightarrow \infty. \quad (42)$$

Accordingly we find for the unscaled form factor<sup>3</sup>  $\tilde{d}(\tilde{k}) = g d(\frac{\tilde{k}}{g})$  the asymptotic form

$$\tilde{d}(\tilde{k}) = \frac{g}{\sqrt{1 - \frac{N_C}{8\pi} \frac{g^2}{\tilde{k}}}}, \quad \tilde{k} \rightarrow \infty. \quad (43)$$

From the Swift relation [3,24]

<sup>2</sup>The ghost integral  $I_d(k)$  is uv-finite as will be shown later.

<sup>3</sup>Below we denote the dimensionful (unscaled) quantities by a "tilde."

$$\tilde{f}(\tilde{k}) = -\frac{1}{g^2} \frac{\partial}{\partial g} \frac{1}{\tilde{d}(\tilde{k})}, \quad (44)$$

we find for the Coulomb form factor

$$f(k) = \frac{1}{\sqrt{1 - \frac{N_C}{8\pi} \frac{1}{k}}} = d(k), \quad k \rightarrow \infty, \quad (45)$$

which obviously has the same asymptotic form as the ghost form factor.

With the asymptotic behavior of the ghost (42), we find for the derivative of the curvature (34)

$$\chi'(k) = -\frac{N_C}{4\pi} \frac{1}{k^3} S(k), \quad k \rightarrow \infty \quad (46)$$

and for the integral  $S(k)$  (35)

$$S(k \rightarrow \infty) \rightarrow \frac{k^2}{2}, \quad (47)$$

so that

$$\chi'(k) = \frac{N_C}{8\pi} \frac{1}{k}, \quad k \rightarrow \infty, \quad (48)$$

i.e.

$$\chi(k) \sim \ln\left(\frac{k}{\mu}\right), \quad k \rightarrow \infty. \quad (49)$$

Accordingly, the ratio

$$\frac{\chi(k)}{\omega(k)} \sim \frac{1}{k} \ln\left(\frac{k}{\mu}\right) \xrightarrow{k \rightarrow \infty} 0 \quad (50)$$

vanishes in the ultraviolet implying that the space of gauge orbits becomes asymptotically flat in accordance with asymptotic freedom.

## B. Renormalization

With the above obtained uv-behavior [see Eqs. (36), (42), and (45)], the integrals  $I_d(k)$  (21) and  $I_f(k)$  (23) are uv-convergent. Thus, contrary to the  $D = 3 + 1$  dimensional case in  $D = 2 + 1$ , the DSE for the ghost and Coulomb form factors do not need renormalization. What needs, however, renormalization is the curvature  $\chi(k)$  and the gap equation. The renormalization of these two equations is carried out in exactly the same way as described in Refs. [3, 15], i.e. basically by subtracting these equations at a renormalization point  $\mu$ , which leads to the renormalized equations [15]

$$\chi(k) = \chi(\mu) + \bar{\chi}(k), \quad \bar{\chi}(k) = I_\chi(k) - I_\chi(\mu) \quad (51)$$

$$\begin{aligned} \omega^2(k) - \bar{\chi}^2(k) &= k^2 + \xi_0 + \Delta I_\omega^{(2)}(k) + 2\bar{\chi}(k)(\xi \\ &+ \Delta I_\omega^{(1)}(k)), \end{aligned} \quad (52)$$

where we have introduced the abbreviation

$$\xi_0 = \omega^2(\mu) - \mu^2 \quad \xi = \chi(\mu) + I_\omega^{(1)}(\mu) \quad (53)$$

$$\Delta I_\omega^{(n)}(k) = I_\omega^{(n)}(k) - I_\omega^{(n)}(\mu) \quad (54)$$

with

$$\begin{aligned} I_\omega^{(n)}(k, \Lambda) &= \frac{N_C}{2} \int^\Lambda \frac{d^2 q}{(2\pi)^2} (\hat{\mathbf{k}} \hat{\mathbf{q}})^2 \cdot \frac{d(\mathbf{k} - \mathbf{q})^2 f(\mathbf{k} - \mathbf{q})}{(\mathbf{k} - \mathbf{q})^2} \\ &\cdot \frac{[\omega(\mathbf{q}) - \bar{\chi}(\mathbf{q})]^n - [\omega(\mathbf{k}) - \bar{\chi}(\mathbf{k})]^n}{\omega(\mathbf{q})}. \end{aligned} \quad (55)$$

Here  $\chi(\mu)$ ,  $\omega(\mu)$ , and  $\xi$  have to be considered as finite renormalization constant. Furthermore in  $D = 2 + 1$  where the coupling constant is dimensionful, there is thus no arbitrary scale and  $\mu$  has to be considered as renormalization parameter as well. Note, since the ghost equation needs no renormalization, there are only four renormalization constants  $\mu$ ,  $\xi_0$ ,  $\chi(\mu)$ , and  $\xi$  and furthermore, the solutions of the coupled ghost and gluon DSEs, Eqs. (20) and (52), do not depend on  $\chi(\mu)$ .<sup>4</sup> [Note the integrals (55) depend only on the finite quantity  $\bar{\chi}(k)$  but not on the renormalization constant  $\chi(\mu)$  [15].] Therefore the coupled DSEs contain only three independent renormalization parameters  $\mu$ ,  $\xi_0$ , and  $\xi$ . One of the three independent renormalization constants,  $\xi_0$ , is used to implement the horizon condition  $d^{-1}(k \rightarrow 0) = 0$ . As will be shown in the next section any consistent solution of the coupled DSEs does satisfy this condition. The remaining two renormalization constants  $\mu$  and  $\xi$  can be chosen at will and determines the infrared limit of the wave functional as will be shown at the end of the next section [see Eq. (88)].

## IV. ANALYTIC RESULTS

The Dyson-Schwinger equations arising from the variational solution of the Schrödinger equation have in principle the same form in  $D = 2 + 1$  as in  $D = 3 + 1$  dimensions. However, due to the fact that the  $2 + 1$  dimensional theory is super-renormalizable some rigorous properties of the solution of the Schwinger-Dyson equations can be derived which are not accessible in  $3 + 1$  dimensions. Below we shall derive some rigorous properties of the solutions of the DSE and discuss their physical implications.

### A. General results

Consider the ghost DSE (20). Since the integral  $I_d(k)$  (21) is convergent in  $D = 2 + 1$  (unlike in  $D = 3 + 1$ ), the ghost DSE needs no renormalization and thus no renormalization constant is introduced by this equation. It is then not difficult to prove the following statement.

If  $d(k)$  is a continuous function in  $k \in [0, \infty)$ , it satisfies

$$d(k) \geq 1. \quad (56)$$

<sup>4</sup>Some observables like the 't Hooft loop (in  $D = 3 + 1$ ) do, however, depend on  $\chi(\mu)$  [15].

We prove the statement by *reductio ad absurdum*: Assume  $d(k) < 0$  for all  $k \in [0, \infty)$ . Then, since  $\omega(k) > 0$  by normalizability of the wave functional it follows from (21)  $I_d(k) \leq 0$  and thus from the ghost DSE (20)

$$\frac{1}{1 - I_d(k)} = d(k) > 0 \quad (57)$$

in contradiction to the assumption. Hence,  $d(k)$  cannot be negative everywhere. Assume now there exists some momentum  $k'$  for which  $d(k') < 0$ . Since  $d(k)$  as a solution of the DSE can be assumed to be continuous and as shown above is not everywhere negative, it must have at least one zero, say  $k_0$ , i.e.  $d(k_0) = 0$ , where  $d(k)$  changes sign. By the ghost DSE, Eq. (20)  $I_d(k)$  has to be singular at  $k \rightarrow k_0$  and change sign at  $k = k_0$ , too. However, changing integration variable in the (convergent) integral  $I_d(k)$  (21) we find

$$I_d(k_0 \pm \epsilon k_0) = \frac{N_C}{2} \int \frac{d^2 q}{(2\pi)^2} \left( 1 - \frac{(\mathbf{k}_0 \cdot (\mathbf{q} \pm \epsilon \mathbf{k}_0))^2}{k_0^2 (\mathbf{q} \pm \epsilon \mathbf{k}_0)^2} \right) \times \frac{d(\mathbf{k}_0 - \mathbf{q})}{(\mathbf{k}_0 - \mathbf{q})^2 \omega(\mathbf{q} - \epsilon \mathbf{k}_0)} \quad (58)$$

and by the positivity of  $\omega(k)$ ,  $I_d(k_0 + \epsilon k_0)$ , and  $I_d(k_0 - \epsilon k_0)$  have the same sign in contradiction to the above assumption. Thus,  $d(k) \geq 0$  holds for all  $k$ . Then from Eq. (21) follows  $I_d(k) \geq 0$  and by the ghost DSE (20)  $d(k) = (1 - I_d(k))^{-1} \geq 1$ . Note, that (56) is also in agreement with the asymptotic uv-behavior found in the previous section.

Assume now that the ghost form factor is bounded from above, i.e. there exist some upperbound  $M > 1$  such that  $d(k) \leq M$  for  $\forall k \in [0, \infty)$ . According to (56)  $d(k)$  is then restricted to the interval  $1 \leq d(k) \leq M$  and the integrand in  $\chi(k)$  (16) is positive definite. Therefore, replacing  $d(k)$  in the curvature (16) by its upper and lower bound, we obtain an upper and lower bound to  $\chi(k)$ :

$$M^2 I \geq \chi(k) \geq I, \quad (59)$$

$$I = \frac{N_C}{2} \int^\Lambda \frac{d^2 q}{(2\pi)^2} \left( 1 - \frac{(\mathbf{k}\mathbf{q})^2}{\mathbf{k}^2 \mathbf{q}^2} \right) \frac{1}{(\mathbf{k} - \mathbf{q})^2},$$

where we have introduced a momentum cutoff  $\Lambda$  since the integral is uv-divergent. The angular integral can be done analytically<sup>5</sup>

$$\int_0^\pi d\varphi \frac{\sin^2 \varphi}{k^2 + q^2 - 2kq \cos \varphi} = \frac{\pi}{2} \begin{cases} \frac{1}{k^2}, & q \leq k \\ \frac{1}{q^2}, & k \leq q \end{cases} \quad (60)$$

yielding

<sup>5</sup>Note that this exact relation is also reproduced by the angular approximation.

$$I = \frac{N_C}{8\pi^2} \left[ \int_0^k dq q \frac{\pi}{k^2} + \int_k^\Lambda dq q \frac{\pi}{q^2} \right] = \frac{N_C}{8\pi} \left[ \frac{1}{2} + \ln \frac{\Lambda}{k} \right]. \quad (61)$$

Inserting this result into Eq. (59), we find for  $\bar{\chi}(k) = \chi(k) - \chi(\mu)$  the bounds

$$-\frac{N_C}{8\pi} M^2 \ln \frac{k}{\mu} \geq \bar{\chi}(k) \geq -\frac{N_C}{8\pi} \ln \frac{k}{\mu}. \quad (62)$$

Note that this relation holds for all  $k$ . In particular, this relation shows that  $\bar{\chi}(k)$  is infrared divergent

$$\bar{\chi}(k) \sim -\ln \frac{k}{\mu}, \quad k \rightarrow 0. \quad (63)$$

For an infrared divergent  $\bar{\chi}(k)$ , the gap equation (51) reduces in the infrared to (see also Ref. [15])

$$\lim_{k \rightarrow 0} \omega(k) - \bar{\chi}(k) = \xi, \quad (64)$$

where  $\xi$  is the renormalization constant introduced in Eq. (53). Equation (64) shows that  $\omega(k)$  has the same infrared behavior as  $\bar{\chi}(k)$ . Thus, if the ghost form factor is bounded  $1 \leq d(k) \leq M$ , the gluon energy is logarithmically infrared divergent  $\omega(k) \sim -\ln \frac{k}{\mu}$ . It is now not difficult to show that with such an infrared behavior of  $\omega(k)$  the ghost Dyson-Schwinger equation does not possess a solution. To show this let us assume that there exist an  $\epsilon > 0$  such that

$$\omega(k) \leq a \left( -\ln \frac{k}{\mu} \right), \quad \forall k \in (0, \epsilon), \quad (65)$$

with some positive constant  $a$ , which includes the case (63). Consider the integral in the ghost Dyson-Schwinger equation (21):

$$I_d(k) = \frac{N_C}{8\pi^2} \int_0^\infty dq \frac{q}{\omega(q)} \int_0^{2\pi} d\varphi \sin^2 \varphi \cdot \frac{d(\mathbf{k} - \mathbf{q})}{(\mathbf{k} - \mathbf{q})^2}. \quad (66)$$

Since  $d(k) \geq 1$ , see Eq. (56), we obtain the following estimate:

$$I_d(k) \geq \frac{N_C}{8\pi^2} \int_0^\infty dq \frac{q}{\omega(q)} \int_0^{2\pi} d\varphi \sin^2 \varphi \cdot \frac{1}{(\mathbf{k} - \mathbf{q})^2} \geq \frac{N_C}{8\pi^2} \int_0^\epsilon dq \frac{q}{\omega(q)} \int_0^{2\pi} d\varphi \sin^2 \varphi \cdot \frac{1}{(\mathbf{k} - \mathbf{q})^2}. \quad (67)$$

The angular integral can be done exactly using Eq. (60) resulting in

$$I_d(k) \geq \frac{N_C}{8\pi^2} \left[ \int_0^k dq \frac{q}{\omega(q)} \frac{\pi}{k^2} + \int_k^\epsilon dq \frac{q}{\omega(q)} \frac{\pi}{q^2} \right] \geq \frac{N_C}{8\pi} \int_k^\epsilon dq \frac{1}{q\omega(q)}. \quad (68)$$

Inserting here Eq. (65) we obtain

$$I_d(k) \geq \frac{N_C}{8\pi} \int_k^\epsilon dq \frac{1}{qa(-\ln \frac{q}{\mu})} = \frac{N_C}{8\pi a} \ln \left| \frac{\ln \frac{k}{\mu}}{\ln \frac{\epsilon}{\mu}} \right| \xrightarrow{k \rightarrow 0} \infty. \quad (69)$$

Thus we find

$$I_d(k) \xrightarrow{k \rightarrow 0} \infty, \quad (70)$$

and from the ghost Dyson-Schwinger equation (20) follows  $d(k \rightarrow 0) = 0$ , which is in contradiction to the rigorous result (56). Thus, we have shown that the coupled ghost and gluon Dyson-Schwinger equations do not allow for a ghost form factor which is bounded from above. We now show that ghost form factor  $d(k)$  is a monotonously decreasing function of  $k$ , i.e.

$$d'(k) < 0 \quad (71)$$

for all finite  $k$ .

As shown above in the ultraviolet analysis  $d'(k) < 0$  for  $k \rightarrow \infty$ . Assume now, as we lower  $k$ , at some finite  $k = k_0$ ,  $d'(k = k_0) = 0$ . Then from the Eq. (30) follows  $I'_d(k_0) < 0$ , which is in contradiction to the ghost DSE from which follows

$$\frac{d'(k)}{d^2(k)} = I'_d(k). \quad (72)$$

Thus (71) holds in the whole momentum range. Since (56)  $d(k) \geq 1$  and  $d(k)$  is monotonously decreasing in the whole momentum range  $k \geq 0$  and furthermore  $d(k)$  must not be bounded from above, it follows that  $d(k)$  is infrared divergent, i.e.

$$d^{-1}(k = 0) = 0, \quad (73)$$

which is the horizon condition.

This is different from the  $3 + 1$  dimensional case where solutions to the Dyson-Schwinger equation exist with an infrared finite ghost form factor.

## B. Infrared analysis

The DSE can be solved analytically in the infrared completely analogous to the  $D = 3 + 1$  dimensional case. For this purpose we make the following power ansätze in the infrared:

$$\omega(k) = \frac{A}{k^\alpha}, \quad d(k) = \frac{B}{k^\beta}, \quad \chi(k) = \frac{C}{k^\gamma}. \quad (74)$$

We will first resort to the angular approximation used already in the uv-analysis. Later on we will present the results obtained without resorting to the angular approximation.

With the infrared ansätze (74), we find for the integrals defined by Eqs. (33) and (35)

$$R(k \rightarrow 0) = \frac{1}{A} \frac{1}{1 + \alpha} k^{2+\alpha}, \quad S(k \rightarrow 0) = \frac{B}{1 - \beta} k^{2-\beta}. \quad (75)$$

From the derivative of the ghost DSE (72) we obtain

$$\frac{A}{B^2} = \frac{N_C}{8\pi} \frac{\beta + 2}{\beta(\alpha + 2)} k^{\alpha-2\beta}, \quad (76)$$

which implies

$$\alpha = 2\beta \quad (77)$$

and

$$\frac{A}{B^2} = \frac{N_C}{8\pi} \frac{\beta + 2}{2\beta(\beta + 1)}. \quad (78)$$

In an analogous fashion we obtain from the derivative of the curvature  $\chi'(k) = I'_\chi(k)$

$$\frac{C}{B^2} = \frac{N_C}{8\pi} \frac{\beta + 2}{\gamma(2 - \beta)} k^{\gamma-2\beta} \quad (79)$$

resulting in

$$\gamma = 2\beta \quad (80)$$

and

$$\frac{C}{B^2} = \frac{N_C}{8\pi} \frac{\beta + 2}{2\beta(2 - \beta)}. \quad (81)$$

From Eqs. (77) and (80) follows

$$\alpha = \gamma = 2\beta, \quad (82)$$

showing that  $\omega(k)$  and  $\chi(k)$  have the same infrared exponents just like in the  $D = 3 + 1$  dimensional case (Ref. [3]). In fact,  $\alpha = \gamma$  follows already from the infrared limit of the gap equation (64). From this equation in addition follows that not only the infrared exponents but also the prefactors of both quantities have to coincide, i.e.

$$A = C. \quad (83)$$

Dividing Eq. (78) by (81) and using (82), we obtain

$$\frac{A}{C} = \frac{(2 - \beta)}{(1 + \beta)} \quad (84)$$

and  $A = C$

$$\beta = \frac{1}{2} \quad \alpha = \gamma = 1. \quad (85)$$

As we will see in Sec. VI, this infrared behavior yields a linearly rising static color Coulomb potential provided we approximate the Coulomb form factor  $f(k)$  by its leading term  $f(k \rightarrow 0) = 1$  which is correct to the order considered in the present paper.

The infrared analysis of the Dyson-Schwinger equations can be also carried out without resorting to the angular approximation. In fact, in Ref. [11] the infrared analysis was carried out for arbitrary dimensions. In that case one finds from the ghost DSE the following sum rule for the

infrared exponents:

$$\alpha = 2\beta + d - 2 \quad (86)$$

due to the nonrenormalization of the ghost-gluon vertex. As shown in Ref. [11], this sum rule guarantees that  $\chi(x)$  and  $\omega(k)$  have the same infrared exponent  $\alpha = \gamma$  as already found above in the angular approximation, and in agreement with the infrared limit of the gap equation (64). Equation (64) together with the ghost DSE in the infrared limit can be solved analytically for the infrared exponents yielding [11]

$$\beta = 0.4 \quad \text{i.e.,} \quad \alpha = \gamma = 0.8, \quad (87)$$

which is somewhat smaller than the infrared exponent found above in the angular approximation (85).

The above given infrared analysis is independent of the so far unfixed renormalization constants  $\xi$  and  $\chi(\mu)$ . [Recall that the coupled set of DSEs (20), (51), and (52) do not depend on  $\chi(\mu)$ .] From Eq. (64) we obtain ( $\chi(k) = \bar{\chi}(k) + \chi(\mu)$ )

$$\lim_{k \rightarrow 0} (\omega(k) - \chi(k)) = c, \quad c = \xi - \chi(\mu). \quad (88)$$

If one uses the infrared expressions for  $\omega(k)$ ,  $d(k)$ ,  $\chi(k)$  [and  $f(k) = 1$ ] defined by Eq. (74), one finds that the energy density is minimized for  $c = 0$  (see Ref. [15]). Using the representation [4]

$$\det J(a) = \exp\left(-\int A \chi A\right), \quad (89)$$

which is correct to the order considered in the present paper,  $c = 0$  implies an infrared limit of the wave functional

$$\Psi(A) = \text{const} \prod_k \Psi(k), \quad \Psi(k \rightarrow 0) = 1. \quad (90)$$

This wave functional describes a stochastic vacuum, where the infrared modes of the gauge field are completely unconstrained.

## V. NUMERICAL RESULTS

The coupled DSEs (16), (20), (22), and (52) were solved numerically in the whole momentum range as described in Refs. [3, 12]. The renormalization constant  $\xi_0$  was fixed by implementing the horizon condition  $d^{-1}(k=0) = 0$ . Furthermore, like in the  $D = 3 + 1$  dimensional case in order to stay consistently in 1-loop approximation we have solved the equation for the Coulomb form factor  $f(k)$  by assuming a bare ghost form factor  $d(k) = 1$  in the DSE for  $f(k)$ .

The numerical results obtained are presented in Figs. 1 and 2. Figure 1 shows the gluon energy  $\omega(k)$  and the curvature  $\chi(k)$ . Both quantities are infrared divergent and approach each other for  $k \rightarrow 0$  in agreement with our infrared analysis given in the previous section. Figure 2

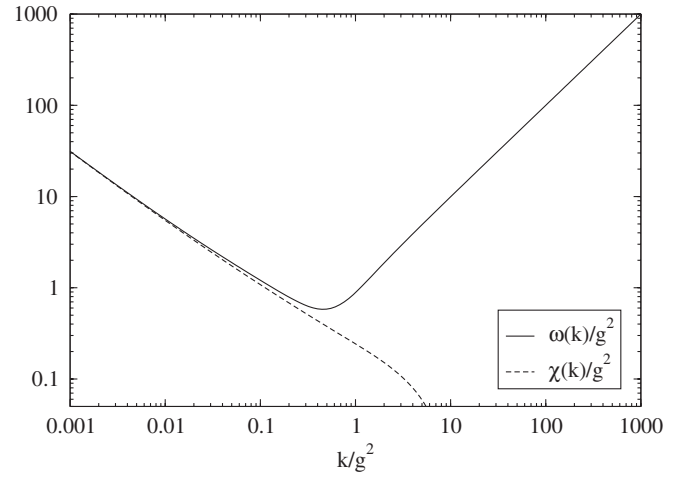


FIG. 1. The gluon energy  $\omega(k)$  and the curvature  $\chi(k)$  obtained from the numerical solution of the DSEs for  $\xi = 1.0g^2$  and  $\mu = 0.322g^2$ .

shows the ghost and the Coulomb form factor. The ghost form factor is of course infrared divergent, since as shown in Sec. IV, the self-consistent solution of the Dyson-Schwinger equations exists only for infrared divergent ghost form factors. The Coulomb form factor is infrared finite and approaches asymptotically for  $k \rightarrow \infty$  the ghost form factor.

In the following we investigate the dependence of the self-consistent solutions of the Dyson-Schwinger equations on the remaining renormalization constants  $\xi$  and  $\mu$ . In Figs. 3 and 4 we show the dependence of  $\omega(k)$  and  $d(k)$  on the choice of the renormalization point  $\mu$ . Finally in Figs. 5 and 6 we show the changes of  $\omega(k)$  and  $d(k)$  with respect to variation of the renormalization constant  $\xi$  in the interval  $[0.5, 1.5]g^2$  for the choice  $\mu = 0.322g^2$ . As can be seen from all these figures, variation of the renormalization

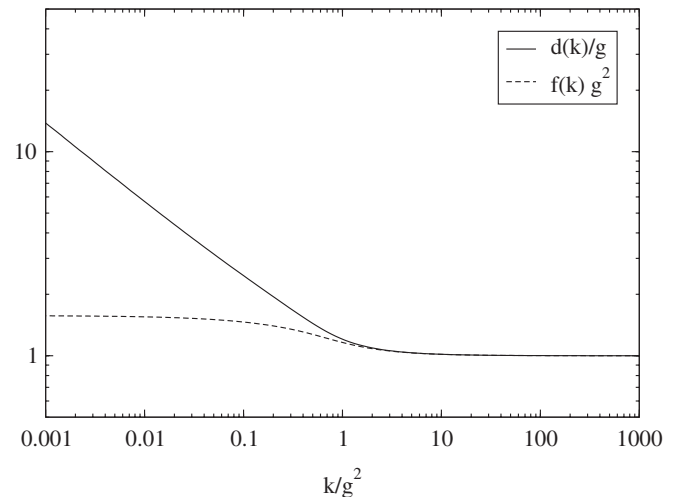
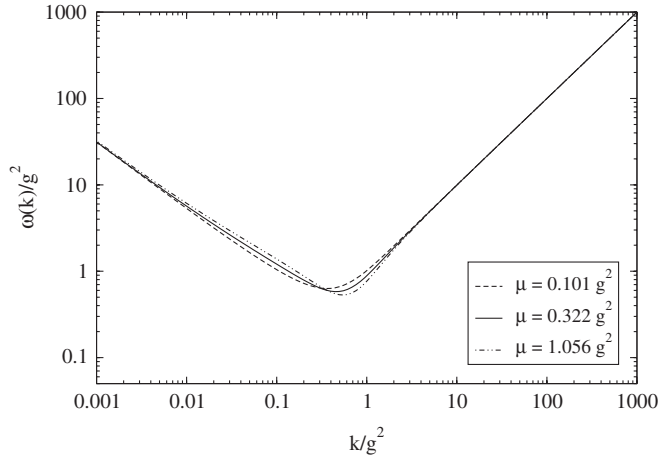


FIG. 2. The ghost form factor  $d(k)$  and the Coulomb form factor  $f(k)$  for  $\xi = 1.0g^2$  and  $\mu = 0.322g^2$ .



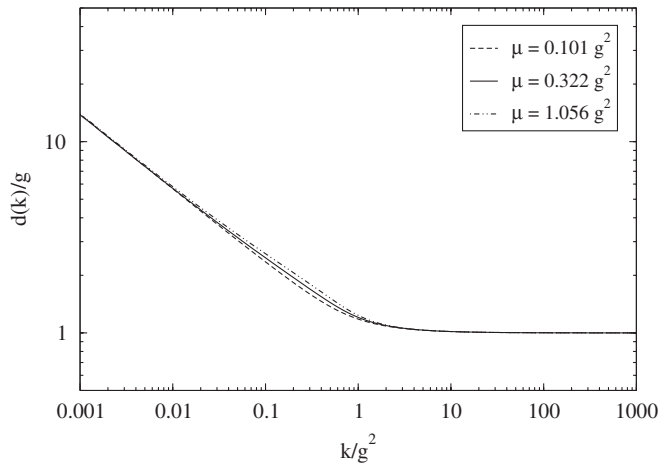
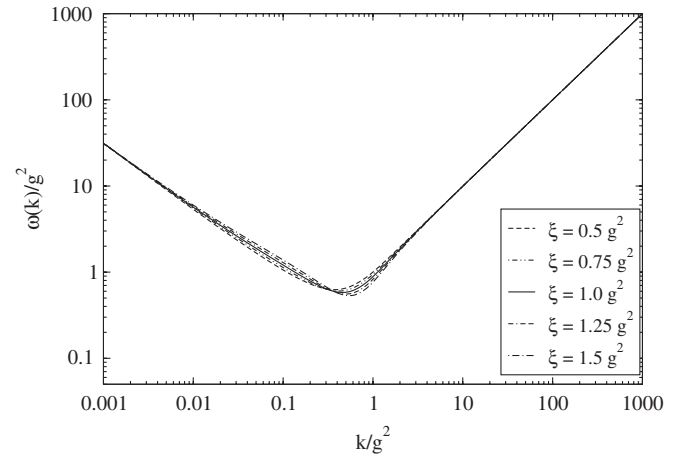

 FIG. 3. The gluon energy  $\omega(k)$  for  $\xi = 1.0g^2$ .

parameters in the considered interval changes the behavior of the solutions only in the intermediate momentum range while the solutions do not change in the infrared and the ultraviolet momentum regime.

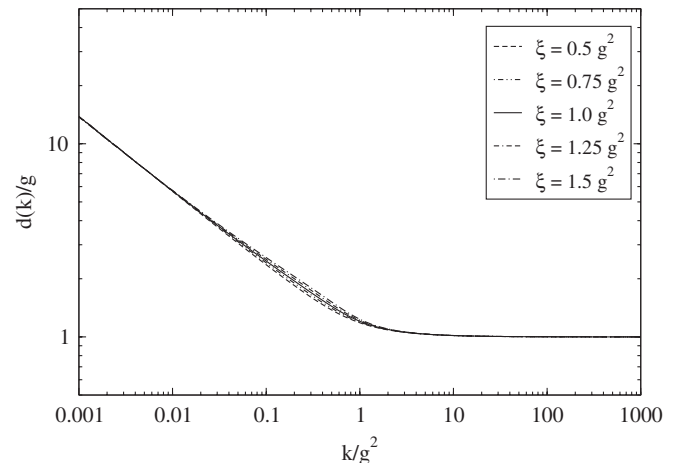
Figure 7 shows the static color Coulomb potential defined by [3]

$$\begin{aligned} V(\mathbf{r}) &= \int \frac{d^2k}{(2\pi)^2} V(k) e^{i\mathbf{k}\mathbf{r}} = \int \frac{d^2k}{(2\pi)^2} \frac{d(k)^2 f(k)}{k^2} e^{i\mathbf{k}\mathbf{r}} \\ &= \frac{1}{2\pi} \int_0^\infty dk \frac{d(k)^2 f(k)}{k^2} J_0(kr), \end{aligned} \quad (91)$$

where  $J_0(k)$  is the (ordinary) zeroth order Bessel function. For the infrared behavior obtained in the angular approximation  $\beta = 1/2$ , see Eq. (74),  $V(k) \sim 1/k^3$ , which leads to a strictly linearly rising quark potential at large distances, while for  $k \rightarrow \infty$ , where  $d(k \rightarrow \infty) = f(k \rightarrow \infty) = 1$  the potential behaves as  $V(k) \sim 1/k^2$  and we obtain the familiar Coulomb potential  $V(r) \sim \ln r/r_0$  in  $D = 2 + 1$  dimen-


 FIG. 4. The ghost form factor  $d(k)$  for  $\xi = 1.0g^2$ .

 FIG. 5. The gluon energy  $\omega(k)$  for  $\mu = 0.322g^2$ .

sions. The infrared analysis carried out without resorting to the angular approximation yields  $V(k) \sim 1/k^{2.8}$ ,  $k \rightarrow 0$ . A careful analysis of our numerical solutions (obtained without the angular approximation) yields  $V(k) \sim 1/k^{2.9}$ ,  $k \rightarrow 0$ , which is in between the analytical results obtained with and without the angular approximation. In Figs. 8–10, we compare our numerical results for the ghost form factor  $d(k)$ , the gluon energy  $\omega(k)$ , and the form factor of the Coulomb potential  $d^2(k)f(k)$  with the lattice data. It has been shown by solving the Dyson-Schwinger equation in Landau gauge on the torus [16,17], that very large lattices are required to capture the correct infrared behavior of the Green's functions of the continuum theory. While these lattice sizes can be reached in 3 dimensions, they are out of reach in  $D = 4$ . The lattice calculations performed in Ref. [9] in  $2 + 1$  dimensions used lattices of the size  $64^3$ , which should be sufficient to extract the correct infrared limit of the Greens functions. Our numerical results obtained by solving the DSE in Coulomb gauge are in quite satisfactory agreement with the lattice data. In particular,


 FIG. 6. The ghost form factor  $d(k)$  for  $\mu = 0.322g^2$ .

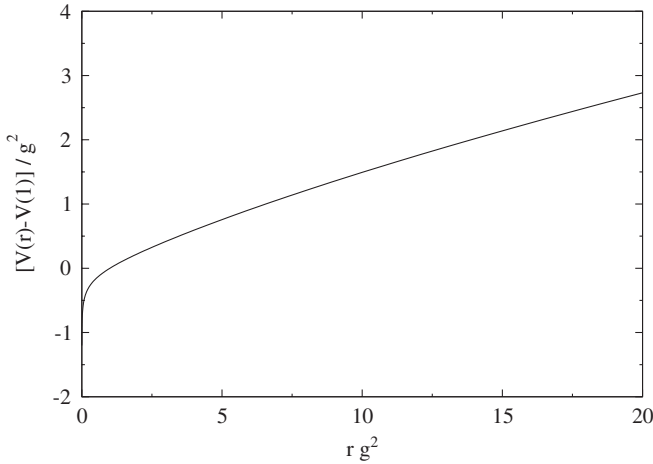


FIG. 7. The static non-Abelian Coulomb potential for  $\xi = 1.0g^2$  and  $\mu = 0.322g^2$ .

the asymptotic ultraviolet and infrared behaviors are quite well reproduced.

### VI. SUMMARY AND CONCLUSIONS

We have performed a variational solution of the Yang-Mills Schrödinger equation in Coulomb gauge in  $D = 2 + 1$ . The Dyson-Schwinger equations resulting from the minimization of the vacuum energy density have been solved analytically in the ultraviolet and in the infrared and in addition some rigorous results of their properties have been derived. In particular, we have shown that the

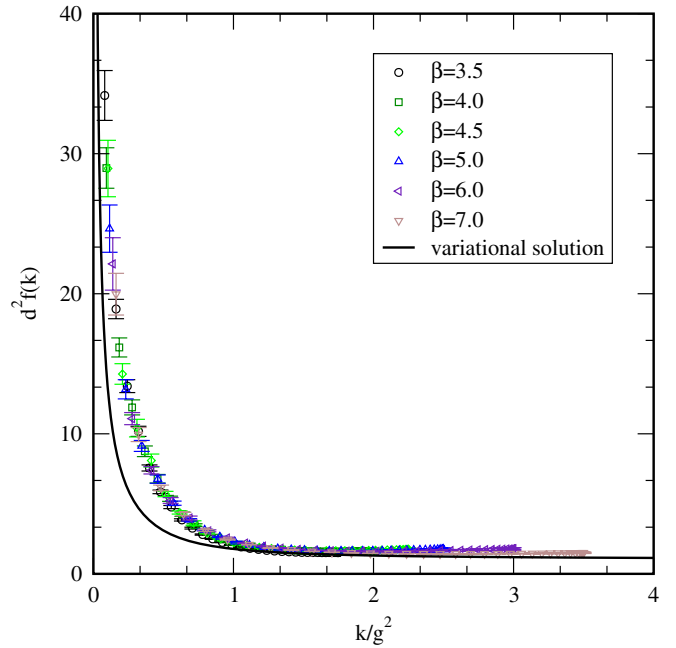


FIG. 10 (color online). The Fourier transform of the static potential obtained from the numerical solution of the DSEs for  $\xi = 1.0g^2$  and  $\mu = 0.322g^2$  and the corresponding lattice data obtained in [9].

ghost form factor as well as the gluon energy have to be infrared divergent, which is different from the  $3 + 1$  dimensional case where solutions of the DSEs exist with these quantities being infrared finite [21]. The static non-

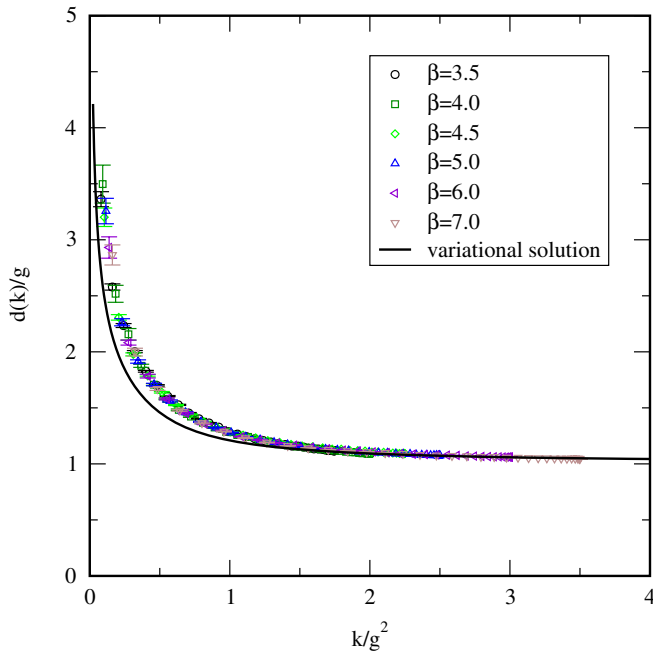


FIG. 8 (color online). Comparison of the ghost form factor  $d(k)$  obtained from the numerical solution of the DSEs for  $\xi = 1.0g^2$  and  $\mu = 0.322g^2$  with the lattice data obtained in [9].

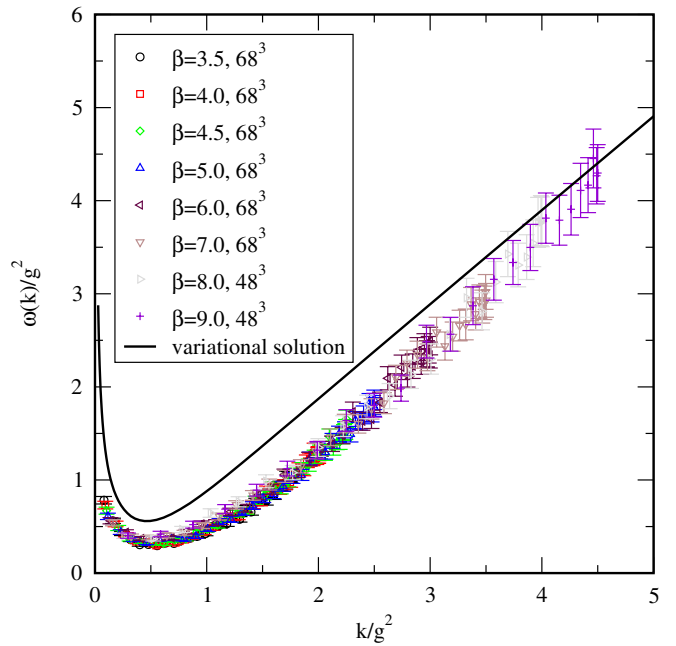


FIG. 9 (color online). The gluon energy  $\omega(k)$  obtained from the numerical solutions of the DSEs for  $\xi = 1.0g^2$  and  $\mu = 0.322g^2$  and the corresponding lattice data obtained in [9].

Abelian Coulomb potential resulting from our numerical solution of the DSE is almost linearly rising. Our numerical results are in satisfactory agreement with the existing lattice data. The lattice calculations performed in  $D = 3 + 1$  so far use too small lattices to give reliable results for the continuum limit, in particular, on the infrared properties of the various Green's functions [16,17].

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